Hopping from Chebyshev polynomials to permutation statistics

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December 31, 2018

Abstract

We prove various formulas which express exponential generating functions counting permutations by the peak number, valley number, double ascent number, and double descent number statistics in terms of the exponential generating function for Chebyshev polynomials, as well as cyclic analogues of these formulas for derangements. We give several applications of these results, including formulas for the (-1)-evaluation of some of these distributions. Our proofs are combinatorial and involve the use of monomino-domino tilings, the modified Foata–Strehl action (a.k.a. valley-hopping), and a cyclic analogue of this action due to Sun and Wang.

Keywords: permutation statistics, Chebyshev polynomials, Pell numbers, modified Foata–Strehl action, derangements

1. Introduction

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation (written in one-line notation) in $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$. We say that π_i (where $i \in [n-1]$) is a *descent* if $\pi_i > \pi_{i+1}$, and that π_i (where $2 \le i \le n-1$) is a *peak* of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$. Define $\mathrm{des}(\pi)$ to be the number of descents of π and $\mathrm{pk}(\pi)$ to be the number of peaks of π. The descent number des and peak number pk are classical permutation statistics whose study dates back to MacMahon [8] and to David and Barton [5], respectively.

The nth *Eulerian polynomial*[1]

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$$

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[1]We note that many works instead define the nth Eulerian polynomial to be $\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1}$.

encodes the distribution of the descent number des over $\mathfrak{S}_n$, and the nth *peak polynomial*

$$P_n^{\mathrm{pk}}(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{pk}(\pi)}$$

is the analogous polynomial for the peak number pk.

It is well-known [6, Théorème 5.6] that the (-1)-evaluation of the Eulerian distribution is given by the formula

$$A_n(-1) = \begin{cases} (-1)^{(n-1)/2} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even,} \end{cases} \tag{1}$$

where E_n is the nth *Euler number* defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec(x) + \tan(x).$$

(The Euler numbers E_n for odd n are called *tangent numbers*, and those for even n are called *secant numbers*.) No combinatorial formula for $P_n^{\mathrm{pk}}(-1)$ is known, although this sequence does appear on the OEIS [13, A006673]. The first several terms of this sequence are given in the following table:

n	1	2	3	4	5	6	7	8	9	10
$P_n^{\mathrm{pk}}(-1)$	1	2	2	-8	-56	-112	848	9088	25216	-310528

The exponential generating functions for $A_n(t)$ and $P_n^{\mathrm{pk}}(t)$ have the following well-known expressions:[2]

$$A(t;x) := \sum_{n=1}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{e^{(1-t)x} - 1}{1 - te^{(1-t)x}};$$

$$P^{\mathrm{pk}}(t;x) := \sum_{n=1}^{\infty} P_n^{\mathrm{pk}}(t) \frac{x^n}{n!} = \frac{1}{\sqrt{1-t}\coth(x\sqrt{1-t}) - 1}.$$

The work in this paper was originally inspired by the curious observation that $A(-1;x)$ and $P^{\mathrm{pk}}(-1;x)$ can be expressed as the logarithmic derivative of the exponential generating function of some non-negative integer sequence. For the Eulerian polynomials, this sequence $\{f_n\}_{n\geq 0}$ is simply $f_n := (n+1) \bmod 2$, i.e., the sequence $1, 0, 1, 0, \ldots$, whose exponential generating function is given by

$$F(x) := 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh(x).$$

For the peak polynomials, this sequence is the sequence of Pell numbers, which has been widely studied in combinatorics and number theory. The Pell numbers $\{g_n\}_{n\geq 0}$ are defined by the recursive formula $g_n := 2g_{n-1} + g_{n-2}$ for $n \geq 2$ with initial values $g_0 = 1$ and $g_1 = 0$. The first several terms of this sequence are below:

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[2]These exponential generating functions are usually given a constant term of 1 in the literature, but it is more convenient to define these without the constant term in this work.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
g_n	1	0	1	2	5	12	29	70	169	408	985	2378	5741

Note that the indexing here is slightly different from the usual indexing of the Pell numbers (see OEIS [13, A000129]). The exponential generating function of $\{g_n\}_{n\geq 0}$ is given by

$$G(x) := \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} = \frac{1}{2}e^x(2\cosh(x\sqrt{2}) - \sqrt{2}\sinh(x\sqrt{2})).$$

Theorem 1. *The exponential generating functions for the Eulerian and peak polynomials evaluated at $t = -1$ can be expressed as the logarithmic derivative of $F(x)$ and $G(x)$, respectively. That is:*

(a) $A(-1; x) = \dfrac{\frac{d}{dx}F(x)}{F(x)}$

(b) $P^{\mathrm{pk}}(-1; x) = \dfrac{\frac{d}{dx}G(x)}{G(x)}$

While Theorem 1 can be proven directly by algebraically manipulating the generating function formulas for $A(t; x)$, $P^{\mathrm{pk}}(t; x)$, $F(x)$, and $G(x)$, one of our goals in this paper is to present a combinatorially-flavored proof. In Section 2, we define several other relevant permutation statistics and introduce a key ingredient of our proof: the modified Foata–Strehl group action (a.k.a. valley-hopping). In Section 3, we define a variant of the Chebyshev polynomials of the second kind which specialize to both the numbers f_n and the Pell numbers g_n. Like the ordinary Chebyshev polynomials of the second kind, our modified Chebyshev polynomials have as a combinatorial model monomino-domino tilings of a rectangle, but with slightly different weights. We present a formula (Theorem 3) involving these modified Chebyshev polynomials for the joint distribution of two statistics: the peak number, and the total number of double ascents and double descents. We give a combinatorial proof of Theorem 3 which involves tilings and valley-hopping, and a special case of this result implies Theorem 1 (b). We transform Theorem 3 into similar results for other permutation statistics, which we then use to prove Theorem 1 (a) and to prove that the (-1)-evaluation of the double descent distribution over $\mathfrak{S}_n$ yields the tangent numbers E_{2n+1}.

