Tighter bounds for online bipartite matching

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Abstract

We study the online bipartite matching problem, introduced by Karp, Vazirani and Vazirani [1990]. For bipartite graphs with matchings of size n, it is known that the *Ranking* randomized algorithm matches at least $(1 - \frac{1}{e})n$ edges in expectation. It is also known that no online algorithm matches more than $(1 - \frac{1}{e})n + O(1)$ edges in expectation, when the input is chosen from a certain distribution that we refer to as D_n. This upper bound also applies to *fractional* matchings. We review the known proofs for this last statement. In passing we observe that the $O(1)$ additive term (in the upper bound for fractional matching) is $\frac{1}{2} - \frac{1}{2e} + O(\frac{1}{n})$, and that this term is tight: the online algorithm known as *Balance* indeed produces a fractional matching of this size. We provide a new proof that exactly characterizes the expected cardinality of the (integral) matching produced by *Ranking* when the input graph comes from the support of D_n. This expectation turns out to be $(1 - \frac{1}{e})n + 1 - \frac{2}{e} + O(\frac{1}{n!})$, and serves as an upper bound on the performance ratio of any online (integral) matching algorithm.

1 Introduction

Given a bipartite graph $G(U, V; E)$, where U and V are the sets of vertices and $E \in U \times V$ is the set of edges, a matching $M \subset E$ is a set of edges such that every vertex is incident with at most one edge of M. Given a matching M, a vertex is referred to as either matched or exposed, depending on whether it is incident with an edge of M. A maximum matching in a graph is a matching of maximum cardinality, and a maximal matching is a matching that is not a proper subset of any other matching. Maximal matchings can easily be found by greedy algorithms, and maximum matchings can also be found by various polynomial time algorithms, using techniques such as alternating paths or linear programming (see [9] and references therein). In every graph, the cardinality of every maximal matching is at least half of that of the maximum matching, because every matched edge can exclude at most two edges from the maximum matching.

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For simplicity of notation, for every n we shall only consider the following class of bipartite graphs, that we shall refer to as G_n. For every $G(U, V; E) \in G_n$ it holds that $|U| = |V| = n$ and that E contains a matching of size n (and hence G has a perfect matching). The vertices of U will be denoted by u_i (for $1 \leq i \leq n$) and the vertices of V will be denoted by v_i (for $1 \leq i \leq n$). All results that we will state for G_n hold without change for all bipartite graphs, provided that n denotes the size of the maximum matching in the graph.

Karp, Vazirani and Vazirani [7] introduced an online version of the maximum bipartite matching problem. This setting can be viewed as a game between two players: a maximizing player who wishes the resulting matching to be as large as possible, and a minimizing player who wishes the matching to be as small as possible. First, the minimizing player chooses $G(U, V; E)$ in private (without the maximizing player seeing E), subject to $G \in G_n$. Thereafter, the structure of G is revealed to the maximizing player in n steps, where at step j (for $1 \leq j \leq n$) the set $N(u_j) \subset V$ of vertices adjacent to u_j is revealed. At every step j, upon seeing $N(u_j)$ (and based on all edges previously seen and all previous matching decisions made), the maximizing player needs to irrevocably either match u_j to a currently exposed vertex in $N(u_j)$, or leave u_j exposed.

There is much recent interest in the online bipartite matching problem and variations and generalizations of it, as such models have applications for allocation problems in certain economic settings, in which buyers (vertices of U) arrive online and are interested in purchasing various items (vertices of V). For more details, see for example the survey by Metha [11].

An algorithm for the maximizing player in the online bipartite matching setting will be called *greedy* if the only vertices of U that it leaves unmatched are those vertices $u \in U$ that upon their arrival did not have an exposed neighbor (and hence could not be matched). It is not difficult to see that every non-greedy algorithm A can be replaced by a greedy algorithm A' that for every graph G matches at least as many vertices as A does. Hence we shall assume that the algorithm for the maximizing player is greedy, and this assumption is made without loss of generality, as far as the results in this manuscript are concerned.

Every greedy algorithm (for the maximizing player) produces a maximal matching, and hence matches at least half the vertices. For every deterministic algorithm, the minimizing player can select a bipartite graph G (that admits a perfect matching) that guarantees that the algorithm matches only half the vertices. (Sketch: The first $\frac{|U|}{2}$ arriving vertices have all of V as their neighbors, and the remaining $\frac{|U|}{2}$ are neighbors only of the $\frac{|V|}{2}$ vertices that the algorithm matched with the first $\frac{|U|}{2}$ vertices.)

To improve the size of the matching beyond $\frac{n}{2}$, Karp, Vazirani and Vazirani [7] considered randomized algorithms for the maximizing player. Specifically, they proposed an algorithm called *Ranking* that works as follows. It first selects uniformly at random a permutation π over the vertices V. Thereafter, upon arrival of a vertex u, it is matched to its earliest (according to π) exposed neighbor if there is one (and left unmatched otherwise). As the maximizing algorithm is randomized (due to the random choice of π), the number of vertices matched is a random variable, and we consider its expectation.

Let A be a randomized algorithm (such as *Ranking*) for the maximizing player. As such, for every bipartite graph G it produces a distribution over matchings. For a bipartite graph $G \in G_n$, we use the following notation:

- $\rho_n(A, G)$ is the expected cardinality of matching produced by A when the input graph is G.
- $\rho_n(A, -)$ is the minimum over all $G \in G_n$ of $\rho_n(A, G)$. Namely, $\rho_n(A, -) = \min_{G \in G_n}[\rho_n(A, G)]$.
- ρ_n is the maximum over all A (randomized online matching algorithms for the maximizing player) of $\rho_n(A, -)$. Namely, $\rho_n = \max_A[\rho_n(A, -)]$. (Showing that the maximum is attained is a technicality that we ignore here.)
- $\rho = \inf_n \frac{\rho_n}{n}$. Namely, ρ is the largest constant (independent of n) such that $\rho \cdot n \leq \rho_n$ for all n. (One might find a definition such as $\rho = \lim_{n \to \infty} \frac{\rho_n}{n}$ more natural, but it turns out that both definitions of ρ give the same value, which will be seen to be $1 - \frac{1}{e}$.)

Karp, Vazirani and Vazirani [7] showed that $\rho_n(Ranking, -) \geq (1 - \frac{1}{e})n - o(n)$, where e is the base of the natural logarithm (and $(1 - \frac{1}{e}) \simeq 0.632$). Unfortunately, that paper had only a conference version and not a journal version, and the proof presented in the conference version appears to have gaps. Later work (e.g., [12, 4, 2]), motivated by extensions of the online matching problem to other problems such as the *adwords* problem, presented alternative proofs, and also established that the $o(n)$ term is not required. There have also been expositions of simpler versions of these proofs. See [1, 10, 3], for example. Summarizing this earlier work, we have:

Theorem 1 *For every bipartite graph $G \in G_n$, the expected cardinality of the matching produced by* Ranking *is at least $(1 - \frac{1}{e})n$. Hence $\rho_n(Ranking, -) \geq (1 - \frac{1}{e})n$, and $\rho \geq 1 - \frac{1}{e} \simeq 0.632$.*

Karp, Vazirani and Vazirani [7] also presented a distribution over G_n, and showed that for every online algorithm, the expected size of the matching produced (expectation taken over random choice of graph from this distribution) is at most $(1 - 1/e)n + o(n)$. This distribution, that we shall refer to as D_n, is defined as follows. Select uniformly at random a permutation τ over V. For every j, the neighbors of vertex u_j are $\{v_{\tau(j)}, \ldots, v_{\tau(n)}\}$. The unique perfect matching M is the set of edges $(u_j, v_{\tau(j)})$ for $1 \leq j \leq n$.

To present the known results regarding D_n more accurately, let as extend previous notation.

