Special values of the Lommel functions and associated integrals

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Abstract

Special values of the Lommel functions allow the calculation of Fresnel like integrals. These closed form expressions along with their asymptotic values are reported.

1 Special values of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$

The Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$ with unrestricted μ,ν satisfy the relations [1]

$$\frac{2\nu}{z}S_{\mu,\nu}(z) = (\mu+\nu-1)S_{\mu-1,\nu-1}(z) - (\mu-\nu-1)S_{\mu-1,\nu+1}(z), \quad (1)$$

$$[(\mu+1)^2-\nu^2]S_{\mu,\nu}(z) + S_{\mu+2,\nu}(z) = z^{\mu+1}, \quad (2)$$

$$\frac{dS_{\mu,\nu}(z)}{dz} + \frac{\nu}{z}S_{\mu,\nu}(z) = (\mu+\nu-1)S_{\mu-1,\nu-1}(z), \quad (3)$$

and the symmetry property

$$S_{\mu,-\nu}(z) = S_{\mu,\nu}(z).$$

In the case where μ is an integer k and $\nu = 1/2$, the recurrence relation (2) viewed as a second-order difference equation is

$$(2k+1)(2k+3)S_{k,1/2}(z) + 4S_{k+2,1/2}(z) = 4z^{k+1}, \quad (4)$$

and can be reduced to first-order equations in the cases of even or odd values of k.

1.1 Case $k = 2m$

With $k = 2m$ equation (4) can be written

$$(4m+1)(4m+3)\, f_m(z) + 4\, f_{m+1}(z) = 4z^{\,2m+1}, \tag{5}$$

where $f_m(z) = S_{2m,1/2}(z)$ or $s_{2m,1/2}(z)$. It has the solution

$$f_m(z) = (-1)^m \Gamma(2m+1/2) \left[\frac{f_0(z)}{\sqrt{\pi}} - z \sum_{j=0}^{m-1} \frac{(-z^2)^{\,j}}{\Gamma(2j+5/2)} \right],$$

the initial values of $f_0(z)$ being either $S_{0,1/2}(z)$ or $s_{0,1/2}(z)$. Using those relations the solutions to (5) become

$$S_{2m,1/2}(z) = (-1)^m \Gamma(2m+1/2) [\tfrac{S_{0,1/2}(z)}{\sqrt{\pi}} - z \sum_{j=0}^{m-1} \tfrac{(-z^2)^{\,j}}{\Gamma(2j+5/2)}], \tag{6a}$$

$$s_{2m,1/2}(z) = (-1)^m \Gamma(2m+1/2) [\tfrac{s_{0,1/2}(z)}{\sqrt{\pi}} - z \sum_{j=0}^{m-1} \tfrac{(-z^2)^{\,j}}{\Gamma(2j+5/2)}], \tag{6b}$$

where $S_{0,1/2}(z)$, and $s_{0,1/2}(z)$ have been given by Magnus, *et al* [2] as

$$\begin{aligned} \frac{1}{\sqrt{\pi}} S_{0,1/2}(z) &= \sqrt{\frac{2}{z}} \left\{ \cos(z)[\frac{1}{2} - S(\chi)] - \sin(z)[\frac{1}{2} - C(\chi)] \right\}, \\ \frac{1}{\sqrt{\pi}} s_{0,1/2}(z) &= \sqrt{\frac{2}{z}} \left\{ \sin(z) C(\chi) - \cos(z) S(\chi) \right\}, \end{aligned}$$

with

$$\chi = \sqrt{\frac{2z}{\pi}},$$

and where S and C are the Fresnel sine and cosine integrals [3]

$$\begin{aligned} S(z) &= \int_0^z \sin(\frac{1}{2}\pi t^2)\, dt, \\ C(z) &= \int_0^z \cos(\frac{1}{2}\pi t^2)\, dt. \end{aligned}$$

1.2 Case $k = 2m + 1$

In the case where $k = 2m+1$, the difference equation (2) becomes

$$(4m+3)(4m+5)\, f_m(z) + 4\, f_{m+1}(z) = 4z^{\,2m+2}, \tag{7}$$

where $f_m(z) = S_{2m+1,1/2}(z)$ or $s_{2m+1,1/2}(z)$. Here the solution is

$$f_m(z) = (-1)^m \Gamma(2m+3/2) \left[\frac{2\, f_0(z)}{\sqrt{\pi}} - z^2 \sum_{j=0}^{m-1} \frac{(-z^2)^{\,j}}{\Gamma(2j+7/2)} \right].$$