In Section 4, we turn our attention to counting derangements by cyclic analogues of the permutation statistics studied in Sections 2–3. Using a variant of valley-hopping due to Sun and Wang [16] for derangements, we prove a cyclic analogue of Theorem 3 and use it to derive formulas relating exponential generating functions counting derangements by cyclic statistics with the exponential generating function for our modified Chebyshev polynomials. We use this to prove a result similar to Theorem 1 for the excedance and cyclic peak distributions over derangements, and to prove that the (-1)-evaluation of the cyclic double descent distribution over derangements yields the secant numbers E_{2n}.

2. Permutation statistics and valley-hopping

Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ in $\mathfrak{S}_n$, we say that π_i (where $i \in [n]$) is:

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- a *valley* if $\pi_{i-1} > \pi_i < \pi_{i+1}$;

- a *double ascent* if $\pi_{i-1} < \pi_i < \pi_{i+1}$;

- a *double descent* if $\pi_{i-1} > \pi_i > \pi_{i+1}$;

where we are using the convention $\pi_0 = \pi_{n+1} = \infty$.[3] Thus, every letter of a permutation is either a peak, valley, double ascent, or double descent. Define $\mathrm{val}(\pi)$, $\mathrm{dasc}(\pi)$, and $\mathrm{ddes}(\pi)$ to be the number of valleys, double ascents, and double descents of π, respectively. We also define $\mathrm{dbl}(\pi) := \mathrm{dasc}(\pi) + \mathrm{ddes}(\pi)$ to be the total number of double ascents and double descents of π.

For a list of statistics $\mathrm{st}_1, \mathrm{st}_2, \ldots, \mathrm{st}_m$ and corresponding variables $t_1, t_2, \ldots, t_m$, we define the polynomials $\{P_n^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m)\}_{n \geq 0}$ by

$$P_n^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m) := \sum_{\pi \in \mathfrak{S}_n} t_1^{\mathrm{st}_1(\pi)} t_2^{\mathrm{st}_2(\pi)} \cdots t_m^{\mathrm{st}_m(\pi)}.$$

and we let

$$P^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m; x) := \sum_{n=1}^{\infty} P_n^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m) \frac{x^n}{n!}$$

be their exponential generating function.[4] For example, we have

$$P_n^{(\mathrm{pk}, \mathrm{dbl})}(s, t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{pk}(\pi)} t^{\mathrm{dbl}(\pi)}$$

and

$$P^{(\mathrm{pk}, \mathrm{dbl})}(s, t; x) = \sum_{n=1}^{\infty} P_n^{(\mathrm{pk}, \mathrm{dbl})}(s, t) \frac{x^n}{n!};$$

we will consider these on the way to proving Theorem 1. Our proof will make use of a bijection based on a group action on $\mathfrak{S}_n$ induced by involutions which toggle between double ascents and double descents; we will spend the remainder of this section defining this action and the associated bijection.

For $\pi \in \mathfrak{S}_n$, fix $k \in [n]$. We may write $\pi = w_1 w_2 k w_4 w_5$ where w_2 is the maximal consecutive subword immediately to the left of k whose letters are all smaller than k, and w_4 is the maximal consecutive subword immediately to the right of k whose letters are all smaller than k. For example, if $\pi = 467125839$ and $k = 5$, then π is the concatenation of $w_1 = 467$, $w_2 = 12$, $k = 5$, the empty word w_4, and $w_5 = 839$.

Define $\varphi_k \colon \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\varphi_k(\pi) = \begin{cases} w_1 w_4 k w_2 w_5, & \text{if } k \text{ is a double ascent or double descent of } \pi, \\ \pi, & \text{if } k \text{ is a peak or valley of } \pi. \end{cases}$$

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[3] We note that many works on permutation enumeration do not use these conventions, and simply restrict the possible positions of valleys, double ascents, and double descents to the interval from 2 to $n - 1$. What we call "valleys" are sometimes called "left-right valleys" or "exterior valleys", what we call "double ascents" are sometimes called "right double ascents", and what we call "double descents" are sometimes called "left double descents". (See, e.g., [18].)

[4] In the case where we have a single statistic st, we write these simply as $P_n^{\mathrm{st}}(t)$ and $P^{\mathrm{st}}(t; x)$.
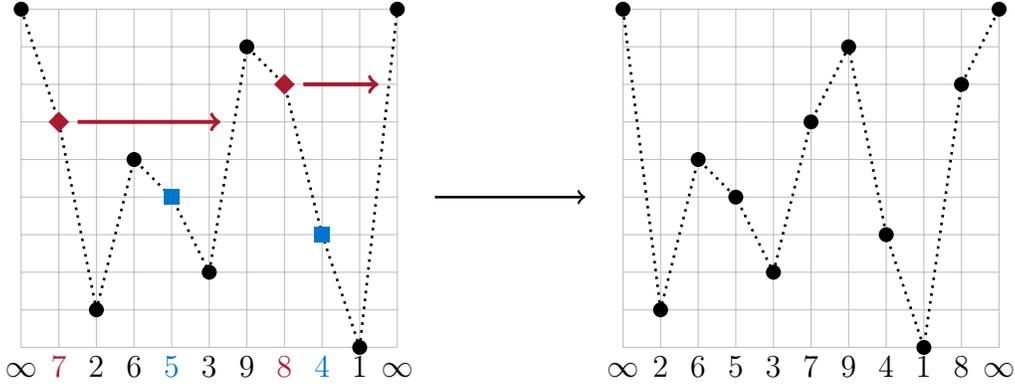
Figure 1: Valley-hopping induces a bijection between $\tilde{\mathfrak{S}}_n$ and $\mathfrak{S}_n$.