- $\rho_n(A, D_n)$ is the expected cardinality of matching produced by A when the input graph G is selected according to distribution D_n. (Hence expectation is taken both over randomness of A and over selection from D_n.)

By definition, for every algorithm A, $\rho_n(A, D_n)$ is an upper bound on $\rho_n(A, -)$.

- $\rho_n(-, D_n)$ is the maximum over all A (randomized online algorithms for the maximizing player) of $\rho_n(A, D_n)$. Namely, $\rho_n(-, D_n) = \max_A[\rho_n(A, D_n)]$. By definition, for every n, $\rho_n(-, D_n)$ is an upper bound on ρ_n.

It is not hard to see (and was shown also in Lemma 13 of [7]) that for every two greedy online algorithms A and A' it holds that $\rho_n(A, D_n) = \rho_n(A', D_n)$. As greedy algorithms are optimal among online algorithms, and *Ranking* is a greedy algorithm, we have the following proposition.

Proposition 2 *For D_n defined as above,*

$$\rho_n(Ranking, D_n) = \rho_n(-, D_n) \geq \rho_n$$

The result of [7] can be stated as showing that $\rho_n(-, D_n) \leq (1 - \frac{1}{e})n + o(n)$. Later analysis (see for example [12], or the lecture notes of Kleinberg [8] or Karlin [6]) replaced the $o(n)$ term by $O(1)$. Moreover, this upper bound holds not only for online randomized integral algorithms (that match edges as a whole), but also for online fractional algorithms (that match fractions of edges). Let us provide more details.

A fractional matching for a bipartite graph $G(U, V; E)$ is a nonnegative weight function w for the edges such that for every vertex $u \in U$ we have $\sum_{v \in N(u)} w(u, v) \leq 1$, and likewise, for every vertex $v \in V$ we have $\sum_{u \in N(v)} w(u, v) \leq 1$. The size of a fractional matching is $\sum_{e \in E} w(e)$. It is well known (see [9], for example) that in bipartite graphs, the size of the maximum fractional matching equals the cardinality of the maximum (integral) matching.

In the online bipartite fractional matching problem, as vertices of U arrive, the maximizing player can add arbitrary positive weights to their incident edges, provided that the result remains a fractional matching. We extend the ρ notation used for the integral case also to the fractional case, by adding a subscript f. Hence for example, $\rho_{f,n}(A, G)$ is the size of the fractional matching produced by an online algorithm A when $G \in G_n$ is the input graph.

It is not hard to see that in the fractional setting, randomization does not help the maximizing player, in the sense that any randomized online algorithm A for fractional matching can replaced by a deterministic algorithm A' that on every input graph produces a fractional matching of at least the same size. (Upon arrival of vertex u, the fractional weight that A' adds to edge (u, v) equals the expected weight that A adds to this edge, where expectation is taken over randomness of A.) Consequently, $\rho_{f,n} \geq \rho_n$, and every upper bound on $\rho_{f,n}$ is also an upper bound on ρ_n.

The following theorem summarizes the known upper bounds on $\rho_{f,n}$, which are also the strongest known upper bounds on ρ_n.

Theorem 3 *For D_n as defined above, $\rho_{f,n}(-, D_n) \leq (1 - \frac{1}{e})n + O(1)$. Consequently, $\rho_n(-, D_n) \leq (1 - \frac{1}{e})n + O(1)$.*

The combination of Theorems 1 and 3 implies the following corollary:

Corollary 4 *Using notation as above, $\rho = 1 - \frac{1}{e}$ and $\rho_f = 1 - \frac{1}{e}$. The* Ranking *algorithm (which produces an integral matching) is asymptotically optimal (for the maximizing player) for online bipartite matching both in the integral and in the fractional case. The distribution D_n is asymptotically optimal for the minimizing player, both in the integral and in the fractional case.*

In this manuscript, we shall be interested not only in the asymptotic ratios ρ and ρ_f, but also in the exact ratios ρ_n and $\rho_{f,n}$. Every (integral) matching is also a fractional matching, hence one may view *Ranking* also as an online algorithm for fractional matching. As such, *Ranking* is easily seen not to be optimal for some n. For example, when $n = 4$, tedious but straighforward analysis shows that a different known algorithm referred to as *Balance* (see Section 2) satisfies $\rho_{f,4}(Balance, -) > \rho_{f,4}(Ranking, -)$ (details omitted). However, for the integral case, it was conjectured in [7] that both *Ranking* and D_n are optimal for every n. Namely, the conjecture is:

Conjecture 5 *$\rho_n = \rho_n(Ranking, D_n)$ for every n.*

The above conjecture, though still open, adds motivation (beyond Proposition 2) to determine the exact value of $\rho_n(Ranking, D_n)$. This is done in the following theorem.

Theorem 6 *Let the function $a(n)$ be such that $\rho_n(Ranking, D_n) = \frac{a(n)}{n!}$ for all n. Then $a(n) = (n+1)! - d(n+1) - d(n)$, where $d(n)$ is the number of derangements (permutations with no fixed points) on the numbers $[1, n]$. Consequently, $\rho_n(-, D_n) = (1 - \frac{1}{e})n + 1 - \frac{2}{e} + O(\frac{1}{n!}) \simeq (1 - \frac{1}{e})n + 0.264$, and this is also an upper bound on ρ_n.*

The rest of this paper is organized as follows. In Section 2 we review a proof of Theorem 3. In doing so, we determine the value of the $O(1)$ term stated in the theorem, and also show that the upper bound is tight. Hence we end up proving the following theorem:

Theorem 7 *For every n,* Balance *is the fractional online algorithm with best approximation ratio, D_n is the distribution over graphs for which the approximation ratio is worst possible, and*

$$\rho_{f,n} = \rho_{f,n}(Balance, D_n) = (1 - \frac{1}{e})n + \frac{1}{2} - \frac{1}{2e} + O(\frac{1}{n}) \simeq (1 - \frac{1}{e})n + 0.316$$

In Section 3 we prove Theorem 6. The combination of Theorems 6 and 7 implies that $\rho_n < \rho_{f,n}$ for sufficiently large n. It also implies that $\rho_{f,n}(Balance, D_n) > \rho_{f,n}(Ranking, D_n)$ for sufficiently large n. Hence Proposition 2 does not extend to online fractional matching.

In an appendix (Section A) we review a proof (due to [3]) of Theorem 1, and derive from it an upper bound of $(1 - \frac{1}{e})n + \frac{1}{e}$ on $\rho_n(Ranking, D_n)$. This last upper bound is weaker than the upper bounds of Theorems 6 and 7, but its proof is different, and hence might turn out useful in attempts to resolve Conjecture 5.

1.1 Preliminaries – *MonotoneG*

When analyzing $\rho_n(Ranking, D_n)$ we shall use the following observation so as to simplify notation. Because *Ranking* is oblivious to names of vertices, the expected size of the matching produced by *Ranking* on every graph in the support of D_n is the same. Hence we shall consider one representative graph from D_n, that we refer to as the monotone graph *MonotoneG*, in which γ (in the definition of D_n) is the identity permutation. The monotone graph $G(U, V; E)$ satisfies $E = \{(u_i, v_j) \mid j \geq i\}$, and its unique perfect matching is $M = \{(u_i, v_i) \mid 1 \leq i \leq n\}$. Statements involving $\rho_n(Ranking, D_n)$ will be replaced by $\rho_n(Ranking, MonotoneG)$, as both expressions have the same value.

Likewise, the algorithm *Balance* is oblivious to names of vertices, and statements involving $\rho_{f,n}(Balance, D_n)$ will be replaced by $\rho_{f,n}(Balance, MonotoneG)$.