In the special case $\mu = -1$ and $\nu = 1/2$ in (2) the functions $f_0(z)$

$$f_0(z) = S_{1,1/2}(z) = 1 + \frac{1}{4} S_{-1,1/2}(z),$$

or

$$f_0(z) = s_{1,1/2}(z) = 1 + \frac{1}{4} s_{-1,1/2}(z).$$

We have

$$\begin{aligned} S_{1,1/2}(z) &= 1 + \sqrt{\frac{\pi}{2z}} \{\cos(z)[\frac{1}{2} - C(\chi)] + \sin(z)[\frac{1}{2} - S(\chi)]\}, \\ s_{1,1/2}(z) &= 1 - \sqrt{\frac{\pi}{2z}} \{\sin(z) S(\chi) + \cos(z) C(\chi)\}, \end{aligned}$$

where values of $S_{-1,1/2}(z)$, $s_{-1,1/2}(z)$ have been given by Magnus, *et al* as

$$\begin{aligned} \frac{1}{2\sqrt{\pi}} S_{-1,1/2}(z) &= \sqrt{\frac{2}{z}} \left\{ \cos(z)[\frac{1}{2} - C(\chi)] + \sin(z)[\frac{1}{2} - S(\chi)] \right\}, \\ \frac{1}{2\sqrt{\pi}} s_{-1,1/2}(z) &= -\sqrt{\frac{2}{z}} \left\{ \sin(z) S(\chi) + \cos(z) C(\chi) \right\}. \end{aligned}$$

Using those relations, the solutions to (7) become after resumming

$$S_{2m+1,1/2}(z) = (-1)^m \Gamma(2m+3/2)[\frac{S_{-1,1/2}(z)}{2\sqrt{\pi}} + \sum_{j=0}^{m} \frac{(-z^2)^{\,j}}{\Gamma(2j+3/2)}], \quad \text{(8a)}$$

$$s_{2m+1,1/2}(z) = (-1)^m \Gamma(2m+3/2)[\frac{s_{-1,1/2}(z)}{2\sqrt{\pi}} + \sum_{j=0}^{m} \frac{(-z^2)^{\,j}}{\Gamma(2j+3/2)}]. \quad \text{(8b)}$$

1.3 Integrals with values containing the Fresnel functions

The integrals

$$\begin{aligned} &\int_0^1 z^{2k} \cos(\lambda z^2) dz, \\ &\int_0^1 z^{2k} \sin(\lambda z^2) dz, \end{aligned}$$

which contain even powers of the variable, can be expressed in terms of the Lommel functions.

In the first instance, Maple gives

$$\int_0^1 z^{2k}\cos(\lambda z^2)dz = [1-\frac{s_{k+1,1/2(\lambda)}}{\lambda^k}]\frac{\cos(\lambda)}{(2k+1)} + [(2k-1)s_{k,3/2}+(2/\lambda)s_{k+1,1/2}]\frac{\sin(\lambda)}{2\lambda^k(2k+1)}. \tag{9}$$

Using (1) and (2) we get the simplified form

$$\int_0^1 z^{2k}\cos(\lambda z^2)dz = \frac{1}{4\lambda^k}[(2k-1)\cos(\lambda)\,s_{k-1,1/2}(\lambda)+2\sin(\lambda)\,s_{k,1/2}(\lambda)]. \tag{10}$$

The values of the integral in the case where $k=2m$ can be obtained from the Lommel expressions above in (6b) and (8b) with m replaced with $m-1$. We get

$$\int_0^1 z^{4m}\cos(\lambda z^2)dz = \frac{(-1)^m\Gamma(2m+1/2)}{2\lambda^{2m}}\{\sqrt{\frac{2}{\lambda}}C(\chi) - \cos(\lambda)\sum_{j=0}^{m-1}\frac{(-\lambda^2)^j}{\Gamma(2j+3/2)} - \lambda\sin(\lambda)\sum_{j=0}^{m-1}\frac{(-\lambda^2)^j}{\Gamma(2j+5/2)}\}.$$

In the case $k=2m+1$ we have

$$\int_0^1 z^{4m+2}\cos(\lambda z^2)dz = \frac{(-1)^{m+1}\Gamma(2m+3/2)}{2\lambda^{2m+1}}\{\sqrt{\frac{2}{\lambda}}S(\chi) + \lambda\cos(\lambda)\sum_{j=0}^{m-1}\frac{(-\lambda^2)^j}{\Gamma(2j+5/2)} - \sin(\lambda)\sum_{j=0}^{m}\frac{(-\lambda^2)^j}{\Gamma(2j+3/2)}\}.$$

With these results we see that all cases of cosine integrals with even powers of the variable have been obtained in terms containing the Fresnel S function.

For the corresponding sine integrals, integration by parts gives

$$\int_0^1 z^{2k}\sin(\lambda z^2)dz = \frac{\sin(\lambda)}{2k+1} - \frac{2\lambda}{2k+1}\int_0^1 z^{2(k+1)}\cos(\lambda z^2)dz.$$

Using (10) we have

$$\int_0^1 z^{2k}\sin(\lambda z^2)dz = \frac{1}{4\lambda^k}[(2k-1)\sin(\lambda)s_{k-1,1/2}(\lambda)-2\cos(\lambda)s_{k,1/2}(\lambda)].$$

With $k = 2m$ we get

$$\int_0^1 z^{4m}\sin(\lambda z^2)dz = \frac{1}{2\lambda^{2m}}[(4m-1)\sin(\lambda)s_{2m-1,1/2}(\lambda) - 2\cos(\lambda)s_{2m,1/2}(\lambda)],$$

which then becomes

$$\begin{aligned}\int_0^1 z^{4m}\sin(\lambda z^2)dz &= \frac{(-1)^m\Gamma(2m+1/2)}{2\lambda^{2m}}\{\sqrt{\frac{2}{\lambda}}S(\chi)\\ &-\sin(\lambda)\sum_{j=0}^{m-1}\frac{(-1)^j\lambda^{2j}}{\Gamma(2j+3/2)}\\ &+\lambda\cos(\lambda)\sum_{j=0}^{m-1}\frac{(-1)^j\lambda^{2j}}{\Gamma(2j+5/2)}\}.\end{aligned}$$