Equivalently, $\varphi_k(\pi) = w_1 w_4 k w_2 w_5$ if exactly one of w_2 and w_4 is nonempty, and $\varphi_k(\pi) = \pi$ otherwise. For any subset $S \subseteq [n]$, we define $\varphi_S \colon \mathfrak{S}_n \to \mathfrak{S}_n$ by $\varphi_S = \prod_{k \in S} \varphi_k$. It is easy to see that φ_S is an involution, and that for all $S, T \subseteq [n]$, the involutions φ_S and φ_T commute with each other. Hence the involutions $\{\varphi_S\}_{S \subseteq [n]}$ define a $\mathbb{Z}_2^n$-action on $\mathfrak{S}_n$ which is often called the *modified Foata–Strehl action* or *valley-hopping*. This action is based on a classical group action of Foata and Strehl [7], was introduced by Shapiro, Woan, and Getu [10], and was later rediscovered by Brändén [2].

Let $\tilde{\mathfrak{S}}_n$ denote the set of permutations of $[n]$ with no double ascents and where each double descent is assigned one of two colors: red or blue.[5] Then valley-hopping induces a map Φ from $\tilde{\mathfrak{S}}_n$ to $\mathfrak{S}_n$ defined in the following way. Given a permutation π in $\tilde{\mathfrak{S}}_n$, let $R(\pi)$ be the set of red double descents in π and let $\bar{\pi}$ be the corresponding permutation of π in $\mathfrak{S}_n$, that is, the permutation obtained by forgetting the colors on the double descents. Then let $\Phi(\pi) = \varphi_{R(\pi)}(\bar{\pi})$. For example, if $\pi = 726539841$, then $\Phi(\pi) = 265379418$. (See Figure 1.)

Lemma 2. *The map* $\Phi \colon \tilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ *is a* $(\mathrm{pk}, \mathrm{dbl})$-*preserving bijection.*

Proof. The inverse Φ^{-1} of the map Φ can be described in the following way. Let $\mathrm{Dasc}(\pi)$ be the set of double ascents of π and $\mathrm{Ddes}(\pi)$ the set of double descents of π. If $S \subseteq \mathrm{Ddes}(\pi)$ and if π has no double ascents, then let π^S be the permutation in $\tilde{\mathfrak{S}}_n$ obtained by coloring the double descents in S blue and all other double descents red. Given a permutation π in $\mathfrak{S}_n$, let $\Phi^{-1}(\pi) = (\varphi_{\mathrm{Dasc}(\pi)}(\pi))^{\mathrm{Ddes}(\pi)}$. Then Φ is a bijection between $\tilde{\mathfrak{S}}_n$ and $\mathfrak{S}_n$. The claim that Φ preserves the pk and dbl statistics follows from the easy fact that valley-hopping preserves these statistics as well. $\qquad\square$

3. Hopping from Chebyshev polynomials to permutation statistics

The numbers $f_n = (n+1) \bmod 2$ and the Pell numbers g_n are specializations of the sequence of polynomials $\{V_n(s,t)\}_{n \geq 0}$ defined by $V_n(s,t) = 2tV_{n-1}(s,t) - sV_{n-2}(s,t)$ for $n \geq 2$ with initial values $V_0(s,t) = 0$ and $V_1(s,t) = 1$. More precisely, $f_n = V_{n-1}(-1,0)$ and $g_n =$

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[5]To be more precise, Davidson College red or Mount Holyoke College blue.

$V_{n-1}(-1, 1)$ for all $n \geq 1$. These polynomials are a variant of the Chebyshev polynomials of the second kind, which can be obtained from the $V_n(s, t)$ by substituting $s = 1$.

The ordinary generating function for these modified Chebyshev polynomials $V_n(s, t)$ is given by the formula

$$\sum_{n=0}^{\infty} V_n(s, t)x^n = \frac{x}{1 - 2tx + sx^2},$$

so $V_n(s, t)$ counts tilings of a $1 \times (n-1)$ rectangle with two types of monominoes, each weighted t, and one type of domino, each weighted $-s$. Also, let

$$V(s, t; x) := \sum_{n=2}^{\infty} V_{n-1}(s, t)\frac{x^n}{n!} = \frac{x^2}{2!} + 2t\frac{x^3}{3!} + (4t^2 - s)\frac{x^4}{4!} + \cdots$$

be the (shifted) exponential generating function for the polynomials $V_n(s, t)$. It can be shown (using the exponential generating function for the usual Chebyshev polynomials of the second kind) that $V(s, t; x)$ has the closed-form expression

$$V(s, t; x) = \frac{1}{s}\left(1 - \cosh(x\sqrt{t^2 - s})e^{tx} + \frac{te^{tx}\sinh(x\sqrt{t^2 - s})}{\sqrt{t^2 - s}}\right). \tag{2}$$

3.1. A Chebyshev formula for the bidistribution $(\mathrm{pk}, \mathrm{dbl})$

We now present our main theorem from this section.

