# A CHARACTERISTICS-BASED APPROXIMATION FOR WAVE SCATTERING FROM AN ARBITRARY OBSTACLE IN ONE DIMENSION *

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**Abstract.** The method of characteristics is extended to solve the Cauchy problem for linear hyperbolic PDEs in one space dimension with arbitrary variation of coefficients. In the presence of continuous variation of coefficients, the number of characteristics that must be dealt with is uncountable. This difficulty is overcome by writing the solution as an infinite series in terms of the number of reflections involved in each characteristic path. We illustrate an interesting combinatorial connection between the traditional reflection and transmission coefficients for a sharp interface to Green's coefficient for transmission through a smoothly-varying region. We prove that the series converges and provide bounds for the truncation error. The effectiveness of the approximation is illustrated with examples.

**Key words.** wave equation, characteristics, hyperbolic PDE

**AMS subject classifications.**

## 1. Introduction and physical setting.

We consider the Cauchy problem for the linear one-dimensional wave equation

$$u_{tt} = \frac{1}{\rho(x)}\left(K(x)u(x,t)\right)_x, \tag{1.1}$$

which can also be written in first-order form as

$$\begin{aligned} p_t(x,t) + K(x)u_x(x,t) &= 0 \\ u_t(x,t) + \frac{1}{\rho(x)}p_x(x,t) &= 0. \end{aligned} \tag{1.2}$$

Here we have used the notation of acoustics: $p$ is pressure, $u$ is velocity, $K$ is the bulk modulus, and $\rho$ is the density. Linear wave equations with the same mathematical structure arise in many other applications, with different interpretations of the material parameters, such as elasticity, electromagnetics, and linearized fluid dynamics or water waves. If the coefficients $(\rho(x), K(x))$ are constant or piecewise-constant, the problem may be solved exactly by the method of characteristics. On the other hand, for more general functions $\rho(x)$ and/or $K(x)$ it is not clear how to apply the method of characteristics since the number of characteristic paths reaching a given point at any time is infinite.

Much is known about scattering of a periodic incident wavefield, particularly for piecewise constant media. We focus instead on scattering of a localized variation in an arbitrary medium including regions of continuous variation. We pay special attention to the scattering of a step function.

In this work we propose and demonstrate a method for approximately solving the general Cauchy problem for (1.2) in the presence of arbitrary variation in $\rho$ and $K$, by grouping characteristic paths according to the number of reflections. Our interest originated in a study of the shoaling of water waves over a continental shelf and our complementary paper [4] contains more discussion of this application and several illustrative examples. Code to reproduce the numerical experiments in this paper is available online. [1]

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[1] https://github.com/ketch/characteristics_rr

In the remainder of this section we briefly review the mathematics of characteristics and reflection in one dimension.

**1.1. The method of characteristics: homogeneous media.** Defining $q = [p, u]^T$, the system (1.2) can be written as $q_t + A(x)q_x = 0$, where $A$ has the eigenvalue decomposition $A = V(x)\Lambda V^{-1}(x)$ with

$$V(x) = \begin{bmatrix} 1 & 1 \\ \frac{-1}{Z(x)} & \frac{1}{Z(x)} \end{bmatrix} \qquad \Lambda(x) = \begin{bmatrix} -c(x) & 0 \\ 0 & c(x) \end{bmatrix}. \tag{1.3}$$

