MULTIPERMUTATION SET THEORETIC SOLUTIONS OF THE YANG-BAXTER EQUATION OF LEVEL 2

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Abstract. We study involutive set-theoretic solutions of the Yang-Baxter equation of multipermutation level 2. These solutions happen to fall into two classes – distributive ones and non-distributive ones. The distributive ones can be effectively constructed using a set of abelian groups and a matrix of constants. Using this construction, we enumerate all distributive involutive solutions up to size 14. The non-distributive solutions can be also easily constructed, using a distributive solution and a permutation. At the end we study left braces giving multipermutation solutions of level 2.

1. Introduction

The Yang-Baxter equation is a fundamental equation occurring in integrable models in statistical mechanics and quantum field theory [22]. Let V be a vector space. A *solution of the Yang–Baxter equation* is a linear mapping $r : V \otimes V \to V \otimes V$ such that

$$(id \otimes r)(r \otimes id)(id \otimes r) = (r \otimes id)(id \otimes r)(r \otimes id).$$

Description of all possible solutions seems to be extremely difficult and therefore there were some simplifications introduced (see e.g. [9]).

Let X be a basis of the space V and let $\sigma : X^2 \to X$ and $\tau : X^2 \to X$ be two mappings. We say that (X, σ, τ) is a *set-theoretic solution of the Yang–Baxter equation* if the mapping $x \otimes y \mapsto \sigma(x, y) \otimes \tau(x, y)$ extends to a solution of the Yang–Baxter equation. It means that $r\colon X^2 \to X^2$, where $r = (\sigma, \tau)$ satisfies the *braid relation*:

$$(id \times r)(r \times id)(id \times r) = (r \times id)(id \times r)(r \times id). \tag{1.1}$$

A solution is called *non-degenerate* if the mappings $\sigma(x, \_)$ and $\tau(\_, y)$ are bijections, for all $x, y \in X$. A solution (X, σ, τ) is *involutive* if $r^2 = \mathrm{id}_{X^2}$, and it is *square free* if $r(x, x) = (x, x)$, for every $x \in X$.

Convention 1.1. All solutions, we study in this paper, are set-theoretic and non-degenerate, so we will call them simply *solutions*. The set X can be of arbitrary cardinality. We investigate here mainly involutive solutions.

It is known (see e.g. [38, 18, 8]) that there is a one-to-one correspondence between (involutive) solutions of the Yang-Baxter equation and (involutive) *biracks* $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ – algebras which have a structure of two one-sided quasigroups $(X, \circ, \backslash_\circ)$ and $(X, \bullet, /_\bullet)$ and satisfy some additional identities (5.1)–(5.3). This fact allows one to characterize solutions of the Yang-Baxter equation applying universal algebra tools.

In [11, Section 3.2] Etingof, Schedler and Soloviev introduced, for each involutive solution (X, σ, τ), the equivalence relation $\sim$ on the set X: for each $x, y \in X$

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$$x \sim y \quad \Leftrightarrow \quad \tau(\_, x) = \tau(\_, y).$$

They showed that the quotient set $X/\!\sim$ can be again endowed with a structure of an involutive solution and they call such a solution the *retraction* of the involutive solution X and denote it by $\mathrm{Ret}(X)$. An involutive solution X is said to be a multipermutation solution of level k, if k is the smallest integer such that $|\mathrm{Ret}^k(X)| = 1$. Since then many results appeared that study multipermutation solutions, often of a small level. Square-free multipermutation solutions are always decomposable [35] and several authors gave descriptions of some of these solutions either as a generalized twisted union [11, 14, 5] or a strong twisted union [16, 18]. We have to say, however, that this approach brings decompositions only and does not offer a direct way how to construct such solutions. In our work we bring a simple-to-use way how to construct multipermutation solutions of level 2 using abelian groups only. Moreover, our approach works for all such solutions, not only for square-free ones.

It was proved by Gateva-Ivanova and Cameron [16, Proposition 8.2] that, for an involutive square-free solution (X, σ, τ), we have $\sigma_x = \sigma(x, \_) \in \mathrm{Aut}(X)$, for all $x \in X$, if and only if the solution X is a multipermutation solution of level 2. In the language of identities this is equivalent to $(X, \circ, \backslash_\circ)$ being left distributive. It turns out, that this property can be characterized by several different identities and the equivalence of these identities holds in more general structures. This is why we in Section 2 study left quasigroups and we establish connections between several identities of binary algebras.

Given an involutive square-free solution of a multipermutation level 2 and the associated birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$, the algebra $(X, \circ, \backslash_\circ)$ turns out to be a medial quandle (see Lemma 3.3). The structure of medial quandles was studied in [19] and one of the main results was a construction of medial quandles based on a set of abelian groups, a matrix of homomorphisms and a matrix of constants. In Section 3, we adapt the construction to the current context (the matrix of homomorphisms is actually not needed here anymore) and we generalize it so that it may include all distributive solutions, not only those square-free ones.

If a solution (X, σ, τ) is an involutive multipermutation solution of level 2 and $e \in X$, then $(X, \sigma\sigma_e^{-1}, \sigma_e\sigma^{-1})$ is a distributive involutive solution [Theorem 7.12]. This phenomenon is a special case of something called an *isotope*. In Section 4 we study these special isotopes on the level of left quasigroups.

In Section 5 we finally get to biracks and we show what results from Section 2 and Section 3 tell us in the world of distributive biracks. Some of these results generalize the latest results by Gateva-Ivanova [15]. We then translate the results into the language of solutions of the Yang-Baxter equation in Section 7. We prove that an involutive solution is of multipermutation level 2 if and only if it is medial [Theorem 7.6] and we show equivalent properties for distributive solutions of multipermutation level 2 [Theorem 7.7]. We also rephrase how to construct any involutive solution of multipermutation level 2. Additionally, we present a short direct proof that each abelian group (of arbitrary order) is an IYB group (Theorem 7.11).

In Section 6 we focus on non-distributive biracks associated to involutive solutions of multipermutation level 2 and the isotopy that transforms them into distributive ones. This way we can effectively construct all involutive solutions of multipermutation level 2, which is used in Section 8 to enumerate small biracks. Distributive biracks are enumerated up to size 14, for the others of multipermutation level 2 we give an upper bound only since we lack an easy-to-use isomorphism criterion.

In the last section we recall the definition of a left brace and we study left braces that give raise to multipermutation solutions of level 2. We prove that λ is, in this case, not only a multiplicative homomorphism but also an additive homomorphism, generalizing the very same result Gateva-Ivanova had for square-free solutions only [15, Theorem 8.2].

2. Left quasigroups

In this preliminary section we introduce some identities that we shall use throughout the text and we show a few examples of left quasigroups with such properties.

Definition 2.1. A *left quasigroup* is an algebra $(X, \circ, \backslash_\circ)$ with two binary operations: the *left multiplication* and the *left division* respectively, satisfying for every $x, y \in X$ the following conditions:

$$x \circ (x \backslash_\circ y) = y = x \backslash_\circ (x \circ y). \tag{2.1}$$

A *right quasigroup* is defined analogously as an algebra $(X, \bullet, /_\bullet)$ with two binary operations of *right multiplication* and the *right division* satisfying for every $x, y \in X$ the conditions:

$$(y /_\bullet x) \bullet x = y = (y \bullet x) /_\bullet x. \tag{2.2}$$

Condition (2.1) simply means that all *left translations* $L_x \colon X \to X$ by x

$$L_x(a) = x \circ a,$$

are bijections, with $L_x^{-1}(a) = x \backslash_\circ a$. Equivalently, that for every $x, y \in X$, the equation $x \circ u = y$ has the unique solution $u = L_x^{-1}(y)$ in X. Similarly, Condition (2.2) gives that all *right translations* $\boldsymbol{R}_x \colon X \to X$ by x; $\boldsymbol{R}_x(a) = a \bullet x$, are bijections with $\boldsymbol{R}_x^{-1}(a) = a /_\bullet x$.

It is obvious that if $(X, \circ, \backslash_\circ)$ is a left quasigroup then $(X, \backslash_\circ, \circ)$ is also a left quasigroup.

The *left multiplication group* of a left quasigroup $(X, \circ, \backslash_\circ)$ is the permutation group generated by left translations, i.e. the group $\mathrm{LMlt}(X) = \langle L_x : \ x \in X \rangle$.

Definition 2.2. Let $m \in \mathbb{N}$. A left quasigroup $(X, \circ, \backslash_\circ)$ is called:

- *left distributive*, if for every $x, y, z \in X$:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z) \quad \Leftrightarrow \quad L_x L_y = L_{x \circ y} L_x \tag{2.3}$$

- *m-reductive*, if for every $x_0, x_1, x_2, \ldots, x_m \in X$:

$$(\ldots((x_0 \circ x_1) \circ x_2) \ldots) \circ x_m = (\ldots((x_1 \circ x_2) \circ x_3) \ldots) \circ x_m \tag{2.4}$$

- *m-permutational*, if for every $x, y, x_1, x_2, \ldots, x_m \in X$:

$$(\ldots((x \circ x_1) \circ x_2) \ldots) \circ x_m = (\ldots((y \circ x_1) \circ x_2) \ldots) \circ x_m \tag{2.5}$$

- *medial*, if for every $x, y, z, t \in X$:

$$(x \circ y) \circ (z \circ t) = (x \circ z) \circ (y \circ t) \quad \Leftrightarrow \quad L_{x \circ y} L_z = L_{x \circ z} L_y \tag{2.6}$$

- *right cyclic*, if it satisfies the *right cyclic* law, i.e. for every $x, y, z \in X$:

$$(x \backslash_\circ y) \backslash_\circ (x \backslash_\circ z) = (y \backslash_\circ x) \backslash_\circ (y \backslash_\circ z) \quad \Leftrightarrow \quad L^{-1}_{x \backslash_\circ y} L^{-1}_x = L^{-1}_{y \backslash_\circ x} L^{-1}_y \quad \Leftrightarrow \quad L_x L_{x \backslash_\circ y} = L_y L_{y \backslash_\circ x} \tag{2.7}$$

- *non-degenerate*, if the mapping

$$T \colon X \to X; \quad x \mapsto x \backslash_\circ x, \tag{2.8}$$

  is a bijection
- *idempotent*, if for every $x \in X$

$$x \circ x = x \quad \Leftrightarrow \quad L_x(x) = x \quad \Leftrightarrow \quad L_x^{-1}(x) = x. \tag{2.9}$$

A left distributive left quasigroup is a *rack*. Idempotent racks are called *quandles*.

The condition of left distributivity is well established in the literature. It appeared in a natural way in such areas as low-dimensional topology – in knot [3] and braid [7] invariants or in the theory of symmetric spaces [24]. Probably at first it was introduced already at the end of 19th century in papers of Peirce [29] and Schröder [37]. Recently, Lebed and Vendramin [23] considered the condition in the context of solutions of the Yang-Baxter equation.

The property of mediality was first investigated as a generalization of the associative law for quasigroups (see Murdoch [26] and Sushkievich [39]). It appears also in the characterization of mean value functions [1]. The first systematic approach to medial groupoids was undertaken by Ježek and Kepka in [21]. Idempotency in the theory of the Yang-Baxter solutions is called *square-freeness*. Idempotent and medial quasigroups are investigated since the middle of 20th century. In the wider context, two monographs [34, 33] of Romanowska and Smith are devoted to idempotent and medial algebras called *modes* which are present in different branches of mathematics and find applications in computer science, economics, physics, and biology.

2-reductive groupoids were considered by Płonka as a special case of *cyclic groupoids* [30]. The more general m-reductive modes were investigated in [31] and [32]. In [19] and [20] m-reductive quandles were characterized.

Right cyclic quasigroups (under the name *cycle sets*) were introduced by Rump in [35]. He showed that there is a correspondence between involutive solutions of the Yang-Baxter equation and non-degenerate cycle sets (see Theorem 5.5).

Finally, Condition (2.5) was defined by Gateva-Ivanova in [15, Remark 4.6] to describe multipermutation solutions of the Yang-Baxter equation (see Theorem 7.3). Earlier, Gateva-Ivanova and Cameron used Condition (2.4) (see [16, Theorem 5.15]). They did not name these properties.

Observation 2.3. *Each m-reductive left quasigroup is m-permutational and each m-permutational, idempotent left quasigroup is m-reductive.*

In this paper we are mainly interested in 2-reductive and 2-permutational left quasigroups. In particular, a left quasigroup $(X, \circ, \backslash_\circ)$ is 2-reductive if, for every $x, y, z \in X$:

$$(x \circ y) \circ z = y \circ z \quad \Leftrightarrow \quad L_{x\circ y} = L_y, \tag{2.10}$$

and it is 2-permutational if for every $x, y, z, t \in X$:

$$(z \circ x) \circ y = (t \circ x) \circ y \quad \Leftrightarrow \quad L_{z\circ x} = L_{t\circ x}. \tag{2.11}$$

Example 2.4. The left quasigroup $(\{0, 1, 2, 3\}, \circ, \backslash_\circ)$ with the following left multiplication:

$\circ$	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	0	1	2	3
3	2	3	0	1

is a 2-reductive rack and, according to Lemma 3.3, it is medial. In this case $L_0 = L_2 = \mathrm{id}$ and $L_1 = L_3 = (02)(13)$.

