THE ADJOINT BRAID ARRANGEMENT AS A COMBINATORIAL LIE ALGEBRA VIA THE STEINMANN RELATIONS

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ABSTRACT. We study the dual action of Lie elements on faces of the adjoint braid arrangement, interpreted as the discrete differentiation of functions on faces across hyperplanes. We encode flags of faces with layered binary trees, allowing for the representation of Lie elements by antisymmetrized layered binary forests. This induces an action of layered binary forests on functions by discrete differentiation, which we call the forest derivative. The forest derivative has antisymmetry and satisfies the Jacobi identity. We show that the restriction of the forest derivative to functions which satisfy the Steinmann relations is additionally delayered, and thus forms a left comodule of the Lie cooperad. Dually, this endows the adjoint braid arrangement modulo the Steinmann relations with the structure of a Lie algebra internal to the category of linear species.

CONTENTS

1. INTRODUCTION

The combinatorial Hopf theory of the braid arrangement is very rich, and can be elegantly realized as structure internal to the category of species. Species are presheafs on the category of finite sets and bijections, and were introduced by André Joyal as a method for studying combinatorial structures in terms of generating functions (see [Joy81], [Joy86], [BLL98]). Species may be viewed as the categorification of formal power series (see [BD01]). The basics of Hopf theory in species has been beautifully described by Aguiar-Mahajan (see [AM10], [AM13]). Aspects of the theory for the braid arrangement have been developed for generic hyperplane arrangements, with a view towards applications to Hopf theory in species (see [AM17]). In this article, we

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develop theory for the adjoint braid arrangement which produces algebraic structure in species. In particular, we construct a Lie algebra $\mathbf{\Gamma}$ internal to the category of (linear) species whose underlying species is the quotient of the adjoint braid arrangement by certain four term relations, which have previously appeared in the foundations of Wightman quantum field theory under the name 'Steinmann identities' (see [Str75, p. 827-828], [Ste60a], [Ste60b]). Note that the importance of combinatorial Hopf theory in the study of renormalization in quantum field theory is well established (see [CK99], [EFK05], [FGB05], [Mor06]).

We obtain the Lie algebra $\mathbf{\Gamma}$ by studying the discrete differentiation of functions on faces of the adjoint braid arrangement across hyperplanes. In order for this derivative to be realized in species, the derivatives of functions must decompose as tensor products of functions. The Steinmann relations say exactly that a function's first derivatives decompose as tensor products. We show that if a function's first derivatives decompose, then all of the function's derivatives decompose. Thus, restricting to functions which satisfy the Steinmann relations is sufficient.

Let us briefly mention the significance of the Lie algebra $\mathbf{\Gamma}$; the universal enveloping algebra of $\mathbf{\Gamma}$ is none other than the combinatorial Hopf algebra of the braid arrangement $\mathbf{\Sigma}$ (this algebra is often called the 'Hopf algebra of compositions', and is defined in [AM13, Section 11.1]). The dual $\mathbf{\Sigma}^* \twoheadrightarrow \mathbf{\Gamma}^*$ of the primitive elements map $\mathbf{\Gamma} \hookrightarrow \mathbf{\Sigma}$ sends the $\mathtt{M}$-basis of $\mathbf{\Sigma}^*$ to signed characteristic functions of permutohedral cones, providing an explanation of the signed quasi-shuffle relations of permutohedral cones observed by Ocneanu. These relations have been studied and generalized by Early (see [Ear17a]). In particular, $\mathbf{\Gamma}^*$ coincides with the span of characteristic functions of permutohedral cones.

We begin in Section 2 by describing some important aspects of the adjoint braid arrangement. Let I be a finite set with cardinality n, and for $P = (S_1|\dots|S_k)$ a partition of I, let

$$\mathbf{A}_P[I] := \Big\{ (x_i)_{i\in I} : x_i \in \mathbb{R} \text{ such that } \sum_{i\in S} x_i = 0 \text{ for all } S \in P \Big\}.$$

Put $\mathbf{A}[I] := \mathbf{A}_{\{I\}}[I]$. The set of points in $\mathbf{A}[I]$ which have integer coordinates forms the root lattice of type A_{n-1}. Notice that $\mathbf{A}_P[I]$ is a hyperplane of $\mathbf{A}[I]$ if P has two blocks. The arrangement of all such hyperplanes in $\mathbf{A}[I]$ is often called the restricted all-subset arrangement, denoted $\mathbf{Br}^\vee[I]$, which is the adjoint of the braid arrangement $\mathbf{Br}[I]$. Equivalently, the hyperplanes of the adjoint braid arrangement are the hyperplanes of $\mathbf{A}[I]$ which can be spanned by roots. We call a subspace of $\mathbf{A}[I]$ an adjoint flat if it is an intersection of hyperplanes of the adjoint braid arrangement. The subspaces of $\mathbf{A}[I]$ which can be spanned by subsets of roots are special examples of adjoint flats, and we call these adjoint flats semisimple. Semisimple flats are exactly the subspaces $\mathbf{A}_P[I]$, for P a partition of I.

The adjoint braid arrangement under $\mathbf{A}_P[I]$, denoted $\mathbf{Br}^\vee_P[I]$, consists of those hyperplanes of $\mathbf{A}_P[I]$ which are adjoint flats of $\mathbf{A}[I]$. The underlying space of $\mathbf{Br}^\vee_P[I]$ may be identified with the underlying space of the product of arrangements $\prod_j \mathbf{Br}^\vee[S_j]$; however, in general $\mathbf{Br}^\vee_P[I]$ has more hyperplanes than the product. The hyperplanes of $\mathbf{Br}^\vee_P[I]$ which come from the product are exactly the semisimple flats, whereas the additional hyperplanes are the adjoint flats which are not semisimple.

Let $\mathbf{Shd}_P[I]$ denote the space of formal linear combinations of chambers of $\mathbf{Br}^\vee_P[I]$, and put $\mathbf{Shd}[I] = \mathbf{Shd}_{\{I\}}[I]$ (we choose this notation since 'shard' will be our name for faces of the adjoint braid arrangement). We obtain a quotient of $\mathbf{Shd}_P[I]$, which is naturally isomorphic to $\bigotimes_j \mathbf{Shd}[S_j]$, by identifying chambers which cannot be distinguished by hyperplanes which are semisimple flats (in Figure 1, this results in the identification of the faces Y_1 and Y_2). In

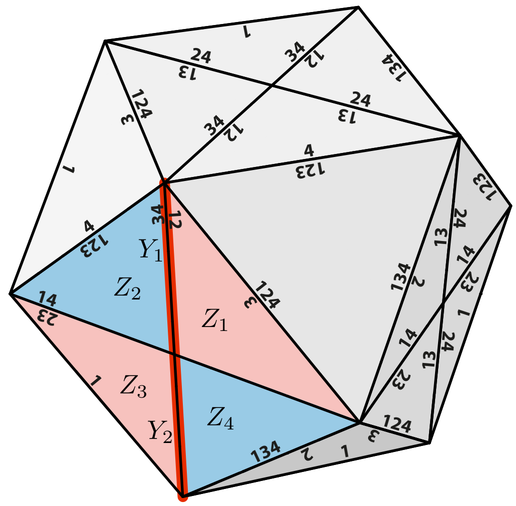

FIGURE 1. The intersection of the adjoint braid arrangement on $I = \{1, 2, 3, 4\}$ with the root polytope of type A_3; Y_1 and Y_2 are codimension one faces, and Z_1, Z_2, Z_3, Z_4 are top dimensional faces. The Lie brackets of Y_1 and Y_2 are $D^*_{[12,34]}Y_1 = Z_1 - Z_2$ and $D^*_{[12,34]}Y_2 = Z_4 - Z_3$ (see Section 6). However, in order for the flat corresponding to the partition $(12|34)$ to have the product structure of type $A_1 \times A_1$, the faces Y_1 and Y_2 must be identified. Therefore we must have $Z_1 - Z_2 = Z_4 - Z_3$, and so $Z_1 - Z_2 + Z_3 - Z_4 = 0$, which is called a Steinmann relation.

Theorem 2.2 we show, in the more general setting of 'R-semisimplicity', that this quotient map coincides with a map we call projection, given in Definition 2.5. By taking the linear dual of this quotient map, we obtain an embedding

$$\bigotimes_j \mathbf{Shd}^*[S_j] \hookrightarrow \mathbf{Shd}^*_P[I],$$

whose image we call semisimple functions. Thus, by taking certain quotients of chambers, or dually by restricting to certain functions, we obtain a product structure on the adjoint braid arrangement under semisimple flats.

In Section 3, we define the category of partitions Lay_I to be the linear category with objects the partitions P of I, and morphisms freely generated by refinements of partitions by choosing a subset of a block. We model the morphisms of this category with labeled layered binary forests $\mathcal{F}$. Let Vec denote the category of finite dimensional vector spaces. We construct a functor

$$\mathsf{Lay}_I \to \mathsf{Vec}, \qquad P \mapsto \mathbf{Shd}_P[I], \quad \mathcal{F} \mapsto \partial^*_{\mathcal{F}},$$

where $\partial^*_{\mathcal{F}}$ is called the dual forest derivative (see Definition 3.4). Using the theory of Lie elements for generic hyperplane arrangements (see [AM17, Chapters 4 and 10]), the dual forest derivative can be obtained by representing certain Lie elements of the adjoint braid arrangement with layered trees, and then letting Lie elements act on faces (see Section 4). The composition of this functor with linear duality is the derivative of functions on faces with respect to forests. This functor sends forests to linear maps which evaluate finite differences of functions across hyperplanes.

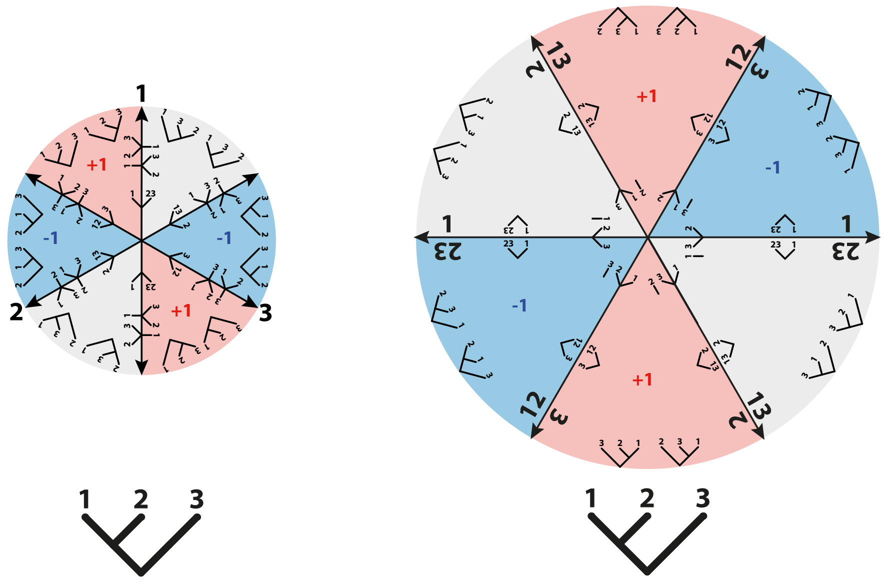

FIGURE 2. The braid and adjoint braid arrangements on $I = \{1, 2, 3\}$, decorated with a schematic for the action of the category of partitions on faces, which interprets layered binary trees as flags of faces. By antisymmetrizing trees, these actions allow us to use layered binary trees to represent Lie elements of both arrangements. Classically, this is only done for the braid arrangement (and with delayered trees), see [Gar90], [Reu03]. The Lie elements represented by the tree $[[1, 2], 3]$ are shown.

In Theorem 3.1, we show that the derivative has antisymmetry and satisfies the Jacobi identity, as interpreted on forests. Antisymmetry is immediate, since we antisymmetrize forests when we define the derivative. The Jacobi identity is a consequence of the fact that the geometry of the adjoint braid arrangement imposes the following 'pre-Lie relations' on trees,

$$[[1, 2], 3] = [[1, 3], 2] \qquad \text{and} \qquad [1, [2, 3]] = [2, [1, 3]]$$

(see Figure 2). To get the Lie operad from the morphisms of Lay_I, and thus structure in species, we also need to delayer the forests; however, the functor $\mathsf{Lay}_I \to \mathsf{Vec}$ is not well defined on delayered forests (see Figure 6).