2 Online fractional matchings

Let us present a specific online fractional matching algorithm that is often referred to as *Balance*, which is the natural fractional analog of an algorithm by the same name introduced in [5]. *Balance* maintains a *load* $\ell(v)$ for every vertex $v \in V$, equal to the sum of weights of edges incident with v. Hence at all times, $0 \leq \ell(v) \leq 1$. Upon arrival of a vertex u with a set of neighbors $N(u)$, *Balance* distributes a weight of $\min[1, |N(u)| - \sum_{v \in N(u)} \ell(v)]$ among the edges incident with u, maintaining the resulting loads as balanced as possible. Namely, one computes a threshold t such that $\sum_{v \in N(u) | \ell(v) < t}(t - \ell(v)) = \min[1, |N(u)| - \sum_{v \in N(u)} \ell(v)]$, and then adds fractional value $t - \ell(v)$ to each edge (u, v) for those vertices $v \in N(u)$ that have load below t.

We first present a proof of Theorem 3 based on previous work. The theorem is restated below, with the additive $O(1)$ term instantiated. Previous work either did not specify the $O(1)$ additive term (e.g., in [6]), or derived an $O(1)$ term that is not tight (e.g., in [8]).

Theorem 8 *For every n it holds that*

$$\rho_{f,n}(-, D_n) = (1 - \frac{1}{e})n + \frac{1}{2} - \frac{1}{2e} + O(\frac{1}{n}) \simeq (1 - \frac{1}{e})n + 0.316$$

Moreover, $\rho_{f,n}(-, D_n) = \rho_{f,n}(Balance, D_n)$.

Proof. For all graphs in the support of D_n, the size of the fractional matching produced by *Balance* is the same (by symmetry). Hence for simplicity of notation, consider the fractional matching produced by *Balance* when the input graph is the monotone graph *MonotoneG* (see Section 1.1). It is not hard to see that when vertex u_i arrives, *Balance* raises the load of each vertex in $\{v_i, \ldots, v_n\}$ by $\frac{1}{n-i+1}$. This can go on until the largest k satisfying $\sum_{i=1}^{k} \frac{1}{n-i+1} \leq 1$. Thereafter, when vertex u_{k+1} arrives, *Balance* can raise the load of its $n-k$ neighbors from $\sum_{i=1}^{k} \frac{1}{n-i+1}$ to 1. Hence altogether the size of the fractional matching is precisely $k + (n-k)(1 - \sum_{i=1}^{k} \frac{1}{n-i+1})$, for k as above.

The value of k can be determined as follows. It is known that the harmonic number $H_n = \sum_{i=1}^{n} \frac{1}{i}$ satisfies $H_n = \ln n + \gamma + \frac{1}{2n} + O(\frac{1}{n^2})$, where $\gamma \simeq 0.577$ is the Euler-Mascheroni constant. k is the largest integer such that $H_n - H_{n-k} \leq 1$. Defining $\alpha \triangleq \frac{n-k}{n}$, we have that

$$H_n - H_{n-k} = \ln n + \gamma + \frac{1}{2n} + O(\frac{1}{n^2}) - \ln \alpha n - \gamma - \frac{1}{2\alpha n} + O(\frac{1}{n^2}) = \ln \frac{1}{\alpha} - \frac{\frac{1}{\alpha} - 1}{2n} + O(\frac{1}{n^2})$$

Choosing $\alpha = \frac{1}{e}$ (and temporarily ignoring the fact that in this case $k = (1-\frac{1}{e})n$ is not an integer), we get that $H_n - H_{n-k} = 1 - \frac{e-1}{2n} + O(\frac{1}{n^2})$. The size of a matching is then

$$(1 - \frac{1}{e})n + \frac{n}{e}(\frac{e-1}{2n} + O(\frac{1}{n^2})) = (1 - \frac{1}{e})n + \frac{1}{2} + \frac{1}{2e} + O(\frac{1}{n})$$

as desired.

The fact that $k = (1 - \frac{1}{e})n$ above was not an integer requires that we round k down to the nearest integer. The effect of this rounding is bounded by the effect of changing the number of neighbors available to u_k and to u_{k+1} by one (compared to the computation without the rounding). Given that the number of neighbors is roughly $\frac{n}{e}$, the overall effect on the size of the fractional matching is at most $O(\frac{1}{n})$.

We conclude that $\rho_{f,n}(Balance, D_n) = (1-\frac{1}{e})n + \frac{1}{2} + \frac{1}{2e} + O(\frac{1}{n})$, implying that $\rho_{f,n}(-, D_n) \geq (1 - \frac{1}{e})n + \frac{1}{2} + \frac{1}{2e} + O(\frac{1}{n})$. In remains to show that $\rho_{f,n}(-, D_n) \leq (1 - \frac{1}{e})n + \frac{1}{2} + \frac{1}{2e} + O(\frac{1}{n})$. This follows because *Balance* is the best possible online algorithm (for fractional bipartite matching) against D_n. Let us provide more details.

Given an input graph from the support of D_n, we shall say that a vertex $v \in V$ is *active* in round i if it is a neighbor of u_i. Initially all vertices are active, and after every round, one more vertex (chosen at random among the active vertices) becomes inactive, and remains inactive forever. Let $a(i)$ denote the number of active vertices at the beginning of round i, and note that $a(i) = n - i + 1$. Consider an arbitrary online algorithm. Let $L(i)$ denote the average load of the active vertices at the beginning of round i. Then in round i, the average load first increases by at most $\frac{1}{a(i)}$ (as long as it does not exceed 1) by raising weights of edges, and thereafter, making one vertex inactive keeps the average load unchanged in expectation (over choice of input from D_n). Hence in expectation, in every round, the average load does not exceed the value of the average load obtained by *Balance*. This means that in every round, in expectation, the amount of unused load of the vertex that became inactive is smallest when the online maximizing algorithm is *Balance*. Summing over all rounds and using the linearity of expectation, *Balance* suffers the smallest sum of unused load, meaning that it maximizes the final expected sum (over all V) of loads. The sum of loads equals the size of the fractional matching. ■

We now prove Theorem 7.

Proof.[Theorem 7] Given Theorem 8, it suffices to show that $\rho_{f,n}(Balance, -) = \rho_{f,n}(Balance, D_n)$, namely, that D_n is the worst possible distribution over input graphs for the algorithm *Balance*. Moreover, given that *Balance* is oblivious to the names of vertices, it suffices to show that *MonotoneG* is the worst possible graph for *Balance*.

Let $G(U, V; E) \in G_n$ be a graph for which $\rho_{f,n}(Balance, G) = \rho_{f,n}(Balance, -)$. As *Balance* is oblivious to the names of vertices, we may assume that $\{(u_i, v_i)|1 \leq i \leq n\}$ is a perfect matching in G.

We use the notation $N(w)$ to denote the set of neighbors of a vertex w in the graph G. When running *Balance* on G, we use the notation $m(i,j)$ to denote the weight that the fractional matching places on edge (u_i, v_j) (and $m(i,j) = 0$ if $(u_i, v_j) \notin E$), and $m_i(j)$ to denote $\sum_{1 \leq \ell \leq i} m(u_\ell, v_j)$. Clearly, $m_i(j)$ is non-decreasing in i. The size of the final fractional matching is $m = \sum_{j=1}^{n} m_n(j)$. When referring to a graph G', we shall use the notation N' and m' instead of N and m.

An edge (u_i, v_j) with $j < i$ is referred to as a *backward* edge.