Example 2.5. Let $(\{0, 1, 2, 3\}, \circ, \backslash_\circ)$ be a left quasigroup with the following left multiplication:

$\circ$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

.

Clearly, $L_0 = L_2 = (01)(23)$ and $L_1 = L_3 = (03)(12)$. One can check that $(\{0, 1, 2, 3\}, \circ, \backslash_\circ)$ is both right cyclic and 2-permutational but neither left distributive nor 2-reductive. Additionally, by Corollary 6.4, $(\{0, 1, 2, 3\}, \circ, \backslash_\circ)$ is medial.

For a left quasigroup $(X, \circ, \backslash_\circ)$, Condition (2.3) means that all left translations for every $x \in X$, are automorphisms of $(X, \circ)$, i.e. for every $x, y, z \in X$

$$L_x(y \circ z) = L_x(y) \circ L_x(z). \tag{2.12}$$

Lemma 2.6. *Let $(X, \circ, \backslash_\circ)$ be a left quasigroup. Then*

- *$(X, \circ, \backslash_\circ)$ is left distributive if and only if $(X, \backslash_\circ, \circ)$ is left distributive.*
- *$(X, \circ, \backslash_\circ)$ is 2-reductive if and only if $(X, \backslash_\circ, \circ)$ is 2-reductive.*
- *$(X, \circ, \backslash_\circ)$ is medial if and only if $(X, \backslash_\circ, \circ)$ is medial.*
- *$(X, \circ, \backslash_\circ)$ is idempotent if and only if $(X, \backslash_\circ, \circ)$ is idempotent.*

Proof. If L_x is an automorphism, then L_x^{-1} is clearly an automorphism as well, giving Property (2.12). Furthermore, for every $x, y, z \in X$:

$$(x \circ y) \circ z = y \circ z \quad \overset{y \mapsto x \backslash_\circ y}{\Longrightarrow} \quad (x \circ (x \backslash_\circ y)) \circ z = (x \backslash_\circ y) \circ z \quad \overset{(2.1)}{\Longleftrightarrow} \quad y \circ z = (x \backslash_\circ y) \circ z \quad \Leftrightarrow$$
$$L_{x \backslash_\circ y}(z) = L_y(z) \quad \Leftrightarrow \quad L^{-1}_{x \backslash_\circ y}(z) = L_y^{-1}(z) \quad \Leftrightarrow \quad (x \backslash_\circ y) \backslash_\circ z = y \backslash_\circ z \overset{y \mapsto x \circ y}{\Longrightarrow}$$
$$(x \backslash_\circ (x \circ y)) \backslash_\circ z = (x \circ y) \backslash_\circ z \quad \overset{(2.1)}{\Longleftrightarrow} y \backslash_\circ z = (x \circ y) \backslash_\circ z \quad \Leftrightarrow$$
$$L_y^{-1}(z) = L^{-1}_{x \circ y}(z) \quad \Leftrightarrow \quad L_y(z) = L_{x \circ y}(z) \quad \Leftrightarrow \quad (x \circ y) \circ z = y \circ z.$$

Similarly, we can show that for every $x, y, z, t \in X$ (see also [33, Exercise 8.6H])

$$(x \circ y) \circ (z \circ t) = (x \circ z) \circ (y \circ t) \quad \Leftrightarrow \quad (x \backslash_\circ y) \backslash_\circ (z \backslash_\circ t) = (x \backslash_\circ z) \backslash_\circ (y \backslash_\circ t).$$ □

Next examples show that, for right cyclic or 2-permutational left quasigroup $(X, \circ, \backslash_\circ)$, the left quasigroup $(X, \backslash_\circ, \circ)$ does not have to be right cyclic or 2-permutational.

Example 2.7. Let $(\{0, 1, 2\}, \circ, \backslash_\circ)$ be a left quasigroup with the following left multiplication and left division:

$\circ$	0	1	2
0	1	0	2
1	2	0	1
2	2	0	1

$\backslash_\circ$	0	1	2
0	1	0	2
1	1	2	0
2	1	2	0

,

or equivalently, $L_0 = (01) = L_0^{-1}$, $L_1 = L_2 = (021)$ and $L_1^{-1} = L_2^{-1} = (012)$. This left quasigroup is 2-permutational, but

$$0 = 0 \backslash_\circ 1 = (0 \backslash_\circ 1) \backslash_\circ 1 \neq (1 \backslash_\circ 1) \backslash_\circ 1 = 2 \backslash_\circ 1 = 2.$$

Example 2.8. Let $(\{0, 1, 2, 3\}, \circ, \backslash_\circ)$ be a left quasigroup with the following left multiplication and left division:

$\circ$	0	1	2	3
0	0	1	3	2
1	2	3	1	0
2	3	2	0	1
3	1	0	2	3

$\backslash_\circ$	0	1	2	3
0	0	1	3	2
1	3	2	0	1
2	2	3	1	0
3	1	0	2	3

,

i.e. $L_0 = (23) = L_0^{-1}$, $L_1 = (0213) = L_2^{-1}$, $L_2 = (0312) = L_1^{-1}$ and $L_3 = (01)(23) = L_3^{-1}$. In this case the left quasigroup is right cyclic, but

$$2 = 1 \circ 0 = (0 \circ 1) \circ (0 \circ 0) \neq (1 \circ 0) \circ (1 \circ 0) = 2 \circ 2 = 0.$$

Directly from (2.3) and Lemma 2.6 we obtain that the left distributivity implies, for every $x, y \in X$,

$$L_{x\circ y} = L_x L_y L_x^{-1} \quad \text{and} \quad L_{x\backslash_\circ y} = L_x^{-1} L_y L_x. \tag{2.13}$$

Note also that, for an arbitrary automorphism α of $(X, \circ)$, we have

$$L_{\alpha(x)}(y) = \alpha(x) \circ y = \alpha(x \circ \alpha^{-1}(y)) = \alpha L_x \alpha^{-1}(y).$$

3. 2-reductive racks

It is known [16, Theorem 5.15] that a square-free involutive multipermutation solutions of level 2 is 2-reductive. It turns out that 2-reductivity has connections to other identities presented in Section 2. We study all these connections on the class of racks which are, after all, an interesting class itself, having many applications, e.g. in knot theory [13], [10, Chapter 5]. Moreover, we can apply existing tools, like a construction using affine meshes which is presented in the second half of this section.

Lemma 3.1. *Let $(X, \circ, \backslash_\circ)$ be a rack. The following conditions are equivalent:*

1. *$(X, \circ, \backslash_\circ)$ is right cyclic;*
2. *the group $\mathrm{LMlt}(X)$ is abelian;*
3. *$(X, \backslash_\circ, \circ)$ is right cyclic;*
4. *$(X, \circ, \backslash_\circ)$ is 2-reductive.*

Proof. In a rack, by (2.13), the conditions (2) and (4) are equivalent:

$$L_y = L_{x\circ y} = L_x L_y L_x^{-1} \quad \Leftrightarrow \quad L_y L_x = L_x L_y.$$

Furthermore, by (2.13) and (2.7) for every $x, y, z \in X$ we have:

$$(x\backslash_\circ y)\backslash_\circ(x\backslash_\circ z) = (y\backslash_\circ x)\backslash_\circ(y\backslash_\circ z) \quad \Leftrightarrow \quad L^{-1}_{x\backslash_\circ y} L_x^{-1} = L^{-1}_{y\backslash_\circ x} L_y^{-1} \quad \Leftrightarrow \quad L_x L_{x\backslash_\circ y} = L_y L_{y\backslash_\circ x}$$
$$\Leftrightarrow \quad L_x L_x^{-1} L_y L_x = L_y L_y^{-1} L_x L_y \quad \Leftrightarrow \quad L_x L_y = L_y L_x \quad \Leftrightarrow \quad L_x L_y L_x^{-1} L_x = L_y L_x L_y^{-1} L_y$$
$$\Leftrightarrow \quad L_{x\circ y} L_x = L_{y\circ x} L_y \quad \Leftrightarrow \quad (x \circ y) \circ (x \circ z) = (y \circ x) \circ (y \circ z),$$

which completes the proof. □

Corollary 3.2. *Let $(X, \circ, \backslash_\circ)$ be a right cyclic left quasigroup. Then the following conditions are equivalent:*

1. *$(X, \circ, \backslash_\circ)$ is a rack;*
2. *$(X, \circ, \backslash_\circ)$ is 2-reductive.*

Proof. If $(X, \circ, \backslash_\circ)$ is a rack then by Lemma 3.1 it is 2-reductive.

Conversely, by 2-reductivity of the right cyclic left quasigroup $(X, \circ, \backslash_\circ)$ we have

$$L_x L_{x\backslash_\circ y} = L_y L_{y\backslash_\circ x} \quad \Rightarrow \quad L_x L_y = L_y L_x = L_{x\circ y} L_x,$$

which shows that $(X, \circ, \backslash_\circ)$ is left distributive. □

Lemma 3.3. *Let $(X, \circ, \backslash_\circ)$ be a 2-reductive left quasigroup. Then the following conditions are equivalent:*

1. *$(X, \circ, \backslash_\circ)$ is left distributive;*
2. *the group $\mathrm{LMlt}(X)$ is abelian;*
3. *$(X, \circ, \backslash_\circ)$ is right cyclic;*
4. *$(X, \backslash_\circ, \circ)$ is right cyclic;*
5. *$(X, \circ, \backslash_\circ)$ is medial.*

Proof. Let $(X, \circ, \backslash_\circ)$ be a 2-reductive left quasigroup. The implications: (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) directly follow by Lemma 3.1.

If the group $\mathrm{LMlt}(X)$ is abelian then

$$L_{x\circ y}L_x = L_yL_x = L_xL_y,$$

which gives (2) $\Rightarrow$ (1).

If $(X, \circ, \backslash_\circ)$ is right cyclic then

$$L_{x\circ y}L_x = L_yL_x = L_yL_{y\backslash_\circ x} = L_xL_{x\backslash_\circ y} = L_xL_y,$$

and similarly, if $(X, \backslash_\circ, \circ)$ is right cyclic then

$$L_{x\circ y}L_x = L_{y\circ x}L_y = L_xL_y,$$

so (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are proved.

Finally,

$$L_{x\circ y}L_z = L_{x\circ z}L_y \quad \Leftrightarrow \quad L_yL_z = L_zL_y \quad \Leftrightarrow \quad L_{z\circ y}L_z = L_zL_y,$$

which shows that (1) $\Leftrightarrow$ (5) and completes the proof. □

In [19, Theorem 3.14] David Stanovský and the authors of this paper presented a general construction of medial quandles. It turned out [19, Theorem 6.9] that the case of 2-reductive quandles is actually much less complicated because 2-reductive quandles are rather combinatorial than algebraic structures. Moreover, the construction of 2-reductive quandles can be easily generalized for 2-reductive racks, as we shall see below.

Definition 3.4. A *trivial affine mesh* over a non-empty set I is the pair

$$\mathcal{A} = ((A_i)_{i\in I},\, (c_{i,j})_{i,j\in I}),$$

where A_i are abelian groups and $c_{i,j} \in A_j$ constants such that $A_j = \langle\{c_{i,j} \mid i \in I\}\rangle$, for every $j \in I$.

If I is a finite set we will usually display a trivial affine mesh as a pair $((A_i)_{i\in I}, C)$, where $C = (c_{i,j})_{i,j\in I}$ is a $|I| \times |I|$ matrix.

Definition 3.5. The *sum of a trivial affine mesh* $\mathcal{A} = ((A_i)_{i\in I},\, (c_{i,j})_{i,j\in I})$ over a set I is an algebra $(\bigcup_{i\in I} A_i, \circ, \backslash_\circ)$ defined on the disjoint union of the sets A_i, with two operations

$$a \circ b = b + c_{i,j},$$
$$a\backslash_\circ b = b - c_{i,j},$$

for every $a \in A_i$ and $b \in A_j$.

Theorem 3.6. *An algebra $(X, \circ, \backslash_\circ)$ is a 2-reductive rack if and only if it is the sum of some trivial affine mesh. The orbits of the action of* $\mathrm{LMlt}(X)$ *then coincide with the groups of the mesh.*

Proof. At first we show that the sum of a trivial affine mesh is a 2-reductive rack with orbits A_i, $i \in I$.

Let $a \in A_i$, $b \in A_j$, $c \in A_k$. Obviously the equation $a \circ x = x + c_{i,j} = b$ has a unique solution $x = b - c_{i,j} \in A_j$. Furthermore,

$$(a \circ b) \circ c = (b + c_{i,j}) \circ c = c + c_{j,k} = b \circ c,$$

and

$$a \circ (b \circ c) = a \circ (c + c_{j,k}) = (c + c_{j,k}) + c_{i,k} = (c + c_{i,k}) + c_{j,k} =$$
$$(b + c_{i,j}) \circ (c + c_{i,k}) = (a \circ b) \circ (a \circ c).$$

For $x \in A_j$ and $a \in A_k$ we have

$$L_a(x) = a \circ x = x + c_{k,j} \in A_j.$$

Thus the group $\mathrm{LMlt}(X)$ acts transitively on A_j if and only if the elements $c_{k,j}$, $k \in I$, generate the group A_j.