Here $Z(x) = \sqrt{K(x)\rho(x)}$ is known as the impedance and $c(x) = \sqrt{K(x)/\rho(x)}$ is the sound speed. If $K(x)$ and $\rho(x)$ are constant (or more generally, if $Z(x)$ is constant) then $V(x)$ is also constant and, setting $w(x,t) = V^{-1}q(x,t)$, (1.2) can be rewritten as

$$w_t + \Lambda(x)w_x = 0. \tag{1.4}$$

System (1.4) consists of two decoupled advection equations, indicating that one component of the solution ($w_1$) travels to the left (with velocity $-c$) while the other ($w_2$) travels to the right (with velocity $+c$). Lines of constant $x + ct$ and $x - ct$ are referred to as characteristics. The solution is simply the sum of the components transmitted along the two characteristic families:

$$p(x,t) = w_1(x + ct, 0) + w_2(x - ct, 0). \tag{1.5}$$

**1.2. Piecewise-constant media: reflection and transmission.** The method of characteristics can also be used to find the exact solution of (1.2) if $K(x)$ and $\rho(x)$ are piecewise-constant functions. Within each constant-coefficient domain the characteristic velocities are $\pm c(x)$. Consider a single interface where the impedance jumps from $Z_-$ on the left to $Z_+$ on the right. Let $v_1^\pm, v_2^\pm$ denote the respective columns of $V(0^\pm)$. For an incident right-going wave, the incident ($p_0$), transmitted ($p_T$), and reflected ($p_R$) wave pressures are related by

$$p_0 v_2^+ = p_T v_2^+ + p_R v_1^-. \tag{1.6}$$

Solving system (1.6) reveals that the transmitted and reflected waves are related to the incident wave by the transmission and reflection coefficients:

$$C_T(Z_-, Z_+) := \frac{p_T}{p_0} = \frac{2Z_+}{Z_- + Z_+}, \tag{1.7}$$

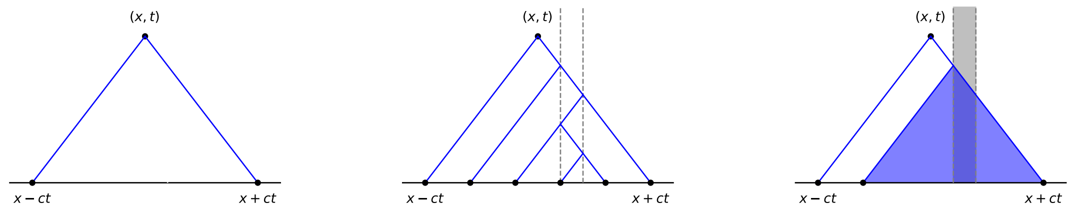
$$C_R(Z_-, Z_+) := \frac{p_R}{p_0} = \frac{Z_+ - Z_-}{Z_- + Z_+}. \tag{1.8}$$

**1.3. Smoothly-varying media.** Wherever the impedance $Z(x)$ is not constant, the system (1.2) cannot be decoupled as in (1.4) because the matrix $V(x)$ that relates $q$ and $w$ varies in space. If $Z(x)$ is differentiable, we have $w_x = (V(x)^{-1}q)_x = V^{-1}(x)q_x + (V^{-1}(x))'q$ and we obtain instead of (1.4) the system

$$\begin{aligned} w_t + \Lambda(x)w_x &= (V^{-1})'q \\ &= (V^{-1})'Vw. \end{aligned} \tag{1.9}$$

Here

$$(V^{-1}(x))'V(x) = \frac{1}{2}\frac{Z'(x)}{Z(x)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{1.10}$$




(a) Homogeneous medium (b) Piecewise-constant medium (c) Continuously-varying medium

FIG. 1. *Characteristics in three different types of media. In the homogeneous medium, the solution at each point is determined by just two characteristics. In the piecewise-constant medium (with material interfaces indicated by dashed lines), the solution at each point is determined by a finite number of characteristics. In the continuously-varying medium (with $Z(x)$ varying throughout the grey-shaded region), the solution at the indicated point depends on all characteristics within the blue-shaded region.*

We see that information is still transmitted along characteristics, but the amplitude of each component is modified by the source terms that couple the characteristic variables through reflection. The coefficient

$$r(x) = \frac{Z'(x)}{2Z(x)} \tag{1.11}$$

gives the amplitude of these reflections and we refer to it as the *infinitesimal reflection coefficient*.