Proposition 9 *Without loss of generality, we may assume that G has no backward edges. Hence $m_i(j) = m_j(j)$ for all $i > j$.*

Proof. Suppose otherwise, and let i be largest so that u_i has backward edges. Modify G by removing all backward edges incident with u_i, thus obtaining a graph G'. Compare the performance of *Balance* against the two graphs, G and G'. On vertices $u_1, \ldots, u_{i-1}$, both graphs produce the same fractional matching. The extent to which u_i is matched is at least as large in G as it is in G' (because also backward edges may participate in the fractional matching). Moreover, for every vertex v_j for $i < j \leq n$, it holds that $m'_i(j) \geq m_i(j)$. It follows that for every vertex u_ℓ for $\ell > i$, its marginal contribution to the fractional matching in G is at least as large as its marginal contribution in G'. Hence the fractional matching produced by *Balance* for G' is not larger than that produced for G. Repeating the above argument, all backward edges can be eliminated from G without increasing the size of the fractional matching. ■

Lemma 10 *Without loss of generality we may assume that:*

1. *$m_i(i) \leq m_j(j)$ (or equivalently, $m_n(i) \leq m_n(j)$) for all $i < j$.*
2. *$m_i(i) \geq m_i(j)$ for all i and j.*

Proof. We first present some useful observations. For $1 \leq i < n$, consider the set $N(u_i)$ of neighbours of u_i in G (and recall that $v_i \in N(u_i)$, and that there are no backward edges). Then without loss of generality we may assume that $m_i(i) \geq m_i(j)$ for all $v_j \in N(u_i)$. This is because if there is some vertex $v_j \in N(u_i)$ with $m_i(j) > m_i(i)$, then it must hold (by the properties of *Balance*) that $m(i,j) = 0$. Hence the run of *Balance* would not change if the edge (u_i, v_j) is removed from G (and then $v_j \notin N(u_i)$).

Moreover, we may assume that $m_i(i) = m_i(j)$ for all $v_j \in N(u_i)$. Suppose otherwise. Then for $v_j \in N(u_i)$ with smallest $m_i(j)$, modify G to a graph G' as follows. For all $\ell < i$, make u_ℓ a neighbor of v_i iff it was a neighbor of v_j, and make u_ℓ a neighbor of v_j iff it was a neighbor of v_i. The final size of the fractional matching in G' (which is $\sum_{j=1}^{n} m'_n(j)$) cannot be larger than in G. This is because $m'_i(i) < m_i(i)$, $m'_i(j) > m_i(j)$ and for $\ell \neq j$ satisfying $\ell > i$ it holds that $m'_i(\ell) = m_i(\ell)$. Moreover, as $m_i(i) < m_i(j) \leq 1$, u_i is fully matched in G and hence also in G', so the total size of fractional matching after step i is the same in both graphs. Thereafter, the marginal increase of the fractional matching at each step cannot be larger in G' than it is in G.

By the same arguments as above we may assume that $m_{i+1}(i+1) = m_{i+i}(j)$ for all $v_j \in N(u_{i+1})$.

Suppose now that item 1 fails to hold. Then for some $1 \leq i \leq n-1$ it holds that $m_i(i) > m_{i+1}(i+1)$. Vertices u_i and u_{i+1} cannot have a common neighbor because if they do (say, v_ℓ) it holds that $m_{i+1}(i+1) = m_{i+1}(\ell) \geq m_i(\ell) = m_i(i_i)$. Hence we may exchange the order of u_i and u_{i+1} (and likewise v_i and v_{i+1}) without affecting the size of the fractional matching produced by *Balance*.

Repeating the above argument whenever needed we prove item 1 of the lemma.

For $j < i$ item 2 holds because $m_i(j) = m_j(j) \leq m_i(i)$ (the last inequality follows from item 1). For $j > i$ item 2 holds because at the first point in time $\ell \leq i$ in which $m_\ell(j) = m_i(j)$ it must be that $m_\ell(j) = m_\ell(\ell)$, and item 1 implies that $m_\ell(\ell) \leq m_i(i)$. ■

It is useful to note that Lemma 10 implies that there is some round number t such that for all $\ell \geq t$ vertex v_ℓ is fully matched (namely, $m_n(\ell) = 1$), and for every $\ell < t$ vertex v_ℓ is not fully matched (namely, $m_n(\ell) < 1$). As to vertices in u, for $\ell < t$ vertex u_ℓ is fully matched, for $\ell > t$ vertex u_ℓ contributes nothing to the fractional matching, and u_t is either partly matched or fully matched. Recalling that m denotes the size of the final fractional matching, we thus have (for t as above):

$$m = t - 1 + \sum_{j \geq t} m(t, j) \tag{1}$$

At every step i, the contribution of vertex v_i towards the fractional matching is finalized at that step, namely, $m_n(i) = m_i(i)$. Lemma 10 implies that for the worst graph G, this vertex v_i is the one with largest m_i value at this given step. Hence $m_i(i) = \max_{j \geq i}[m_i(j)]$ and we have:

$$m = \sum_{i=1}^{n} m_n(i) = \sum_{i=1}^{n} m_i(i) = \sum_{i=1}^{n} \max_{j \geq i}[m_i(j)].$$

At this point it is intuitively clear why *MonotoneG* is the graph in G_n on which *Balance* produces the smallest fractional matching. This is because with *MonotoneG*, at each step i the fractional matching gets credited a value $m_i(i)$

that is the average of the values $m_i(j)$ for $j \geq i$, whereas for G its gets credited the maximum of these values. Below we make this argument rigorous.

Consider an alternative *averaging process* replacing algorithm *Balance*. It uses the same fractional matching as in *Balance* and the same $m(i,j)$ values, but maintains values $m'_i(i)$ that may differ from $m_i(i)$. At round 1, instead of being credited the maximum $m_1(1) = \max_{j \geq 1}[m_1(j)]$, the process is credited only the average $m'_i(1) = \frac{1}{n}\sum_{j=1}^{n} m_i(j)$. The remaining $\max_{j \geq 1}[m_1(j)] - \frac{1}{n}\sum_{j=1}^{n} m_1(j)$ is referred to as the *slackness* $s(1)$. More generally, at every round $i > 1$, instead of being credited by $\max_{j \geq i}[m_i(j)]$ at step i, the averaging process gets credit from two sources. One part of the credit is the average $\frac{1}{n-i+1}\sum_{j=i}^{n} m_i(j)$, where $s(i) = \max_{j \geq i}[m_i(j)] - \frac{1}{n-i+1}\sum_{j=i}^{n} m_i(j)$ is the slackness generated at round i. In addition, the process gets credit also for the slackness accumulated in previous rounds $\ell < i$, in such a way that each slackness variable $s(\ell)$ gets distributed evenly among the $n - \ell$ rounds that follow it. Hence we set

$$m'_i(i) = \frac{1}{n-i+1}\sum_{j=i}^{n} m_i(j) + \sum_{\ell=1}^{i-1} \frac{s(\ell)}{n-\ell}. \tag{2}$$

The averaging process continues until the first round t' at which $m'_{t'}(t') \geq 1$, at which point $m'_j(j)$ is set to 1 for all $j \geq t'$, and the process ends. The size of the fractional matching associated with the averaging process is $m' = \sum_{i=1}^{n} m'_i(i)$. Computing m' using the contributions of the vertices from U, for t' as above, we get that:

$$m' = t' - 1 + \sum_{j \geq t'} m(t', j) \tag{3}$$

Proposition 11 *For the graph G, the size of the fractional matching produced by the averaging process is no larger than that produced by* Balance. *Namely, $m' \leq m$.*

Proof. Compare Equations (1) and (3). If $t' = t$ then $m' = m$, and if $t' < t$ then $m' < m$. Hence it suffices to show that the assumption $t' \geq t$ implies that $t' = t$. This follows because $m_t(j) = 1$ for all $j \leq t$ (as noted above), and so:

$$m'_t(t) = \frac{1}{n-t+1}\sum_{j=t}^{n} m_t(j) + \sum_{\ell=1}^{t-1} \frac{s(\ell)}{n-\ell} = 1 + \sum_{\ell=1}^{t-1} \frac{s(\ell)}{n-\ell} \geq 1$$

where the last inequality holds because all slackness variables $s(\ell)$ are non-negative. ■

Proposition 12 *For* MonotoneG*, running the averaging process and running* Balance *are exactly the same process, giving $m'(MonotoneG) = m(MonotoneG)$.*

Proof. This is because when running *Balance* on *MonotoneG*, at every round i we have that $m_i(i) = m_i(j)$ for all $j > i$. Hence there is no difference between the average and the maximum of the $m_i(j)$ for $j \geq i$. ■