Now let $(X, \circ, \backslash_\circ)$ be a 2-reductive rack, and choose a transversal E to the orbit decomposition. By Lemma 3.3, the group $\mathrm{LMlt}(X)$ is abelian. Hence for every $e \in E$, the orbit $Xe = \{\alpha(e) \mid \alpha \in \mathrm{LMlt}(X)\}$ is an abelian group $(Xe, +, -, e)$ with $\alpha(e) + \beta(e) = \alpha\beta(e)$ and $-\alpha(e) = \alpha^{-1}(e)$, for $\alpha, \beta \in \mathrm{LMlt}(X)$.

Let for every $e, f \in E$

$$c_{e,f} := e \circ f = L_e(f) \in Xf.$$

Since $\mathrm{LMlt}(X)$ is abelian, and each $\alpha \in \mathrm{LMlt}(X)$ is an automorphism of $(X, \circ)$, we have $\alpha(e) \circ f = L_{\alpha(e)}(f) = \alpha L_e \alpha^{-1}(f) = L_e(f) = e \circ f$. This implies that the set

$$\{c_{e,f} \mid e \in E\} = \{e \circ f \mid e \in E\} = \{\alpha(e) \circ f \mid \alpha \in \mathrm{LMlt}(X), e \in E\} = \{L_a(f) \mid a \in X\}$$

generates the group $(Xf, +, -, f)$. This shows that $(X, \circ, \backslash_\circ)$ is the sum of the trivial affine mesh $((Xe)_{e \in E},\ (c_{e,f})_{e,f \in E})$ over the set E.

Finally, let $a = \alpha(e) \in Xe$ and $b = \beta(f) \in Xf$ with $\alpha, \beta \in \mathrm{LMlt}(X)$. Therefore we obtain

$$a \circ b = L_a(b) = L_{\alpha(e)}\beta(f) = L_e\beta(f) = L_e(f) + \beta(f) = c_{e,f} + b.$$

So we verified that the sum of $((Xe)_{e \in E},\ (c_{e,f})_{e,f \in E})$ yields the original rack $(X, \circ, \backslash_\circ)$. □

Note that the sum of such trivial affine mesh is idempotent if and only if $c_{i,i} = 0$, for each $i \in I$.

Theorem 3.7. *Let $\mathcal{A} = ((A_i)_{i \in I},\ (a_{i,j})_{i,j \in I})$ and $\mathcal{B} = ((B_i)_{i \in I},\ (b_{i,j})_{i,j \in I})$ be two trivial affine meshes, over the same index set I. Then the sums of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic 2-reductive racks if and only if there is a bijection π of the set I and group isomorphisms $\psi_i \colon A_i \to B_{\pi(i)}$ such that $\psi_j(a_{i,j}) = b_{\pi(i),\pi(j)}$, for every $i, j \in I$.*

Proof. The proof goes in the same way as the proof of [19, Theorem 4.2] for medial quandles in the case of 2-reductive ones. □

Example 3.8. Up to isomorphism, there are exactly five 2-reductive racks of size 3. They are the sums of the following trivial affine meshes:

- One orbit: $((\mathbb{Z}_3), (1))$.
- Two orbits: $((\mathbb{Z}_2, \mathbb{Z}_1), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_1), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$ and $((\mathbb{Z}_2, \mathbb{Z}_1), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$.
- Three orbits: $((\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right))$.

Theorems 3.6 and 3.7 allow us to enumerate 2-reductive racks, up to isomorphism. The numbers are presented in Table 1 in Section 8.

4. 2-permutational left quasigroups

Our goal in this paper is to study involutive multipermutation solutions of level 2. In the language of identities they are 2-permutational, see Theorem 7.3. This is why we focus on 2-permutational left quasigroups. In particular, we link them via a permutation to 2-reductive left quasigroups studied in the previous section. We start with a few auxiliary lemmas.

Lemma 4.1. *Let $(X, \circ, \backslash_\circ)$ be a 2-permutational left quasigroup. Then for every $x, y, z \in X$*

$$L_{L_y L_z^{-1}(x)} = L_x. \tag{4.1}$$

Proof. By (2.11) and (2.1), we have

$$L_{L_y L_z^{-1}(x)}(t) = (y \circ (z \backslash_\circ x)) \circ t = (z \circ (z \backslash_\circ x)) \circ t = x \circ t = L_x(t).$$ □

Lemma 4.2. *Let $(X, \circ, \backslash_\circ)$ be a medial left quasigroup. Then for every $x, y, z \in X$*

(1) $L_{z\circ x} = L_{z\circ z} L_x L_z^{-1}$;
(2) $L_{z\backslash_\circ x} = L_{z\circ z}^{-1} L_x L_z$;
(3) $L_x L_z^{-1} L_y = L_y L_z^{-1} L_x$.

Proof. Directly by mediality we have

$$L_{z\circ x} L_z = L_{z\circ z} L_x \quad \text{and} \quad L_{z\circ z} L_{z\backslash_\circ x} = L_{z\circ(z\backslash_\circ x)} L_z = L_x L_z.$$

Further, by (2.6)

$$L_{z\circ z} L_x L_z^{-1} L_y = L_{z\circ x} L_y = L_{z\circ y} L_x = L_{z\circ y} L_z L_z^{-1} L_x = L_{z\circ z} L_y L_z^{-1} L_x,$$

which implies

$$L_x L_z^{-1} L_y = L_y L_z^{-1} L_x.$$ □

As we noticed in Examples 2.7 and 2.8, for right cyclic or 2-permutational left quasigroups $(X, \circ, \backslash_\circ)$, the left quasigroup $(X, \backslash_\circ, \circ)$ need not be right cyclic nor 2-permutational. But under some additional assumptions, they are.

Lemma 4.3. *Let $(X, \circ, \backslash_\circ)$ be a 2-permutational medial left quasigroup. Then both left quasigroups $(X, \circ, \backslash_\circ)$ and $(X, \backslash_\circ, \circ)$ are right cyclic.*

Proof. Let $x, y \in X$. Then

$$L_x L_{x\backslash_\circ y} \overset{(2.1)}{=} L_{y\circ(y\backslash_\circ x)} L_{x\backslash_\circ y} \overset{(2.6)}{=} L_{y\circ(x\backslash_\circ y)} L_{y\backslash_\circ x} \overset{(2.11)}{=} L_{x\circ(x\backslash_\circ y)} L_{y\backslash_\circ x} \overset{(2.1)}{=} L_y L_{y\backslash_\circ x} \quad \text{and}$$
$$L_{x\circ y} L_x \overset{(2.1)}{=} L_{x\circ y} L_{y\backslash_\circ(y\circ x)} \overset{(2.6)}{=} L_{x\circ(y\backslash_\circ(y\circ x))} L_y \overset{(2.11)}{=} L_{y\circ(y\backslash_\circ(y\circ x))} L_y \overset{(2.1)}{=} L_{y\circ x} L_y.$$ □

Lemma 4.4. *Let $(X, \circ, \backslash_\circ)$ be a right cyclic medial left quasigroup. Then both $(X, \circ, \backslash_\circ)$ and $(X, \backslash_\circ, \circ)$ are 2-permutational.*

Proof. We prove the claim first for $(X, \backslash_\circ, \circ)$ using Lemma 2.6. Note that Condition (2.6) for $(X, \backslash_\circ, \circ)$ means that for $x, y, z \in X$

$$L_{x\backslash_\circ y}^{-1} L_z^{-1} = L_{x\backslash_\circ z}^{-1} L_y^{-1} \quad \Leftrightarrow \quad L_{x\backslash_\circ y}^{-1} L_z^{-1} L_y = L_{x\backslash_\circ z}^{-1}. \tag{4.2}$$

Hence,

$$L_{y\backslash_\circ x}^{-1} \overset{(2.7)}{=} L_{x\backslash_\circ y}^{-1} L_x^{-1} L_y \overset{(4.2)}{=} L_{x\backslash_\circ x}^{-1}$$

and the right-hand side does not depend on y. Now, for $(X, \circ, \backslash_\circ)$, we notice that substituting $y \mapsto x\backslash_\circ y$ in (2.6) we get $L_{x\circ z} = L_y L_z L_{x\backslash_\circ y}^{-1}$ which we use in

$$L_{x\circ y} \overset{(2.6)}{=} L_{x\circ z} L_y L_z^{-1} = L_y L_z L_{x\backslash_\circ y}^{-1} L_y L_z^{-1} = L_y L_z L_{y\backslash_\circ y}^{-1} L_y L_z^{-1},$$

where the last equality follows from $(X, \backslash_\circ, \circ)$ being 2-permutational. Again, the right-hand side does not depend on x, which finishes the proof. □

In the theory of quasigroups (see e.g. [28, Section II.2]), there is a standard method, called *isotopy*, how to derive a quasigroup from another quasigroup. We do not need this notion here in the full generality, we shall present here a special case only.

Definition 4.5. Let $(X, \circ, \backslash_\circ)$ be a left quasigroup and π be a bijection of the set X. Define on the set X new binary operations:

$$x * y := x \circ \pi(y) = L_x\pi(y) \quad \text{and} \tag{4.3}$$

$$x \backslash_* y := \pi^{-1}(x \backslash_\circ y) = \pi^{-1}L_x^{-1}(y). \tag{4.4}$$

The algebra $(X, *, \backslash_*)$ is called the π-*isotope* of $(X, \circ, \backslash_\circ)$.

Remark 4.6. It is easy to note that

$$x * (x \backslash_* y) = L_x\pi\pi^{-1}L_x^{-1}(y) = y \quad \text{and}$$
$$x \backslash_* (x * y) = \pi^{-1}L_x^{-1}L_x\pi(y) = y.$$

Therefore $(X, *, \backslash_*)$ is also a left quasigroup. To obtain the multiplication table of $*$ for a π-isotope of a *finite* left quasigroup $(X, \circ, \backslash_\circ)$, one should permute all columns of the multiplication table of $\circ$ using the permutation π.

Lemma 4.7. *Let $(X, \circ, \backslash_\circ)$ be a left quasigroup and π be a bijection of the set X. Then the π-isotope of $(X, \circ, \backslash_\circ)$ is*

(1) 2*-reductive if and only if, for every $x, y \in X$,*

$$L_{L_x\pi(y)} = L_y, \tag{4.5}$$

(2) 2*-permutational if and only if, for every $x, y, z \in X$,*

$$L_{L_x\pi(z)} = L_{L_y\pi(z)}, \tag{4.6}$$

(3) *left distributive if and only if, for every $x, y, z \in X$,*

$$L_{L_x\pi(y)}\pi L_x = L_x\pi L_y. \tag{4.7}$$

Proof. Let $(X, *, \backslash_*)$ be the π-isotope of $(X, \circ, \backslash_\circ)$. Hence for every $x, y, z \in X$

$$(x * y) * z = y * z \quad \Leftrightarrow \quad L_{L_x\pi(y)}\pi(z) = L_y\pi(z),$$
$$(x * z) * t = (y * z) * t \quad \Leftrightarrow \quad L_{L_x\pi(z)}\pi(t) = L_{L_y\pi(z)}\pi(t),$$
$$(x * y) * (x * z) = x * (y * z) \quad \Leftrightarrow \quad L_{x*y}\pi L_x\pi(z) = L_{L_x\pi(y)}\pi L_x\pi(z) = L_x\pi L_y\pi(z).$$

□

Remark 4.8. A left quasigroup $(X, \circ, \backslash_\circ)$ is 2-permutational if and only if $(X, \circ, \backslash_\circ)$ satisfies Condition (4.6) for a bijection π of the set X. Indeed, for every $x, y \in X$ we have

$$L_{L_x\pi(z)} = L_{L_y\pi(z)} \overset{z\mapsto\pi^{-1}(z)}{\Rightarrow} L_{L_x(z)} = L_{L_y(z)}.$$

Lemma 4.9. *Let $(X, \circ, \backslash_\circ)$ be a 2-reductive left quasigroup and ϱ be a bijection on the set X. Then the ϱ-isotope of $(X, \circ, \backslash_\circ)$ is a 2-permutational left quasigroup.*

Proof. By Lemma 4.7(2) it is sufficient to show that the ϱ-isotope of $(X, \circ, \backslash_\circ)$ satisfies Condition (4.6). For $x, y, z \in X$ we have

$$L_{L_x\varrho(y)} \overset{(2.10)}{=} L_{\varrho(y)} \overset{(2.10)}{=} L_{L_z\varrho(y)},$$

which finishes the proof. □

The idea of the next theorem is the following: we already know how to construct 2-reductive racks, using the construction from Section 3. Now, according to Lemma 4.9, ϱ-isotopes of these 2-reductive racks are 2-permutational. And we want these ϱ-isotopes to be right cyclic.