The infinitesimal reflection ceofficient $r(x)$ is related to the traditional reflection coefficient $R(Z_-, Z_+)$; if we take $Z(x)$ to be a continuous function with value $Z_-$ at $x$ and value $Z_+$ at $x+\Delta x$, the ratio $R/\Delta x$ approaches $r(x)$ as $\Delta x$ tends to zero:

$$\frac{R}{\Delta x} \approx \frac{1}{\Delta x}\frac{Z(x+\Delta x) - Z(x)}{Z(x+\Delta x) + Z(x)} \approx \frac{1}{2}\frac{Z'(x)}{Z(x)}. \tag{1.12}$$

Characteristics for each of the three classes of media just discussed are illustrated in Figure 1. We see that in the presence of constant or piecewise-constant impedance, the number of characteristics that must be accounted for to compute the solution at a given point is finite. On the other hand, if $Z(x)$ varies continuously then there are in general infinitely many characteristic paths passing through a given point. The technique developed in the rest of this work is based on the hypothesis that the dominant contributions to the solution come from accounting for characteristic paths with relatively few reflections. This hypothesis is clearly reasonable when $|r(x)| < 1$, since then each reflection must diminish the significance of the corresponding characteristic path. The motivation for this hypothesis more generally is given in Section 4.2.

**2. Characteristics in continuously-varying media.** In this section we develop an approximate solution to (1.2) in the form of an infinite series. We focus on the case of a finite region of variation in the spatial coefficients, as illustrated in Figure 2:

$$(K(x), \rho(x)) = \begin{cases} (K_-, \rho_-) & x < 0 \\ (K(x), \rho(x)) & 0 \le x \le x_+ \\ (K_+, \rho_+) & x > x_+. \end{cases} \tag{2.1}$$

Here $x_+$ is the width of the region of varying coefficients, and need not be small. For simplicity we consider the case of a right-going disturbance that is initially confined to $x < 0$, and investigate the

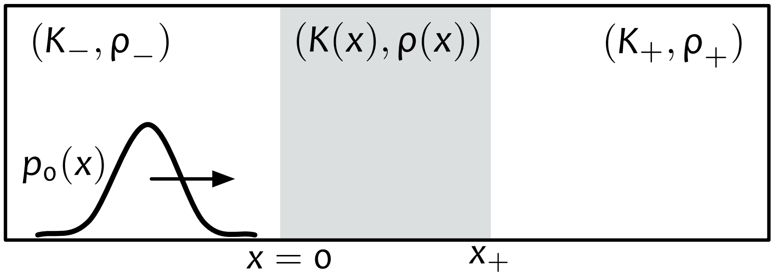



FIG. 2. *The setting for most of the paper.*

resulting reflected and transmitted disturbances. Thus

$$\begin{bmatrix} p(x,0) \\ u(x,0) \end{bmatrix} = \begin{cases} p_0(x) \begin{bmatrix} 1 \\ 1/Z_- \end{bmatrix} & x < 0 \\ 0 & x \geq 0. \end{cases} \tag{2.2}$$

We assume for simplicity that $Z(x)$ is continuous. Our method and results can be generalized in a natural way to arbitrary initial data and piecewise continuous media.

Outside of the region $[0, x_+]$, characteristics are straight lines in the $x$–$t$ plane. Let $X(t)$ denote the characteristic path starting from $x = 0$ at time zero; i.e., the solution of the initial value ODE

$$X'(t) = c(X(t)) \qquad X(0) = 0 \qquad t \in [0, t_+]. \tag{2.3}$$

Here $t_+$ is the crossing time so that $X(t_+) = x_+$. It is convenient in what follows to extend $X(t)$ by defining $X(t) = 0$ for $t < 0$ and $X(t) = x_+$ for $t > t_+$.