Proposition 13 *The size of the fractional matching produced by the averaging process for graph G is not smaller than the size it produces for* MonotoneG. *Namely, $m'(G) \geq m'(MonotoneG)$.*

Proof. Running the averaging process on graph G, we claim that for every round $i < t'$ we have that:

$$m'_i(i) = \sum_{k \leq i} \frac{1}{n-k+1} \tag{4}$$

The equality can be proved by induction. For $i = 1$ both sides of the equality are $\frac{1}{n}$. For the inductive step, recalling Equation 2 one can infer that

$$m'_{i+1}(i+1) = \frac{1}{n-i}\left((n-i+1)m'_i(i) - m'_i(i) + 1\right)$$

where the $+1$ term is because $i < t'$. Likewise, the right hand side develops in the same way:

$$\sum_{k \leq i+1} \frac{1}{n-k+1} = \frac{1}{n-i}\left((n-i+1)\sum_{k \leq i} \frac{1}{n-k+1} - \sum_{k \leq i} \frac{1}{n-k+1} + 1\right)$$

The left hand side of Equation (4) concerns graph G. Observe that $m'_i(i)$ for *MonotoneG* exactly equals the right hand side of Equation (4). It follows that the averaging process ends at the same step t' both on the graph G and on *MonotoneG*, and up to step t' the accumulated fractional matching m' is identical. For rounds $j \geq t'$ we have that $m'_j(j) = 1$ for G and it cannot be larger than 1 for *MonotoneG*, proving the proposition. ■

Combining the three propositions above we get that:

$$m(G) \geq m'(G) \geq m'(MonotoneG) = m(MonotoneG)$$

This completes the proof of Theorem 7. ■

3 Online integral matching

The first part of Theorem 6 is restated in the following theorem (recall the definition of the monotone graph *MonotoneG* in Section 1.1).

Theorem 14 *Let the function $a(n)$ be such that $\rho_n(Ranking, MonotoneG) = \frac{a(n)}{n!}$ for all n. Then $a(n) = (n+1)! - d(n+1) - d(n)$, where $d(n)$ is the number of derangements (permutations with no fixed points) on the numbers $[1, n]$.*

Proof. When the input is *MonotoneG*, then for every permutation π used by *Ranking*, the matching M' produced satisfies the following two properties:

- All vertices in some prefix of U are matched, and then no vertices in the resulting suffix are matched. This is because all neighbors of u_{j+1} are also neighbors of u_j, so if u_{j+1} is matched then so is u_j.
- The order in which vertices of V are matched is consistent with the order π (for those vertices that are matched – some vertices of V may remain unmatched). In other words, if two vertices v_i and v_j are matched and $\pi(i) < \pi(j)$, then the vertex $u \in U$ matched with v_i arrived earlier (has smaller index) than the vertex $u' \in U$ matched with v_j.

Some arguments in the proof that follows make use of the above properties, without explicitly referring to them.

Fix n and *MonotoneG* as input. Let Π_n denote the set of all permutations over V. Hence $|\Pi_n| = n!$. *Ranking* picks one permutation $\pi \in \Pi_n$ uniformly at random. Recall our notation that $\pi(i)$ is the rank of v_i under π. We shall use π_i to denote the item of rank i in π (namely, $\pi_i = \pi^{-1}(i)$). For $i \leq n$, let $a(n, i)$ denote the number of permutations $\pi \in \Pi_n$ under which π_i is matched.

Proposition 15 *For $a(n)$ as defined in Theorem 14 and $a(n, i)$ as defined above, it holds that $a(n) = \sum_{i=1}^{n} a(n, i)$.*

Proof. For a permutation $\pi \in \Pi_n$, let $x(\pi)$ denote the size of the greedy matching produced when *Ranking* uses π and the input graph in *MonotoneG*. Then by definition:

$$a(n) = \sum_{\pi \in \Pi_n} x(\pi).$$

By changing the order of summation:

$$\sum_{\pi \in \Pi_n} x(\pi) = \sum_{i=1}^{n} a(n, i).$$

Combining the above equalities proves the proposition. ■

Proposition 15 motivates the study of the function $a(n, i)$.

Lemma 16 *The function $a(n, i)$ satisfies the following:*

1. *$a(n, 1) = n!$ for every $n \geq 1$.*
2. *$a(n, i) = a(n, i+1) + a(n-1, i)$ for every $1 \leq i < n$.*

Proof. The first statement in the lemma holds because in every permutation π, the item π_1 is matched with u_1. Hence it remains to prove the second statement.

Fixing $n > 1$ and $i < n$, consider the following bijection $B_i : \Pi_n \longrightarrow \Pi_n$, where given a permutation $\pi \in \Pi_n$, $B_i(\pi)$ flips the order between locations i and $i + 1$. Namely, $B_i(\pi)_i = \pi_{i+1}$ and $B_i(\pi)_{i+1} = \pi_i$ (we use $B_i(\pi)_i$ as shorthand notation for $(B_i(\pi))_i$). We compare the events that π_i is matched by the greedy matching when *Ranking* uses π with the event that $B_i(\pi)_{i+1}$ is matched by the greedy matching when *Ranking* uses $B_i(\pi)$.

There are four possible events:

1. Both π_i and $B_i(\pi)_{i+1}$ are matched.

2. Neither π_i nor $B_i(\pi)_{i+1}$ are matched.

3. π_i is matched but $B_i(\pi)_{i+1}$ is not matched.

4. π_i is not matched but $B_i(\pi)_{i+1}$ is matched.

Though any of the first three events may happen, the fourth event cannot possibly happen. This is because the item in location $i + 1$ in $B_i(\pi)$ is moved forward to location i in π, so if the greedy algorithm matches it (say to u_j) in $B_i(\pi)$, then the greedy algorithm must match it (either to the same u_j or to the earlier u_{j-1}) in π.

It follows that $a(n, i) - a(n, i+1)$ exactly equals the number of permutations in which the third event happens. Hence we characterize the conditions under which the third event happens. Let u_j be the vertex matched with π_i in π. Up to the arrival of u_j, the behavior of *Ranking* on $B_i(\pi)$ and π is identical. Thereafter, for u_j not to be matched to $B_i(\pi)_{i+1} = \pi_i$, it must be matched to the earlier $B_i(\pi)_i$. Thereafter, for u_{j+1} not to be matched to $B_i(\pi)_{i+1}$, it must be that $B_i(\pi)_{i+1}$ is not a neighbor of u_{j+1}. But $B_i(\pi)_{i+1} = \pi_i$ is a neighbor of u_j (it was matched to u_j under π), and hence it must be that $\pi_i = v_j$. Summarizing, the third event happens if and only if the permutation $B_i(\pi)$ comes from the following class $\hat{\Pi}$, where permutations $\hat{\pi} \in \hat{\Pi}$ are those that have the property that $\hat{\pi}_i$ is matched, and $\hat{\pi}_{i+1} = v_j$, for the same j for which u_j is the vertex matched with $\hat{\pi}_i$. Consequently, $a(n, i) = a(n, i + 1) + |\hat{\Pi}|$.

To complete the proof of the lemma, it remains to show that $|\hat{\Pi}| = a(n-1, i)$. Let $\Pi' \subset |\Pi_{n-1}|$ be the set of these permutations $\pi' \in \Pi_{n-1}$ under which *Ranking* (when $|U| = |V| = n - 1$) matches the item π'_i.