Theorem 4.10. *Let $(X, \circ, \backslash_\circ)$ be a 2-reductive left quasigroup and ϱ be a bijection on the set X such that for every $x, y \in X$*

$$L_{\varrho(y)}\varrho L_x = L_{\varrho(x)}\varrho L_y \quad \Leftrightarrow \quad \forall z \in X \quad \varrho(y) \circ \varrho(x \circ z) = \varrho(x) \circ \varrho(y \circ z). \tag{4.8}$$

Then the ϱ-isotope of $(X, \circ, \backslash_\circ)$ is a 2-permutational right cyclic left quasigroup.

Proof. Let $(X, *, \backslash_*)$ be the ϱ-isotope of $(X, \circ, \backslash_\circ)$. By Lemma 4.9, $(X, *, \backslash_*)$ is 2-permutational. Further, Condition (4.8) is equivalent to the following one:

$$L_x^{-1}\varrho^{-1}L_{\varrho(y)}^{-1} = L_y^{-1}\varrho^{-1}L_{\varrho(x)}^{-1}. \tag{4.9}$$

Substituting x by $\varrho^{-1}(x)$ and y by $\varrho^{-1}(y)$ in (4.9) we obtain:

$$L_{\varrho^{-1}(x)}^{-1}\varrho^{-1}L_y^{-1} = L_{\varrho^{-1}(y)}^{-1}\varrho^{-1}L_x^{-1}. \tag{4.10}$$

Together with Lemma 2.6 this implies that for $x, y, z \in X$

$$(x\backslash_* y)\backslash_*(x\backslash_* z) = \varrho^{-1}L_x^{-1}(y)\backslash_*\varrho^{-1}L_x^{-1}(z) = \varrho^{-1}L_{\varrho^{-1}L_x^{-1}(y)}^{-1}\varrho^{-1}L_x^{-1}(z) \overset{(4.10)}{=}$$
$$\varrho^{-1}L_{\varrho^{-1}(x)}^{-1}\varrho^{-1}L_{L_x^{-1}(y)}^{-1}(z) \overset{(2.10)}{=} \varrho^{-1}L_{\varrho^{-1}(x)}^{-1}\varrho^{-1}L_y^{-1}(z) \overset{(4.10)}{=} \varrho^{-1}L_{\varrho^{-1}(y)}^{-1}\varrho^{-1}L_x^{-1}(z) \overset{(2.10)}{=}$$
$$\varrho^{-1}L_{\varrho^{-1}(y)}^{-1}\varrho^{-1}L_{L_y^{-1}(x)}^{-1}(z) \overset{(4.10)}{=} \varrho^{-1}L_{\varrho^{-1}L_y^{-1}(x)}^{-1}\varrho^{-1}L_y^{-1}(z) = \varrho^{-1}L_y^{-1}(x)\backslash_*\varrho^{-1}L_y^{-1}(z) =$$
$$(y\backslash_* x)\backslash_*(y\backslash_* z),$$

which shows that the left quasigroup $(X, *, \backslash_*)$ is right cyclic. □

On the other hand, each 2-permutational medial left quasigroup has as an isotope that is a 2-reductive rack.

Theorem 4.11. *Let $(X, \circ, \backslash_\circ)$ be a left quasigroup and π be a bijection on the set X which satisfies Condition (4.5) and such that for each $x, y \in X$*

$$L_x\pi L_y = L_y\pi L_x. \tag{4.11}$$

Then the π-isotope of $(X, \circ, \backslash_\circ)$ is a 2-reductive rack.

Proof. Let $(X, *, \backslash_*)$ be the π-isotope of $(X, \circ, \backslash_\circ)$. By Lemma 4.7(1), $(X, *, \backslash_*)$ is 2-reductive. Moreover, for $x, y \in X$ we have

$$L_{L_x\pi(y)}\pi L_x \overset{(4.5)}{=} L_y\pi L_x \overset{(4.11)}{=} L_x\pi L_y.$$

By Lemma 4.7(3) the left quasigroup $(X, *, \backslash_*)$ is left distributive, and in consequence 2-reductive rack. □

Corollary 4.12. *Let $(X, \circ, \backslash_\circ)$ be a 2-permutational medial left quasigroup and $e \in X$. Then the L_e^{-1}-isotope of $(X, \circ, \backslash_\circ)$ is a 2-reductive rack.*

Proof. By Lemmas 4.1 and 4.2, for each $x, y \in X$

$$L_{L_xL_e^{-1}(y)} = L_y \quad \text{and} \quad L_xL_e^{-1}L_y = L_yL_e^{-1}L_x,$$

which shows that Conditions (4.5) and (4.11) are satisfied for $\pi = L_e^{-1}$. Corollary follows by Theorem 4.11. □

Example 4.13. Let $(X, \circ, \backslash_\circ)$ be the 2-permutational medial left quasigroup from Example 2.5 and let $e = 0$. Then $(X, *, \backslash_*)$, with $x * y = x \circ L_0^{-1}(y)$ and $x\backslash_* y = L_0(x\backslash_\circ y)$, is a 2-reductive rack with the $*$-multiplication table presented in Example 2.4.

The next example shows that the assumption of mediality in Corollary 4.12 is not always needed.

Example 4.14. Let $(\{0,1,2\},\circ,\backslash_\circ)$ be a left quasigroup with the following left multiplication:

$$\begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & 2 & 0 \end{array},$$

i.e. $L_0=L_1=(12)$ and $L_2=(012)$. This left quasigroup is 2-permutational, but not medial

$$0=0\circ 0=(0\circ 0)\circ(1\circ 0)\neq(0\circ 1)\circ(0\circ 0)=2\circ 0=1.$$

But for $\pi=L_0^{-1}=(12)$ Condition (4.11) is satisfied and the π-isotope of $(\{0,1,2\},\circ,\backslash_\circ)$

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \end{array}$$

is 2-reductive rack $((\mathbb{Z}_2,\mathbb{Z}_1),(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}))$.

It is worth emphasizing that all results from Sections 2 – 4 established for left quasigroups are also true for right quasigroups, when using their dual versions.

5. Left distributive biracks

In the previous three sections we prepared tools that we shall be now using on biracks – universal algebraic incarnations of set-theoretic solutions of the Yang-Baxter equation. Originally, biracks are algebras studied in low-dimensional topology [12, 10]. The equational definition of a birack we use here was given first in [38].

Definition 5.1. An algebra $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ with four binary operations is called a *birack*, if $(X,\circ,\backslash_\circ)$ is a left quasigroup, $(X,\bullet,/_\bullet)$ is a right quasigroup and the following holds for any $x,y,z\in X$:

$$x\circ(y\circ z)=(x\circ y)\circ((x\bullet y)\circ z), \tag{5.1}$$

$$(x\circ y)\bullet((x\bullet y)\circ z)=(x\bullet(y\circ z))\circ(y\bullet z), \tag{5.2}$$

$$(x\bullet y)\bullet z=(x\bullet(y\circ z))\bullet(y\bullet z). \tag{5.3}$$

We will say that a birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is *left distributive*, if $(X,\circ,\backslash_\circ)$ is a rack, is *right distributive*, if for every $x,y,z\in X$

$$(y\bullet z)\bullet x=(y\bullet x)\bullet(z\bullet x),$$

i.e. the right quasigroup $(X,\bullet,/_\bullet)$ is *right distributive*. The birack is *distributive* if it is left and right distributive. It is evident that all properties of left distributive biracks stay true in its dual form for right distributive ones.

Lemma 5.2. *Let $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ be a birack. The following is equivalent:*

(1) *$(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is left distributive;*
(2) *$(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ satisfies, for every $x,y\in X$,*

$$L_x=L_{x\bullet y}; \tag{5.4}$$

(3) *$(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ satisfies, for every $x,y\in X$,*

$$L_x=L_{x/_\bullet y}. \tag{5.5}$$

Proof. Indeed, by (5.1) we have for $x, y, z \in X$

$$(x \circ y) \circ (x \circ z) = x \circ (y \circ z) \quad \Leftrightarrow \quad (x \circ y) \circ (x \circ z) = (x \circ y) \circ ((x \bullet y) \circ z) \quad \Leftrightarrow$$
$$x \circ z = (x \bullet y) \circ z \;\Leftrightarrow\; L_x = L_{x \bullet y}.$$

Additionally, by (2.2), substituting of x by $x /_{\bullet} y$ in (5.4) we immediately obtain

$$L_x = L_{x/_{\bullet} y}.$$

Similarly, substituting of x by $x \bullet y$ in (5.5) we have

$$L_{x \bullet y} = L_x. \qquad \square$$

Analogously we can show that

Remark 5.3. Let $(X, \circ, \backslash_{\circ}, \bullet, /_{\bullet})$ be a birack. The following is equivalent:

(1) $(X, \circ, \backslash_{\circ}, \bullet, /_{\bullet})$ is right distributive;
(2) for every $x, y \in X$, $\boldsymbol{R}_x = \boldsymbol{R}_{y \backslash_{\circ} x}$;
(3) for every $x, y \in X$, $\boldsymbol{R}_x = \boldsymbol{R}_{y \circ x}$.

Example 5.4. Let X be a non-empty set and let $f, g \colon X \to X$ be two bijections with $fg = gf$. An algebra $(X, \circ, \backslash_{\circ}, \bullet, /_{\bullet})$ such that for every $x, y \in X$,

$$x \circ y = f(y), \quad x \backslash_{\circ} y = f^{-1}(y),$$
$$x \bullet y = g(x), \quad x /_{\bullet} y = g^{-1}(x)$$

is a birack called 1-*permutational* (since both quasigroups are 1-permutational). If $f, g = \mathrm{id}$, 1-permutational birack is called a *projection birack*.

Each 1-permutational birack is left distributive, since for every $x, y, z \in X$

$$x \circ (y \circ z) = L_x L_y(z) = f^2(z) = L_{x \circ y} L_x(z) = (x \circ y) \circ (x \circ z).$$

A birack is *idempotent* if both one-sided quasigroups $(X, \circ, \backslash_{\circ})$ and $(X, \bullet, /_{\bullet})$ are idempotent. And a birack is *involutive* if it additionally satisfies, for every $x, y \in X$:

$$(x \circ y) \circ (x \bullet y) = x, \tag{5.6}$$
$$(x \circ y) \bullet (x \bullet y) = y. \tag{5.7}$$

Note that Conditions (5.6) and (5.7) give, for every $x, y \in X$,

$$x \bullet y = L^{-1}_{x \circ y}(x) = (x \circ y) \backslash_{\circ} x \quad \text{and} \quad x \circ y = \boldsymbol{R}^{-1}_{x \bullet y}(y) = y /_{\bullet} (x \bullet y). \tag{5.8}$$

It follows then, that an involutive birack is idempotent if $(X, \circ, \backslash_{\circ})$ *or* $(X, \bullet, /_{\bullet})$ is idempotent. We shall see (Corollary 5.9) that an involutive birack is left distributive if and only if it is right distributive.

The next, well known result (see [35, Proposition 1], [8, Proposition 1.5], [20, Section 4.2]) is crucial for our considerations.

Theorem 5.5. *An algebra $(X, \circ, \backslash_{\circ}, \bullet, /_{\bullet})$ is an involutive birack if and only if $(X, \circ, \backslash_{\circ})$ is a non-degenerate right cyclic left quasigroup.*

Recall, if $(X, \circ, \backslash_{\circ})$ is a non-degenerate right cyclic left quasigroup then defining for every $x, y \in X$, $x \bullet y = \boldsymbol{R}_y(x) = (x \circ y) \backslash_{\circ} x$, and $x /_{\bullet} y = \boldsymbol{R}^{-1}_y(x)$, the algebra $(X, \circ, \backslash_{\circ}, \bullet, /_{\bullet})$ is an involutive birack.

Remark 5.6. Conditions (5.1) – (5.3) and (5.6) – (5.7) are dual with respect to operations $\circ$ and $\bullet$. Thus Theorem 5.5 immediately implies (see [8], [35] or [20, Section 4.2]) that in an involutive birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$, the right quasigroup $(X,\bullet,/_\bullet)$ is non-degenerate and left cyclic i.e. for every $x,y,z\in X$

$$(z\bullet x)\bullet(y\bullet x)=(z\bullet y)\bullet(x\bullet y),$$

and the mapping

$$S\colon X\to X;\quad x\mapsto x/_\bullet x$$

is a bijection. Moreover (see [35] and [20, Section 2]), operations $\backslash_\circ$ and $/_\bullet$ are connected by

$$(x\backslash_\circ x)/_\bullet(x\backslash_\circ x)=x\quad\text{and}\quad(x/_\bullet x)\backslash_\circ(x/_\bullet x)=x,$$

which is equivalent to the fact that the mappings S and $T\colon X\to X; x\mapsto x\backslash_\circ x$ are mutually inverse. It simply means that each involutive birack is a *biquandle* (see [38]).