Theorem 3. $P^{(\mathrm{pk},\mathrm{dbl})}(s, t; x) = \dfrac{\frac{\partial}{\partial x}V(s, t; x)}{1 - sV(s, t; x)}$

Setting $s = -1$ and $t = 1$ in Theorem 3 yields Theorem 1 (b).

Proof. From the combinatorial interpretation of multiplication of exponential generating functions (see, e.g., [15, Proposition 5.1.3]), it suffices to show that

$$P_n^{(\mathrm{pk},\mathrm{dbl})}(s, t) = \sum_{k=0}^{\lfloor(n-1)/2\rfloor} s^k \sum_{B} V_{|B_0|}(s, t)V_{|B_1|-1}(s, t)\cdots V_{|B_k|-1}(s, t) \tag{3}$$

where the second sum is over all ordered set partitions B of $[n]$ into blocks $B_0, B_1, \ldots, B_k$. Thus, the right-hand side of Equation (3) counts these set partitions together with:

- a tiling of an $1 \times (|B_0| - 1)$ rectangle with two types of monominoes (colored red and blue), each weighted t, and one type of domino, each weighted $-s$;

- for each $1 \leq i \leq k$, a tiling of an $1 \times (|B_i| - 2)$ rectangle with the same types of shapes and weights as above;

and each block (other than B_0) is given an additional weight of s. We place an ∞ in the first block, write out each block in decreasing order, and separate adjacent blocks with a bar, as in

$$\infty > \pi_1 > \pi_2 > \cdots > \pi_{|B_0|} \mid \pi_{|B_0|+1} > \pi_{|B_0|+2} > \cdots > \pi_{|B_0|+|B_1|} \mid \cdots \mid \pi_{n-|B_k|+1} > \cdots > \pi_n.$$

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$$\infty \quad \boxed{7} \quad \boxed{6 \quad 5} \quad \langle 3 \rangle \quad 2 \mid 9 \quad \langle 8 \rangle \quad \boxed{4} \quad 1$$

$$\infty \quad \boxed{7} \quad 6 \mid 5 \quad \langle 3 \rangle \quad 2 \mid 9 \quad \langle 8 \rangle \quad \boxed{4} \quad 1$$

Figure 2: Example of the sign-reversing involution.

Here, we consider the tiling on each block as being a tiling on all but the first and last elements of the block. Now we define a sign-reversing involution on these objects in the following way: Find the first pair of elements (π_i, π_{i+1}) where there is a domino, or where π_i and π_{i+1} are in separate blocks and $\pi_i > \pi_{i+1}$. If (π_i, π_{i+1}) is covered by a domino, then we remove the domino and insert a new bar in between π_i and π_{i+1}, thus splitting their block into two blocks. If π_i and π_{i+1} are in separate blocks and $\pi_i > \pi_{i+1}$, then we merge the two blocks and cover (π_i, π_{i+1}) with a domino. (See Figure 2.) This involution swaps a domino (weighted $-s$) with an additional block (weighted s), and after cancellation we are left with those objects with no dominoes and such that $\pi_i < \pi_{i+1}$ whenever π_i and π_{i+1} are in separate blocks.

If we treat any one of these remaining objects $\pi = \pi_1 \pi_2 \cdots \pi_n$ as a permutation, we see that π has no double ascents and has each double descent colored either red or blue (depending on the color of the corresponding monomino). Hence, π belongs to $\tilde{\mathfrak{S}}_n$ and contributes a weight of $s^{\mathrm{pk}(\pi)} t^{\mathrm{dbl}(\pi)}$ to the right-hand side of Equation (3). The result then follows from applying the $(\mathrm{pk}, \mathrm{dbl})$-preserving bijection Φ. $\qquad\square$

3.2. A Chebyshev formula for the quadruple distribution $(\mathrm{pk}, \mathrm{val}, \mathrm{dasc}, \mathrm{ddes})$

We shall now derive from Theorem 3 an analogous result for the joint distribution of the four statistics pk, val, dasc, and ddes.

Theorem 4. $P^{(\mathrm{pk},\mathrm{val},\mathrm{dasc},\mathrm{ddes})}(s, t, u, v; x) = \dfrac{t \frac{\partial}{\partial x} V(st, (u+v)/2; x)}{1 - st V(st, (u+v)/2; x)}$

Proof. First, observe that $\mathrm{val}(\pi) = \mathrm{pk}(\pi) + 1$ for all permutations π, and that the reversal $r(\pi) := \pi_n \pi_{n-1} \cdots \pi_1$ of a permutation is a $(\mathrm{pk}, \mathrm{val})$-preserving involution on $\mathfrak{S}_n$ that switches double ascents with double descents. Thus, we have

$$P_n^{(\mathrm{pk},\mathrm{val},\mathrm{dasc},\mathrm{ddes})}(s, t, u, v) = t P_n^{(\mathrm{pk},\mathrm{dbl})}\left(st, \frac{1}{2}(u + v)\right)$$

for all $n \geq 1$, which proves the result in light of Theorem 3. $\qquad\square$

Theorem 4 and the formula (2) for $V(s, t; x)$ can be used together to derive the closed-form formula

$$P^{(\mathrm{pk},\mathrm{val},\mathrm{dasc},\mathrm{ddes})}(s, t, u, v; x) = \frac{2t}{\alpha \coth(\frac{1}{2}\alpha x) - u - v}$$

where $\alpha = \sqrt{(u+v)^2 - 4st}$. This is equivalent to a classical formula of Carlitz and Scoville [3]; see also [14, Exercise 1.61a].

The following corollary states several specializations of Theorem 4.

Corollary 5.