For $k = 2m+1$ the sine integrals are given by Maple as

$$\int_0^1 z^{4m+2}\sin(\lambda z^2)dz = \frac{1}{4\lambda^{2m+1}}[\sin(\lambda)(4m+1)s_{2m,1/2}(\lambda) - 2\cos(\lambda)s_{2m+1,1/2}(\lambda)].$$

Using the values of Lommel functions given above we have

$$\begin{aligned}\int_0^1 z^{4m+2}\sin(\lambda z^2)dz &= \frac{(-1)^m\Gamma(2m+3/2)}{2\lambda^{2m+1}}\{\sqrt{\frac{2}{\lambda}}C(\chi)\\ &-\lambda\sin(\lambda)\sum_{j=0}^{m-1}\frac{(-1)^j\lambda^{2j}}{\Gamma(2j+5/2)}\\ &-\cos(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{\Gamma(2j+3/2)}\}.\end{aligned}$$

With these results we see that all values of the sine integrals which contain *even* powers of z requires the presence of the Fresnel integrals $C(\chi)$.

2 Integrals with values containing elementary functions

We next consider integrals which contain *odd* powers of z i.e.

$$\int_0^1 z^{2k+1}\cos(\lambda z^2)dz,$$
$$\int_0^1 z^{2k+1}\sin(\lambda z^2)dz.$$

In the first instance Maple gives

$$\begin{aligned}\int_0^1 z^{2k+1}\cos(\lambda z^2)dz &= \frac{1}{2(k+1)}\{(1-\frac{s_{k+3/2,1/2}(\lambda)}{\lambda^{k+1/2}})\cos(\lambda)\\ &+(k\,s_{k+1/2,3/2}(\lambda)) + \frac{s_{k+3/2,1/2}(\lambda)}{\lambda})\frac{\sin(\lambda)}{\lambda^{k+1/2}}\},\end{aligned}$$

which reduces to

$$\int_0^1 z^{2k+1}\cos(\lambda z^2)dz = \frac{1}{2\lambda^{k+1/2}}[k\cos(\lambda)s_{k-1/2,1/2}(\lambda)+\sin(\lambda)s_{k+1/2,1/2}(\lambda)] \tag{11}$$

using (1) and (2). Where $k=2m$ we have

$$\int_0^1 z^{4m+1}\cos(\lambda z^2)dz = \frac{1}{2\lambda^{2m+1/2}}[2m\cos(\lambda)s_{2m-1/2,1/2}(\lambda)+\sin(\lambda)s_{2m+1/2,1/2}(\lambda)]. \tag{12}$$

We see that this expression requires the Lommel functions $s_{2m-1/2,1/2}(\lambda)$ and $s_{2m+1/2,1/2}(\lambda)$. In the first case we have from (2) and the initial condition $s_{3/2,1/2}(\lambda)=\sqrt{\lambda}\,[1-\sin(\lambda)/\lambda]$, the Lommel function $s_{2m-1/2,1/2}(\lambda)$ as

$$s_{2m-1/2,1/2}(\lambda) = \frac{(-1)^m(2m-1)!}{\sqrt{\lambda}}\left[\sin(\lambda)-\sum_{j=0}^{m-1}\frac{(-1)^j\lambda^{2j+1}}{(2j+1)!}\right].$$

The function $s_{2m+1,1/2}(\lambda)$ then can be obtained from the differential-difference equation in (3). That is to say

$$\frac{d\,s_{2m+1/2,1/2}(\lambda)}{d\lambda}+\frac{1}{2\lambda}s_{2m+1/2,1/2}(\lambda) = 2m\,s_{2m-1/2,1/2}(\lambda). \tag{13}$$

We get with the initial condition $s_{2m+1/2,1/2}(0)=0$, the solution to (13) as

$$s_{2m+1/2,1/2}(\lambda) = \frac{2m}{\sqrt{\lambda}}\int_0^\lambda \sqrt{z}s_{2m-1/2,1/2}(z)\,dz, \tag{14}$$

which immediately gives

$$s_{2m+1/2,1/2}(\lambda) = \frac{(-1)^{m+1}(2m)!}{\sqrt{\lambda}}\left[\cos(\lambda)-\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j)!}\right].$$

We have obtained in these cases closed forms for the Lommel functions which contain only elementary functions. As a result, we get

$$\begin{aligned}\int_0^1 z^{4m+1}\cos(\lambda z^2)dz &= \frac{(-1)^m(2m)!}{2\lambda^{2m+1}}\{\sin(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j)!} \\ &\quad -\lambda\cos(\lambda)\sum_{j=0}^{m-1}\frac{(-1)^j\lambda^{2j}}{(2j+1)!}\}.\end{aligned}$$