An involutive birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is 2-*reductive* if the left quasigroup $(X,\circ,\backslash_\circ)$ is 2-reductive. By Theorem 5.5 and Corollary 3.2 we directly obtain the following.

Corollary 5.7. *An involutive birack is left distributive if and only if it is* 2*-reductive.*

From now on, we will use both terms: a *(left) distributive* involutive birack and a 2*-reductive* involutive birack, interchangeably.

In some cases in a birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$, the left multiplication $\circ$ and the right multiplication $\bullet$ are mutually inverse, i.e. for every $x,y\in X$, the following condition is satisfied:

(5.9) $$(x\circ y)\bullet x=y=x\circ(y\bullet x)\quad\Leftrightarrow\quad L_x=\boldsymbol{R}_x^{-1}.$$

Condition (5.9) was called **lri** in [18, Definition 2.17].

For example, Condition (5.9) is satisfied in idempotent involutive biracks [18, Corollary 2.33]. Moreover, Gateva-Ivanova showed that also 2-reductive involutive biracks satisfy this condition. Below we present a shorter alternative proof of this fact.

Lemma 5.8. [15, Lemma 7.1] *An involutive* 2*-reductive birack satisfies Condition* (5.9)*.*

Proof. Let $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ be an involutive 2-reductive birack. Then, for each $x,y\in X$ we obtain

$$x\circ(y\bullet x)\overset{(5.8)}{=}x\circ L^{-1}_{y\circ x}(y)\overset{(2.10)}{=}x\circ L_x^{-1}(y)=x\circ(x\backslash_\circ y)\overset{(2.1)}{=}y$$

and

$$(x\circ y)\bullet x\overset{(5.8)}{=}L^{-1}_{(x\circ y)\circ x}(x\circ y)\overset{(2.10)}{=}L_x^{-1}L_x(y)=y.$$ □

Corollary 5.9. *An involutive birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is left distributive if and only if it is right distributive, i.e. for every $x,y\in X$*

$$L_x=L_{x\bullet y}\quad\Leftrightarrow\quad\boldsymbol{R}_x=\boldsymbol{R}_{y\backslash_\circ x}\quad\textit{and}$$
$$L_x=L_{x/_\bullet y}\quad\Leftrightarrow\quad\boldsymbol{R}_x=\boldsymbol{R}_{y\circ x}.$$

Hence, an involutive left (or right) distributive birack is distributive.

Proof. By Corollary 5.7, an involutive left distributive birack is 2-reductive and by Lemma 5.8 it satisfies Condition (5.9). Hence, for every $x,y\in X$, we have:

$$\boldsymbol{R}_x=\boldsymbol{R}_{y\backslash_\circ x}\quad\Leftrightarrow\quad L_x^{-1}=L^{-1}_{L_y^{-1}(x)}=L^{-1}_{\boldsymbol{R}_y(x)}=L^{-1}_{x\bullet y}\quad\Leftrightarrow\quad L_x=L_{x\bullet y},\quad\text{and}$$
$$\boldsymbol{R}_x=\boldsymbol{R}_{y\circ x}\quad\Leftrightarrow\quad L_x^{-1}=L^{-1}_{L_y(x)}=L^{-1}_{\boldsymbol{R}_y^{-1}(x)}=L^{-1}_{x/_\bullet y}\quad\Leftrightarrow\quad L_x=L_{x/_\bullet y}.$$

The proof in the opposite direction follows by the fact that a right distributive right quasigroup satisfies dual 2-reductive law, and in consequence it also satisfies Condition (5.9).

□

Moreover, if $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is an involutive distributive birack then the left quasigroup $(X, \circ, \backslash_\circ)$ and the right quasigroup $(X, \bullet, /_\bullet)$ are *mutually orthogonal*, i.e. for every $a, b \in X$, the pair of equations

$$a = x \circ y \text{ and } b = x \bullet y$$

has a unique solution: $x = a \circ b$ and $y = b \backslash_\circ a$. Indeed, by Corollary 5.7 the left quasigroup $(X, \circ, \backslash_\circ)$ is 2-reductive. Therefore, we have

$$x \circ y = (a \circ b) \circ (b \backslash_\circ a) = L_{a\circ b} L_b^{-1}(a) \overset{(2.10)}{=} L_b L_b^{-1}(a) = a.$$

Further, by Lemma 5.8, $x \bullet y = y \backslash_\circ x$ and $x /_\bullet y = y \circ x$. Hence,

$$x \bullet y = (a \circ b) \bullet (b \backslash_\circ a) = (b \backslash_\circ a) \backslash_\circ (a \circ b) = L_{b\backslash_\circ a}^{-1} L_a(b) \overset{(2.10)}{=} L_a^{-1} L_a(b) = b.$$

Since by Corollary 5.7, for an involutive distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$, the left quasigroup $(X, \circ, \backslash_\circ)$ is 2-reductive, Theorem 3.6 immediately implies

Theorem 5.10. *Each involutive distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a disjoint union, over a set I, of abelian groups $A_j = \langle \{a_{i,j} \mid i \in I\} \rangle$, for every $j \in I$, with operations:*

$$\begin{aligned} &x \circ y = y + a_{i,j} \quad \text{and} \quad x \backslash_\circ y = y - a_{i,j}, \\ &x \bullet y = (x \circ y) \backslash_\circ x = (y + a_{i,j}) \backslash_\circ x = x - a_{j,i} \quad \text{and} \quad x /_\bullet y = x + a_{j,i}, \end{aligned}$$

for $x \in A_i$ and $y \in A_j$.

Taking the notion from 2-reductive racks, we will shortly say that the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is the sum of a trivial affine mesh $\mathcal{A} = ((A_i)_{i\in I}, (a_{i,j})_{i,j\in I})$ over a set I. Note that each orbit is a 1-permutational birack.

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a birack. Etingof, Schedler and Soloviev defined in [11] the relation

$$a \sim b \quad \Leftrightarrow \quad L_a = L_b \quad \Leftrightarrow \quad \forall x \in X \quad a \circ x = b \circ x. \tag{5.10}$$

By their results, the relation $\sim$ is a congruence of involutive biracks, i.e. an equivalence relation on the set X preserving all four operations in a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$. In the case of non-involutive biracks, the equivalence $\sim$ need not be a congruence (see [20, Example 3.4]) but it is so if the birack is left distributive.

Theorem 5.11. *Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a left distributive birack. Then the relation* (5.10) *is a congruence of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.*

Proof. By (2.1), (2.13) and (5.4) the proof is straightforward. Let $a \sim x$ and $b \sim y$. Then

$$\begin{aligned} &L_{a\circ b} \overset{(2.13)}{=} L_a L_b L_a^{-1} = L_x L_y L_x^{-1} = L_{x\circ y} \quad \Rightarrow \quad a \circ b \sim x \circ y, \\ &L_{a\backslash_\circ b} \overset{(2.13)}{=} L_a^{-1} L_b L_a = L_x^{-1} L_y L_x = L_{x\backslash_\circ y} \quad \Rightarrow \quad a \backslash_\circ b \sim x \backslash_\circ y, \\ &L_{a\bullet b} \overset{(5.4)}{=} L_a = L_x = L_{x\bullet y} \quad \Rightarrow \quad a \bullet b \sim x \bullet y, \\ &L_{a/_\bullet b} \overset{(5.4)}{=} L_{(a/_\bullet b)\bullet b} \overset{(2.1)}{=} L_a = L_x = L_{(x/_\bullet y)\bullet y} = L_{x/_\bullet y} \quad \Rightarrow \quad a /_\bullet b \sim x /_\bullet y. \end{aligned}$$

□

Example 5.12. Let $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ be an involutive distributive birack. By Theorem 5.10, the birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is the sum of a trivial affine mesh $((A_i)_{i\in I},\,(c_{i,j})_{i,j\in I})$ over a set I. For $a\in A_i$ and $b\in A_j$

$$a\sim b \quad \Leftrightarrow \quad \forall k\in I \quad \forall x\in A_k \quad x+c_{i,k}=a\circ x=b\circ x=x+c_{j,k} \quad \Leftrightarrow \quad \forall k\in I \quad c_{i,k}=c_{j,k}.$$

Lemma 5.13. *Let $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ be a left distributive birack. Then the quotient $(X/{\sim},\circ,\backslash_\circ,\bullet,/_\bullet)$ is idempotent. If additionally, $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is involutive, the quotient $(X/{\sim},\circ,\backslash_\circ,\bullet,/_\bullet)$ is a projection birack.*

Proof. By the left distributivity and (5.4), for every $x,y\in X$,

$$x\sim x\bullet y.$$

Furthermore,

$$L_x=L_xL_xL_x^{-1}=L_{x\circ x},$$

which shows that $x\sim x\circ x$ and $(X/{\sim},\circ,\backslash_\circ,\bullet,/_\bullet)$ is idempotent.

If a left distributive birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is involutive then, by Corollary 5.7, it is 2-reductive. In consequence, $x\circ y\sim y$, which finishes the proof. □

For an involutive solution of the Yang-Baxter equation, Gateva-Ivanova considered in [15, Definition 4.3] the Condition $(*)$ which in the language of biracks means that

$$(*) \qquad \forall x\in X \quad \exists a\in X \quad a\circ x=x.$$

It is evident, that each idempotent birack satisfies Condition $(*)$. On the other hand involutive distributive biracks without fixed points are examples of biracks which do not satisfy the condition. The representation of involutive distributive birack as the sum of a trivial affine mesh allows one to verify quickly Condition $(*)$.

Remark 5.14. Let $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ be an involutive distributive birack. By Theorem 5.10, $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is the sum of a trivial affine mesh $((A_i)_{i\in I},\,(c_{i,j})_{i,j\in I})$ over a set I. Then $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ satisfies Condition $(*)$ if and only if

$$\forall i\in I \quad \forall x\in A_i \quad \exists j\in I \quad \exists a\in A_j \quad a\circ x=x+c_{j,i}=x \quad \Leftrightarrow \quad \forall i\in I \quad \exists j\in I \quad c_{j,i}=0.$$

Remark 5.14 says that an involutive distributive birack satisfies Condition $(*)$ if and only if in each column in the matrix of constants there is at least one 0.

Example 5.15. Let a birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ be the sum of the trivial affine mesh $((\mathbb{Z}_4,\mathbb{Z}_4),(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}))$. Then $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is distributive but does not satisfy Condition $(*)$. This birack is also not 1-permutational.

By Observation 2.3 we have that each idempotent m-permutational left quasigroup is m-reductive, for arbitrary $m\in\mathbb{N}$. The same is also true for m-permutational left quasigroups which satisfy Condition $(*)$ (see also [15, Proposition 4.7]).

Lemma 5.16. *Let $(X,\circ,\backslash_\circ)$ be a left quasigroup which satisfies Condition $(*)$ and $m\in\mathbb{N}$. Then $(X,\circ,\backslash_\circ)$ is m-permutational if and only if it is m-reductive.*

Proof. We have only to prove that each m-permutational left quasigroup which satisfies Condition $(*)$ is m-reductive. But it is evident. By Condition $(*)$ for each $x\in X$ there exists $a_x\in X$ such that $a_x\circ x=x$. Then for every $x_0,x_1,x_2,\ldots,x_m\in X$ we have:

$$(\ldots((x_0\circ x_1)\circ x_2)\ldots)\circ x_m \overset{(2.5)}{=} (\ldots((a_{x_1}\circ x_1)\circ x_2)\ldots)\circ x_m=(\ldots((x_1\circ x_2)\circ x_3)\ldots)\circ x_m,$$

which completes the proof. □

6. 2-permutational biracks

Lemma 5.13 shows that, for each involutive distributive birack, its quotient by the relation (5.10) is a projection birack. There are also not distributive involutive biracks such that the quotient is a 1-permutational birack.

Example 6.1. Let $(X = \{0, 1, 2, 3\}, \circ, \backslash_\circ, \bullet, /_\bullet)$ be the following involutive birack:

$\circ$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

$\bullet$	0	1	2	3
0	3	1	3	1
1	2	0	2	0
2	1	3	1	3
3	0	2	0	2

,

i.e. $L_0 = L_2 = \boldsymbol{R}_1 = \boldsymbol{R}_3 = (01)(23)$ and $L_1 = L_3 = \boldsymbol{R}_0 = \boldsymbol{R}_2 = (03)(12)$. Example 2.5 shows that the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is not left distributive, but the left quasigroup $(X, \circ, \backslash_\circ)$ is 2-permutational. Clearly, the quotient $(X/{\sim}, \circ, \backslash_\circ, \bullet, /_\bullet)$

$\circ$	$0/{\sim}$	$1/{\sim}$
$0/{\sim}$	$1/{\sim}$	$0/{\sim}$
$1/{\sim}$	$1/{\sim}$	$0/{\sim}$

$\bullet$	$0/{\sim}$	$1/{\sim}$
$0/{\sim}$	$1/{\sim}$	$1/{\sim}$
$1/{\sim}$	$0/{\sim}$	$0/{\sim}$

.

is a 1-permutational, but not a projection birack.