(a) $A(t; x) = \dfrac{\frac{\partial}{\partial x} V(t, (1+t)/2; x)}{1 - tV(t, (1+t)/2; x)}$

(b) $P^{\mathrm{pk}}(t; x) = \dfrac{\frac{\partial}{\partial x} V(t, 1; x)}{1 - tV(t, 1; x)}$

(c) $P^{\mathrm{ddes}}(t; x) = \dfrac{\frac{\partial}{\partial x} V(1, (1+t)/2; x)}{1 - V(1, (1+t)/2; x)}$

Further specializing Corollary 5 (a) at $t = -1$ implies Theorem 1 (a).

Next we show that the (-1)-evaluation of the double descent distribution over $\mathfrak{S}_{2n+1}$ gives the tangent numbers E_{2n+1}.

Theorem 6. *For all $n \geq 1$, we have*

$$P_n^{\mathrm{ddes}}(-1) = \begin{cases} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Thus $P^{\mathrm{ddes}}(-1; x) = \tan(x)$.

Proof. Comparing Theorem 3 with Corollary 5 (c), we have $P^{\mathrm{ddes}}(-1; x) = P^{(\mathrm{pk,dbl})}(1, 0; x)$, which implies $P_n^{\mathrm{ddes}}(-1) = P_n^{(\mathrm{pk,dbl})}(1, 0)$ for all $n \geq 1$. Observe that $P_n^{(\mathrm{pk,dbl})}(1, 0)$ is the number of permutations in $\mathfrak{S}_n$ with no double ascents or double descents. It is easy to see that there are no such permutations for even n, and the only such permutations for odd n are *alternating permutations*: permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ satisfying $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots < \pi_n$. It is well known that there are E_n alternating permutations in $\mathfrak{S}_n$, and the proof follows. $\qquad\square$

Similar reasoning can be used to prove the formula (1) for Eulerian polynomials evaluated at $t = -1$.

4. Counting derangements by cyclic statistics

Recall that a *derangement* is a permutation with no fixed points, i.e., a permutation for which $\pi_i \neq i$ for all i. Let $\mathfrak{D}_n$ be the set of derangements in $\mathfrak{S}_n$. Our goal in this section is to provide an analogous treatment of the material from the previous section but for counting derangements with respect to several "cyclic statistics" that we will define shortly.

When writing permutations in cycle notation, we adopt the convention of writing each cycle with its largest letter in the first position, and writing the cycles from left-to-right in increasing order of their largest letters. (This convention is sometimes called *canonical cycle representation*.) For example, the permutation $\pi = 649237185$ in one-line notation is written as $\pi = (42)(716)(8)(953)$ in cycle notation.

Given $\pi = \pi_1 \pi_2 \cdots \pi_n$, we say that π_i is:

- a *cyclic peak* if $i < \pi_i > \pi_{\pi_i}$;

- a *cyclic valley* if $i > \pi_i < \pi_{\pi_i}$;

- a *cyclic double ascent* if $i < \pi_i < \pi_{\pi_i}$;

- a *cyclic double descent* if $i > \pi_i > \pi_{\pi_i}$.

Every letter of a derangement is either a cyclic peak, cyclic valley, cyclic double ascent, or cyclic double descent. Define $\mathrm{cpk}(\pi)$, $\mathrm{cval}(\pi)$, $\mathrm{cdasc}(\pi)$, and $\mathrm{cddes}(\pi)$ to be the number of cyclic peaks, cyclic valleys, cyclic double ascents, and cyclic double descents of π, respectively.

These "cyclic statistics" were studied earlier by, e.g., Zeng [17], Shin and Zeng [12], and Sun and Wang [16].[6] These statistics are also closely related to a classical permutation statistic, the excedance number. We say that $i \in [n]$ is an *excedance* of π if $i < \pi_i$ and let $\mathrm{exc}(\pi)$ denote the number of excedances of π. Then i is an excedance of π if and only if i is a cyclic valley or cyclic double ascent of π, and it is well-known that the excedance number exc and the descent number des are equidistributed over $\mathfrak{S}_n$.

Define the map $o\colon \mathfrak{S}_n \to \mathfrak{S}_n$, where the input is a permutation in canonical cycle representation and the output is a permutation in one-line notation, by erasing the parentheses. Continuing the example with $\pi = (42)(716)(8)(953)$, we have $o(\pi) = 427168953$. It is easy to see that this map is a bijection; we can recover the cycles of π by noting the left-to-right maxima of $o(\pi)$: given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, we say that σ_i is a *left-to-right maximum* of σ if $\sigma_j < \sigma_i$ for all $1 \le j < i$.

Our work in this section will rely on a cyclic variant of valley-hopping introduced in [16]. Define $\theta_k\colon \mathfrak{D}_n \to \mathfrak{D}_n$ by $\theta_k(\pi) = o^{-1}(\varphi_k(o(\pi)))$, where the 0th letter of $o(\pi)$ is treated as 0 rather than ∞. Similarly, for a subset $S \subseteq [n]$, define $\theta_S\colon \mathfrak{D}_n \to \mathfrak{D}_n$ by $\theta_S = \prod_{k \in S} \theta_k$. Then the *cyclic modified Foata–Strehl action* (or *cyclic valley-hopping*) is the $\mathbb{Z}_2^n$-action defined by the involutions θ_S. It is easy to see that cyclic valley-hopping toggles between cyclic double ascents and cyclic double descents, but does not change cyclic peaks or cyclic valleys.