The cosine integral containing powers $4m+3$ is given by

$$\int_0^1 z^{4m+3}\cos(\lambda z^2)dz = \frac{1}{2\lambda^{2m+3/2}}[(2m+1)\cos(\lambda)\,s_{2m+1/2,1/2}(\lambda)+\sin(\lambda)s_{2m+3/2,1/2}(\lambda)].$$

In the latter expression the quantity $s_{2m+3/2,1/2}(\lambda)$ can be obtained from (3) and we have

$$\int_0^1 z^{4m+3}\cos(\lambda z^2)dz = \frac{(-1)^{m+1}(2m+1)!}{2\lambda^{2m+2}}\{1-\cos(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j)!} -\lambda\sin(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j+1)!}\}.$$

An expression for the sine integrals with odd powers i.e.

$$\int_0^1 z^{2k+1}\sin(\lambda z^2)dz = \frac{1}{2\lambda^{k+1/2}}[k\sin(\lambda)\ s_{k-1/2,1/2}(\lambda)-\cos(\lambda)s_{k+1/2,1/2}(\lambda)],$$

has been obtained using integration by parts of the corresponding $\cos(\lambda z^2)$ integral together with the latter's integrated form. Then in the case where $k = 2m$ we have

$$\int_0^1 z^{4m+1}\sin(\lambda z^2)dz = \frac{(-1)^{m}(2m)!}{2\lambda^{2m+1}}\{1-\cos(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j)!} -\lambda\sin(\lambda)\sum_{j=0}^{m-1}\frac{(-1)^j\lambda^{2j}}{(2j+1)!}\}.$$

Where $k = 2m + 1$ we have

$$\int_0^1 z^{4m+3}\sin(\lambda z^2)dz = \frac{1}{2\lambda^{2m+3/2}}[(2m+1)\sin(\lambda)\ s_{2m+1/2,1/2}(\lambda)-\cos(\lambda)s_{2m+3/2,1/2}(\lambda).$$

In the latter expression the quantity $s_{2m+3/2,1/2}(\lambda)$ can also be obtained from (3) and the integral being sought has the value

$$\int_0^1 z^{4m+3}\sin(\lambda z^2)dz = \frac{(-1)^{m}(2m+1)!}{2\lambda^{2m+2}}\{-\lambda\cos(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j+1)!} +\sin(\lambda)\sum_{j=0}^{m}\frac{(-1)^j\lambda^{2j}}{(2j)!}\}.$$

3 Asymptotic forms for the integrals containing $S(\chi)$ and $C(\chi)$

It is useful to provide values of the integrals evaluated above in section 2 where the parameter λ is large. Initially we consider the case of the integrals with even powers of the integration variable z i.e.

$$\int_0^1 z^{4m}\cos(\lambda z^2)dz,$$
$$\int_0^1 z^{4m}\sin(\lambda z^2)dz,$$
$$\int_0^1 z^{4m+2}\cos(\lambda z^2)dz,$$
$$\int_0^1 z^{4m+2}\sin(\lambda z^2)dz,$$

which we have seen contain the Fresnel integrals. The asymptotic forms for the functions $S(\chi)$ and $C(\chi)$ can be obtained from the expressions [4]

$$\begin{aligned} S\left(\sqrt{\frac{2\lambda}{\pi}}\right) &= \frac{1}{2} - f\,(\sqrt{\frac{2\lambda}{\pi}})\cos(\lambda) - g(\sqrt{\frac{2\lambda}{\pi}})\sin(\lambda), \\ C\left(\sqrt{\frac{2\lambda}{\pi}}\right) &= \frac{1}{2} + f\,(\sqrt{\frac{2\lambda}{\pi}})\sin(\lambda) - g(\sqrt{\frac{2\lambda}{\pi}})\cos(\lambda), \end{aligned}$$

where $f(z)$ and $g(z)$ are the Fresnel auxiliary functions. Their asymptotic forms with (cut off $N \geq 1$) are given by

$$\begin{aligned} f\,(\sqrt{\frac{2\lambda}{\pi}}) &\sim \frac{1}{\sqrt{2}\,\pi\lambda^{1/2}}\sum_{j=0}^{N-1}\frac{(-1)^j}{\lambda^{2j}}\Gamma(2j+1/2), \\ g\,(\sqrt{\frac{2\lambda}{\pi}}) &\sim \frac{1}{\sqrt{2}\,\pi\lambda^{3/2}}\sum_{j=0}^{N-1}\frac{(-1)^j}{\lambda^{2j}}\Gamma(2j+3/2). \end{aligned}$$