In Section 5, we show that the derivative of a tensor product of functions decomposes as a tensor product of derivatives. Therefore, if we restrict to functions whose derivatives are all semisimple, called semisimply differentiable functions, then the derivative does not depend upon the layering of forests (see Corollary 5.1.1). Let Lie_I denote the quotient of Lay_I by antisymmetry, the Jacobi identity, and delayering. Then a new functor $\mathsf{Lay}_I \to \mathsf{Vec}$, obtained by restricting the derivative to semisimply differentiable functions, factors through the quotient map $\mathsf{Lay}_I \twoheadrightarrow \mathsf{Lie}_I$. This new functor then provides the data for a Lie algebra in species.

In Theorem 5.3, we show that if a function's first derivatives are semisimple, which is equivalent to the function satisfying the Steinmann relations, then the function is semisimply differentiable. One can consider the functions which satisfy the Steinmann relations as differentiable functions, and semisimply differentiable functions as smooth functions. Thus, Theorem 5.3 is an analog of the result in complex analysis that a differentiable function is analytic. We will study a discrete analog of Taylor series in future work.

In Section 6, we translate the data of the restricted derivative into a Lie algebra internal to the category of species. For simplicity, we identify now the two maps which we denote by ∂_Λ and D_Λ in Section 6. First, we realize the derivative as a left coaction of the Lie cooperad $\mathbf{Lie}^*$ on the species $\mathbf{\Gamma}^*$ of semisimply differentiable functions on chambers (equivalently functions which satisfy the Steinman relations),

$$\mathbf{\Gamma}^* \to \mathbf{Lie}^* \circ \mathbf{\Gamma}^*, \qquad f \mapsto \bigoplus_P \sum_{\Lambda \in \mathbf{Lay}[P]} \Lambda^* \otimes \partial_\Lambda f,$$

where '$\circ$' is the composition of species, and $\partial_\Lambda f$ denotes the derivative of f with respect to the tree Λ. Dualizing this, we obtain a left action of the Lie operad $\mathbf{Lie}$ on the species of chambers of the adjoint braid arrangement modulo the Steinmann relations,

$$\mathbf{Lie} \circ \mathbf{\Gamma} \to \mathbf{\Gamma}, \qquad \Lambda \otimes Z \mapsto \partial^*_\Lambda Z.$$

Note that left $\mathbf{Lie}$-modules in species are equivalent to Lie algebra in species (see [AM10, Section B.5]). The corresponding Lie algebra $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} \cdot \mathbf{\Gamma} \to \mathbf{\Gamma}, \qquad Z \mapsto \partial^*_{[S,T]} Z,$$

where '$\cdot$' is the Cauchy product of species, S and T are finite sets, $Z \in \mathbf{\Gamma}[S] \otimes \mathbf{\Gamma}[T]$, and $\partial^*_{[S,T]}$ is the dual derivative with respect to the tree $[S,T]$. Note that algebras in species with respect to the Cauchy product also go by the name 'twisted algebras' (see [Bar78], [Sto93], [PR04], [Aub10], [Aub10]); however, following Aguiar-Mahajan, we do not use this name.

It appears that structures related to our Lie algebra, and its relationship to the Tits algebra of the braid arrangement, are used in quantum field theory (see [EGS75], [Eps16], [Eva91], [Eva94]). An up operator on the species of the adjoint braid arrangement plays a central role in the algebraic formalism developed in [EGS75] for the study of the generalized retarded functions, although the authors do not use species. We leave the development of these connections with quantum field theory to future work.

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This paper was inspired by part of a lecture course given by the third author at Harvard University in the fall of 2017 (see [Ocn18]). The lecture course is available on YouTube (see video playlist), and supplementary materials are in preparation for publication. The relevant lectures are numbered 33, 34, and 35, in which Ocneanu defines a map on characteristic functions of permutohedral cones into layered binary trees by letting trees encode boundary flags. He considers finite differences of functions across hyperplanes to prove by induction that a certain map from trees to functions is the inverse of his map from characteristic functions of permutohedral cones to trees. He later observed that this inductive process was described precisely by his layered trees, giving rise to the notion of the derivative of a function with respect to a tree.

2. The Adjoint Braid Arrangement

We define the adjoint braid arrangement and describe some of its key aspects. In particular, we identify the flats which are spanned by subsets of roots as being particularly important. We give a combinatorial description of both orthogonal projections of faces, and its linear dual, which is a product of functions on faces. We show that by taking a certain quotient of faces, or dually by restricting to certain functions, projections and products become bijections. This gives a product structure on the adjoint braid arrangement under flats which are spanned by subsets of roots. This product structure is required in order to obtain algebraic structure in species.

2.1. **Flats of the Adjoint Braid Arrangement.** Let I be a finite set with cardinality n. We will often let $I = \{1, \dots, n\}$. A *partition* $P = (S_1 | \dots | S_k)$ of I of *rank* $n-k$ is a (unordered) set of k disjoint nonempty *blocks* $S_j \subseteq I$ whose union is I. For partitions P and Q of I, we say that Q is *finer* than P if every block of Q is a subset of some block of P. Let

$$\mathbb{R}I := \big\{ (x_i)_{i \in I} : x_i \in \mathbb{R} \big\}.$$

For $x \in \mathbb{R}I$ and $S \subseteq I$, let

$$x_S := \sum_{i \in S} x_i.$$

Let $\mathbf{A}[I]$ be the hyperplane of $\mathbb{R}I$ on which the sum of coordinate values is zero,

$$\mathbf{A}[I] := \big\{ x \in \mathbb{R}I : x_I = 0 \big\}.$$

Let $(e_i)_{i \in I}$ be the standard basis of $\mathbb{R}I$. Then $\mathbf{A}[I]$, together with *roots*

$$e_{i_1} - e_{i_2}, \qquad i_1, i_2 \in I, \quad i_1 \neq i_2$$

is the root system of type A_{n-1}.

Definition 2.1. A *semisimple flat* is a subspace of $\mathbf{A}[I]$ which can be spanned by a subset of the roots of A_{n-1}.

We associate to each partition $P = (S_1 | \dots | S_k)$ of I the semisimple flat $\mathbf{A}_P[I]$ given by

$$\mathbf{A}_P[I] := \big\{ x \in \mathbb{R}I : x_{S_j} = 0 \text{ for } 1 \leq j \leq k \big\}.$$

Conversely, for each semisimple flat $V \leq \mathbf{A}[I]$ there exists a unique partition P of I such that $V = \mathbf{A}_P[I]$. Therefore semisimple flats of $\mathbf{A}[I]$ are in one-to-one correspondence with partitions of I. The dimension of $\mathbf{A}_P[I]$ is the rank of P. For partitions P and Q of I, $\mathbf{A}_Q[I]$ is a subspace of $\mathbf{A}_P[I]$ if and only if Q is finer than P.

We call the semisimple flat $\mathbf{A}_P[I]$ a *simple flat* if exactly one of the blocks of P is not a singleton. For $S \subseteq I$, let $\mathbf{A}_S[I]$ be the subspace of $\mathbf{A}[I]$ given by

$$\mathbf{A}_S[I] := \big\{ x \in \mathbb{R}I : x_S = 0, \ x_i = 0 \text{ for all } i \notin S \big\}.$$

We have a natural isomorphism $\mathbf{A}_S[I] \cong \mathbf{A}[S]$. The simple flats of $\mathbf{A}[I]$ are the subspaces $\mathbf{A}_S[I]$ with $|S| \geq 2$, where the partition corresponding to $\mathbf{A}_S[I]$ is the completion of S with singletons.

For $P = (S_1|\dots|S_k)$, the semisimple flat $\mathbf{A}_P[I]$ orthogonally decomposes into simple flats as follows,

$$(*) \qquad \mathbf{A}_P[I] = \bigoplus_{|S_j|\geq 2} \mathbf{A}_{S_j}[I] \cong \bigoplus_{|S_j|\geq 2} \mathbf{A}[S_j].$$

The subspace $\mathbf{A}_P[I]$ together with the roots of $\mathbf{A}[I]$ which are contained in $\mathbf{A}_P[I]$ forms the root system of type $\prod_j A_{|S_j|-1}$. The decomposition of $\mathbf{A}_P[I]$ into simple flats is the decomposition of this root system into irreducible root systems.

Definition 2.2. An *adjoint hyperplane* is a semisimple flat which has codimension one in $\mathbf{A}[I]$. An *adjoint flat* is a subspace of $\mathbf{A}[I]$ which is an intersection of a set of adjoint hyperplanes of $\mathbf{A}[I]$.

The arrangement consisting of the adjoint hyperplanes in $\mathbf{A}[I]$ is the adjoint $\mathbf{Br}^\vee[I]$ of the braid arrangement $\mathbf{Br}[I]$ (adjoint in the sense of [AM17, Section 1.9.2]). The adjoint braid arrangement is often called the restricted all-subset arrangement (for example, see [KTT11], [KTT12]). Notice that semisimple flats are adjoint flats; if $P = (S_1|\dots|S_k)$ and $T_j = I \setminus S_j$, then

$$\mathbf{A}_P[I] = \bigcap_{j=1}^{k} \mathbf{A}_{(S_j|T_j)}[I].$$

However, the set of semisimple flats is not closed under intersection, and so there exist adjoint flats which are not semisimple.

Definition 2.3. The adjoint braid arrangement *under* $\mathbf{A}_P[I]$, denoted $\mathbf{Br}^\vee_P[I]$, is the hyperplane arrangement in $\mathbf{A}_P[I]$ consisting of all the adjoint flats of $\mathbf{A}[I]$ which are hyperplanes of $\mathbf{A}_P[I]$.

Let $\mathbf{Br}^\vee_S[I]$ denote the adjoint braid arrangement under $\mathbf{A}_S[I]$. A natural isomorphism $\mathbf{Br}^\vee_S[I] \cong \mathbf{Br}^\vee[S]$ is induced by the natural isomorphism of their underlying spaces. For $P = (S_1|\dots|S_k)$, the hyperplanes of $\mathbf{Br}^\vee_P[I]$ which are semisimple flats of $\mathbf{A}[I]$ are in natural bijection with the hyperplanes of the $\mathbf{Br}^\vee_{S_j}[I]$; however, if P has at least two blocks which are not singletons, then $\mathbf{Br}^\vee_P[I]$ will have additional hyperplanes which are not semisimple. Therefore $(*)$ does not hold at the level of hyperplane arrangements.

Let $P = (S_1|\dots|S_k)$ be a partition of I. The hyperplanes of $\mathbf{Br}^\vee_P[I]$ can be ranked according to how 'bad' they are. In general, a hyperplane of $\mathbf{Br}^\vee_P[I]$ is obtained by choosing some proper and nonempty subset $E \subset I$ which is not a union of blocks of P, and taking the subspace of $\mathbf{A}_P[I]$ which satisfies $x_E = 0$. Let $[E]_P$ denote the collection of subsets of I which are obtained by adding or subtracting blocks of P to E and its compliment in I. The hyperplanes of $\mathbf{Br}^\vee_P[I]$ are in natural bijection with the collections $[E]_P$, as E ranges over proper and nonempty subsets of I.

Definition 2.4. Let $P = (S_1|\dots|S_k)$ be a partition of I, and let R be a partition of I such that P is finer than R. Let $E \subset I$ be a proper and nonempty subset. Then E, $[E]_P$, and the hyperplane of $[E]_P$ are called *R-semisimple* if the blocks S_j such that $E \cap S_j \notin \{\emptyset, S_j\}$ are contained in a single block of R.

Notice that this is indeed well defined for $[E]_P$, and therefore also the corresponding hyperplane. In case $R = P$, we recover the definition of a semisimple hyperplane of $\mathbf{Br}^\vee_P[I]$. The partition R is the threshold of 'badness' for hyperplanes of $\mathbf{Br}^\vee_P[I]$, with a finer choice corresponding to a higher threshold.