Let $\tilde{\mathfrak{D}}_n$ denote the set of derangements of $[n]$ with no cyclic double ascents and where each cyclic double descent is assigned one of two colors: red or blue. Then cyclic valley-hopping induces a map $\mathring{\Phi}$ from $\tilde{\mathfrak{D}}_n$ to $\mathfrak{D}_n$ defined in the analogous way as the map Φ from Section 2, but with $R(\pi)$ being the set of red cyclic double descents. It then follows from the same reasoning as in the proof of Lemma 2 that $\mathring{\Phi}$ is a $(\mathrm{cpk}, \mathrm{cdbl})$-preserving bijection, where $\mathrm{cdbl}(\pi) := \mathrm{cdasc}(\pi) + \mathrm{cddes}(\pi)$ is the total number of cyclic double ascents and cyclic double descents of π.

4.1. A cyclic analogue of Theorem 3 for derangements

For permutation statistics $\mathrm{st}_1, \mathrm{st}_2, \ldots, \mathrm{st}_m$ and variables $t_1, t_2, \ldots, t_m$, we define the polynomials $\{D_n^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m)\}_{n \ge 0}$ by

$$D_n^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m) := \sum_{\pi \in \mathfrak{D}_n} t_1^{\mathrm{st}_1(\pi)} t_2^{\mathrm{st}_2(\pi)} \cdots t_m^{\mathrm{st}_m(\pi)}.$$

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[6]Chow et al. [4] also derived various formulas for counting permutations by cyclic peaks and cyclic valleys, but their definition of these statistics differ from ours in that they do not allow the first or last letter of a cycle to be a cyclic peak or cyclic valley.

and we let

$$D^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m; x) := 1 + \sum_{n=1}^{\infty} D_n^{(\mathrm{st}_1, \mathrm{st}_2, \ldots \mathrm{st}_m)}(t_1, t_2, \ldots, t_m) \frac{x^n}{n!}$$

be their exponential generating function.[7] These encode the distributions of permutation statistics over derangements.

We now present a cyclic analogue of Theorem 3 for derangements.

Theorem 7. $D^{(\mathrm{cpk}, \mathrm{cdbl})}(s, t; x) = \dfrac{1}{1 - sV(s, t; x)}$

Proof. It suffices to show that

$$D_n^{(\mathrm{cpk}, \mathrm{cdbl})}(s, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} s^k \sum_{B} V_{|B_1|-1}(s, t) \cdots V_{|B_k|-1}(s, t) \tag{4}$$

where the second sum is over all ordered set partitions B of $[n]$ into parts $B_1, \ldots, B_k$. We interpret the right-hand side of (4) as in the proof of Theorem 3 (without the initial block B_0 with an ∞) and apply the same sign-reversing involution; the objects that remain after cancellation are of the form

$$c_1 > c_2 > \cdots > c_{|B_0|} \mid c_{|B_0|+1} > c_{|B_0|+2} > \cdots > c_{|B_0|+|B_1|} \mid \cdots \mid c_{n-|B_k|+1} > \cdots > c_n$$

with no dominoes and such that $c_i < c_{i+1}$ whenever c_i and c_{i+1} are in separate blocks.

Now, rather than treating these remaining objects as permutations in one-line notation, we want to treat them as permutations in cycle notation with blocks corresponding to cycles. In doing so, we merge two adjacent blocks whenever the first element of the second block in the pair is not larger than all elements from all preceding blocks, i.e., whenever the element is not a left-to-right maximum of the underlying permutation written in one-line notation; this guarantees that the resulting permutations are correctly written in canonical cycle representation and is clearly reversible.

Moreover, these permutations are derangements because each block has size at least 2, and they have no cyclic double ascents and have each cyclic double descent colored either red or blue (depending on the color of the corresponding monomino). In other words, these permutations π are precisely the elements of $\widetilde{\mathfrak{D}}_n$ and each contributes a weight of $s^{\mathrm{cpk}(\pi)} t^{\mathrm{cdbl}(\pi)}$ to the right-hand side of Equation (4). The result then follows from applying the (cpk, cdbl)-preserving bijection $\mathring{\Phi}$. $\qquad \square$

We extend Theorem 7 to an analogous result for the joint distribution of the statistics cpk, cval, cdasc, and cddes over $\mathfrak{D}_n$.

Theorem 8. $D^{(\mathrm{cpk}, \mathrm{cval}, \mathrm{cdasc}, \mathrm{cddes})}(s, t, u, v; x) = \dfrac{1}{1 - stV(st, (u+v)/2; x)}$

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[7]As before, if we have a single statistic st, we write these simply as $D_n^{\mathrm{st}}(t)$ and $D^{\mathrm{st}}(t; x)$.

Proof. First, observe that $\mathrm{cpk}(\pi) = \mathrm{cval}(\pi)$ for all derangements π. The cyclic valley-hopping map θ_S, where S is the set containing all cyclic double ascents and cyclic double descents of π, is a $(\mathrm{cpk}, \mathrm{cval})$-preserving involution on $\mathfrak{D}_n$ that switches cyclic double ascents with cyclic double descents. Thus, we have

$$D_n^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes})}(s, t, u, v) = D_n^{(\mathrm{pk},\mathrm{dbl})}\left(st, \frac{1}{2}(u + v)\right)$$

for all $n \geq 1$, which along with Theorem 7 proves the result. $\qquad\square$