**2.1. Amplification or attenuation along characteristics: Greens Law.** In general the pressure is given by $p = w_1 + w_2$; for the case of a pure right-going pulse (2.2), for which $w_1$ is zero, we have $p(x, 0) = w_2(x, 0) = p_0$. According to (1.9), along the path $X(t)$ the value of $w_2$ (and hence the value of $p$) satisfies the ODE

$$p'(X(t)) = \frac{Z'(X(t))}{2Z(X(t))} p(X(t)) \tag{2.4}$$

with solution

$$p(X) = \left(\frac{Z(X)}{Z_-}\right)^{1/2} p_0. \tag{2.5}$$

In particular, at $x = x_+$ we have

$$\frac{p_+}{p_0} = \left(\frac{Z_+}{Z_-}\right)^{1/2} = C_G. \tag{2.6}$$

Thus the amplitude of the unreflected part of the wave (for $x \geq x_+$) is $C_G p_0$ for any smoothly varying $Z(x)$, and depends only on the values $Z_-$ and $Z_+$; it is independent of how $Z$ varies over $[0, x_+]$. As we will see, (2.6) represents the first term in an infinite series that sums to the transmission coefficient $C_T$.

*Remark* 2.1. We use $C_G$ for the quantity defined in (2.6) since this is the amplification factor given by Green's law in the context of shoaling, as we discuss in more detail in [4]. The linearized shallow

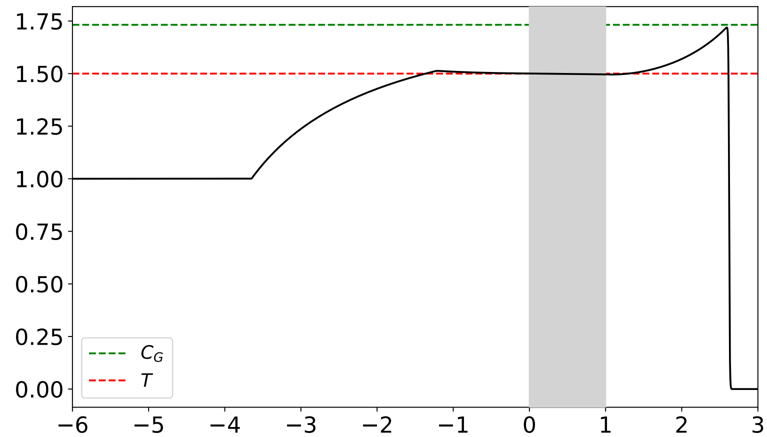



FIG. 3. *Transmission and reflection of an initial right-going step. The leading edge of the transmitted part has amplitude $C_G$ (the amplitude in the absence of reflections), then tends to $C_T$ at later times as multiply reflected components contribute.*

water equations used there can be put in the form (1.2) by introducing $p(x,t)$ as the depth perturbation of a small amplitude long wave on a background water depth $h(x)$, and $\mu(x,t)$ as the momentum perturbation. Then the linearized shallow water equations can be written in the nonconservative form

$$\begin{aligned} p_t(x,t) + \mu_x(x,t) &= 0 \\ \mu_t(x,t) + gh(x)p_x(x,t) &= 0, \end{aligned} \tag{2.7}$$

where $g$ is the gravitational constant. This differs from the conservative form used in [4], and has the same form as (1.2) if we set $K(x) \equiv 1$ and $\rho(x) = 1/(gh(x))$. Then the wave speed is $c(x) = \sqrt{gh(x)}$, the impedance is $Z(x) = 1/\sqrt{gh(x)}$, and $C_G = (h_-/h_+)^{1/4}$. This is the standard form of Green's law used to estimate the amplification of a shoaling wave as it passes into shallower water, in which case $h_- > h_+$. Note that this particular application is a special case in that there is only a single variable coefficient $h(x)$, so it is not possible to vary the wave speed and impedance separately.