Example 2.1. Let $I = \{1, \dots, 9\}$, and let

$$P = (12|34|56|78|9), \qquad R_1 = (12|3456|789), \qquad R_2 = (123456|789).$$

Then $E_1 = \{3, 5\}$ is both R_1-semisimple and R_2-semisimple, however $E_2 = \{1, 3, 5\}$ is R_2-semisimple but not R_1-semisimple.

2.2. **Shards.** We define a *shard* to be (the interior of) a face of the adjoint braid arrangement. Equivalently, a shard is a maximal region Y of $\mathbf{A}[I]$ which has the property that for each subset $S \subseteq I$, the value of x_S is either positive for all $x \in Y$, negative for all $x \in Y$, or zero for all $x \in Y$. If we take the intersection of shards with the unit ball $\mathbb{S}^{n-2}$ in $\mathbf{A}[I]$, then we obtain a regular pure cell complex, sometimes called the Steinmann planet or Steinmann sphere by physicists (for example, see [Eps16, p. 168]). This cell complex can be given the piecewise Euclidean metric of the boundary of the root polytope of type A_{n-1} (see Figure 1). Let 2^I denote the set of all subsets of I. The *sign* σ_Y of a shard Y is the function on 2^I given by

$$\sigma_Y : 2^I \to \{+, -, 0\}, \qquad S \mapsto \operatorname{sign}(x_S), \quad x \in Y.$$

This is different to the usual definition of the sign sequence of a face of a hyperplane arrangement, since proper and nonempty subsets of I count half-spaces, not hyperplanes. We call the top dimensional shards *maximal shards*, which are the shards Y with $\sigma_Y(S) = 0$ if and only if $S \in \{\emptyset, I\}$. Maximal shards are called 'geometric cells' in [Eps16]. It is proved in [BMM+12] that the number of maximal shards, which is sequence A034997 in the OEIS, grows superexponentially with n. The *support* $\operatorname{supp}(Y)$ of a shard Y is the adjoint flat given by

$$\operatorname{supp}(Y) := \bigcap_{\sigma_Y(S)=0} \mathbf{A}_{(S|T)}[I] = \big\{x \in \mathbf{A}[I] : x_S = 0 \text{ for all } \sigma_Y(S) = 0\big\}.$$

Since Y is a nonempty convex open set of $\operatorname{supp}(Y)$, the support of Y is equivalently the linear span of Y. The set of shards which have support some adjoint flat $V \leq \mathbf{A}[I]$ are the connected components of the complement in V of the adjoint flats which are hyperplanes of V. In particular, maximal shards are the connected components of the complement in $\mathbf{A}[I]$ of the adjoint hyperplanes. A *wall* of a shard Y is an adjoint flat of $\mathbf{A}[I]$ which is the support of a facet of Y.

For P a partition of I, let $\mathbf{Shd}_P[I]$ denote the real vector space of formal linear combinations of shards with support $\mathbf{A}_P[I]$, and put $\mathbf{Shd}[I] := \mathbf{Shd}_{\{I\}}[I]$. If P' is a partition of a subset $S \subseteq I$, let $\mathbf{Shd}_{P'}[I]$ denote $\mathbf{Shd}_P[I]$ for P equal to the completion of P' with singletons to a partition of I. For a proper subset $S \subset I$, let $\mathbf{Shd}_S[I]$ denote the space of shards with support $\mathbf{A}_S[I]$. We have a natural isomorphism $\mathbf{Shd}_S[I] \cong \mathbf{Shd}[S]$.

Let $R = (T_1| \dots |T_k)$ be a partition of I, and let P be a partition of I which is finer than R. For $1 \leq j \leq k$, let P_j denote the partition of T_j which is the restriction of P. For each shard $Y \in \mathbf{Shd}_P[I]$ and $1 \leq j \leq k$, let $\Delta_j(Y)$ be the shard in $\mathbf{Shd}_{P_j}[I]$ given by

$$\sigma_{\Delta_j(Y)}(S) := \sigma_Y(S \cap T_j).$$

To see that the shard $\Delta_j(Y)$ exists, notice that the orthogonal projection of a point in Y onto $\mathbf{A}_{P_j}[I]$ satisfies the equations and inequalities which define $\Delta_j(Y)$.

Definition 2.5. Let $R = (T_1| \dots |T_k)$ be a partition of I, and let P be a partition of I which is finer than R. The *projection* $\Delta_R(Y)$ of $Y \in \mathbf{Shd}_P[I]$ with respect to R is the element of the abstract tensor product $\bigotimes_j \mathbf{Shd}_{P_j}[I]$ given by

$$\Delta_R(Y) := \bigotimes_j \Delta_j(Y).$$

Proposition 2.1. Let $R = (T_1|\dots|T_k)$ be a partition of I, and let P be a partition of I which is finer than R. The map

$$\Delta_R : \mathbf{Shd}_P[I] \to \bigotimes_j \mathbf{Shd}_{P_j}[I], \qquad Y \mapsto \Delta_R(Y)$$

is surjective.

Proof. Suppose that we have a family of shards $Y_j \in \mathbf{Shd}_{P_j}[I]$, and let $x_j \in Y_j$ such that $\sum_j x_j$ does not lie on any hyperplane of $\mathbf{Br}^\vee_P[I]$. We can do so because the sum of the Y_j is a Cartesian product of open sets, and so it is also open. Therefore it will not be contained in the union of the hyperplanes. Then the shard containing $\sum_j x_j$ is in $\mathbf{Shd}_P[I]$, and has Δ_j-image the shard Y_j. □

2.3. **Products of Functions on Shards.** Let $\mathbf{Shd}^*_P[I]$ denote the linear dual of $\mathbf{Shd}_P[I]$. We call a function on shards $f \in \mathbf{Shd}^*_P[I]$ *simple* if it is supported by a simple subspace, i.e. if $\mathbf{Shd}^*_P[I]$ is of the form $\mathbf{Shd}^*_S[I]$ for some subset $S \subset I$ with $|S| \geq 2$. As before, let $R = (T_1|\dots|T_k)$ be a partition of I, let P be a partition of I which is finer than R, and let P_j denote the partition of T_j which is the restriction of P. Consider an abstract tensor product of functions,

$$\vec{f} = f_1 \otimes \cdots \otimes f_k \in \bigotimes_j \mathbf{Shd}^*_{P_j}[I].$$

The *product* $\mu_R(\vec{f})$ of $\vec{f}$ over R is the function in $\mathbf{Shd}^*_P[I]$ whose value taken on each shard $Y \in \mathbf{Shd}_P[I]$ is the product of the values taken by the f_j on the projections of Y onto $\mathbf{Shd}_{P_j}[I]$, thus

$$\mu_R(\vec{f})(Y) := \prod_j f_j\big(\Delta_j(Y)\big), \qquad Y \in \mathbf{Shd}_P[I].$$

Notice that μ_R is just the linear dual of Δ_R. Therefore we obtain an injective linear map

$$\mu_R : \bigotimes_j \mathbf{Shd}^*_{P_j}[I] \hookrightarrow \mathbf{Shd}^*_P[I], \qquad \vec{f} \mapsto \mu_R(\vec{f}).$$

A function $f \in \mathbf{Shd}^*_P[I]$ is called *R-semisimple* if there exists $\vec{f} \in \bigotimes_j \mathbf{Shd}^*_{P_j}[I]$ with $\mu_R(\vec{f}) = f$. If $R = P$, then $\vec{f} = f_1 \otimes \cdots \otimes f_k$ is a tensor product of simple functions with $f_j \in \mathbf{Shd}^*_{S_j}[I]$. In this case, we can make the natural identification $\mathbf{Shd}^*_{S_j}[I] \cong \mathbf{Shd}^*[S_j]$ to define

$$\mu : \bigotimes_j \mathbf{Shd}^*[S_j] \hookrightarrow \mathbf{Shd}^*_P[I], \qquad \vec{f} \mapsto \mu(\vec{f}) = \mu_P(\vec{f}).$$

A function $f \in \mathbf{Shd}^*_P[I]$ is called *semisimple* if it is P-semisimple, i.e. if f is a linear combination of products of simple functions. Let

$$\mathbf{Shd}_{P|R}[I] = {}^{\textstyle \mathbf{Shd}_P[I]} \big/ {}_{\ker \Delta_R}.$$

Then Δ_R and μ_R induce isomorphisms

$$\mathbf{Shd}_{P|R}[I] \cong \bigotimes_j \mathbf{Shd}_{P_j}[I], \qquad \mathbf{Shd}^*_{P|R}[I] \cong \bigotimes_j \mathbf{Shd}^*_{P_j}[I].$$

In particular, we may identify the space of R-semisimple functions with $\mathbf{Shd}^*_{P|R}[I]$.

We now give an explicit description of ker Δ_R. Continue to let R and P be partitions of I with P finer than R, and let $Y_1, Y_2 \in \mathbf{Shd}_P[I]$ be distinct shards. We call Y_1 and Y_2 *Steinmann R-adjacent* if they have a common facet whose support is not R-semisimple. Equivalently, Y_1 and Y_2 are Steinmann R-adjacent if there exists a family of subsets $[E]_P$ which is not R-semisimple, such that σ_{Y_2} is obtained from σ_{Y_1} by switching the sign taken on $[E]_P$ only,

$$\sigma_{Y_2}(T) = \begin{cases} -\sigma_{Y_1}(S) & \text{if } S \in [E]_P \\ \ \ \sigma_{Y_1}(S) & \text{otherwise.} \end{cases}$$

We call Y_1 and Y_2 *Steinmann R-equivalent* if there exists a sequence of consecutively Steinmann R-adjacent shards starting with Y_1 and terminating with Y_2. For Steinmann P-adjacency and Steinmann P-equivalence we just say *Steinmann adjacent* and *Steinmann equivalent* respectively.

Theorem 2.2. Let R and P be partitions of I with P finer than R. Let $Y_1, Y_2 \in \mathbf{Shd}_P[I]$ be shards. Then Y_1 and Y_2 are Steinmann R-equivalent if and only if

$$\Delta_R(Y_1) = \Delta_R(Y_2).$$

In other words, Steinmann R-adjacency generates ker Δ_R, and so $\mathbf{Shd}_{P|R}[I]$ is the quotient of $\mathbf{Shd}_P[I]$ by Steinmann R-adjacency.

Proof. Notice that $\Delta_R(Y_1) = \Delta_R(Y_2)$ if and only if the restrictions of σ_{Y_1} and σ_{Y_2} to subsets which are not R-semisimple are equal. The signs of Steinmann R-adjacent shards must agree on subsets which are not R-semisimple because, in the definition of Steinmann R-adjacency, the sign was altered only on subsets which are not R-semisimple. Therefore Steinmann R-equivalence implies the same projections.

Conversely, suppose that σ_{Y_1} and σ_{Y_2} agree on subsets which are not R-semisimple. If $Y_1 = Y_2$ the result follows, so assume that $Y_1 \neq Y_2$. Then there must exist a wall of Y_1 which is not R-semisimple and which separates Y_1 and Y_2, since the shards are distinct and yet are not separated by any R-semisimple hyperplanes. Let this separating wall correspond to some family $[E]_P$, and move to the shard obtained from Y_1 by switching the sign on $[E]_P$ only. This new shard is Steinmann R-adjacent to Y_1. We repeat this process until the newly obtained shard is Y_2. This produces a sequence of consecutively Steinmann R-adjacent shards from Y_1 to Y_2, and so Y_1 is Steinmann R-equivalent to Y_2. □

Corollary 2.2.1. A function on shards is R-semisimple if and only if it is constant on Steinmann R-equivalence classes of shards.

In terms of finite differences of functions across hyperplanes, which we study next, Corollary 2.2.1 characterizes R-semisimple functions as functions whose value does not change across hyperplanes which are not R-semisimple. In particular, a semisimple function is equivalently a function whose value changes only across semisimple hyperplanes.