We can use Theorem 8 and Equation (2) to derive the formula

$$D^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes})}(s, t, u, v; x) = \frac{\alpha e^{-\frac{1}{2}(u+v)x}}{\alpha \cosh(\frac{1}{2}\alpha x) - (u + v) \sinh(\frac{1}{2}\alpha x)} \tag{5}$$

where $\alpha = \sqrt{(u + v)^2 - 4st}$. Furthermore, given any permutation π, let $\mathrm{cyc}(\pi)$ denote the number of cycles of π and let $\mathrm{fix}(\pi)$ be the number of fixed points of π. Standard applications of the exponential formula (see [15, Section 5.1]) yield the identities

$$D^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes},\mathrm{cyc})}(s, t, u, v, w; x) = D^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes})}(s, t, u, v; x)^w \tag{6}$$

and

$$1 + P^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes},\mathrm{cyc},\mathrm{fix})}(s, t, u, v, w, y; x) = e^{wyx} D^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes},\mathrm{cyc})}(s, t, u, v, w; x). \tag{7}$$

Then, combining Equations (5), (6), and (7) yields the following exponential generating function formula for the sextuple distribution $(\mathrm{cpk}, \mathrm{cval}, \mathrm{cdasc}, \mathrm{cddes}, \mathrm{cyc}, \mathrm{fix})$ over all permutations:

$$1 + P^{(\mathrm{cpk},\mathrm{cval},\mathrm{cdasc},\mathrm{cddes},\mathrm{cyc},\mathrm{fix})}(s, t, u, v, w, y; x) = \left(\frac{\alpha e^{(y-\frac{1}{2}(u+v))x}}{\alpha \cosh(\frac{1}{2}\alpha x) - (u + v) \sinh(\frac{1}{2}\alpha x)}\right)^w.$$

An equivalent form of this formula was proven earlier by Zeng [17, Théorème 1].

4.2. Counting derangements by excedances

In the remainder of this section, we examine specializations of Theorem 8 that give rise to formulas for individual cyclic statistics, beginning with the excedance number.

The excedance polynomials $D_n(t) := D_n^{\mathrm{exc}}(t)$ have been well-studied; for example, it is known that they have exponential generating function

$$D(t; x) := D^{\mathrm{exc}}(t; x) = \frac{(1 - t)e^{-x}}{e^{-(1-t)x} - t}$$

[9] and are γ-positive [1, 12, 16]. From Theorem 8 we obtain the following.

Corollary 9. $D(t; x) = \dfrac{1}{1 - tV(t, (1 + t)/2; x)}$

It follows from Corollary 9 that the exponential generating function for the excedance polynomials $D(t; x)$ evaluated at $t = -1$ is the reciprocal of the exponential generating function $F(x) = \cosh(x)$ for the sequence $1, 0, 1, 0, \ldots$.

Corollary 10. $D(-1; x) = F(x)^{-1}$

This identity can be used to rederive the classical result due to Roselle [9] that

$$D_n(-1) = \begin{cases} (-1)^{n/2} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Let $W_n(t) := V_n(t, (1+t)/2)$. By the definition of $V_n(t)$, the polynomials $W_n(t)$ satisfy the recurrence $W_n(t) = (1+t)W_{n-1}(t) - tW_{n-2}(t)$ for $n \geq 2$ with initial values $W_0(t) = 0$ and $W_1(t) = 1$. From this recurrence, it is easily seen that

$$W_n(t) = 1 + t + \cdots + t^{n-1} \tag{8}$$

for $n \geq 1$. By using the characterization of $W_n(t)$ provided by Equation (8), Corollary 9 can be obtained as a specialization of a formula of Shareshian and Wachs [11, Equation (1.4)] involving Eulerian quasisymmetric functions.

4.3. Counting derangements by cyclic peaks

Next, we examine the distribution of the cyclic peak number over derangements.

Corollary 11. $D^{\text{cpk}}(t; x) = \dfrac{1}{1 - tV(t, 1; x)}$

By specializing (5) appropriately, we derive the formula

$$D^{\text{cpk}}(t; x) = \frac{\sqrt{1-t}\, e^{-x}}{\sqrt{1-t} \cosh(x\sqrt{1-t}) - \sinh(x\sqrt{1-t})}.$$

The first several polynomials $D_n^{\text{cpk}}(t)$ are given in the following table.

n	$D_n^{\text{cpk}}(t)$	n	$D_n^{\text{cpk}}(t)$
1	0	5	$8t + 36t^2$
2	t	6	$16t + 188t^2 + 61t^3$
3	$2t$	7	$32t + 864t^2 + 958t^3$
4	$4t + 5t^2$	8	$64t + 3728t^2 + 9656t^3 + 1385t^4$

Note that the coefficient of t in $D_n^{\text{cpk}}(t)$ seems to be 2^{n-2} for all $n \geq 2$; this is easy to explain combinatorially.

Proposition 12. *For all $n \geq 2$, the number of derangements in $\mathfrak{D}_n$ with exactly one cyclic peak is 2^{n-2}.*

Proof. It is easy to see that every derangement π of $[n]$ with exactly one cyclic peak has exactly one cycle and can be written in the form

$$(c_1 c_2 \cdots c_k c_{k+1} \cdots c_n)$$

where $c_1 = n$ is the only cyclic peak of π, the sequence $c_2 \cdots c_k$ is decreasing (with $c_k = 1$), and the sequence $c_{k+1} \cdots c_n$ is increasing. Thus, for every letter i between 2 and $n-1$, either i belongs to the decreasing sequence or the increasing sequence, and these $n-2$ choices completely determine the derangement π. □

It follows from Corollary 11 that the exponential generating function for the numbers $D_n^{\mathrm{cpk}}(-1)$ is the reciprocal of the exponential generating function $G(x)$ for the Pell numbers.