We get for the integrals in question, having used the relation

$$\frac{1}{\Gamma(-z)} = -\frac{\Gamma(z+1)}{\pi}\sin(\pi z),$$

in the sine and cosine sums, the expressions

$$\begin{aligned} \int_0^1 z^{4m}\cos(\lambda z^2)dz \sim & \frac{\Gamma(2m+1/2)}{2}\{\frac{(-1)^m}{\sqrt{2}\lambda^{2m+1/2}} \\ & +\frac{\cos(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^j}{\lambda^{2j}}\Gamma(2j-2m-1/2) \\ & +\frac{\sin(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^{j+1}}{\lambda^{2j-1}}\Gamma(2j-2m-3/2)\}, \end{aligned}$$

$$\int_0^1 z^{4m}\sin(\lambda z^2)dz \sim \frac{\Gamma(2m+1/2)}{2}\{\frac{(-1)^m}{\sqrt{2}\lambda^{2m+1/2}} + \frac{\cos(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^j}{\lambda^{2j-1}}\Gamma(2j-2m-3/2) + \frac{\sin(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^j}{\lambda^{2j}}\Gamma(2j-2m-1/2)\},$$

$$\int_0^1 z^{4m+2}\cos(\lambda z^2)dz \sim \frac{\Gamma(2m+3/2)}{2}\{\frac{(-1)^{m+1}}{\sqrt{2}\lambda^{2m+3/2}} + \frac{\cos(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^{j+1}}{\lambda^{2j}}\Gamma(2j-2m-3/2) + \frac{\sin(\lambda)}{\pi}\sum_{j=1}^{m+N+1}\frac{(-1)^j}{\lambda^{2j-1}}\Gamma(2j-2m-5/2)\},$$

$$\int_0^1 z^{4m+2}\sin(\lambda z^2)dz \sim \frac{\Gamma(2m+3/2)}{2}\{\frac{(-1)^m}{\sqrt{2}\lambda^{2m+3/2}} + \frac{\cos(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^{j+1}}{\lambda^{2j+1}}\Gamma(2j-2m-1/2) + \frac{\sin(\lambda)}{\pi}\sum_{j=1}^{m+N}\frac{(-1)^{j+1}}{\lambda^{2j}}\Gamma(2j-2m-3/2)\}.$$

We see that the terms in these asymptotic expressions contain the sine and cosine sums along with the additional and significant contributions from the Fresnel integrals.

3.1 Asymptotic values for Fresnel related integrals

In extensions of the Schwinger-Englert semi-classical theory [5] of atomic structure, asymptotic forms for the integrals

$$\int_0^1 \frac{\sin(\lambda z^2)}{(1+a\,z^2)}dz,$$
$$\int_0^1 \frac{\cos(\lambda z^2)}{(1+a\,z^2)}dz,$$

arise with $\lambda \to \infty$ and $0 < a < 1$. Here we give estimates for those forms. Expanding the denominators of the integrals above we have

$$\int_0^1 \frac{\sin(\lambda z^2)}{(1+a\,z^2)}dz = \sum_{k=0}^{\infty} a^{2k}\int_0^1 z^{4k}\sin(\lambda z^2) - a\sum_{k=0}^{\infty} a^{2k}\int_0^1 z^{4k+2}\sin(\lambda z^2),$$

using the results obtained above we get (where $z!!$ is the double factorial function and N is the order of truncation)

$$\begin{aligned}\int_0^1 \frac{\sin(\lambda z^2)}{(1+a\,z^2)}dz \quad &\sim \quad \frac{1}{2}\sqrt{\frac{\pi}{2\lambda}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a}{2\lambda}\right)^{2k}\left[(4k-1)!!-\frac{a}{2\lambda}(4k+1)!!\right]\\ &+\frac{\cos(\lambda)}{2}\{\sum_{k=0}^{\infty}a^{2k}(4k-1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-5)!!}{(2\lambda)^{2j-1}}\\ &+\frac{a}{\lambda}\sum_{k=0}^{\infty}a^{2k}(4k+1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-3)!!}{(2\lambda)^{2j}}\}\\ &+\frac{\sin(\lambda)}{2}\{\sum_{k=0}^{\infty}a^{2k}(4k-1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-3)!!}{(2\lambda)^{2j}}\\ &+\frac{a}{\lambda}\sum_{k=0}^{\infty}a^{2k}(4k+1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-5)!!}{(2\lambda)^{2j-1}}\}.\end{aligned}$$

In lowest order in a and $1/\lambda$ we have

$$\int_0^1 \frac{\sin(\lambda z^2)}{(1+a\,z^2)}dz \sim \frac{1}{4}\sqrt{\frac{2\pi}{\lambda}}\left(1-\frac{a}{2\lambda}\right)-\frac{\cos(\lambda)}{2\lambda}(1-a)-\frac{\sin(\lambda)}{(2\lambda)^2}(1+a)+\cdots, \quad (15)$$

and in the case of the cosine integral we have

$$\begin{aligned}\int_0^1 \frac{\cos(\lambda z^2)}{(1+a\,z^2)}dz \quad &\sim \quad \frac{1}{2}\sqrt{\frac{\pi}{2\,\lambda}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a}{2\lambda}\right)^{2k}\left[(4k-1)!!+\frac{a}{2\lambda}(4k+1)!!\right]\\ &+\cos(\lambda)\{\sum_{k=0}^{\infty}a^{2k}(4k-1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-3)!!}{(2\lambda)^{2j}}\\ &+\sum_{k=0}^{\infty}a^{2k+1}(4k+1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-5)!!}{(2\lambda)^{2j}}\}\\ &-\sin(\lambda)\{\sum_{k=0}^{\infty}a^{2k}(4k-1)!!\sum_{j=1}^{k+N}\frac{(-1)^j(4j-4k-5)!!}{(2\lambda)^{2j-1}}\\ &+\sum_{k=0}^{\infty}a^{2k+1}(4k+1)!!\sum_{j=1}^{k+N+1}\frac{(-1)^j(4j-4k-7)!!}{(2\lambda)^{2j-1}}\}.\end{aligned}$$