3. The Forest Derivative

We define the notion of 'tree' and 'forest' we shall be using. We describe a way of composing forests. This gives forests the structure of a category, which we call the category of partitions. We define the antisymmetrization of forests, which is an endofunctor of the category of partitions. We associate linear maps to forests which evaluate finite differences of functions on shards across semisimple flats. We show that this association has antisymmetry and satisfies the Jacobi identity, as interpreted on forests. Forests cannot be used to consider finite differences of functions across flats which are not semisimple, because forests can only 'see' semisimple flats.

3.1. **Trees and Forests.** A *tree* over a finite set S is a rooted full binary tree whose leaves are labeled bijectively with the blocks of a partition of S.

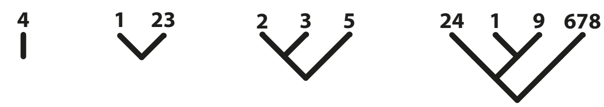


FIGURE 3. Trees over sets of integers.

A *layered tree* Λ over a finite set S_Λ is a tree over S_Λ together with the structure of a linear ordering of the nodes of Λ such that if $v \in \Lambda$ is a node on the geodesic from the root of Λ to another node $u \in \Lambda$, then $v < u$. We say a layered tree is *unlumped* if its leaves are labeled with singletons. An unlumped layered tree over S_Λ corresponds to a choice of Weyl chamber of $\mathbf{A}[S_\Lambda]$, namely the order of the leaves as they appear from left to right, together with a permutation of the vertices of the associated Dynkin diagram (see Figure 4). In particular, if $|S_\Lambda| = n$, then there are $n!(n-1)!$ unlumped layered trees over S_Λ.

FIGURE 4. Schematic representations of two layered trees over $\{1, 2, 3, 4\}$ which have the same underlying delayered tree. Their corresponding Weyl chambers are both $(1, 2, 3, 4)$. The permutations of the vertices $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ of the corresponding Dynkin diagram are $(\mathbf{1}, \mathbf{2}, \mathbf{3}) \mapsto (\mathbf{1}, \mathbf{3}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{2}, \mathbf{3}) \mapsto (\mathbf{3}, \mathbf{1}, \mathbf{2})$ respectively.

We let $|\Lambda|$ denote the number of leaves of Λ. A *stick* is a tree Λ with $|\Lambda| = 1$.

Let $P = (S_1| \dots |S_k)$ be a partition of I. A *layered forest* $\mathcal{F} = \{\Lambda_1, \dots, \Lambda_k\}$ over P is a set of trees such that Λ_j is a tree over S_j, together with the structure of a linear ordering of the nodes of the trees of $\mathcal{F}$ such that the restriction to each tree is a layered tree.

We denote layered trees by nested products of sets $[\,\cdot\,,\,\cdot\,]$ when there is no ambiguity regarding the layering. For example, the trees in Figure 3 have unique layerings and may be denoted

$$[4] \qquad [1, 23] \qquad [[2, 3], 5] \qquad [[24, [1, 9]], 678].$$

We denote layered forests by sets of trees. For example, the forests in Figure 5 are denoted by $\{[13], [24, 5], [6]\}$ and $\{[13], [5, 24], [6]\}$ respectively.

For Q a partition of I which is finer than P, let $\mathbf{Lay}^Q_P[I]$ denote the real vector space of formal linear combinations of layered forests over P whose trees are labeled by blocks of Q. Put $\mathbf{Lay}[I] := \mathbf{Lay}^I_{\{I\}}[I]$, which is the space of unlumped layered trees over I. We write $\mathcal{F} : P \leftarrow Q$ to mean $\mathcal{F} \in \mathbf{Lay}^Q_P[I]$.

Let $P = (S_1| \dots |S_k)$ be a partition of I, and let Λ be a tree over S_m for some $1 \leq m \leq k$. The Λ-*forest* over P is the forest obtained by completing Λ with sticks labeled by the S_j, $j \neq m$. In contexts where there is no ambiguity, we denote this forest by Λ. The Λ-forests with $|\Lambda| = 2$ are called *cuts*, and are denoted by $\mathcal{V}$. The *complement* $\mathcal{V}^-$ of $\mathcal{V}$ is the cut obtained by switching the left and right branches of $\mathcal{V}$.

FIGURE 5. The cuts $\mathcal{V} = [24, 5]$ and $\mathcal{V}^- = [5, 24]$ over the partition $(13|245|6)$

Definition 3.1. Given layered forests $\mathcal{F}_1 : P \leftarrow Q$ and $\mathcal{F}_2 : Q \leftarrow R$, their *composition*

$$\mathcal{F}_1 \circ \mathcal{F}_2 : P \leftarrow R$$

is the layered forest obtained by identifying the leaf of $\mathcal{F}_1$ labeled by S_j with the root node of the tree of $\mathcal{F}_2$ over S_j, requiring that v_1 is less than v_2 for all nodes $v_1 \in \mathcal{F}_1$ and $v_2 \in \mathcal{F}_2$.

Every layered forest $\mathcal{F}$ has a unique decomposition into cuts, corresponding to the linear ordering of the nodes of $\mathcal{F}$,

$$\mathcal{F} = \mathcal{V}_1 \circ \cdots \circ \mathcal{V}_l.$$

The *category of partitions* over I, denoted by Lay_I, is the linear one-way category with objects the partitions of I, hom-spaces formal linear combinations of layered forests,

$$\mathrm{Hom}_{\mathsf{Lay}_I}(Q, P) = \mathbf{Lay}^Q_P[I],$$

and morphism composition the linearization of layered forest composition,

$$\mathbf{Lay}^Q_R[I] \otimes \mathbf{Lay}^P_Q[I] \to \mathbf{Lay}^P_R[I], \qquad (\mathcal{F}_2, \mathcal{F}_1) \mapsto \mathcal{F}_1 \circ \mathcal{F}_2.$$

The category of partitions is freely generated by cuts $\mathcal{V}$, which follows from the fact that every layered forest has a unique decomposition into cuts. See Section 4 for an important interpretation of this category in terms of the braid arrangement. We also show in Section 4 that the category of partitions acts on faces and top-lunes of both the braid arrangement and the adjoint braid arrangement.

Let $P = (S_1| \dots |S_k)$ be a partition of I, and let C be a proper and nonempty subset of a block $S_m \in P$ for some $1 \leq m \leq k$. Let $C^- = S_m \setminus C$. Let Q be the refinement of P obtained by replacing the block S_m with the blocks C and C^-. Then the forests $\mathcal{F}$ of the form $\mathcal{F} : P \leftarrow Q$ are the cuts

$$\mathcal{V} = [C, C^-] \quad \text{and} \quad \mathcal{V}^- = [C^-, C].$$

We think of $\mathcal{V}$ as refining P by choosing C, and of $\mathcal{V}^-$ as refining P by choosing C^-. In this way, any layered forest $\mathcal{F} : P \leftarrow Q$ describes a process of refining P to give Q by cutting blocks such that each time a block is cut, one of the two new blocks is favored. The favored block appears on the left branch of the forest, whereas the unfavored block appears on the right branch. However, notice that morphism composition is in the direction of fusing blocks back together.

Definition 3.2. For $\mathcal{F} \in \mathsf{Lay}_I$ a layered forest, the *antisymmetrization* $\mathcal{A}_I(\mathcal{F})$ of $\mathcal{F}$ is the alternating sum of all layered forests obtained by switching left and right branches at nodes of $\mathcal{F}$, with sign the parity of the number of switches.

The antisymmetrization of layered forests defines a functor, which is an endofunctor on the category of partitions,

$$\mathsf{Lay}_I \to \mathsf{Lay}_I, \qquad \mathcal{F} \mapsto \mathcal{A}_I(\mathcal{F}).$$

The group $(\mathbb{Z}/2)^{|Q|-|P|}$ acts freely on $\mathbf{Lay}^Q_P[I]$ by switching left and right branches. An easy dimension argument shows that the kernel of antisymmetrization is spanned by relations of the form

$$\gamma \cdot \mathcal{F} = -\mathcal{F},$$

for $\gamma = (0, \dots, 0, 1, 0, \dots, 0) \in (\mathbb{Z}/2)^{|Q|-|P|}$.

Example 3.1. We have

$$\mathcal{A}_I\big([[1,2],3]\big) = [[1,2],3] - [[2,1],3] - [3,[1,2]] + [3,[2,1]].$$

and

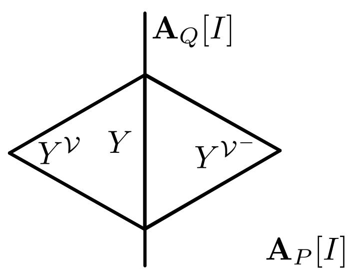
$$\mathcal{A}_I\big(\{[1,2],[3,4]\}\big) = \{[1,2],[3,4]\} - \{[2,1],[3,4]\} - \{[1,2],[4,3]\} + \{[2,1],[4,3]\}.$$

Remark 3.1. Categories of forests of this kind, subject to various relations, can often be interpreted as the data for operads, with morphism composition providing the operadic composition. In the case of layered forests, the crucial structure preventing an operadic structure is the layering. Operads cannot 'see' layering because an operad models a forest as a tensor product of trees.

3.2. **The Definition of the Forest Derivative.** For a cut $\mathcal{V} = [C, C^-] : P \leftarrow Q$ and a shard $Y \in \mathbf{Shd}_Q[I]$, let us denote by $Y^{\mathcal{V}}$ the shard in $\mathbf{Shd}_P[I]$ which is given by

$$\sigma_{Y^{\mathcal{V}}}(S) := \begin{cases} \sigma_Y(S) & \text{if } S \notin \{C, C^-\} \\ + & S = C \\ - & S = C^-. \end{cases}$$

Notice that $\mathbf{A}_Q[I]$ is the hyperplane of $\mathbf{A}_P[I]$ which contains Y as a top dimensional shard, and $Y^{\mathcal{V}}$ and $Y^{\mathcal{V}^-}$ are the two shards with support $\mathbf{A}_P[I]$ for which Y is a facet.



Definition 3.3. Let $f \in \mathbf{Shd}^*_P[I]$ be a function on shards. The *first derivative* $\partial_{\mathcal{V}} f$ of f with respect to the cut $\mathcal{V} = [C, C^-] : P \leftarrow Q$ is the function in $\mathbf{Shd}^*_Q[I]$ given by

$$\partial_{\mathcal{V}} f(Y) := f(Y^{\mathcal{V}}) - f(Y^{\mathcal{V}^-}).$$

More generally, let $\mathcal{F} : P \leftarrow Q$ be any layered forest with decomposition into cuts

$$\mathcal{F} = \mathcal{V}_1 \circ \cdots \circ \mathcal{V}_l.$$

The *forest derivative* $\partial_{\mathcal{F}} f$ of $f \in \mathbf{Shd}^*_P[I]$ with respect to $\mathcal{F}$ is the function in $\mathbf{Shd}^*_Q[I]$ given by the following composition of derivatives with respect to cuts,

$$\partial_{\mathcal{F}} f := \partial_{\mathcal{V}_l}(\partial_{\mathcal{V}_{l-1}}(\dots(\partial_{\mathcal{V}_2}(\partial_{\mathcal{V}_1} f))\dots)).$$

See Section 4 for a more abstract definition of the forest derivative, which uses the category of Lie elements of the adjoint braid arrangement. We linearize $\partial_{\mathcal{F}}$ to obtain a map of functions on shards,

$$\partial_{\mathcal{F}} : \mathbf{Shd}^*_P[I] \to \mathbf{Shd}^*_Q[I], \qquad f \mapsto \partial_{\mathcal{F}} f.$$

It is a direct consequence of the definition that the derivative respects forest composition; we have

$$\partial_{\mathcal{F}_1 \circ \mathcal{F}_2} = \partial_{\mathcal{F}_2} \circ \partial_{\mathcal{F}_1}.$$

The identities of Lay_I are the forests of sticks. If $\mathcal{F}$ is a forest of sticks, then the decomposition of $\mathcal{F}$ into cuts is empty, and $\partial_{\mathcal{F}}$ is the identity linear map. Therefore the forest derivative defines a contravariant linear functor on the category of partitions into the category of vector spaces, given covariantly by

$$\mathsf{Lay}_I^{op} \to \mathsf{Vec}, \qquad P \mapsto \mathbf{Shd}^*_P[I], \quad \mathcal{F} \mapsto \partial_{\mathcal{F}}.$$

Let $P = (S_1|\dots|S_k)$ be a partition of I, and let Λ be a tree over S_m for some $1 \leq m \leq k$. Then we let ∂_Λ denote the derivative with respect to the Λ-forest over P, i.e. the completion of Λ with sticks labeled by the blocks of P.