Claim 17 *For $\hat{\Pi}$ and Π' as defined above it holds that $|\hat{\Pi}| = |\Pi'|$.*

Proof. We first show a mapping from $\hat{\Pi}$ to Π'. Given $\hat{\pi} \in \hat{\Pi}$, let $v_j = \hat{\pi}_{i+1}$. To obtain permutation $\pi' \in \Pi_{n-1}$ from $\hat{\pi}$, remove v_j from $\hat{\pi}$, identify location k in $\hat{\pi}$ with location $k - 1$ in π' (for $i + 2 \leq k \leq n$), and identify item v_ℓ of $\hat{\pi}$ with item $v_{\ell-1}$ of π' (for $j + 1 \leq \ell \leq n$). We show now that $\pi' \in \Pi'$ (namely, π'_i is matched, when the input graph is *MonotoneG* with $|U| = |V| = n - 1$).

The vertices $u_1, \ldots, u_{j-1}$ are matched to exactly the same locations in π' and in $\hat{\pi}$, because the only vertices whose indices were decremented had index

$\ell \geq j+1$, and are neighbors of $u_1, \ldots, u_{j-1}$ both before and after the decrement. Let $v_k = \hat{\pi}_i$ and note that $k > j$, because v_k is matched to u_j and it is not $v_j = \hat{\pi}_{i+1}$. Hence $\pi'_i = v_{k-1}$ and it too is a neighbor of u_j, because $j \leq k-1$. Hence π'_i will be matched to u_j.

Conversely, we have the following mapping from Π' to $\hat{\Pi}$. Given $\pi' \in \Pi'$, let u_j be the vertex matched π'_i. To obtain permutation $\hat{\pi} \in \hat{\Pi}$ from π', identify location k in $\hat{\pi}$ with location $k-1$ in π' (for $i+2 \leq k \leq n$), identify item v_ℓ of $\hat{\pi}$ with item $v_{\ell-1}$ of π' (for $j+1 \leq \ell \leq n$), and set $\hat{\pi}_{i+1} = v_j$. We show now that $\hat{\pi} \in \hat{\Pi}$.

As in the first mapping, the vertices $u_1, \ldots, u_{j-1}$ are matched to exactly the same locations in π' and in π. Let $v_k = \pi'_i$ and note that $k \geq j$, because v_k was matched to u_j. Hence $\hat{\pi}_i = v_{k+1}$ is neighbor of u_j, and will be matched to u_j. On the other hand, $\hat{\pi}_{i+1} = v_j$ will not be matched because it is not a neighbor of any of $[u_{j+1}, u_n]$. Hence $\hat{\pi} \in \hat{\Pi}$.

Given the two mappings described above (one is the inverse of the other) we have a bijection between Π' and $\hat{\Pi}$, proving the claim. ■

The claim above implies that $|\hat{\Pi}| = |\Pi'| = a(n-1, i)$, and consequently that $a(n,i) = a(n, i+1) + a(n-1, i)$, proving the lemma. ■

In passing, we note the following corollary.

Corollary 18 *For $a(n,i)$ and $a(n)$ as defined above, $a(n) = (n+1)! - a(n+1, n+1)$.*

Proof. Using item 1 of Lemma 16 we have that $a(n+1, 1) = (n+1)!$. Applying item 2 of Lemma 16 iteratively for all $1 \leq i \leq n$ we have that $a(n+1,1) - a(n+1, n+1) = \sum_{i=1}^{n} a(n,i)$. Proposition 15 shows that $\sum_{i=1}^{n} a(n,i) = a(n)$. Combining these three equalities we obtain $a(n) = (n+1)! - a(n+1, n+1)$, as desired. ■

Corollary 18 can also be proved directly, without reference to Lemma 16. See Appendix B for details.

To obtain expressions for the values $a(n,i)$, let us introduce additional notation. A *fixpoint* (or *fixed point*) in a permutation π is an item that does not change its location under π (namely, $\pi(i) = i$). For $n \geq 1$ and $1 \leq i \leq n$ define $d(n,i)$ be the number of permutations over $[n]$ in which the only fixpoints (if any) are among the first i items. For example, $d(3,1) = 3$ due to the permutations 132 (only 1 is a fixed point) 231 (no fixpoints) and 312 (no fixpoints).

Lemma 19 *The function $d(n,i)$ satisfies the following:*

1. *$d(n,n) = n!$ for every $n \geq 1$.*
2. *$d(n, i+1) = d(n,i) + d(n-1, i)$ for every $1 \leq i < n$.*

Proof. $d(n, n)$ denotes the number of permutations on $[n]$ with no restrictions, and hence $d(n, n) = n!$, which is the first statement of the lemma.

Consider now the second statement of the lemma. Let $\Pi_{n,i}$ denote the set of permutations in which the only fixpoints (if any) are among the first i items. Then the second statement asserts that $|\Pi_{n,i+1}| = |\Pi_{n,i}| + |\Pi_{n-1,i}|$. The set $\Pi_{n,i+1}$ can be partitioned in two. In one part $i + 1$ is not a fixpoint. This part is precisely $\Pi_{n,i}$. In the second part, $i + 1$ is a fixpoint. To specify a permutation in this part we need to specify the location of the remaining $n - 1$ items, where the only fixpoints allowed are among the first i items. The number of permutations satisfying these constraints is $\Pi_{n-1,i}$, by definition. Hence indeed $|\Pi_{n,i+1}| = |\Pi_{n,i}| + |\Pi_{n-1,i}|$, proving the lemma. ■

Corollary 20 *For every $n \geq 1$ and $1 \leq i \leq n$ it holds that $a(n, i) = d(n, n + 1 - i)$.*

Proof. The proof is by induction on n, and for every value of n, by induction on i.

For the base case $n = 1$, necessarily $i = 1$ (and hence also $n + 1 - i = 1$) and indeed we have $a(1, 1) = 1 = d(1, 1)$. Fixing $n > 1$, the base case for i is $i = 1$ (and $n + 1 - 1 = n$) and indeed we have that $a(n, 1) = n! = d(n, n)$. For the inductive step, consider $a(n, i)$ with $n > 1$ and $1 < i \leq n$, and assume the inductive hypothesis for $n' < n$ and the inductive hypothesis for n and $i' < i$. Then we have:

$$a(n, i) = a(n, i-1) - a(n-1, i-1) = d(n, n-i+2) - d(n-1, n-i+1) = d(n, n-i+1)$$

The first equality is by Lemma 16, the second equality is by the inductive hypothesis, and the third equality is by Lemma 19. ■

We can now complete the proof of Theorem 14. By Corollary 18 we have that $a(n) = (n+1)! - a(n+1, n+1)$. By Corollary 20 we have that $a(n+1, n+1) = d(n + 1, 1)$. By definition, $d(n + 1, 1)$ is the number of permutations on $[n + 1]$ in which only item 1 is allowed to be a fixpoint. This number is precisely $d(n + 1) + d(n)$ (where $d(j)$ are the derangement numbers), where the term $d(n + 1)$ counts those permutations in which there is no fixpoint, and the term $d(n)$ counts those permutations in which item 1 is the only fixpoint. ■

The second part of Theorem 6 is restated in the following Corollary.

Corollary 21 *For every n,*

$$\rho_n(Ranking, MonotoneG) = (1 + \frac{1}{e})n + (1 - \frac{2}{e}) + \nu(n)$$

where $|\nu(n)| < \frac{1}{n!}$.