Definition 6.2. An involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-*permutational* (*medial*) if the left quasigroup $(X, \circ, \backslash_\circ)$ is 2-permutational (medial).

Proposition 6.3. *An involutive birack is* 2*-permutational if and only if it is medial.*

Proof. Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be an involutive 2-permutational birack. Since the relation (5.10) is a congruence of an involutive birack then by (2.1) and (2.11) for every $x, y, z \in X$ we have:

$$z\backslash_\circ y = z\backslash_\circ(x \circ (x\backslash_\circ y)) \sim z\backslash_\circ(z \circ (x\backslash_\circ y)) = x\backslash_\circ y,$$

which implies

$$L_{x\backslash_\circ y} = L_{z\backslash_\circ y}.$$

By Theorem 5.5, the left quasigroup $(X, \circ, \backslash_\circ)$ is right cyclic. Hence for $x, y, a, b \in X$ we obtain

$$L_x L_{a\backslash_\circ y} = L_x L_{x\backslash_\circ y} \overset{(2.7)}{=} L_y L_{y\backslash_\circ x} = L_y L_{b\backslash_\circ x}.$$

Substitution of x by $b \circ x$ and y by $a \circ y$ gives that the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is medial

$$L_{b\circ x} L_y = L_{a\circ y} L_x \overset{(2.11)}{=} L_{b\circ y} L_x.$$

Lemma 4.4 completes the proof. □

Rump showed in [35, Theorem 2] that each finite right cyclic left quasigroup is non-degenerate (see also [20, Proposition 4.7]). Therefore, directly by Theorem 5.5 and Proposition 6.3, we obtain

Corollary 6.4. *Each finite* 2*-permutational right cyclic left quasigroup is medial.*

But the following question is still open.

Question 6.5. *Is it true that every infinite* 2*-permutational right cyclic left quasigroup is medial?*

Corollary 6.6. *An involutive* 2*-permutational birack* $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is distributive if and only if the quotient* $(X/{\sim}, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is idempotent.*

Proof. By Proposition 6.3 the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is medial. Let $(X/\!\sim, \circ, \backslash_\circ, \bullet, /_\bullet)$ be idempotent. This implies that for each $x \in X$, $x \sim x \circ x$. Therefore, by Lemma 4.2, for every $e, x \in X$,

$$L_{e\circ x} = L_{e\circ e} L_x L_e^{-1} = L_e L_x L_e^{-1}.$$

Lemma 5.13 completes the proof. □

In Section 4 we presented the notion of a π-isotope. This construction allows us to tie distributive and 2-permutational biracks.

Theorem 6.7. *Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be an involutive birack and π be a bijection on the set X. We define the following four operations on the set X:*

$$x * y := L_x\pi(y) \quad \text{and} \quad x\backslash_* y := \pi^{-1}L_x^{-1}(y),$$
$$x \diamond y := \pi^{-1}L_y^{-1}(x) = y\backslash_* x \quad \text{and} \quad x/_\diamond y := L_y\pi(x) = y * x.$$

Then,

(1) *if π satisfies Conditions* (4.5) *and* (4.11)*, then $(X, *, \backslash_*, \diamond, /_\diamond)$ is a distributive involutive birack;*

(2) *if $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is distributive and π satisfies Condition* (4.8)*, then $(X, *, \backslash_*, \diamond, /_\diamond)$ is a 2-permutational involutive birack.*

Proof. By Remark 4.5, $(X, *, \backslash_*)$ is the π-isotope of $(X, \circ, \backslash_\circ)$. Further, for every $x, y \in X$

$$(y/_\diamond x) \diamond x = (x * y) \diamond x = x\backslash_*(x * y) = y,$$

and

$$(y \diamond x)/_\diamond x = (x\backslash_* y)/_\diamond x = x * (x\backslash_* y) = y,$$

which implies that $(X, \diamond, /_\diamond)$ is a right quasigroup.

By Theorem 5.5, the left quasigroup $(X, \circ, \backslash_\circ)$ is right cyclic and non-degenerate. It means that the mapping

$$T\colon X \to X; \quad x \mapsto L_x^{-1}(x) = x\backslash_\circ x,$$

is a bijection. This implies that the mapping

$$T_\pi\colon X \to X; \quad x \mapsto \pi^{-1}L_x^{-1}(x) = x\backslash_* x,$$

is a bijection, too. This proves that the left quasigroup $(X, *, \backslash_*)$ is non-degenerate.

(1) Let π satisfy Conditions (4.5) and (4.11). By Theorem 4.11, the left quasigroup $(X, *, \backslash_*)$ is a 2-reductive rack. Finally, Corollary 5.7 implies that $(X, *, \backslash_*, \diamond, /_\diamond)$ is an involutive distributive birack.

(2) Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be distributive and π satisfy Condition (4.8). By Corollary 5.7 and Theorem 4.10, $(X, *, \backslash_*)$ is a 2-permutational right cyclic left quasigroup. Hence, by Theorem 5.5, $(X, *, \backslash_*, \diamond, /_\diamond)$ is a 2-permutational involutive birack. □

Note that for the involutive birack $(X, *, \backslash_*, \diamond, /_\diamond)$ constructed in Theorem 6.7, the left quasigroup $(X, *, \backslash_*)$ is the π-isotope of the left quasigroup $(X, \circ, \backslash_\circ)$. Thus this justifies the following definition.

Definition 6.8. Let $(X, *, \backslash_*, \diamond, /_\diamond)$ and $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be two involutive biracks. The birack $(X, *, \backslash_*, \diamond, /_\diamond)$ is a π-*isotope* of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ if, for some bijection π of the set X, the left quasigroup $(X, *, \backslash_*)$ is the π-isotope of $(X, \circ, \backslash_\circ)$.

Let an involutive birack $(X, *, \backslash_*, \diamond, /_\diamond)$ be a π-isotope of an involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$, for some bijection π of the set X. Then $(X, *, \backslash_*, \diamond, /_\diamond)$ satisfies Condition (5.9). Moreover, if the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is finite, then the multiplication table of $*$ is obtained by a permuting columns of the multiplication table of $\circ$ and the multiplication table of $\diamond$ is obtained by a permuting rows of the multiplication table of $\bullet$.

Example 6.9. Let $(\{0,1,2,3\},\circ,\backslash_\circ,\bullet,/_\bullet)$ be the 2-permutational involutive birack with its left quasigroup $(\{0,1,2,3\},\circ,\backslash_\circ)$ from Example 2.5. Then, by Proposition 6.3 and Example 4.13, the L_0^{-1}-isotope $(\{0,1,2,3\},*,\backslash_*,\diamond,/_\diamond)$, with $x*y=x\circ L_0^{-1}(y)$, $x\backslash_* y=0\circ L_x^{-1}(y)$, $x\diamond y=0\circ L_y^{-1}(x)$ and $x/_\diamond y:=y\circ L_0^{-1}(x)$ is an involutive distributive birack with the $*$-table presented in Example 2.4.

Example 6.10. Let $(\{0,1,2,3\},\circ,\backslash_\circ,\bullet,/_\bullet)$ be the distributive involutive birack with the left quasigroup $(\{0,1,2,3\},\circ,\backslash_\circ)$ defined in Example 2.4. Note that the permutation $\pi=(01)(23)$ satisfies Condition (4.8). Then constructing the π-isotope of $(\{0,1,2,3\},\circ,\backslash_\circ,\bullet,/_\bullet)$ we obtain the birack $(\{0,1,2,3\},*,\backslash_*,\diamond,/_\diamond)$ with the $*$-table presented in Example 2.5.

Note that different choices of a bijection in Theorem 6.7 may give non isomorphic biracks.

Example 6.11. Let $(\{0,1,2,3,4\},\circ,\backslash_\circ,\bullet,/_\bullet)$ be the 2-permutational involutive birack with multiplication $\circ$

$\circ$	0	1	2	3	4
0	0	2	1	4	3
1	3	2	1	0	4
2	4	2	1	3	0
3	0	2	1	4	3
4	0	2	1	4	3

i.e. $L_0=L_3=L_4=(12)(34)$, $L_1=(03)(12)$ and $L_2=(04)(12)$. Then L_i^{-1}-isotopes, for $i\in\{0,1\}$, of $(\{0,1,2,3,4\},\circ,\backslash_\circ,\bullet,/_\bullet)$ have the following multiplication tables of $*_i$

$*_0$	0	1	2	3	4
0	0	1	2	3	4
1	3	1	2	4	0
2	4	1	2	0	3
3	0	1	2	3	4
4	0	1	2	3	4

and

$*_1$	0	1	2	3	4
0	4	1	2	0	3
1	0	1	2	3	4
2	3	1	2	4	0
3	4	1	2	0	3
4	4	1	2	0	3

Both isotopes are distributive. It is clear that these two biracks are not isomorphic, as the L_0^{-1}-isotope is idempotent, whereas the L_1^{-1}-isotope is not.

Example 6.12. In Example 6.10 we showed that the birack $(\{0,1,2,3\},*,\backslash_*,\diamond,/_\diamond)$ with the $*$-table presented in Example 2.5 is the π-isotope, for $\pi=(01)(23)$, of the distributive birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ with the left quasigroup $(X,\circ,\backslash_\circ)$ defined in Example 2.4. Nevertheless, there is another choice of a permutation that yields another birack. If we take $\gamma=(0123)$ then this γ satisfies Condition (4.8) as well and we obtain the involutive birack with multiplication $*_1$:

$*_1$	0	1	2	3
0	1	2	3	0
1	3	0	1	2
2	1	2	3	0
3	3	0	1	2

or in other words

$$L_0=L_2=\mathbf{R}_0^{-1}=\mathbf{R}_2^{-1}=(0123)$$
$$L_1=L_3=\mathbf{R}_1^{-1}=\mathbf{R}_3^{-1}=(3210)$$

which is clearly not isomorphic to the birack $(\{0,1,2,3\},*,\backslash_*,\diamond,/_\diamond)$.

Theorem below shows that each 2-permutational involutive birack originates from an involutive distributive birack.

Theorem 6.13. *Each 2-permutational involutive birack is a π-isotope of a distributive one, for some bijection π.*

Proof. Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a 2-permutational involutive birack. By Proposition 6.3, the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is medial. Now let $e \in X$ and $(X, *, \backslash_*, \diamond, /_\diamond)$ be the L_e^{-1}-isotope of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$. By Corollary 4.12 and Theorem 6.7, $(X, *, \backslash_*, \diamond, /_\diamond)$ is a distributive involutive birack.

Let $\varrho = L_e$. By Lemma 4.2(3) we have for each $x, y, z \in X$

$$\varrho(y) * \varrho(x * z) = L_{L_e(y)} L_e^{-1} L_e L_x L_e^{-1}(z) = L_{L_e(y)} L_x L_e^{-1}(z) = L_{L_e(e)} L_y L_e^{-1} L_x L_e^{-1}(z) = L_{L_e(e)} L_x L_e^{-1} L_y L_e^{-1}(z) = L_{L_e(x)} L_y L_e^{-1}(z) = L_{L_e(x)} L_e^{-1} L_e L_y L_e^{-1}(z) = \varrho(x) * \varrho(y * z),$$

which shows that the left quasigroup $(X, *, \backslash_*)$ satisfies Condition (4.8), for $\varrho = L_e$.

Moreover, for each $x, y \in X$

$$x * \varrho(y) = L_x L_e^{-1} L_e(y) = x \circ y \quad \text{and} \quad \varrho^{-1}(x \backslash_* y) = L_e^{-1} L_e L_x^{-1}(y) = x \backslash_\circ y,$$
$$\varrho^{-1}(y \backslash_* x) = L_e^{-1} L_e L_y^{-1}(x) = y \backslash_\circ x \quad \text{and} \quad y * \varrho(x) = L_y L_e^{-1} L_e(x) = y \circ x,$$

which shows that $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is the L_e-isotope of the involutive distributive birack $(X, *, \backslash_*, \diamond, /_\diamond)$. □

At the end we collect some useful facts about bijections satisfying Conditions (4.5) and (4.8).