Definition 3.4. Let $\mathcal{V} : P \leftarrow Q$ be a cut of a partition P of I, and let $Y \in \mathbf{Shd}_Q[I]$ be a shard. The *dual first derivative* $\partial^*_{\mathcal{V}} Y$ of Y with respect to the cut $\mathcal{V}$ is the vector in $\mathbf{Shd}_P[I]$ given by

$$\partial^*_{\mathcal{V}} Y := Y^{\mathcal{V}} - Y^{\mathcal{V}^-}.$$

More generally, let

$$\mathcal{F} = \mathcal{V}_1 \circ \dots \circ \mathcal{V}_l : P \leftarrow Q$$

be a forest and let $Y \in \mathbf{Shd}_Q[I]$ be a shard. The *dual forest derivative* $\partial^*_{\mathcal{F}} Y$ of Y with respect to $\mathcal{F}$ is the vector in $\mathbf{Shd}_P[I]$ given by the following composition of dual derivatives with respect to cuts,

$$\partial^*_{\mathcal{F}} Y := \partial^*_{\mathcal{V}_1}(\partial^*_{\mathcal{V}_2} \dots (\partial^*_{\mathcal{V}_{l-1}}(\partial^*_{\mathcal{V}_l}(Y)))).$$

We then linearize $\partial^*_{\mathcal{F}}$ to obtain a map of formal linear combinations of shards,

$$\partial^*_{\mathcal{F}} : \mathbf{Shd}_Q[I] \to \mathbf{Shd}_P[I], \qquad Y \mapsto \partial^*_{\mathcal{F}} Y.$$

It is a direct consequence of the definition that dual derivative respects forest composition,

$$\partial^*_{\mathcal{F}_1 \circ \mathcal{F}_2} = \partial^*_{\mathcal{F}_1} \circ \partial^*_{\mathcal{F}_2}.$$

Notice that $\partial^*_{\mathcal{F}}$ is just the linear dual of $\partial_{\mathcal{F}}$; we have

$$\partial_{\mathcal{F}} f(Y) = f(\partial^*_{\mathcal{F}} Y).$$

The dual forest derivative defines a linear functor on the category of partitions into the category of vector spaces, given by

$$\mathsf{Lay}_I \to \mathsf{Vec}, \qquad P \mapsto \mathbf{Shd}_P[I], \quad \mathcal{F} \mapsto \partial^*_{\mathcal{F}}.$$

We have the following description of the dual derivative; for $\mathcal{F} = \mathcal{V}_1 \circ \dots \circ \mathcal{V}_l$, put

$$(\star) \qquad Y^{\mathcal{F}} := \big((Y^{\mathcal{V}_l})^{\cdots}\big)^{\mathcal{V}_1}.$$

Let us extend the definition of $Y^{\mathcal{F}}$ linearly to formal linear combinations of forests. Then directly from the definition of $\partial^*_{\mathcal{F}}$, we see that

$$\partial^*_{\mathcal{F}} Y = Y^{\mathcal{A}_I(\mathcal{F})}.$$

In particular, we have

$$\partial_{\mathcal{F}} f(Y) = f(Y^{\mathcal{A}_I(\mathcal{F})}).$$

Note that the derivative depends upon the layering of forests (see Figure 6).

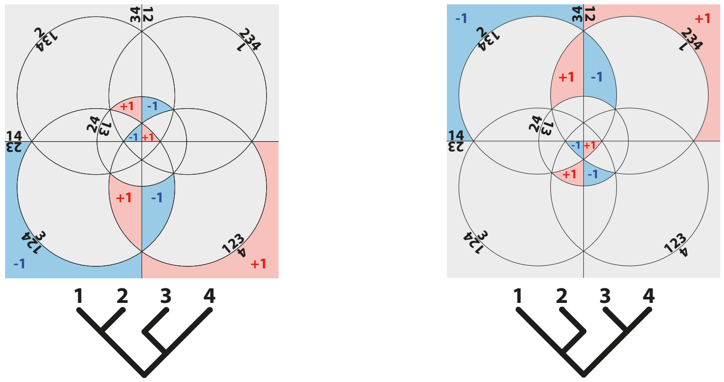


FIGURE 6. The dual forest derivative $\partial^*_{\mathcal{F}} Y$, for Y equal to the shard with $\sigma_Y(S) = 0$ for all $S \subseteq I$, and $\mathcal{F}$ equal to the two layerings of $[[1, 2], [3, 4]]$, depicted on the stereographic projection of the Steinmann planet. In this case, $\partial^*_{\mathcal{F}} Y$ coincides with the Lie element of $\mathcal{F}$ (see Section 4).

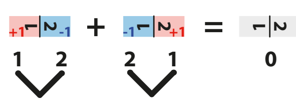
3.3. **The Lie Properties of the Forest Derivative.**

We now show that the forest derivative satisfies the Lie axioms of antisymmetry and the Jacobi identity, as interpreted on layered forests. We first give two examples, the first showing antisymmetry holding for $n = 2$, and the second showing the Jacobi identity holding for $n = 3$.

Example 3.2. Let $I = \{1, 2\}$. Let Y be the shard with $\sigma_Y(S) = 0$ for all $S \subseteq I$. Then

$$\partial^*_{[1,2]} Y + \partial^*_{[2,1]} Y = 0.$$

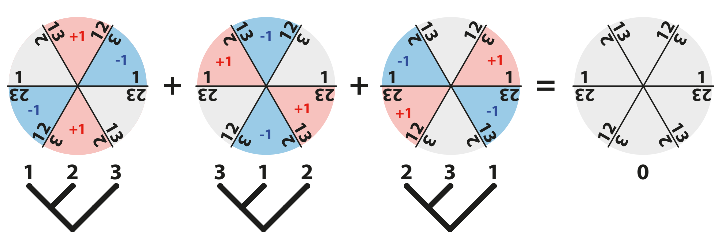
Schematically, we have



Example 3.3. Let $I = \{1, 2, 3\}$. Again let Y be the shard with $\sigma_Y(S) = 0$ for all $S \subseteq I$. Then

$$\partial^*_{[[1,2],3]} Y + \partial^*_{[[3,1],2]} Y + \partial^*_{[[2,3],1]} Y = 0.$$

Schematically, we have



Theorem 3.1. Let $P = (S_1 | \dots | S_k)$ be a partition of I. For trees Λ_1 and Λ_2 such that $(S_{\Lambda_1} | S_{\Lambda_2})$ is a partitions of S_m for some $1 \leq m \leq k$, we have

$$\partial_{[\Lambda_1, \Lambda_2]} + \partial_{[\Lambda_2, \Lambda_1]} = 0.$$

For trees Λ_1, Λ_2 and Λ_3 such that $(S_{\Lambda_1}|S_{\Lambda_2}|S_{\Lambda_3})$ is a partition of S_m for some $1 \leq m \leq k$, we have

$$\partial_{[[\Lambda_1,\Lambda_2],\Lambda_3]} + \partial_{[[\Lambda_3,\Lambda_1],\Lambda_2]} + \partial_{[[\Lambda_2,\Lambda_3],\Lambda_1]} = 0.$$

Proof. We first prove antisymmetry. Since the derivative respects forest composition, we have

$$\partial_{[\Lambda_1,\Lambda_2]} + \partial_{[\Lambda_2,\Lambda_1]} = (\partial_{[S_{\Lambda_1},S_{\Lambda_2}]} + \partial_{[S_{\Lambda_2},S_{\Lambda_1}]}) \circ \partial_{\{\Lambda_1,\Lambda_2\}}.$$

Therefore it is enough to check the case where $[\Lambda_1, \Lambda_2]$ is a cut $\mathcal{V}$. Then, rewriting in terms of the dual derivative, we have

$$(\partial_{\mathcal{V}} + \partial_{\mathcal{V}^-})f(Y) = f(\partial^*_{\mathcal{V}} Y + \partial^*_{\mathcal{V}^-} Y) = f(Y^{\mathcal{V}} - Y^{\mathcal{V}^-} + Y^{\mathcal{V}^-} - Y^{\mathcal{V}}) = f(0) = 0.$$

We now prove the Jacobi identity. Since the derivative respects forest composition, we have

$$\begin{aligned}&\partial_{[[\Lambda_1,\Lambda_2],\Lambda_3]} + \partial_{[[\Lambda_3,\Lambda_1],\Lambda_2]} + \partial_{[[\Lambda_2,\Lambda_3],\Lambda_1]}\\ =&(\partial_{[[S_{\Lambda_1},S_{\Lambda_2}],S_{\Lambda_3}]} + \partial_{[[S_{\Lambda_3},S_{\Lambda_1}],S_{\Lambda_2}]} + \partial_{[[S_{\Lambda_2},S_{\Lambda_3}],S_{\Lambda_1}]}) \circ \partial_{\{\Lambda_1,\Lambda_2,\Lambda_3\}}.\end{aligned}$$

Therefore it is enough to check the case where $[[\Lambda_1, \Lambda_2], \Lambda_3] = [[A, B], C]$, for $A, B, C \subset I$. Then, rewriting in terms of the dual derivative, we have

$$\begin{aligned}&(\partial_{[[A,B],C]} + \partial_{[[C,A],B]} + \partial_{[[B,C],A]})f(Y)\\ &= f(\partial^*_{[[A,B],C]} Y + \partial^*_{[[C,A],B]} Y + \partial^*_{[[B,C],A]} Y)\\ &= f(Y^{[[A,B],C]} - Y^{[[B,A],C]} - Y^{[C,[A,B]]} + Y^{[C,[B,A]]}\\ &\quad +Y^{[[C,A],B]} - Y^{[[A,C],B]} - Y^{[B,[C,A]]} + Y^{[B,[A,C]]}\\ &\quad +Y^{[[B,C],A]} - Y^{[[C,B],A]} - Y^{[A,[B,C]]} + Y^{[A,[C,B]]}).\end{aligned}$$

However, by checking the signs of shards, we see that

$$Y^{[[A,B],C]} = Y^{[[A,C],B]}, \quad Y^{[B,[A,C]]} = Y^{[A,[B,C]]}, \quad Y^{[[B,A],C]} = Y^{[[B,C],A]},$$

$$Y^{[C,[B,A]]} = Y^{[B,[C,A]]}, \quad Y^{[[C,A],B]} = Y^{[[C,B],A]}, \quad Y^{[A,[C,B]]} = Y^{[C,[A,B]]}.$$

And so we obtain zero as required. □

4. The Action of Lie Elements on Faces

We describe the forest derivative of Definition 3.3 in the context of general theory. To do this, we assume some knowledge of the theory of hyperplane arrangements, in particular the theory of Lie elements for which [AM17] is a good reference, and the theory of linear species for which [AM10] and [AM13] are good references. We show that the forest derivative is obtained by representing certain Lie elements of the adjoint braid arrangement with layered trees, and then composing with a hom-functor which constructs the action of Lie elements on faces. We show that the forest derivative is a geometric analog of the standard right action of the Lie operad on the associative operad.