Proof. Theorem 14 shows that $a(n) = (n+1)! - d(n+1) - d(n)$, where $d(n)$ are the derangement numbers. It is known that $d(n) = \frac{n!}{e}$ rounded to the nearest integer. Hence $|d(n) - \frac{n!}{e}| < \frac{1}{2}$ and $|d(n+1) + d(n) - \frac{(n+1)!}{e} - \frac{n!}{e}| < 1$. Hence $|a(n) - (1 - \frac{1}{e})(n+1)! - \frac{n!}{e}| < 1$. Dividing by $n!$ and replacing $(1 - \frac{1}{e})(n+1)$ by $(1 - \frac{1}{e})n + 1 - \frac{1}{e}$ the corollary is proved. ■

3.1 Some related sequences

To illustrate the values of some of the parameters involved in the proof of Theorem 14, consider a triangular table T where row n has n columns. The entries (for $1 \leq i \leq n$) are $d(n,i)$, as defined prior to Lemma 19. Recall that $d(n,i) = a(n, n+1-i)$, hence the table also provides the $a(n,i)$ values. We initialize the diagonal of the table by $d(n,n) = n!$. Thereafter we fill the remaining cells of table row by row, by using the relation $d(n,i) = d(n,i+1) - d(n-1,i)$, implied by Lemma 19. Finally, compute $a(n) = \sum_{i=1}^{n} a(n,i) = \sum_{i=1}^{n} d(n,i)$ by summing up each row. The table below shows the computation of $a(n)$ for $n \leq 6$.

n	d(n,1)=a(n,n)	d(n,2)	d(n,3)	d(n,4)	d(n,5)	d(n,6)	a(n)
1	1						1
2	1	2					3
3	3	4	6				13
4	11	14	18	24			67
5	53	64	78	96	120		411
6	309	362	426	504	600	720	2921

The table T is identical in its definition to Sequence A116853, named *Difference triangle of factorial numbers read by upward diagonals*, in *The Online Encyclopedia of Integer Sequences* [13]. The row sums (and hence $a(n)$) in this table give Sequence A180191 (with an offset of 1 in the value of n), named *Number of permutations of* $[n]$ *having at least one succession*. The first column (which equals $a(n,n)$) is the sequence A000255. These relations between $a(n)$ and the various sequences in [13] helped guide the statement and proof of Theorem 14.

The derangement numbers $d(n)$ (which form the sequence A000166) can be easily computed by the recurrence $d(n) = n \cdot d(n-1) + (-1)^n$ (due to Euler). The table below shows the computation of $a(n) = (n+1)! - d(n+1) - d(n)$ for $n \leq 7$.

n	n!	d(n)	a(n)
1	1	0	1
2	2	1	3
3	6	2	13
4	24	9	67
5	120	44	411
6	720	265	2921
7	5040	1854	23633
8	40320	14833	

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A A performance guarantee for *Ranking*

For completeness, we review here a proof of Theorem 1. The proof that we present uses essentially the same mathematical expressions as the proof presented in [2]. A simple presentation of the proof of [2] appeared in a blog post of Claire Mathieu [10] (with further slight simplifications made possible by a comment provided there by Pushkar Tripathi). We shall give an arguably even simpler presentation, due to Eden, Feldman, Fiat and Segal [3]. The proofs in [2, 10] make use of linear programming duality. The proof below is based on an economic interpretation, and a proof technique that splits *welfare* into the sum of *utility* and *revenue*. These last two terms turn out to be scaled versions of the dual variables used in [2, 10], but the proof does not need to make use of LP duality.

Proof.[Theorem 1] Fix an arbitrary perfect matching M in G. Given a vertex $v \in V$, we use $M(v)$ to denote the vertex in U matched with v under M.

Recall that *Ranking* chooses a random permutation π over V. Equivalently, we may assume that every vertex $v_i \in V$ chooses independently uniformly at random a real valued weight $w_i \in [0, 1]$, and then the vertices of V are sorted in order of increasing weight (lowest weight first). This gives a random permutation π. The same permutation π is also obtained if each weight w_i is replaced by a "price" $p_i = e^{w_i - 1}$ and vertices are sorted by prices (because e^{x-1} is a monotonically increasing function in x). Observe that $p_i \in [\frac{1}{e}, 1]$, though it is not uniformly distributed in that range. The expected price that *Ranking* assigns to an item is:

$$E[p_i] = \int_0^1 e^{w_i - 1} dw_i = \frac{1}{e}(e-1) = 1 - \frac{1}{e} \qquad (5)$$

It is convenient to think of the vertices of U as *buyers* and the vertices of V as *items*. Suppose that given $G(U, V; E)$, each vertex (buyer) $u \in U$ desires only items $v \in V$ that are neighbors of u (namely, u desires v iff $(u, v) \in E$), is willing to pay 1 for any such item, and wishes to buy exactly one item. The seller holding the items is offering to sell each item v_i for a price of p_i. Then given G, the matching produced by executing the *Ranking* algorithm is the same as the one that would be produced in a setting in which each buyer u_j, upon arrival, buys its cheapest exposed desired item, if there is any. If p_i is the price of the purchased item v_i, then the *revenue* that the seller extracts from the sale of v_i to u_j is $r(v_i) = p_i$, whereas the *utility* that the buyer extracts is $y(u_i) = 1 - p_i$. Consequently, the revenue plus utility extracted from a sale is 1, and the total revenue extracted from all sales plus the total utility sum up to exactly the cardinality of the matching.

To lower bound the expected cardinality of the matching, we consider each edge $(M(v_i), v_i) \in M$ separately, and consider the expectation $E[r(v_i)+y(M(v_i))]$, where expectation is taken over the choice of π. Using the linearity of the expectation, we will have that $\rho_n(Ranking, G) = \sum_{v_i \in V} E[r(v_i) + y(M(v_i))]$.

Lemma 22 *For every $v_i \in V$ it holds that $E[r(v_i) + y(M(v_i))] \geq 1 - \frac{1}{e}$. Moreover, this holds even if expectation is taken only over the choice of random weight w_i (and hence of random price p_i) of item v_i, without need to consider other aspects of the random permutation π.*

Proof. Fix an arbitrary graph $G(U, V; E) \in G_n$, an arbitrary perfect matching M, and arbitrary prices $p_j \in [\frac{1}{e}, 1]$ for all items $v_j \neq v_i$. The price p_i for item v_i is set at random. Let M' denote the greedy matching produced by this realization of the *Ranking* algorithm (where each buyer upon its arrival is matched to the exposed vertex of lowest price among its neighbors, if there is any). Suppose as a thought experiment that item v_i is removed from V, and consider the greedy matching M'_{-i} that would have been produced in this setting. Let p denote the price of the item in V matched to $M(v_i)$ under M'_{-i}, and set $p = 1$ if $M(v_i)$ is left unmatched under M'_{-i}. Now we make two easy claims.

1. *If $p_i < p$, then v_i is matched in M'.* This follows because at the time that $M(v_i)$ arrived, either v_i was already matched (as desired), or it was available for matching with $M(v_i)$ and preferable (in terms of price) over all other items that $M(v_i)$ desires (as all have price at least $p > p_i$).

2. *The utility of $M(v_i)$ in M' satisfies $y(M(v_i)) \geq 1-p$.* This follows because under M'_{-i} the utility of $M(v_i)$ is $1 - p$, and under the greedy algorithm

considered, the introduction of an additional item (the item v_i when considering M') cannot decrease the utility of any agent. (At every step of the arrival process, the set of exposed vertices under M' contains the set of exposed vertices under M'_{-i}, and one more vertex.)

Using the above two claims and taking z to be the value satisfying $p = e^{z-1}$, we have:

$$E[y(M(v_i))+r(v_i)] \geq 1-p+Pr[p_i < p]p_i = 1-e^{z-1}+\int_{w_i=0}^{z} e^{w_i-1}dw_i = 1-\frac{e^z}{e}+\frac{e^z-1}{e} = 1-\frac{1}{e}$$

This completes the proof of Lemma 22. ■

Using the linearity of the expectation, we have that

$$\rho_n(Ranking, G) = \sum_{v_i \in V} E[r(v_i) + y(M(v_i))] \geq (1-\frac{1}{e})n$$

This completes the proof of Theorem 1. ■

One can adapt the proof presented above to the special case in which the input graph is *MonotoneG* (or more generally, comes from the distribution D_n). In this case one can upper bound the slackness involved in the proof of Theorem 1, and infer the following theorem.