Remark 6.14. Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be an involutive birack and let ρ be a bijection on the set X which satisfies Condition (4.8). Then, for every $x, y \in X$,

$$x \sim y \quad \Leftrightarrow \quad \varrho(x) \sim \varrho(y).$$

Indeed, by Definition 5.10 we have

$$x \sim y \;\Leftrightarrow\; L_x = L_y \;\Rightarrow\; L_{\varrho(y)} \varrho L_x \overset{(4.8)}{=} L_{\varrho(x)} \varrho L_y = L_{\varrho(x)} \varrho L_x \;\Rightarrow\; L_{\varrho(x)} = L_{\varrho(y)} \;\Leftrightarrow\; \varrho(x) \sim \varrho(y).$$

On the other hand,

$$\varrho(x) \sim \varrho(y) \;\Leftrightarrow\; L_{\varrho(x)} = L_{\varrho(y)} \;\Rightarrow\; L_{\varrho(y)} \varrho L_x \overset{(4.8)}{=} L_{\varrho(x)} \varrho L_y = L_{\varrho(y)} \varrho L_y \;\Rightarrow\; L_x = L_y \;\Leftrightarrow\; x \sim y.$$

Remark 6.15. Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a 2-reductive involutive birack. By Lemma 4.7(1), π-isotope of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-reductive if and only if $L_{L_x \pi(y)} = L_y$, for every $x, y \in X$. Since

$$L_{L_x \pi(y)} \overset{(2.10)}{=} L_{\pi(y)},$$

this shows that the π-isotope of a 2-reductive involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-reductive if and only if for every $x \in X$

$$L_{\pi(x)} = L_x. \tag{6.1}$$

If π happens to be an automorphism of the left quasigroup $(X, \circ, \backslash_\circ)$, then

$$L_x = L_{\pi(x)} = \pi L_x \pi^{-1} \quad \Leftrightarrow \quad \pi L_x = L_x \pi.$$

Hence, in this case the π-isotope of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-reductive if and only if the automorphism π commutes with each left translation. In particular, the π-isotope is 2-reductive for any choice $\pi = L_e$ or $\pi = L_e^{-1}$, with $e \in X$.

Note that even for a distributive involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ and π being an automorphism of the left quasigroup $(X, \circ, \backslash_\circ)$, the π-isotope does not need to be isomorphic to the birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$; consider, e.g., Example 6.12: both permutations (01)(23) and (0123) are actually automorphisms of the 2-reductive involutive birack.

7. Solutions

As it was written in Section 1, each non-degenerate (involutive) solution (X, σ, τ) of the Yang-Baxter equation yields an (involutive) birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$. And conversely, if $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is an (involutive) birack, then defining

$$r(x, y) = (\sigma(x, y), \tau(x, y)) = (x \circ y, x \bullet y) = (L_x(y), \boldsymbol{R}_y(x)),$$

we obtain a non-degenerate (involutive) solution $(X, L, \boldsymbol{R})$ of the Yang-Baxter equation.

Such an equivalence allows us to treat each non-degenerate (involutive) solution as a birack and formulate results from Sections 5 and 6 in the language of solutions. In particular, 1-permutational birack corresponds to a *permutation solution* and the projection birack corresponds to the *trivial solution*.

Etingof et al. reasoned that the quotient set $X/\sim$, by the relation (5.10), has a structure of an involutive solution $(X/\sim, \overline{\sigma}, \overline{\tau})$ with $\overline{\sigma}(x/\sim, y/\sim) = \sigma(x, y)/\sim$ and $\overline{\tau}(x/\sim, y/\sim) = \tau(x, y)/\sim$ for $x/\sim, y/\sim \in X/\sim$ and $x \in x/\sim, y \in y/\sim$. They called such solution the *retraction* of (X, σ, τ) and denoted it by $\mathrm{Ret}(X, \sigma, \tau)$. The birack corresponding to the retraction solution $\mathrm{Ret}(X, \sigma, \tau)$ is the quotient birack $(X/\sim, \circ, \backslash_\circ, \bullet, /_\bullet)$.

Among involutive solutions, an important role is played by *multipermutation solutions*, see e.g. [5, 16, 40]. Let (X, σ, τ) be an involutive solution. One defines *iterated retraction* in the following way: $\mathrm{Ret}^0(X, \sigma, \tau) := (X, \sigma, \tau)$ and $\mathrm{Ret}^k(X, \sigma, \tau) := \mathrm{Ret}(\mathrm{Ret}^{k-1}(X, \sigma, \tau))$, for any natural number $k > 1$. A solution (X, σ, τ) is called a *multipermutation solution of level m* if m is the least nonnegative integer such that

$$|\mathrm{Ret}^m(X, \sigma, \tau)| = 1.$$

In the language of an involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ this means that applying m times the congruence $\sim$ to the subsequent quotient biracks, one obtains the one-element birack.

Let us consider $(X/\sim, \circ, \backslash_\circ, \bullet, /_\bullet)$, the quotient birack of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ and denote it by $\mathrm{Ret}(X, \circ, \bullet)$. Let $\mathrm{Ret}^0(X, \circ, \bullet) := (X, \circ, \bullet)$ and $\mathrm{Ret}^k(X, \circ, \bullet) := \mathrm{Ret}(\mathrm{Ret}^{k-1}(X, \circ, \bullet))$, for any natural number $k > 1$.

Definition 7.1. An involutive birack is a *multipermutation birack* if there exists a positive integer m such that $\mathrm{Ret}^{m-1}(X, \circ, \bullet)$ is a 1-permutational birack. A birack of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called a *multipermutation birack of level m* if m is the least nonnegative integer m such that

$$|\mathrm{Ret}^m(X, \circ, \bullet)| = 1.$$

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is *irretractable* if $\mathrm{Ret}(X, \circ, \bullet) := (X, \circ, \bullet)$, i.e. $\sim$ is the trivial relation.

Observation 7.2. [17, Section 3] *Let $|X| \geq 2$. A square-free solution (X, σ, τ) is a multipermutation solution of level m if and only if $\mathrm{Ret}^{m-1}(X, \sigma, \tau)$ is a trivial solution.*

Theorem 7.3. [15, Proposition 4.7] *Let (X, σ, τ) be a solution and $|X| \geq 2$. (X, σ, τ) is a multipermutation solution of level $0 \leq m$ if and only if Condition (2.5) holds for the corresponding birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.*

Definition 7.4. An involutive solution is *distributive* (*2-reductive*, *2-permutational*, *medial*, respectively), if it corresponds to a distributive (2-reductive, 2-permutational, medial, respectively) involutive birack.

Fact 7.5. [16, Proposition 8.2], [15, Proposition 4.7] *A square-free involutive solution (X, σ, τ) is multipermutation of level 2 if and only if it is distributive. In this case it has an abelian permutation group $\langle \sigma_x \colon x \in X \rangle$. More generally, if an involutive solution satisfies Condition $(*)$ then it is a multipermutation solution of level 2 if and only if it is 2-reductive.*

By Corollary 5.7, Proposition 6.3, Corollary 6.6 and Theorem 7.3, we can generalize some results given in [18, 16, 15].

Theorem 7.6. *Let (X, σ, τ) be an involutive solution. Then*

(1) *(X, σ, τ) is a multipermutation solution of level 2 if and only if it is medial.*
(2) *If (X, σ, τ) is distributive then it is a multipermutation solution of level 2.*

Theorem 7.7. *Let (X, σ, τ) be an involutive solution. The following conditions are equivalent:*

(1) *(X, σ, τ) is distributive,*
(2) *(X, σ, τ) is 2-reductive,*
(3) *$\mathrm{Ret}(X, \sigma, \tau)$ is the trivial solution.*

By Theorem 5.10 we can completely describe all involutive distributive solutions.

Theorem 7.8. *Each involutive distributive solution (X, σ, τ) is a disjoint union, over a set I, of abelian groups $A_j = \langle \{a_{i,j} \mid i \in I\} \rangle$, for every $j \in I$, with*

$$\sigma_x(y) = y + a_{i,j} \quad \text{and} \quad \tau_y(x) = x - a_{j,i}, \tag{7.1}$$

where $x \in A_i$ and $y \in A_j$.

By Corollary 5.9 each involutive distributive solution satisfies Condition **stu**, introduced in [18, Definition 5.1], which means that it is trivially a strong twisted union of abelian groups A_j.

Example 7.9. Let I be a (finite or infinite) index set and let A_i, for $i \in I$, be cyclic groups. Let $(a_{i,j})_{i,j \in I}$ be constants such that $a_{i,j} \in A_j$, for all $i, j \in I$, and, for each $j \in I$, there exists at least one $i \in I$, such that $a_{i,j}$ is a generator of the group A_j. Then $(\bigcup A_i, \sigma, \tau)$, with σ and τ defined in (7.1), is an involutive distributive solution.

We can construct all distributive solutions of size n using the following algorithm:

Algorithm 7.10. Outputs all distributive solutions of size n:

(1) For all partitionings $n = n_1 + n_2 + \cdots + n_k$ do (2)–(4).
(2) For all abelian groups $A_1, \ldots, A_k$ of size $|A_i| = n_i$ do (3)–(4).
(3) For all constants $a_{i,j} \in A_j$ do (4).
(4) If, for all $1 \leq j \leq k$, we have $A_j = \langle \{a_{i,j} \mid i \in I\} \rangle$ then construct a solution $(\bigcup A_i, \sigma, \tau)$ using (7.1).

When all solutions are constructed, we can get rid of isomorphic copies using Theorem 3.7.

In [4] the permutation group $\langle \sigma_x \colon x \in X \rangle$ of a *finite* involutive solution (X, σ, τ) was called *the involutive Yang-Baxter group* (IYB group) associated to the solution (X, σ, τ). In particular, Cedo et al. showed in [4, Corollary 3.11] that each finite nilpotent group of class 2 (and thus each finite abelian group) is an IYB group. Here, using the construction of the sum of a trivial affine mesh, we present short direct proof of this fact for an abelian group of an arbitrary cardinality.

Theorem 7.11. *Let A be an abelian group. Then there exists an involutive solution (X, σ, τ) with its permutation group isomorphic to A.*

Proof. Let A be generated by a (finite or infinite) set $\{g_i \colon i \in I\}$. We construct the solution (X, σ, τ) as the sum of the trivial affine mesh $((A_i)_{i \in I}, (c_{i,j})_{i,j \in I})$ over I, with $A_i = A$ and $c_{i,j} = g_i$, for all $i, j \in I$.

By construction, $L_a(b) = b + c_{i,j} = b + g_i$, for all $a \in A_i$ and $b \in A_j$. Therefore the permutation group consists solely of mappings $b_j \mapsto b_j + c$, for each $j \in I$ and some $c \in A$. This means that the group $\langle \sigma_x \colon x \in X \rangle$ naturally embeds into A. Moreover, the permutation group is generated by L_a, for $a \in A$, and hence it is isomorphic to A. □

By Corollary 4.12 and Theorem 6.7 each involutive multipermutation solution of level 2 defines a distributive one.

Theorem 7.12. *Let (X, σ, τ) be an involutive multipermutation solution of level 2 and $e \in X$. Then $(X, \sigma\sigma_e^{-1}, \sigma_e\sigma^{-1})$ is an involutive distributive solution.*

On the other side, Theorem 6.13 shows that each involutive multipermutation solution of level 2 originates from an involutive distributive solution. We have even more. The theorem gives a procedure how to obtain all involutive multipermutation solutions of level 2 from distributive ones.

We have to take all involutive distributive solutions (X, σ, τ) such that there exists $a \in X$ with $L_a = \mathrm{id}$ and, for each of them, all permutations π of the set X which satisfy Condition (4.8) i.e.

$$\sigma_{\pi(y)}\pi\sigma_x = \sigma_{\pi(x)}\pi\sigma_y.$$

Then $(X, \sigma\pi, \pi^{-1}\sigma^{-1})$ will be involutive multipermutation solutions of level 2. Obviously, each involutive multipermutation solutions of level 2 satisfies Condition **lri** (Condition (5.9)).

By Remarks 6.14 and 6.15 we can construct all non-distributive involutive solutions of multipermutation level 2 of size n.

Algorithm 7.13. Outputs all non-distributive involutive solutions of multipermutation level 2 of size n:

(1) For every distributive solution (X, σ, τ) of size n do (2)–(7).
(2) If there exist no $x \in X$ such that $\sigma_x = \mathrm{id}$ return to (1).
(3) For every permutation $\pi \in S_X$ do (4)–(7).
(4) If $\pi \in \langle \sigma_x : x \in X \rangle$, return to (3).
(5) If π does not send classes of $\sim$ onto classes of $\sim$, return to (3).
(6) If π permutes classes of $\sim$, that means if π satisfies (6.1), return to (3).
(7) If π does not satisfy (4.8), return to (3).
(8) Construct the solution $(X, \sigma\pi, \pi^{-1}\sigma^{-1})$.

Unlike in the case of distributive solutions, we do not have any efficient criterion to test isomorphisms. As Example 6.11 shows, the same solutions can be obtained from different distributive solutions.

8. Enumeration

In this section we enumerate involutive solutions of multipermutation level 2 for small sizes and we estimate, for all sizes, how many racks and involutive solutions are there, up to isomorphism.

Using the characterization by sums of trivial affine meshes we can straightforwardly describe all involutive solutions of small sizes. The size 4 can be done manually.

Example 8.1. By results of [11], there are 23 involutive solutions of size 4, up to isomorphism. Two of them are irretractable. Exactly 17 of them are distributive. They are the sums of the following trivial affine meshes:

- One orbit: $((\mathbb{Z}_4), (1))$.
- Two orbits: $((\mathbb{Z}_3, \mathbb{Z}_1), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_3, \mathbb{Z}_1), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_3, \mathbb{Z}_1), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_3, \mathbb{Z}_1), \left(\begin{smallmatrix} 2 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_2), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_2), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_2), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_2), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_2), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_2), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right))$.
- Three orbits: $((\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$, $((\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix})$, $((\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1), \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix})$, $((\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix})$, $((\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1), \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix})$.