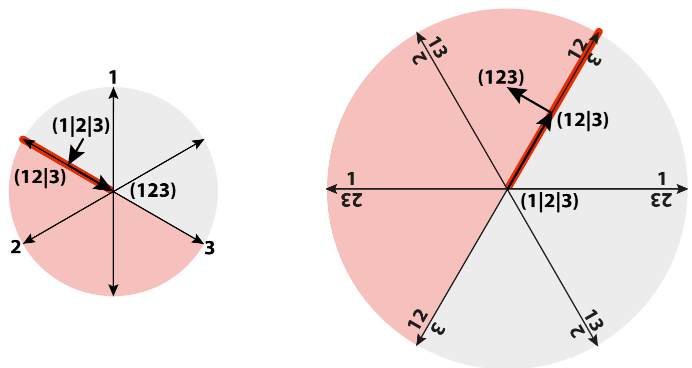


FIGURE 7. The image of $[12,3] \circ \big\{[1,2],[3]\big\}$ in Ass_I and Shd_I respectively, showing our convention for the direction of morphisms. The morphism has been positioned to shows its action on the zero dimensional face (the action on faces of the braid arrangement is contravariant).

4.1. **The Category of Partitions and the Category of Lunes.** For P a partition of I, let

$$\mathbf{A}^P[I] := \big\{x \in \mathbb{R}I : x_{i_1} = x_{i_2} \text{ for } i_1 \sim_P i_2\big\},$$

where $i_1 \sim_P i_2$ means that i_1 and i_2 are in the same block of P. Notice that $\mathbf{A}[I]^P$ is the subspace of $\mathbf{A}[I]$ which is orthogonal to $\mathbf{A}_P[I]$. The subspaces $\mathbf{A}[I]^P$, as P ranges over partitions of I, are the flats of the braid arrangement $\mathbf{Br}[I]$. We associate to the cut

$$\mathcal{V} = [C, C^-] : P \leftarrow Q$$

the half-space of $\mathbf{A}^Q[I]$ consisting of those points $x \in \mathbf{A}^Q[I]$ with $x_{i_1} \geq x_{i_2}$, for $i_1 \in C$ and $i_2 \in C^-$. Similarly, associated to $\mathcal{V}^-$ is the complementary half-space. Under this association, the category of partitions Lay_I becomes the category freely generated by half-flats of the braid arrangement $\mathbf{Br}[I]$. We also associate to $\mathcal{V}$ the half-space of $\mathbf{A}_P[I]$ consisting of those points $x \in \mathbf{A}_P[I]$ with $x_C \geq 0$. Similarly, associated to $\mathcal{V}^-$ is the complementary half-space.

See [AM17, Section 4.8.2] for the definition of the category of lunes of a generic hyperplane arrangement. Let Ass_I denote the opposite category of lunes of the braid arrangement $\mathbf{Br}[I]$, and let Shd_I denote the category of lunes of the adjoint braid arrangement $\mathbf{Br}^\vee[I]$. The associations of half-spaces to cuts, just defined, are maps on the free generators of Lay_I into lunes of slack-1, and so define two functors

$$\pi_I : \mathsf{Lay}_I \to \mathsf{Ass}_I \qquad \text{and} \qquad \pi_I^\vee : \mathsf{Lay}_I \to \mathsf{Shd}_I.$$

We call a half-flat of the adjoint braid arrangement *semisimple* if both its support and boundary flat are semisimple flats of $\mathbf{A}[I]$ (recall semisimple means 'can be spanned by roots'). The kernel of π_I is spanned by delayering and debracketing, i.e. π_I sends a layered forest $\mathcal{F}$ to the lune corresponding to the composite ordered partition of I which forms the canopy of $\mathcal{F}$. In particular, π_I is surjective. The image of $\pi_I^\vee$ is generated by semisimple half-flats.

4.2. **The Category of Partitions and the Category of Lie Elements.** See [AM17, Section 10.6] for the definition of the category of Lie elements of a generic hyperplane arrangement. Let us denote by Lie_I and LLie_I the image of $\pi_I \circ \mathcal{A}_I$ and $\pi_I^\vee \circ \mathcal{A}_I$ respectively. Then Lie_I is the (opposite) category of Lie elements of $\mathbf{Br}[I]$, and LLie_I is the subcategory of the category of Lie elements of $\mathbf{Br}^\vee[I]$ which is generated by differences of complimentary semisimple half-flats.

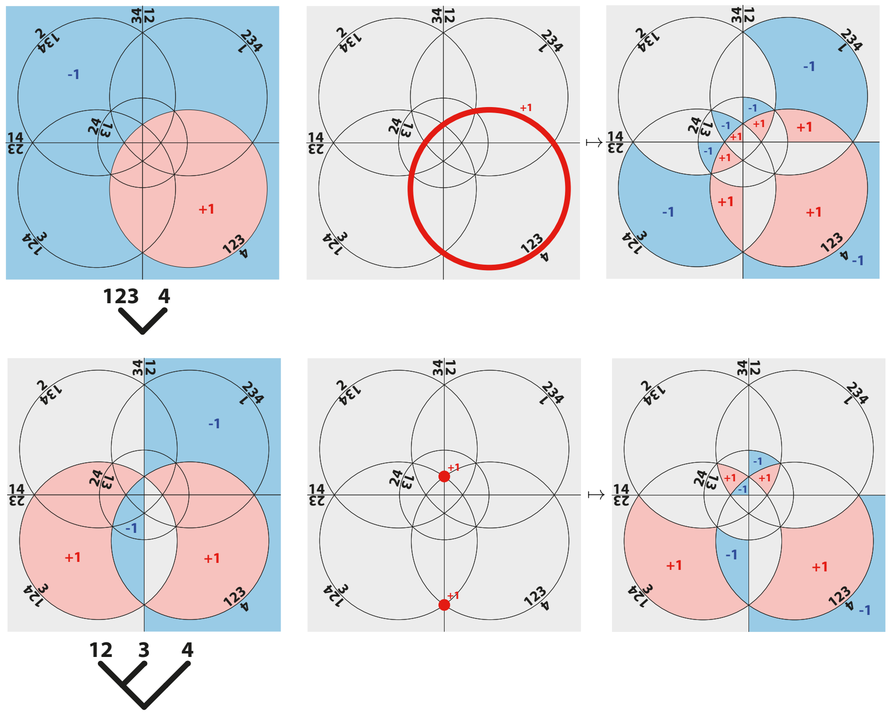


FIGURE 8. The action of the Lie elements of $[123, 4]$ and $[[12, 3], 4]$ on flats, i.e. if a Lie element has source the partition P, then we take the action of the Lie element on all the shards with support $\mathbf{A}_P[I]$. The resulting linear combination of shards was called the *antiderivative* of a tree by Ocneanu in [Ocn18], which expresses the tree derivative ∂_Λ as an inner product in the case $\Lambda \in \mathbf{Lay}[I]$.

Taking hom-functors on Ass_I or Shd_I at the flats corresponding to I and $\{I\}$ define two actions of lunes, one on chambers under flats, and one on top-lunes over flats. We only consider the actions on chambers under flats. We collect all our functors in the following diagram,

$$\begin{array}{ccccccccc} \mathsf{Vec} & \xleftarrow{\mathrm{Hom}(-,\{I\})} & \mathsf{Ass}_I & \xleftarrow{\pi_I} & \mathsf{Lay}_I & \xrightarrow{\pi_I^\vee} & \mathsf{Shd}_I & \xrightarrow{\mathrm{Hom}(I,-)} & \mathsf{Vec} \\ & & \uparrow & & \downarrow \mathcal{A}_I & & \uparrow & & \\ & & \mathsf{Lie}_I & \overset{\pi_I}{\twoheadleftarrow} & \mathsf{Lay}_I & \overset{\pi_I^\vee}{\twoheadrightarrow} & \mathsf{LLie}_I & & \end{array}$$

It follows directly from the definitions that the following coincide,

$$\mathsf{Lay}_I \xrightarrow{\pi_I^\vee} \mathsf{Shd}_I \xrightarrow{\mathrm{Hom}(I,-)} \mathsf{Vec}, \qquad \mathcal{F} \mapsto (Y \mapsto Y^{\mathcal{F}})$$

$$\mathsf{Lay}_I \overset{\pi_I^\vee \circ \mathcal{A}_I}{\twoheadrightarrow} \mathsf{LLie}_I \xrightarrow{\mathrm{Hom}(I,-)} \mathsf{Vec}, \qquad \mathcal{F} \mapsto \partial^*_{\mathcal{F}}.$$

The kernel of $\pi_I \circ \mathcal{A}_I$ is spanned by antisymmetry, the Jacobi identity, and delayering. The action obtained by composing π_I, respectively $\pi_I \circ \mathcal{A}_I$, with $\mathrm{Hom}(-,\{I\})$ is the data for the standard right action of the operad **Ass**, respectively **Lie**, on **Ass**. For the standard left actions of **Ass** and **Lie** on **Ass**, one should compose with $\mathrm{Hom}(I, -)$, which is the action on top-lunes.

On the adjoint side, in Theorem 3.1 we showed that $\pi_I^\vee \circ \mathcal{A}_I$ factors through the Lie axioms of antisymmetry and the Jacobi identity, but not delayering. We obtain delayering next, by restricting to functions whose derivatives are semisimple.

5. Semisimple Differentiability and the Steinmann Relations

We study the relationship between the forest derivative and the property of semisimplicity for functions on shards. The derivative does not preserve semisimplicity, however we do show that the derivative of a product of functions decomposes as a product of derivatives. Crucially, by restricting to functions whose derivatives are semisimple, we are able to delayer forests, and thus obtain algebraic structure in species. The Steinmann relations are easily seen to be equivalent to the property that the first derivatives of functions are semisimple. We show that the Steinmann relations are equivalent to the property that all derivatives are semisimple, i.e. to conclude that all of a functions derivatives are semisimple, it is enough to check the first derivatives.

5.1. **Derivatives of Products of Functions.** Let $\mathcal{F} = \{\Lambda_1, \dots, \Lambda_k\} : P \leftarrow Q$ be a layered forest. For $1 \leq j \leq k$, let Q_j denote the partition of S_j which is the restriction of Q to S_j. Let us denote by

$$\partial_j : \mathbf{Shd}^*_{S_j}[I] \to \mathbf{Shd}^*_{Q_j}[I]$$

the forest derivative with respect to the completion of Λ_j with singleton sticks. Notice that under the identification $\mathbf{Shd}^*_{S_j}[I] \cong \mathbf{Shd}^*[S_j]$, the map ∂_j is the derivative

$$\partial_{\Lambda_j} : \mathbf{Shd}^*[S_j] \to \mathbf{Shd}^*_{Q_j}[S_j].$$

Let us denote by $\otimes\partial_\mathcal{F}$ the tensor product of maps $\bigotimes_j \partial_j$, thus

$$\otimes\partial_\mathcal{F} : \bigotimes_j \mathbf{Shd}^*_{S_j}[I] \to \bigotimes_j \mathbf{Shd}^*_{Q_j}[I], \qquad f_1 \otimes \cdots \otimes f_k \mapsto \partial_1 f_1 \otimes \cdots \otimes \partial_k f_k.$$

Theorem 5.1. Let $\mathcal{F} = \{\Lambda_1, \dots, \Lambda_k\} : P \leftarrow Q$ be a layered forest. Then the following diagram commutes,

$$\begin{array}{ccc} \bigotimes_j \mathbf{Shd}^*_{Q_j}[I] & \xrightarrow{\mu_P} & \mathbf{Shd}^*_Q[I] \\ {\scriptstyle \otimes\partial_\mathcal{F}}\uparrow & & \uparrow{\scriptstyle \partial_\mathcal{F}} \\ \bigotimes_j \mathbf{Shd}^*_{S_j}[I] & \xrightarrow[\mu_P]{} & \mathbf{Shd}^*_P[I] \end{array}$$

Proof. It follows directly from the definition of $\otimes\partial_\mathcal{F}$ that

$$\otimes\partial_{\mathcal{F}_1 \circ \mathcal{F}_2} = \otimes\partial_{\mathcal{F}_1} \circ \otimes\partial_{\mathcal{F}_2}.$$

Therefore, since every forest is a composition of cuts, it is enough to consider the case where $\mathcal{F}$ is a cut $\mathcal{V} = [C, C^-] : P \leftarrow Q$. Let

$$\vec{f} = f_1 \otimes \cdots \otimes f_k \in \bigotimes_j \mathbf{Shd}^*_{S_j}[I]$$

and put $f = \mu_P(\vec{f})$. Then, for each shard $Y \in \mathbf{Shd}_Q$, we have

$$\partial_\mathcal{V} f(Y) = f(Y^\mathcal{V}) - f(Y^{\mathcal{V}^-}) = \prod_j f_j(\Delta_j(Y^\mathcal{V})) - \prod_j f_j(\Delta_j(Y^{\mathcal{V}^-})).$$

Notice that if $j \neq m$, then $C \cap S_j = \emptyset$ and $C^- \cap S_j = \emptyset$, and so

$$\Delta_j(Y^\mathcal{V}) = \Delta_j(Y) = \Delta_j(Y^{\mathcal{V}^-}).$$

Therefore we can factor out the terms $j \neq m$, to obtain

$$\begin{aligned}\partial_{\mathcal{V}}\big(\mu_P(\vec{f})\big)(Y) = \partial_{\mathcal{V}} f(Y) &= \prod_{j\neq m} f_j(\Delta_j(Y)) \cdot \big(f_m(\Delta_m(Y^{\mathcal{V}})) - f_m(\Delta_m(Y^{\mathcal{V}^-}))\big)\\ &= \mu_P\Big(\bigotimes_{j\neq m} f_j \otimes \partial_m f_m\Big)(Y)\\ &= \mu_P\big(\otimes\, \partial_{\mathcal{V}} \vec{f}\,\big)(Y).\end{aligned}$$

Since the derivative is linear, the result then extends from products to semisimple functions. □

Definition 5.1. A function $f \in \mathbf{Shd}^*_P[I]$ is called *semisimply differentiable* if the derivative $\partial_{\mathcal{F}} f$ is a semisimple function for all forests $\mathcal{F}$ over P.

