Equivalence of OEIS A007729 and A174868

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Abstract

We verify the conjecture that the sixth binary partition function [2] is equal (aside from the initial zero term) to the partial sums of the Stern-Brocot sequence [3]:

$$0,1,2,4,5,8,10,13,14,18,21,26,28,33,36,40,41,46\dots$$

.

Let b'_k be the *sixth binary partition function*, which is the number of ways to write k as a sum

$$k=\sum_{i\geq 0}\varepsilon_i 2^i$$

with $\varepsilon_i \in \{0,1,2,3,4,5\}$. We obtain

$$b'_{2k}=b'_k+b'_{k-1}+b'_{k-2} \tag{1}$$

by counting the number of representations of $2k$ with $\varepsilon_0=0,2$ or 4; in each case we have a representation $\hat{\varepsilon}$ of $\frac{2k-\varepsilon_0}{2}$ by taking $\hat{\varepsilon}_i=\varepsilon_{i+1}$, and the correspondence is clearly one-to-one. Also $b'_{2k+1}=b'_{2k}$, since we can get a representation of $2k+1$ only by taking a representation of $2k$ and adding 1 to ε_0. Thus

$$\begin{aligned} b'_{2k} &= 2b'_{k-1}+b'_k \quad (k \text{ even}) \\ b'_{2k} &= 2b'_{k-1}+b'_{k-2} \quad (k \text{ odd}) \end{aligned} \tag{2}$$

since b'_{k-1} equals either b'_k or b'_{k-2}.

We eliminate the even/odd repetition by defining $b_k=b'_{2k}$. Then $b_0=1, b_1=2$ and

$$\begin{aligned} b_{2k} &= 2b'_{2k-1}+b'_{2k}=2b_{k-1}+b_k \\ b_{2k+1} &= 2b'_{2k}+b'_{2k-1}=2b_k+b_{k-1}\ . \end{aligned} \tag{3}$$

This is A007729. If we prepend a zero, defining $\hat{b}_0 = 0$ and $\hat{b}_k = b_{k-1}$ we obtain

$$\begin{aligned} \hat{b}_{2k} &= 2\hat{b}_k + \hat{b}_{k-1} \\ \hat{b}_{2k+1} &= 2\hat{b}_k + \hat{b}_{k+1} \ . \end{aligned} \tag{4}$$

The same recurrence with the same initial conditions gives A174868, the partial sums of the Stern-Brocot sequence [1]. The Stern-Brocot sequence itself can be defined by $s_0 = 0, s_1 = 1$, and

$$\begin{aligned} s_{2k} &= s_k \\ s_{2k+1} &= s_k + s_{k+1} \ . \end{aligned} \tag{5}$$

The partial sums are $\sigma_k = \sum_{0 \le i \le k} s_k$. Letting $\ell_j = s_{2j-1} + s_{2j} = 2s_j + s_{j-1}$ we get

$$\sigma_{2k} = \sum_{1 \le j \le k} \ell_j = 2\sigma_k + \sigma_{k-1}$$

and similarly with $\ell'_j = s_{2j} + s_{2j+1} = 2s_j + s_{j+1}$,

$$\sigma_{2k+1} = \sum_{0 \le j \le k} \ell'_j = 2\sigma_k + \sigma_{k+1} \ .$$

References

[1] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, http://oeis.org/a002487, 2018.

[2] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, http://oeis.org/a007729, 2018.

[3] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, http://oeis.org/a174868, 2018.