- Four orbits: $((\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1), \left(\begin{smallmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{smallmatrix}\right))$.

There remain four multipermutation solutions of size 4 that are not distributive. Two of them are of level 2, both of them described in Example 6.12. Two of them are of level 3 - the corresponding biracks have the following tables of $\circ$-multiplication:

$\circ$	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	3	2
3	1	0	3	2

and

$\circ$	0	1	2	3
0	1	0	2	3
1	1	0	2	3
2	0	1	3	2
3	1	0	3	2

.

They are not isomorphic since the first one has two idempotent elements, whereas the other one has none.

The same way as we did it for size 4, we can compute other small sizes, on a computer of course. We start with the numbers of small racks. In Table 1, we compare the numbers of isomorphism classes of all racks (see OEIS sequence A181770 [27]) and 2-reductive racks. Computing 2-reductive racks directly using Theorems 3.6 and 3.7 is hopeless for larger numbers. Hence the numbers of 2-reductive racks were computed using Burnside's lemma, see [19] for more details.

As we can see, the numbers of 2-reductive racks grow really fast, actually, according to Blackburn [2], there are at least $2^{n^2/4-O(n\log n)}$ 2-reductive racks of size n. We can also give an upper bound, which is not far from the lower bound. The proof is exactly the same as the proof of [19, Theorem 8.2].

Theorem 8.2. *There are at most $2^{(1/4+o(1))n^2}$ 2-reductive racks of size n, up to isomorphism.*

The numbers in Table 1 suggest that the vast majority of all racks are 2-reductive. However, we do not have any proof of this fact. Hence we can only conjecture that:

Conjecture 8.3. *There are $2^{(1/4+o(1))n^2}$ racks of size n, up to isomorphism.*

In Table 2, we can see the numbers of involutive solutions. The total numbers of solutions is taken from [11], the numbers of 2-reductive solutions are the same as the numbers of 2-reductive racks. The numbers of 2-permutational solutions (i.e. multipermutation solutions of level 2) that are not 2-reductive were computed by a brute force search algorithm using the Mace4 software [25].

Theorem 8.4. *There are at most $2^{(1/4+o(1))n^2}$ involutive multipermutation solutions of level 2 of size n, up to isomorphism.*

Proof. As was shown in Theorem 7.12, every multipermutation solution of level 2 can be obtained as an isotope of a 2-reductive solution using a permutation. Hence, using Theorem 8.2, the number of 2-permutational solutions is less than $2^{(1/4+o(1))n^2} \cdot n! = 2^{(1/4+o(1))n^2}$. □

In the case of solutions, Table 2 suggests that the numbers of all involutive solutions grow faster than the numbers of multipermutation solutions of level 2 but not much faster. We can therefore conjecture:

Conjecture 8.5. *There are $2^{O(n^2)}$ involutive solutions of size n, up to isomorphism.*

9. Braces

A popular way how to approach involutive set-theoretic solutions of the Yang-Baxter equation are left braces. In this section we look at left braces corresponding to multipermutation solutions of level 2.

n	1	2	3	4	5	6	7	8	9	10	11
racks	1	2	6	19	74	353	2080	16023			
2-reductive	1	2	5	17	65	323	1960	15421	155889	2064688	35982357

n	12	13	14
racks			
2-reductive racks	832698007	25731050861	1067863092309

TABLE 1. The number of racks and 2-reductive racks of size n, up to isomorphism.

n	1	2	3	4	5	6	7	8
involutive solutions	1	2	5	23	88	595	3456	34528
multipermutation of level 2	1	2	5	19	70	359	2095	16332
2-reductive	1	2	5	17	65	323	1960	15421
2-permutational, not 2-reductive	0	0	0	2	5	36	135	911

TABLE 2. The number of involutive solutions of size n, up to isomorphism.

Definition 9.1. An algebra $(B,+,\cdot,0)$ is called a *left brace* if $(B,+,0)$ is an abelian group, $(B,\cdot,0)$ is a group and the operations satisfy for all $a,b,c\in B$

$$a\cdot b+a\cdot c=a\cdot(b+c)+a. \tag{9.1}$$

A left brace is called *trivial* if $a+b=a\cdot b$, for all $a,b\in B$.

Given a left brace $(B,+,\cdot,0)$, we can define $\lambda_a(b)=a\cdot b-a$ and

$$a\circ b=\lambda_a(b),\qquad a\backslash_\circ b=\lambda_a^{-1}(b),\quad a\bullet b=\lambda^{-1}_{\lambda_a(b)}(a),\quad a/_\bullet b=\lambda_{\lambda_a(b)}(a).$$

It turns out that $(B,\circ,\backslash_\circ,\bullet,/_\bullet)$ is an involutive birack [36]. We shall write $\mathcal{L}_a(b)=\lambda_a(b)$ and $\mathcal{R}_b(a)=\lambda^{-1}_{\lambda_a(b)}(a)=\mathcal{L}^{-1}_{\mathcal{L}_a(b)}(a)$.

For left braces, it can be shown that kernels of homomorphisms are subsets called ideals. An *ideal* of a left brace $(B,+,\cdot,0)$ is a normal subgroup of $(B,\cdot)$ closed on λ_a, for each $a\in B$. An important ideal is the *socle* defined as $\mathrm{Soc}(B)=\{a\in B\colon \lambda_a=\mathrm{id}\}$.

The following characterization of left braces corresponding to multipermutation solution of level 2 effectively generalizes an analogous result by Gateva-Ivanova [15, Theorem 8.2] for square-free solutions:

Theorem 9.2. *Let $(B,+,\cdot,0)$ be a left brace. Then the following conditions are equivalent:*

(i) *$(B,\mathcal{L},\mathcal{R})$ is a multipermutation solution of level at most 2;*
(ii) *$(B,\mathcal{L},\mathcal{R})$ is a 2-reductive solution;*
(iii) *the factor left brace $B/\mathrm{Soc}(B)$ is trivial;*
(iv) *the mapping $\lambda\colon a\mapsto\lambda_a$ is a homomorphism from $(B,+,0)$ onto $\langle\mathcal{L}_a,\mathcal{R}_a\colon\ a\in B\rangle$;*
(v) *we have $\lambda_{a+b}=\lambda_{a\cdot b}$, for all $a,b\in B$.*

Proof. The equivalence (i)⇔(iii) is well known, see [35]. Since $\lambda_0=id$, the solution $(B,\mathcal{L},\mathcal{R})$ satisfies Condition $(*)$ and the equivalence (i)⇔(ii) comes from Lemma 5.16.

We remark now that Condition (iii) can be equivalently reformulated as follows:

$$(a+b)(ab)^{-1}\in\mathrm{Soc}(B),\ \text{for all } a,b\in B,$$

since the element $(a+b)(ab)^{-1}$ is zero in every trivial left brace. We know that λ is a homomorphism from $(B, \cdot, 0)$ (see [6]) and therefore

$$\lambda_{(a+b)(ab)^{-1}} = \lambda_{a+b}\lambda_{ab}^{-1}.$$

The left hand side is the identity mapping under Condition (iii) and the right-hand side is the identity mapping under Condition (v), proving thus (iii)$\Leftrightarrow$(v).

Now (iv)$\Rightarrow$(v) is obtained by

$$\lambda_{a+b} = \lambda_a\lambda_b = \lambda_{a\cdot b}$$

and (v)$\Rightarrow$(iv) by

$$\lambda_{a+b} = \lambda_{a\cdot b} = \lambda_a\lambda_b.$$

□

Example 9.3. Consider the following left brace: $(\mathbb{Z}_6, +, \cdot, 0)$ where $+$ is the usual operation in $\mathbb{Z}_6$ and $a \cdot b = a + (-1)^a b$. Here we have $\lambda_0 = \lambda_2 = \lambda_4 = \mathrm{id}$ and $\lambda_1 = \lambda_3 = \lambda_5 = (15)(24)$. Since λ here is a homomorphism from $(\mathbb{Z}_6, +, 0)$ onto a two-element group, the corresponding solution is 2-reductive. Indeed, we have $0 \sim 2 \sim 4$ and $1 \sim 3 \sim 5$ and the retraction is a trivial 2-element solution. We can obtain the 6-element solution coming from the left brace as the sum of the following affine mesh:

$$((\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2), \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&1\\ 0&1&0&1 \end{pmatrix})$$

where the first two components correspond to the two orbits of the permutation group that lie in the socle, namely $\{0\}$ and $\{2, 4\}$ and the other two correspond to the orbits outside of the socle, namely $\{3\}$ and $\{1, 5\}$.

Every involutive solution embeds into a solution coming from a left brace. There is a standard construction of such a left brace. Let (X, σ, τ) be an involutive solution and $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ its corresponding involutive brace. The abelian group can be taken to be the free X-generated $\mathbb{Z}$-module, that means $(\mathbb{Z}^X, +, 0)$. We define $\lambda_a \in \mathrm{Aut}(\mathbb{Z}^X)$, for each $a = \sum_{x\in X} c_x x \in \mathbb{Z}^X$ by induction on $|a| = \sum_{x\in X} |c_x|$, following [11, Subsection 2.3]:

$$\begin{aligned}
\lambda_0 &= \mathrm{id} && \\
\lambda_x &= \eta\sigma_x && \text{for } x \in X\\
\lambda_{-x} &= \eta\sigma^{-1}_{x\backslash_\circ x} && \text{for } x \in X\\
\lambda_{a+x} &= \lambda_{\lambda_a(x)}\lambda_a && \text{for } |x| = 1 \text{ and } |a + x| = |a| + 1
\end{aligned}$$

where $\eta : S_X \to \mathrm{Aut}(\mathbb{Z}^X)$ is the homomorphism that extends uniquely a permutation of the basis X to an automorphism. It can be proved that, by defining

$$a \cdot b = \lambda_a(b) + a,$$

the algebra $(\mathbb{Z}^X, \cdot, 0)$ is a group [11]. By construction, for all $x, y \in X$, we have $x \circ y = \sigma_x(y) = \eta\sigma_x(y) = \lambda_x(y) = \mathcal{L}_x(y)$ and therefore (X, σ, τ) is a proper subsolution of $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$.

It is a question, which properties of (X, σ, τ) hold in $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$ as well. We know that the solution $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$ has always an element with trivial permutations, since $\mathcal{L}_0(a) = \lambda_0(a) = a$, for each $a \in \mathbb{Z}^X$. This means, for instance, that $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$ can never be a non-trivial permutation solution, even if (X, σ, τ) is. In the case of distributive solutions we can prove that they embed into distributive solutions coming from left braces.

Theorem 9.4. *Let (X, σ, τ) be an involutive solution. Let $(\mathbb{Z}^X, +, \cdot, 0)$ be the corresponding left brace and let $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$ be the solution formed from the left brace. Then the following conditions are equivalent:*

(i) *(X, σ, τ) is a 2-reductive solution;*

(ii) $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$ *is a 2-reductive solution;*
(iii) *the group* $\mathrm{LMlt}(\mathbb{Z}^X) = \langle \mathcal{L}_a \colon a \in \mathbb{Z}^X \rangle$ *is abelian and* $\mathcal{L}_a = \eta \prod_{x \in X} \sigma_x^{c_x}$*, for all* $a \in \mathbb{Z}^X$ *written as* $a = \sum_{x \in X} c_x x$*.*

Proof. (i) $\Rightarrow$ (iii) The group $\mathrm{LMlt}(X)$ is abelian, according to Lemma 3.3, and $\mathrm{LMlt}(\mathbb{Z}^X)$ is clearly isomorphic to $\mathrm{LMlt}(X)$ since $\mathrm{LMlt}(\mathbb{Z}^X)$ is generated by automorphisms $\eta\sigma_x$, for $x \in X$.

The structure of $\mathcal{L}_a$ is evident for $a = 0$ and $a = x \in X$. Now, $\mathcal{L}_{-x} = \lambda_{-x} = \eta\sigma^{-1}_{x \backslash_\circ x} \overset{(2.10)}{=} \eta\sigma_x^{-1}$. Next, we suppose that the claim holds for $b = \sum_{y \in X} c_y y$ and we prove it for $a = b + x$.

$$\mathcal{L}_a = \lambda_a = \lambda_{b+x} = \lambda_{\lambda_b(x)}\lambda_b = (\eta\sigma_{\prod \sigma_y^{c_y}(x)})(\eta \prod {\sigma_y}^{c_y}) = \eta\sigma_x \prod {L_y}^{c_y},$$

where we used $\sigma_{\prod \sigma_y^{c_y}(x)} = L_x$ due to 2-reductivity.

(iii)$\Rightarrow$(ii) follows from Theorem 9.2 since such λ is an additive homomorphism. And (ii)$\Rightarrow$(i) is trivial since (X, σ, τ) is a subsolution of $(\mathbb{Z}^X, \mathcal{L}, \mathcal{R})$. ☐

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