Notice that a semisimply differentiable function is semisimple since, for $\mathcal{F}$ a forest of sticks, the derivative with respect to $\mathcal{F}$ is the identity. Let $\mathbf{\Gamma}^*_P[I]$ be the subspace of $\mathbf{Shd}^*_P[I]$ of semisimply differentiable functions. We denote by $\mathbf{\Gamma}_P[I]$ the quotient of $\mathbf{Shd}_P[I]$ which is the linear dual of $\mathbf{\Gamma}^*_P[I]$.

Corollary 5.1.1. Let P be a partition of I. Let $f \in \mathbf{\Gamma}^*_P[I]$, and let $\mathcal{F}$ be a forest over P. Then $\partial_{\mathcal{F}} f$ does not depend upon the layering of $\mathcal{F}$.

Proof. The operator $\otimes\partial_{\mathcal{F}}$ is invariant of the layering between the trees of $\mathcal{F}$. □

Let Lie_I be as in Section 4, i.e Lie_I is the linear category which is the quotient of Lay_I by the Lie axioms of antisymmetry and the Jacobi identity, and identifying forests which differ only by their layerings.

Corollary 5.1.2. Let $[\mathcal{F}]$ denote the image of $\mathcal{F} \in \mathsf{Lay}_I$ in the quotient Lie_I. Then

$$\mathsf{Lie}_I^{op} \to \mathsf{Vec}, \qquad P \mapsto \mathbf{\Gamma}^*_P[I], \quad [\mathcal{F}] \mapsto \partial_{\mathcal{F}}$$

and

$$\mathsf{Lie}_I \to \mathsf{Vec}, \qquad P \mapsto \mathbf{\Gamma}_P[I], \quad [\mathcal{F}] \mapsto \partial^*_{\mathcal{F}}$$

are well defined linear functors.

Either of these functors provides the data for a Lie (co)algebra in species (see Section 6).

Theorem 5.2. Let $P = (S_1|\dots|S_k)$ be a partition of I. Then

$$\bigotimes_j \mathbf{\Gamma}^*_{S_j}[I] \to \mathbf{\Gamma}^*_P[I], \qquad f \mapsto \mu_P(f)$$

is well defined and is an isomorphism.

Proof. The map is well defined since for any $\vec{f} \in \bigotimes_j \mathbf{\Gamma}^*_{S_j}[I]$, the product $\mu_P(\vec{f})$ is semisimply differentiable by Theorem 5.1. We have already seen that μ_P is injective. For surjectivity, let $f \in \mathbf{\Gamma}^*_P[I]$, and let us assume that f is nonzero. In particular, f is semisimple. We may assume that f is a product of simple functions, because semisimple functions are spanned by products of simple functions. Thus, let

$$\vec{f} = f_1 \otimes \cdots \otimes f_k \in \bigotimes_j \mathbf{Shd}^*_{S_j}[I] \quad \text{such that} \quad \mu_P(\vec{f}) = f.$$

Let Λ_m be a tree over some block S_m of P. By Theorem 5.1, we have

$$\partial_{\Lambda_m} f = \partial_{\Lambda_m}\mu_P(\vec{f}) = \mu_P(f_1 \otimes \cdots \otimes f_{m-1} \otimes \partial_{\Lambda_m} f_m \otimes f_{m+1} \otimes \cdots \otimes f_k).$$

Let P_m denote the partition of I which is the completion of the labels of the leaves of Λ_m with singletons. Towards a contradiction, suppose that $\partial_{\Lambda_m} f_m \in \mathbf{Shd}^*_{P_m}[I]$ is not semisimple. Then there exist shards $Y_1, Y_2 \in \mathbf{Shd}_{P_m}[I]$ with

$$\Delta_{P_m}(Y_1) = \Delta_{P_m}(Y_2) \qquad \text{and} \qquad \partial_{\Lambda_m} f_m(Y_1) \neq \partial_{\Lambda_m} f_m(Y_2).$$

We have $f_j \neq 0$ since f is nonzero. Therefore there exist shards $Z_j \in \mathbf{Shd}_{S_j}[I]$ with $f_j(Z_j) \neq 0$. Let Q denote the partition of I which is the completion of the labels of the leaves of Λ_m with the blocks of P. Let $\mathcal{Y}_1 \in \mathbf{Shd}_Q[I]$ be any shard such that

$$\Delta_P(\mathcal{Y}_1) = Z_1 \otimes \cdots \otimes Z_{m-1} \otimes Y_1 \otimes Z_{m+1} \otimes \cdots \otimes Z_m$$

and let $\mathcal{Y}_2 \in \mathbf{Shd}_Q[I]$ be any shard such that

$$\Delta_P(\mathcal{Y}_2) = Z_1 \otimes \cdots \otimes Z_{m-1} \otimes Y_2 \otimes Z_{m+1} \otimes \cdots \otimes Z_m.$$

Then

$$\begin{aligned}\Delta_Q(\mathcal{Y}_1) =& Z_1 \otimes \cdots \otimes Z_{m-1} \otimes \Delta_{P_m}(Y_1) \otimes Z_{m+1} \otimes \cdots \otimes Z_m \\ =& Z_1 \otimes \cdots \otimes Z_{m-1} \otimes \Delta_{P_m}(Y_2) \otimes Z_{m+1} \otimes \cdots \otimes Z_m = \Delta_Q(\mathcal{Y}_2)\end{aligned}$$

and

$$\begin{aligned}\partial_{\Lambda_m} f(\mathcal{Y}_1) =& f_1(Z_1) \dots f_{m-1}(Z_{m-1}) \cdot \partial_{\Lambda_m} f_m(Y_1) \cdot f_{m+1}(Z_{m+1}) \dots f_k(Z_k) \\ \neq& f_1(Z_1) \dots f_{m-1}(Z_{m-1}) \cdot \partial_{\Lambda_m} f_m(Y_2) \cdot f_{m+1}(Z_{m+1}) \dots f_k(Z_k) = \partial_{\Lambda_m} f(\mathcal{Y}_2).\end{aligned}$$

But f is semisimply differentiable, and so $\partial_{\Lambda_m} f$ must be semisimple, a contradiction. Therefore $f_m \in \mathbf{\Gamma}^*_{S_m}[I]$ for all $1 \leq m \leq k$, and so f is in the μ_P-image of $\bigotimes_j \mathbf{\Gamma}^*_{S_j}[I]$. ☐

Remark 5.1. The main argument used in this proof shows that the product of a nonzero semisimple function with a nonzero function which is not semisimple is not semisimple.

Corollary 5.2.1. Let $P = (S_1 | \dots | S_k)$ be a partition of I. Then

$$\mathbf{\Gamma}_P[I] \to \bigotimes_j \mathbf{\Gamma}_{S_j}[I], \qquad Z \mapsto \Delta_P(Z)$$

is well defined and is an isomorphism.

Proof. This is the linear dual of Theorem 5.2. ☐

5.2. **The Steinmann Relations.** We now characterize the subspace of semisimply differentiable functions $\mathbf{\Gamma}^*[I] \hookrightarrow \mathbf{Shd}^*[I]$ by describing a set of relations which generate the kernel of its linear dual $\mathbf{Shd}[I] \twoheadrightarrow \mathbf{\Gamma}[I]$.

Definition 5.2. Let $\mathcal{V} : I \leftarrow Q$ be a cut of I, and let $Y_1, Y_2 \in \mathbf{Shd}_Q[I]$ be Steinmann adjacent shards. We call a relation of the form

$$Y_1^{\mathcal{V}} - Y_1^{\mathcal{V}^-} + Y_2^{\mathcal{V}^-} - Y_2^{\mathcal{V}} = 0$$

a *Steinmann relation* over I.

This coincides with the definition of Steinmann relations in axiomatic quantum field theory (for example, see [Str75, p. 827-828]). For $f \in \mathbf{Shd}^*[I]$, directly from the definitions we see that $\partial_{\mathcal{V}} f$ is semisimple if and only if

$$f(Y_1^{\mathcal{V}} - Y_1^{\mathcal{V}^-} + Y_2^{\mathcal{V}^-} - Y_2^{\mathcal{V}}) = 0$$

for all Steinmann adjacent shards $Y_1, Y_2 \in \mathbf{Shd}_Q[I]$. Let

$$\mathbf{Stein}[I] := \big\langle Y_1^{\mathcal{V}} - Y_1^{\mathcal{V}^-} + Y_2^{\mathcal{V}^-} - Y_2^{\mathcal{V}} \big\rangle,$$

where $\mathcal{V} : I \leftarrow Q$ ranges over cuts of I, and $Y_1, Y_2 \in \mathbf{Shd}_Q[I]$ are Steinmann adjacent shards. Then $f \in \mathbf{Shd}^*[I]$ has semisimple first derivatives if and only if

$$f \in \Big({}^{\mathbf{Shd}[I]} \big/ {}_{\mathbf{Stein}[I]} \Big)^*.$$

The following result shows that this is sufficient to conclude that f is semisimply differentiable. In other words, the derivative preserves the property of having semisimple first derivatives.