Corollary 13. $D^{\mathrm{cpk}}(-1; x) = G(x)^{-1}$

We do not have a combinatorial interpretation for the numbers $D_n^{\mathrm{cpk}}(-1)$ themselves. The first several of these numbers appear in the following table.

n	1	2	3	4	5	6	7	8	9	10
$D_n^{\mathrm{cpk}}(-1)$	0	-1	-2	1	28	111	-126	-4067	-26280	53663

4.4. Counting derangements by cyclic double descents

Lastly, we study the distribution of the cyclic double descent number over derangements.

Corollary 14. $D^{\mathrm{cddes}}(t; x) = \dfrac{1}{1 - V(1, (1+t)/2; x)}$

The exponential generating function formula

$$D^{\mathrm{cddes}}(t; x) = \frac{\beta e^{-\frac{1}{2}(1+t)x}}{\beta \cosh(\frac{1}{2}\beta x) - (1+t)\sinh(\frac{1}{2}\beta x)},$$

where $\beta = \sqrt{(t+3)(t-1)}$, can be obtained by specializing (5). The first several of the polynomials $D_n^{\mathrm{cddes}}(t)$ appear in the following table.

n	$D_n^{\mathrm{cddes}}(t)$	n	$D_n^{\mathrm{cddes}}(t)$
1	0	5	$19 + 21t + 3t^2 + t^3$
2	1	6	$109 + 98t + 53t^2 + 4t^3 + t^4$
3	$1 + t$	7	$588 + 808t + 334t^2 + 118t^3 + 5t^4 + t^5$
4	$6 + 2t + t^2$	8	$4033 + 5766t + 3827t^2 + 952t^3 + 248t^4 + 6t^5 + t^6$

An *increasing run* of a permutation π is a maximal increasing consecutive subsequence of π (in one-line notation). For example, the increasing runs of $\pi = 467192685$ are 467, 19, 268, and 5. We call increasing runs of length 1 *short runs*. The sequence of constant coefficients of $D_n^{\mathrm{cddes}}(t)$ matches the OEIS sequence [13, A097899] for the number of permutations of $[n]$ with no short runs; this can be verified by comparing the evaluation $D^{\mathrm{cddes}}(0; x)$ with the exponential generating function of this OEIS entry, but we give a bijective proof below.

For the purpose of this proof, let us temporarily modify our convention for cycle notation so that we write each cycle with its smallest letter in the first position, and write the cycles from left-to-right in decreasing order of their smallest letters. For example, whereas we previously wrote $\pi = 649237185$ as $\pi = (42)(716)(8)(953)$ in canonical cycle representation, now we write π as $\pi = (8)(395)(24)(167)$. Let $o' \colon \mathfrak{S}_n \to \mathfrak{S}_n$ be the map defined by taking a permutation in cycle notation under this new convention and erasing the parentheses, yielding a permutation in one-line notation. This is a bijection; we can recover the cycles of π by noting the left-to-right minima of $o(\pi)$: given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, we say that σ_i is a *left-to-right minimum* of σ if $\sigma_j > \sigma_i$ for all $1 \le j < i$.

Proposition 15. *An letter $i \in [n]$ is a fixed point or cyclic double descent of $\pi \in \mathfrak{S}_n$ if and only if i is a short run of $o'(\pi)$.*

In particular, this proposition implies that the number of derangements of $[n]$ with no cyclic double descents is equal to the number of permutations of $[n]$ with no short runs.

Proof. Fix a permutation $\pi \in \mathfrak{S}_n$ and a letter $i \in [n]$. Let us write $o'(\pi) = \sigma_1 \sigma_2 \cdots \sigma_n$ in one-line notation, and take $\sigma_j = i$. By taking as convention $\sigma_0 = \infty$ and $\sigma_{n+1} = 0$, it is easy to see that $\sigma_j = i$ is a short run of $o'(\pi)$ if and only if $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$.

We divide into cases. First, suppose that $i \in [n]$ is a fixed point of π. Then σ_j and σ_{j+1} are both left-to-right minima, so $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$. Now suppose that $i \in [n]$ is a cyclic double descent of π. Note that the first letter of a cycle (under our current convention) cannot be a cyclic double descent. If i is neither the first nor last letter of its cycle in π, then $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$. Otherwise, if i is the last letter of its cycle in π, then $\sigma_{j-1} > \sigma_j$ and σ_{j+1} is a left-to-right minimum; thus $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$. In each case, it follows that $\sigma_j = i$ is a short run of $o'(\pi)$. Hence, every fixed point and cyclic double descent of π is a short run of $o'(\pi)$; the reverse direction is similar. □

Finally, we give a cyclic analogue of Theorem 6 for derangements.

Theorem 16. *For all $n \ge 1$, we have*

$$D_n^{\mathrm{cddes}}(-1) = \begin{cases} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Thus $D^{\mathrm{cddes}}(-1; x) = \sec(x)$.

In other words, the (-1)-evaluation of the cyclic double descent distribution over $\mathfrak{D}_{2n}$ gives the secant numbers E_{2n}.

Proof. By comparing Theorem 7 with Corollary 14, we have $D_n^{\mathrm{cddes}}(-1) = D_n^{(\mathrm{cpk,cdbl})}(1,0)$ for all $n \ge 1$. Observe that $D_n^{(\mathrm{cpk,cdbl})}(1,0)$ is the number of derangements of $[n]$ with no cyclic double ascents or cyclic double descents. Because the number of cyclic peaks of any permutation is equal to its number of cyclic valleys, it is evident that there are no such permutations for odd n, and it is easy to see that the map o defined earlier is a bijection between such permutations for an even n and *reverse-alternating permutations* of $[n]$: permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ satisfying $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots > \pi_n$. Since there are E_n reverse-alternating permutations in $\mathfrak{S}_n$, the proof follows. □

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