Theorem 23 *For every n it holds that $\rho_n(Ranking, MonotoneG) \leq (1-\frac{1}{e})n+\frac{1}{e}$.*

Proof. Recall the two properties mentioned in the beginning of the proof of Theorem 14. Recall also that the analysis of *Ranking* in the proof of Theorem 1 (within Lemma 22) involved the matching M' and other matchings M'_{-i}, and two claims. Let us analyse the slackness involved in these claims when the input is the monotone graph. The claims are restated with $M(v_i)$ replaced by u_i, because for the monotone graph $M(v_i) = u_i$.

The first claim stated that *if $p_i < p$, then v_i is matched in M'.* When the input is the monotone graph, then a converse also holds: if $p_i > p$, then v_i is not matched in M'. This follows because up to the time that u_i arrives and is matched, only vertices of V priced at most p are matched, and thereafter, no other vertex in U desires v_i. The event that $p_i = p$ has probability 0. Hence there is no slackness involved in the first claim – it is an *if and only if* statement.

The second claim stated that *the utility of u_i in M' is $y(u_i) \geq 1-p$.* This inequality is not tight. Rather, the utility of u_i in M'_{-i} is $1-p$, and $y(u_i)$ is not smaller. Let us quantify the slackness involved in this inequality by introducing slackness variables $s(u)$. For a vertex $u \in U$ we shall use the notation $y(u)$ to denote the utility of u under *Ranking*, and $y_{-v}(u)$ for the utility of u when vertex $v \in V$ is removed. The *slackness* $s(u_i)$ of vertex u_i is defined as $s(u_i) = y(u_i) - y_{-v_i}(u_i)$.

Lemma 24 *For the monotone graph and an arbitrary vertex $u_j \in U$, the expected utility of u_j (expectation taken over choices of w_i for all $1 \le i \le n$ by the* Ranking *algorithm) is identical in the following two settings: when v_j is removed, and when v_n is removed. Namely, $E[y_{-v_j}(u_j)] = E[y_{-v_n}(u_j)]$.*

Proof. Both v_j and v_n are neighbors of all vertices u_k arriving up to u_j (for $1 \le k \le j$). Hence whichever of the two vertices, v_j or v_n, is removed, the distributions of the outcomes of *Ranking* on the first j arriving vertices (including u_j) are the same. ■

As a consequence of Lemma 24 we deduce that for the monotone graph, the expected slackness of every vertex $u \in U$ satisfies $E[s(u)] = E[y(u)] - E[y_{-v_n}(u)]$.

Lemma 25 *For the monotone graph and arbitrary setting of prices for the items (as chosen at random by* Ranking*), $\sum_{u \in U} s(u) \le 1 - p_n$. Consequently, $\sum_{u \in U} E[s(u)] \le \frac{1}{e}$, where expectation is taken over choice of weights w_i for vertices in V.*

Proof. Fix the prices p_i (hence π). Let $u_1, \dots, u_k$ be the vertices of U matched under *Ranking*, and let $m(u_1), \dots, m(u_k)$ be the vertices in V to which they are matched. Observe that the prices $p(m(u_i))$ (where $1 \le i \le k$) of these vertices form a monotonically increasing sequence. Necessarily, v_n is one of the matched vertices, because it is a neighbor of all vertices in U. Let j be such that $v_n = m(u_j)$.

Consider now what happens when v_n is removed. The vertices $u_1, \dots, u_{j-1}$ are matched to $m(u_1), \dots, m(u_j - 1)$ as before. As to the vertices $u_j, \dots, u_{k-1}$, they can be matched to $m(u_{j+1}), \dots, m(u_k)$, hence the algorithm will match them to vertices of no higher price. Specifically, for every i in the range $j \le i \le k-1$, vertex u_i will be matched either to $m(u_{i+1})$ or to an earlier vertex, though not earlier than $m(u_i)$. The vertex u_k may either be matched or be left unmatched. For simplicity of notation, we say that u_k is matched to either $m(u_{k+1})$ or to an earlier vertex, where $m(u_{k+1})$ is an auxiliary vertex of price 1 than indicates that u_k is left unmatched.

Note that:

$$\sum_{u \in U} y(u) = \sum_{i=1}^{k} y(u_i) = k - \sum_{i=1}^{k} p(m(u_i))$$

and that:

$$\sum_{u \in U} y_{-v_n}(u) = \sum_{i=1}^{k} y_{-v_n}(u_i) \ge k - \sum_{i=1}^{j-1} p(m(u_i)) - \sum_{i=j+1}^{k+1} p(m(u_i))$$

Hence we have that:

$$\sum_{u\in U} s(u) = \sum_{u\in U} y(u) - \sum_{u\in U} y_{-v_n}(u) \leq p(m(u_{k+1})) - p(v_n)$$

Finally, noting that $p(m(u_{k+1})) \leq 1$ and that $E[p(v_n)] = 1 - \frac{1}{e}$ (see Equation (5)), the lemma is proved. ■

As in the proof of Theorem 1 we have:

$$E[y(u_i) + r(v_i)] = 1 - p + s(u_i) + Pr[p_i < p]p_i = 1 - \frac{1}{e} + s(u_i)$$

Using the linearity of the expectation and Lemma 25 we have that:

$$\rho_n(Ranking, MonotoneG) = \sum_{v_i\in V} E[r(v_i)+y(u_i)] = (1-\frac{1}{e})n+\sum_{u\in U} E[s(u)] \leq (1-\frac{1}{e})n+\frac{1}{e}$$

This completes the proof of Theorem 23. ■

B An alternative proof of a combinatorial identity

We present a proof of Corollary 18 that does not make use of Lemma 16.
Proof.[Corollary 18] Let $\Pi_{\not\sim(n+1)}$ denote those permutations $\pi' \in \Pi_{n+1}$ such that if *Ranking* uses π' when the input is *MonotoneG* (with $|U| = |V| = n+1$), then π'_{n+1} (the last item in π') is not matched. By definition of $a(n,i)$ the expression $(n+1)!-a(n+1,n+1)$ can be interpreted as $|\Pi_{\not\sim(n+1)}|$. We describe a bijection B between $(\Pi_n, [n+1])$ and Π_{n+1}. The bijection will have the property that given a pair $(\pi \in \Pi_n, i \in [n+1])$ the resulting permutation $B(\pi, i) \in \Pi_{n+1}$ belongs to $\Pi_{\not\sim(n+1)}$ if and only if u_i is matched, thus proving the Corollary.

We now describe the bijection for a given $\pi \in \Pi_n$ and $i \in [n+1]$:

- $B(\pi, n+1)$: place v_{n+1} at location $n+1$. This gives one permutation that we call $\pi_{\to(n+1)}$.
- $B(\pi, i)$ for $1 \leq i \leq n$: place v_{n+1} at location i and place v_i at location $n+1$. This gives n additional permutations, named $\pi_{\leftrightarrow i}$ (the notation $\leftrightarrow$ indicates that v_{n+1} is swapped with v_i).

There are three cases to consider:

- $i = n+1$. In $\pi_{\to(n+1)}$ the item v_{n+1} at location $n+1$ is matched, because it is a neighbor of all vertices in U.

- $u_i \in U$ is matched in π. Then all vertices up to u_i are also matched in $\pi_{\leftrightarrow i}$, and to items at locations no later than n. This is because the only differences between π and $\pi_{\leftrightarrow i}$ involve vertices v_i and v_{n+1}, and both of them are neighbors of all arriving vertices up to and including u_i. None of the vertices $u_{i+1}, \ldots, u_{n+1}$ is a neighbor of v_i, hence in $\pi_{\leftrightarrow i}$ the item $v_i \in V$ at location $n+1$ is not matched.
- $u_i \in U$ is not matched in π. In this case u_i will not be matched to any of the first n items of $\pi_{\leftrightarrow i}$ (again, because the only differences between π and $\pi_{\leftrightarrow i}$ involve vertices v_i and v_{n+1}, and both of them are neighbors of all arriving vertices up to and including u_i). Consequently, u_i will be matched to v_i that is at location $n+1$ in $\pi_{\leftrightarrow i}$.

■