In lowest order in a and $1/\lambda$ we have

$$\int_0^1 \frac{\cos(\lambda z^2)}{(1+a\,z^2)}dz \sim \frac{1}{4}\sqrt{\frac{2\pi}{\lambda}}\left(1+\frac{a}{2\lambda}\right)-\frac{\cos(\lambda)}{(2\lambda)^2}(1+a)+\frac{\sin(\lambda)}{2\lambda}(1-a)+\cdots \quad (16)$$

It is interesting to note that the integrals with infinite range i.e.

$$\int_0^\infty \frac{\sin(\lambda z^2)}{(1+a\,z^2)} = -\frac{\pi}{2\sqrt{a}}\sin(\lambda/a) + \frac{\pi}{2\sqrt{a}}\left\{\begin{array}{c} C(\sqrt{\frac{2\lambda}{\pi a}})\left[\sin(\frac{\lambda}{a})+\cos(\frac{\lambda}{a})\right] \\ +S(\sqrt{\frac{2\lambda}{\pi a}})\left[\sin(\frac{\lambda}{a})-\cos(\frac{\lambda}{a})\right]\end{array}\right\}, \tag{17}$$

and

$$\int_0^\infty \frac{\cos(\lambda z^2)}{(1+a\,z^2)} = \frac{\pi}{2\sqrt{a}}\cos(\lambda/a) - \frac{\pi}{2\sqrt{a}}\left\{\begin{array}{c} C(\sqrt{\frac{2\lambda}{\pi a}})\left[\cos(\frac{\lambda}{a})-\sin(\frac{\lambda}{a})\right] \\ +S(\sqrt{\frac{2\lambda}{\pi a}})\left[\cos(\frac{\lambda}{a}))+\sin(\frac{\lambda}{a})\right]\end{array}\right\}, \tag{18}$$

which have the exact values shown above, posses the same forms for large λ as the integrals (albeit without oscillations) with finite range. That is to say

$$\int_0^\infty \frac{\sin(\lambda z^2)}{(1+a\,z^2)} \sim \frac{1}{4}\sqrt{\frac{2\pi}{\lambda}}\left(1-\frac{a}{2\lambda}\right)+\cdots, \tag{19}$$

$$\int_0^\infty \frac{\cos(\lambda z^2)}{(1+a\,z^2)}dz \sim \frac{1}{4}\sqrt{\frac{2\pi}{\lambda}}\left(1+\frac{a}{2\lambda}\right)+\cdots \tag{20}$$

It is also worth noting that asymptotic values of the integrals which contain higher powers of the trigonometric function described above and with higher powers of the denominators i.e.

$$\int_0^1 \frac{\sin^n(\lambda z^2)}{(1+a\,z^2)^{\nu}}, \qquad \int_0^1 \frac{\cos^n(\lambda z^2)}{(1+a\,z^2)^{\nu}},$$

and

$$\int_0^\infty \frac{\sin^n(\lambda z^2)}{(1+a\,z^2)^{\nu}}, \qquad \int_0^\infty \frac{\cos^n(\lambda z^2)}{(1+a\,z^2)^{\nu}},$$

can be expressed in terms of the forms given in (15,16) and (19,20). By way of example, writing

$$I_\nu^{(\eta)}(a,\lambda) = \int_0^\infty \frac{\cos^{\eta}(\lambda z^2)}{(1+a\,z^2)^{\nu}},$$

(noting the scaling property)

$$I_\nu^{(\eta)}(a,\lambda) = \frac{1}{\sqrt{a}} I_\nu^{(\eta)}(1,\lambda/a),$$

we get the differential-difference equation

$$I_{\nu+1}^{(\eta)}(a,\lambda) = I_\nu^{(\eta)}(a,\lambda) + \left(\frac{a}{\nu}\right)\frac{d\,I_\nu^{(\eta)}(a,\lambda)}{da}. \tag{21}$$

In the case where $\eta = 1$ and $\nu = 1/2$ we have

$$\begin{aligned} I_{1/2}^{(1)}(a,\lambda) &= \int_0^\infty \frac{\cos(\lambda z^2)}{(1+a\,z^2)^{1/2}} \\ &= \frac{\pi}{4\sqrt{a}}\left\{\sin(\frac{\lambda}{2a})\,J_0(\frac{\lambda}{2a}) - \cos(\frac{\lambda}{2a})Y_0(\frac{\lambda}{2a})\right\}, \end{aligned}$$

a closed form expression where $J_0(z)$ and $Y_0(z)$ are Bessel functions of the first and second kind. The value for this integral for large $\frac{\lambda}{2a}$ using Hankel's [6] asymptotic expressions for the Bessel functions is

$$I_{1/2}^{(1)}(a,\lambda) \sim \frac{1}{4}\sqrt{\frac{2\pi}{\lambda}}\left\{1+\frac{1}{4}\left(\frac{a}{\lambda}\right)-\frac{9}{32}\left(\frac{a}{\lambda}\right)^2-\frac{75}{128}\left(\frac{a}{\lambda}\right)^3+\cdots\right\}.$$