Theorem 5.3. Let $f \in \mathbf{Shd}^*[I]$ be a function on maximal shards. Then f is semisimply differentiable if (and only if) the first derivatives of f are semisimple. Thus,

$$\mathbf{\Gamma}[I] = {}^{\mathbf{Shd}[I]} \big/ {}_{\mathbf{Stein}[I]}.$$

Proof. Let us assume that $f \in \mathbf{Shd}^*[I]$ has semisimple first derivatives, i.e. $\partial_{\mathcal{V}} f$ is semisimple for all cuts $\mathcal{V}$ of I. Consider a second derivative of f, i.e. a first derivative of some $\partial_{\mathcal{V}} f$. Up to antisymmetry, this second derivative will be of the form $\partial_{[[A,B],C]} f$, for some $A, B, C \subset I$. Let Q denote the partition $(A|B|C)$, and let P denote the partition $(A \cup B|C)$. By extending linearly, it is enough to consider the case when the first derivative $\partial_{[A \cup B, C]} f$ is a product; so let $\partial_{[A \cup B, C]} f = \mu_P(f_1 \otimes f_2)$. Then, by Theorem 5.2, we have

$$\partial_{[[A,B],C]} f = \partial_{[A,B]}\ \mu_P(f_1 \otimes f_2) = \mu_P(\partial_{[A,B]} f_1 \otimes f_2).$$

Let $P_{A|B}$ be the partition of I which is the completion of the blocks A and B with singletons. In particular, we have $\partial_{[A,B]} f_1 \in \mathbf{Shd}^*_{P_{A|B}}[I]$. Towards a contradiction, suppose that $\partial_{[[A,B],C]} f$ is not semisimple. A product of semisimple functions is clearly semisimple; therefore, since f_2 is simple, we have that $\partial_{[A,B]} f_1$ is not semisimple. So there exist shards $Y_1, Y_2 \in \mathbf{Shd}_{P_{A|B}}$ with

$$\Delta_{P_{A|B}}(Y_1) = \Delta_{P_{A|B}}(Y_2) \quad \text{and} \quad \partial_{[A,B]} f_1(Y_1) \neq \partial_{[A,B]} f_1(Y_2).$$

We must have $f_2 \neq 0$, because otherwise $\partial_{[[A,B],C]} f = 0$, which is trivially semisimple. So let Z be any shard such that $f_2(Z) \neq 0$. Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathbf{Shd}_Q[I]$ be shards such that

$$\Delta_P(\mathcal{Y}_1) = Y_1 \otimes Z \quad \text{and} \quad \Delta_P(\mathcal{Y}_2) = Y_2 \otimes Z.$$

Then

$$\partial_{[[A,B],C]} f(\mathcal{Y}_1) = \partial_{[A,B]} f_1(Y_1) \cdot f_2(Z) \neq \partial_{[A,B]} f_1(Y_2) \cdot f_2(Z) = \partial_{[[A,B],C]} f(\mathcal{Y}_2).$$

Recall that the derivative satisfies the Jacobi identity, and so

$$\partial_{[[A,B],C]} = -\partial_{[[C,A],B]} - \partial_{[[B,C],A]}.$$

Therefore, we have

$$(1) \qquad (-\partial_{[[C,A],B]} - \partial_{[[B,C],A]}) f(\mathcal{Y}_1) \neq (-\partial_{[[C,A],B]} - \partial_{[[B,C],A]}) f(\mathcal{Y}_2).$$

However, by the definition of the derivative, for $\partial_{[[C,A],B]}$ we have

$$(2) \qquad \partial_{[[C,A],B]} f(\mathcal{Y}_1) = \partial_{[C \cup A, B]} f(\mathcal{Y}_1^{[C,A]}) - \partial_{[C \cup A, B]} f(\mathcal{Y}_1^{[A,C]})$$

and

$$(3) \qquad \partial_{[[C,A],B]} f(\mathcal{Y}_2) = \partial_{[C \cup A, B]} f(\mathcal{Y}_2^{[C,A]}) - \partial_{[C \cup A, B]} f(\mathcal{Y}_2^{[A,C]}).$$

Let P_{CA} denote the partition $(C \cup A|B)$. In particular, we have $\partial_{[C\cup A,B]} f \in \mathbf{Shd}^*_{P_{CA}}[I]$. Notice that

$$\Delta_{P_{CA}}(\mathcal{Y}_1^{[C,A]}) = \Delta_{P_{CA}}(\mathcal{Y}_2^{[C,A]}) \quad \text{and} \quad \Delta_{P_{CA}}(\mathcal{Y}_1^{[A,C]}) = \Delta_{P_{CA}}(\mathcal{Y}_2^{[A,C]}).$$

Then, since $\partial_{[C\cup A,B]} f$ is a first derivative of f and so must be semisimple, we have

$$\partial_{[C\cup A,B]} f(\mathcal{Y}_1^{[C,A]}) = \partial_{[C\cup A,B]} f(\mathcal{Y}_2^{[C,A]}) \quad \text{and} \quad \partial_{[C\cup A,B]} f(\mathcal{Y}_1^{[A,C]}) = \partial_{[C\cup A,B]} f(\mathcal{Y}_2^{[A,C]}).$$

Together with (2) and (3), this implies

$$(\neg 1a) \qquad \partial_{[[C,A],B]} f(\mathcal{Y}_1) = \partial_{[[C,A],B]} f(\mathcal{Y}_2).$$

The following similar equality for $\partial_{[[B,C],A]}$ is obtained by the same method,

$$(\neg 1b) \qquad \partial_{[[B,C],A]} f(\mathcal{Y}_1) = \partial_{[[B,C],A]} f(\mathcal{Y}_2).$$

Then $(\neg 1a)$ and $(\neg 1b)$ contradict (1), and so $\partial_{[[A,B],C]} f$ must be semisimple. Thus, we have shown that if all the first derivatives of f are semisimple, then all the second derivatives of f are semisimple. The result then follows by induction on the order of the derivative. □

In [Ocn18], Ocneanu gave an interesting alternative proof of this result for the case $n \leq 5$, which features an analysis of the structure of shards in five coordinates. This proof may generalize to all n.

Corollary 5.3.1. Let P be a partition of I, and let $f \in \mathbf{Shd}^*_P[I]$ be a function on shards. Then f is semisimply differentiable if (and only if) f is semisimple and has semisimple first derivatives.

Proof. This follows from Theorem 5.3 and Theorem 5.2. □

6. A Lie Algebra in Species

We now show that the forest derivative of semisimply differentiable functions is the data of a comodule of the Lie cooperad, internal to the category of species. Dually, this endows the adjoint braid arrangement modulo the Steinmann relations with the structure of a Lie algebra in species. For species and operads, we follow the references [AM10] and [AM13].

We now make the identification $\mathbf{\Gamma}^*_S[I] = \mathbf{\Gamma}^*[S]$, and only write $\mathbf{\Gamma}^*[S]$ from now on. By Theorem 5.2, we can restrict μ to obtain a bijection

$$\mu|_{\mathbf{\Gamma}} : \bigotimes_j \mathbf{\Gamma}^*[S_j] \to \mathbf{\Gamma}^*_P[I], \qquad f \mapsto \mu(f).$$

However, we continue to make a conceptual distinction between the abstract tensor products of functions $\bigotimes_j \mathbf{\Gamma}^*[S_j]$, and geometrically realized functions on shards $\mathbf{\Gamma}^*_P[I]$. Dually, we have the bijection

$$\Delta|_{\mathbf{\Gamma}} : \mathbf{\Gamma}_P[I] \to \bigotimes_j \mathbf{\Gamma}[S_j], \qquad f \mapsto \Delta(f).$$

In the definitions of various structures in species, we will need to compose derivatives and dual derivatives with inverse products $\mu|_{\mathbf{\Gamma}}^{-1}$ and inverse projections $\Delta|_{\mathbf{\Gamma}}^{-1}$ respectively. To simplify notation, let us put

$$D_\Lambda := \mu|_{\mathbf{\Gamma}}^{-1} \circ \partial_\Lambda \quad \text{and} \quad D^*_\Lambda := \partial^*_\Lambda \circ \Delta|_{\mathbf{\Gamma}}^{-1}.$$

The map D^*_Λ is the linear dual of D_Λ. Let $\mathsf{Set}^\times$ denote the category of finite sets and bijections. We have the species $\mathbf{Lay}$ of layered trees

$$\mathbf{Lay} : \mathsf{Set}^\times \to \mathsf{Vec}, \qquad I \mapsto \mathbf{Lay}[I].$$

We also have the species $\mathbf{\Gamma}$ of maximal shards modulo the Steinmann relations

$$\mathbf{\Gamma} : \mathsf{Set}^{\times} \to \mathsf{Vec}, \qquad I \mapsto \mathbf{\Gamma}[I].$$

We denote the respective dual species by $\mathbf{Lay}^*$ and $\mathbf{\Gamma}^*$. The category of species is equipped with a monoidal product '$\circ$' called *composition*. Monoids internal to species, constructed with respect to composition, are operads by another name. Let us write $P \vdash I$ to mean that P is a partition of I. The composition of $\mathbf{Lay}^*$ with $\mathbf{\Gamma}^*$ is given by

$$\mathbf{Lay}^* \circ \mathbf{\Gamma}^*[I] = \bigoplus_{P \vdash I} \Big(\mathbf{Lay}^*[P] \otimes \bigotimes_j \mathbf{\Gamma}^*[S_j] \Big).$$

For each tree $\Lambda \in \mathbf{Lay}[P]$, let $\Lambda^* \in \mathbf{Lay}^*[P]$ be defined by $\Lambda^*(\Lambda') := \delta_{\Lambda,\Lambda'}$. For $f \in \mathbf{\Gamma}^*[I]$, let

$$\gamma_P(f) := \sum_{\Lambda \in \mathbf{Lay}[P]} \Lambda^* \otimes D_\Lambda f.$$

Let

$$\gamma : \mathbf{\Gamma}^* \to \mathbf{Lay}^* \circ \mathbf{\Gamma}^*, \qquad \gamma(f) := \bigoplus_P \gamma_P(f).$$

These linear maps are natural, and so define a morphism of species. Let $\mathbf{Lie}$ denote the Lie operad, represented using layered trees quotiented by the relations of antisymmetry, the Jacobi identity, and delayering. Our representation of $\mathbf{Lie}$ induces an embedding $\mathbf{Lie}^* \hookrightarrow \mathbf{Lay}^*$, which in turn induces an embedding $\mathbf{Lie}^* \circ \mathbf{\Gamma}^* \hookrightarrow \mathbf{Lay}^* \circ \mathbf{\Gamma}^*$.

Proposition 6.1. The image of γ is contained in the image of $\mathbf{Lie}^* \circ \mathbf{\Gamma}^* \hookrightarrow \mathbf{Lay}^* \circ \mathbf{\Gamma}^*$.

Proof. This is a direct consequence of Theorem 3.1 and Corollary 5.1.1, i.e. the differentiation of semisimply differentiable functions satisfies the Lie axioms and does not depend upon the layering of trees. □

By restricting the image of γ, we obtain

$$\gamma|_{\mathbf{Lie}} : \mathbf{\Gamma}^* \to \mathbf{Lie}^* \circ \mathbf{\Gamma}^*, \qquad f \mapsto \bigoplus_P \gamma_P(f).$$

We then take the dual of $\gamma|_{\mathbf{Lie}}$, to obtain

$$\gamma|^*_{\mathbf{Lie}} : \mathbf{Lie} \circ \mathbf{\Gamma} \to \mathbf{\Gamma}, \qquad \Lambda \otimes Z \mapsto D^*_\Lambda Z.$$

Theorem 6.2. The morphism $\gamma^*|_{\mathbf{Lie}}$ is a left $\mathbf{Lie}$-module.

Proof. The unit of the Lie operad is the stick. The fact that $\gamma^*|_{\mathbf{Lie}}$ is unital then follows from the fact that $\partial_{\mathcal{F}}$ is the identity when $\mathcal{F}$ is a forest of sticks. The morphism $\gamma^*|_{\mathbf{Lie}}$ is an action since

$$\gamma^*|_{\mathbf{Lie}}\big((\Lambda \circ \mathcal{F}) \otimes Z\big) = D^*_{\Lambda \circ \mathcal{F}} Z = D^*_\Lambda (D^*_{\mathcal{F}} Z) = \gamma^*|_{\mathbf{Lie}}\big(\Lambda \otimes (D^*_{\mathcal{F}} Z)\big). \qquad \square$$

Corollary 6.2.1. The morphism $\gamma|_{\mathbf{Lie}}$ is a left $\mathbf{Lie}^*$-comodule.

Proof. This is the dual of Theorem 6.2. □

Left $\mathbf{Lie}$-modules in species with respect to composition are equivalent to Lie algebras in species with respect to the Cauchy product '$\cdot$' (see [AM10, Appendix B.5]). The Lie algebra corresponding to $\gamma^*|_{\mathbf{Lie}}$ is given by

$$[-] : \mathbf{\Gamma} \cdot \mathbf{\Gamma} \to \mathbf{\Gamma}, \qquad [Z] := D^*_{[S,T]} Z,$$

where $Z \in \mathbf{\Gamma}[S] \otimes \mathbf{\Gamma}[T]$. Its dual Lie coalgebra has cobracket the discrete differentiation of functions on faces across hyperplanes,

$$[-]^* : \mathbf{\Gamma}^* \to \mathbf{\Gamma}^* \cdot \mathbf{\Gamma}^*, \qquad [f]^*_{(S,T)} := D_{[S,T]} f.$$

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