In the case of the integral $I_1^{(2)}(a,\lambda)$, its exact value is

$$\begin{aligned} I_1^{(2)}(a,\lambda) &= \int_0^\infty \frac{\cos^2(\lambda z^2)}{(1+a\,z^2)} \\ &= \frac{\pi}{2\sqrt{a}}\cos^2(\lambda/a)-\frac{\pi}{4\sqrt{a}}\left\{\begin{array}{c} C(\sqrt{\frac{4\lambda}{\pi a}})\left[\cos(\frac{2\lambda}{a})-\sin(\frac{2\lambda}{a})\right] \\ +S(\sqrt{\frac{4\lambda}{\pi a}})\left[\cos(\frac{2\lambda}{a}))+\sin(\frac{2\lambda}{a})\right] \end{array}\right\}, \end{aligned}$$

or rewriting it in terms of the Fresnel auxiliary functions [7] $f(\chi')$ and $g(\chi')$ referred to above (here with $\chi' = 2\sqrt{\lambda/\pi a}$) we get

$$\int_0^\infty \frac{\cos^2(\lambda z^2)}{(1+a\,z^2)} = \frac{\pi}{4\sqrt{a}}\left\{1+f(\chi')+g(\chi')\right\}.$$

The asymptotic values of $f(\chi')$ and $g(\chi')$ for large λ [8] give the result

$$\begin{aligned} \int_0^\infty \frac{\cos^2(\lambda z^2)}{(1+a\,z^2)} \quad \sim \quad & \frac{\pi}{4\sqrt{a}}+\frac{1}{8}\sqrt{\frac{\pi}{\lambda}}\left\{1-\frac{3}{16}\left(\frac{a}{\lambda}\right)^2+\cdots\right\} \\ & +\frac{1}{32}\sqrt{\frac{\pi}{\lambda}}\left\{\frac{a}{\lambda}-\frac{15}{16}\left(\frac{a}{\lambda}\right)^3+\cdots\right\}. \end{aligned}$$

The first of the higher order integrals i.e. $I_2^{(2)}$ has the closed form expression

$$\begin{aligned} I_2^{(2)} &= \int_0^\infty \frac{\cos^2(\lambda z^2)}{(1+a\,z^2)^2} \\ &= \frac{\pi}{8\sqrt{a}}\left\{\begin{array}{c} 2\cos^2(\frac{\lambda}{a})+4(\frac{\lambda}{a})\sin(\frac{2\lambda}{a})+\sqrt{\frac{4\lambda}{\pi a}} \\ +C(\sqrt{\frac{4\lambda}{\pi a}})[\sin(\frac{2\lambda}{a})(1-4\frac{\lambda}{a})-\cos(\frac{2\lambda}{a})(1+4\frac{\lambda}{a})] \\ -S(\sqrt{\frac{4\lambda}{\pi a}})[\sin(\frac{2\lambda}{a})(1+4\frac{\lambda}{a})+\cos(\frac{\lambda}{a})(1-4\frac{2\lambda}{a}] \end{array}\right\}. \end{aligned}$$

The higher order integrals i.e. those containing powers of the cosine function i.e. $\cos^\eta(\lambda x^2)$ are expressible in terms of the integrals in (18). Using the trigonometric relations

$$\cos^\eta(\lambda x^2) = \frac{1}{2^\eta}\sum_{j=0}^{\eta}\binom{\eta}{j}\cos([\eta-2j]\lambda x^2),$$

we have

$$
\begin{aligned}
I_\nu^{(2\eta)}(a,\lambda) &= \sqrt{\frac{\pi}{a}}\frac{\Gamma(2\eta-1/2)}{2^{\,2\eta}\,\eta!\,(\eta-1)!} + \frac{1}{2^{2\eta-1}}\sum_{j=1}^{\eta}\binom{2\,\eta}{\eta-j} I_\nu^{(1)}(a,2j\,\lambda), \\
I_\nu^{(2\eta+1)}(a,\lambda) &= \frac{1}{2^{2\eta}}\sum_{j=0}^{\eta}\binom{2\eta+1}{\eta-j} I_\nu^{(1)}(a,(2j+1)\,\lambda).
\end{aligned}
$$

With results from Maple, it is possible using an incomplete form of induction to give general expressions for the integrals $I_\nu^{(1)}(a,\lambda)$ with even and odd values of ν. The integrals are found to contain the Anger functions [9] $\mathbf{J}_{1/2}(z)$ and $\mathbf{J}_{3/2}(z)$ i.e.

$$
I_{2n}^{(1)}(a,\lambda) = \frac{1}{\sqrt{a}\,2^{\,\epsilon(n)+n}}\left\{\begin{array}{c}
2\sqrt{\frac{2\pi a}{\lambda}}\sum_{k=0}^{2n-1} a_{k,2n}\left(\frac{\lambda}{a}\right)^k \\
+2\pi\cos(\lambda/a)\sum_{k=0}^{n-1} c_{k,2n}\left(\frac{\lambda}{a}\right)^{2k} \\
+4\pi\sin(\lambda/a)\sum_{k=0}^{n-1} d_{k,2n}\left(\frac{\lambda}{a}\right)^{2k+1} \\
-\pi\sqrt{\frac{2\pi a}{\lambda}}\mathbf{J}_{1/2}(\lambda/a)\sum_{k=0}^{n} e_{k,2n}\left(\frac{\lambda}{a}\right)^{2k} \\
+\pi\sqrt{\frac{2\pi a}{\lambda}}\mathbf{J}_{3/2}(\lambda/a)\sum_{k=0}^{n-1} f_{k,2n}\left(\frac{\lambda}{a}\right)^{2k+1},
\end{array}\right\}
$$

where the Greubel eta function $\epsilon(n)$ is defined here as

$$
\epsilon(n) = \lfloor\sqrt{2}n\rfloor + \lfloor\sqrt{3/2}n\rfloor\ ,
$$

and $\lfloor z\rfloor$ is the floor function. The sequence of integers produced by the eta function has been studied by G. C. Greubel and others [10]. The expression for $I_{2n+1}^{(1)}(a,\lambda)$ is

$$
I_{2n+1}^{(1)}(a,\lambda) = \frac{1}{\sqrt{a}\,2^{\,\epsilon(n+1)+n}}\left\{\begin{array}{c}
2\sqrt{\frac{2\pi a}{\lambda}}\sum_{k=0}^{2n-1} a_{k,2n+1}\left(\frac{\lambda}{a}\right)^k \\
+2\pi\cos(\lambda/a)\sum_{k=0}^{n} c_{k,2n+1}\left(\frac{\lambda}{a}\right)^{2k} \\
+2\pi\sin(\lambda/a)\sum_{k=0}^{n-1} d_{k,2n+1}\left(\frac{\lambda}{a}\right)^{2k+1} \\
-\pi\sqrt{\frac{2\pi a}{\lambda}}\mathbf{J}_{1/2}(\lambda/a)\sum_{k=0}^{n} e_{k,2n+1}\left(\frac{\lambda}{a}\right)^{2k} \\
+\pi\sqrt{\frac{2\pi a}{\lambda}}\mathbf{J}_{3/2}(\lambda/a)\sum_{k=0}^{n} f_{k,2n+1}\left(\frac{\lambda}{a}\right)^{2k+1}
\end{array}\right\}.
$$

Alternatively, in keeping with results given above the Anger functions can be written in terms of the Fresnel functions appearing in $I_1^{(2)}$, and $I_2^{(2)}$, using the relations

$$
\begin{aligned}
\mathbf{J}_{1/2}(z) &= \sqrt{\frac{2}{\pi z}}[C(\sqrt{\frac{2z}{\pi}})\{\sin(z)+\cos(z)\} + S(\sqrt{\frac{2z}{\pi}})\{\sin(z)-\cos(z)\}], \\
\mathbf{J}_{3/2}(z) &= -\frac{2}{\pi z} + \sqrt{\frac{2}{\pi z}}C(\sqrt{\frac{2z}{\pi}})\{\sin(z)\,(1+1/z) - \cos(z)\,(1-1/z)\} \\
&\quad -\sqrt{\frac{2}{\pi z}}S(\sqrt{\frac{2z}{\pi}})\{\sin(z)\,(1-1/z) + \cos(z)\,(1+1/z)\}.
\end{aligned}
$$

The coefficients $a_{k,\nu}, c_{k,\nu}, d_{k,\nu}, e_{k,\nu}, f_{k,\nu}$ occurring in the $I_\nu^{(1)}$ integrals are interrelated by (21) from which we obtain the connection formulas

$$\begin{aligned}
\frac{4n}{2^{\Delta\,\epsilon(n)}} a_{2n,2n+1} &= -e_{n,2n}, \qquad (k=n), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} a_{2k,2n+1} + (4k-4n)\, a_{2k,2n} &= -e_{k,2n}, \qquad (k<n), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} a_{2k+1,2n+1} + (4k+2-4n)\, a_{2k+1,2n} &= f_{k,2n}, \qquad (k<n), \\
\frac{n}{2^{\Delta\,\epsilon(n)}} c_{n,2n+1} &= -d_{n-1,2n}, \qquad (k=n), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} c_{k,2n+1} + (4k+1-4n)\, c_{k,2n} &= -4\, d_{k-1,2n}, \qquad (k<n), \\
2n\, d_{k,2n+1} + (4k+3-4n)\, d_{k,2n} &= c_{k,2n}, \qquad (k \le n-1), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} e_{0,2n+1} &= (4n-1) e_{0,2n}, \qquad (k=n), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} e_{k,2n+1} + (4k+1-4n)\, e_{k,2n} &= 2 f_{k-1,2n}, \qquad (k>0), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} f_{n,2n+1} &= -2 e_{n,2n}, \qquad (k=n), \\
\frac{4n}{2^{\Delta\,\epsilon(n)}} f_{k,2n+1} + (4k-1-4n)\, f_{k,2n} &= -2 e_{k,2n}, \qquad (k<n),
\end{aligned}$$

with $\Delta\epsilon(n) = \epsilon(n+1) - \epsilon(n)$. These relations do not appear to be useful except as an internal check on the values of the coefficients.

References

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[2] Reference 1, p. 112.

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