Both the amplification factor $C_G$ and the transmission coefficent $C_T$ defined in (1.7) are related to the amplitude of transmitted waves. Their differing roles are illustrated in Figure 3, where we consider the propagation of a step function (taking $p_0(x) = 1$, and with an impedance that grows linearly from $Z_- = 1$ to $Z_+ = 3$ in the region $[0, 1]$). Since $C_G$ governs the amplification of unreflected characteristics, the leading part of the transmitted wave (which is unaffected by characteristics that undergo reflection, since they will emerge at later times) has amplitude $C_G$. Meanwhile, $C_T$ accounts for the cumulative effect of all characteristics (including those that have been reflected one or more times), and so the amplitude of the transmitted wave at long times approaches $C_T$.

In Figure 4 we consider what happens as $x_+$ tends to zero, for fixed values of $Z_-, Z_+$. We again take a step function as the initial condition (plotted as a dashed line). In this case the solution is invariant if $x_+$ and $t$ are scaled by the same factor, but in Figure 4 we have plotted solutions for different values of $x_+$ all at the same time $t$. We see that as the region $[0, x_+]$ shrinks, the width of the transmitted peak becomes increasingly narrow until in the limit $x_+ = 0$ (for which the impedance is discontinuous) the peak is gone and we have a single intermediate state dictated by the transmission coefficient. One way to think about this is that since the discontinuity thought has infinitesimally small width, all the characteristics must arrive in infinitesimally short time. The combinatoric relation between the transmission/reflection coefficients and $C_G$ is further explored in Section 4.1.

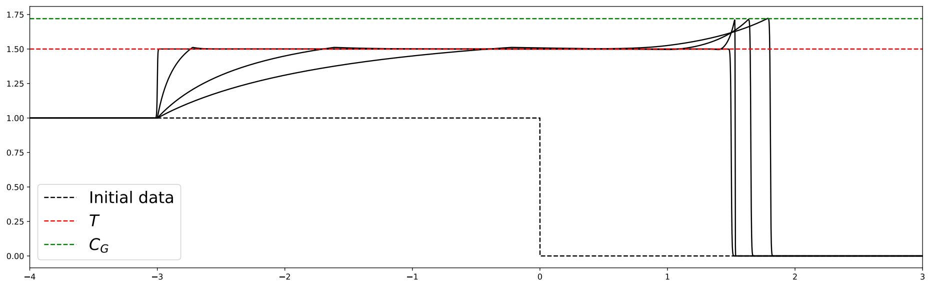



FIG. 4. *A sequence of solutions with differing values of $x_+$ (the width of the variable region). In all cases the initial data is a step function with unit amplitude (dashed line) and the impedance increases linearly from $Z_- = 1$ to $Z_+ = 3$. The width of the region of varying impedance is taken to be* $[1, 1/2, 1/10, 0]$.

**2.2. Approximating the reflected wave.** Let us now turn our attention to the reflected wave in Figure 3. The main contribution to this wave comes from characteristics that are reflected exactly once, as illustrated in Figure 5. The figure on the left shows two characteristics that emerge at $x = 0$ at the same time but started from different initial points and were reflected at different points. It is evident that at any time $t$ there will be such a characteristic reaching $x = 0$ that was reflected from point $x$ for each $x \in (0, X(t/2))$, since the characteristic reflected from the rightmost point (the red characteristic in the figure) must have traveled from $x = 0$ to the point of reflection in time $t/2$. For the initial condition $p_0(x) = 1$, the solution along each of these characteristics has the same initial amplitude. In this case the combined amplitude of these reflected waves is

$$\int_0^{X(t/2)} r(x_1)dx_1 = \frac{1}{2}\log\left(\frac{Z(X(t/2))}{Z_-}\right), \tag{2.8}$$

where $r(x)$ is defined in (1.11). Figure 5 shows this diagrammatically. Initially, the reflected wave only contains the contribution of characteristics reflected near $x = 0$. After some time $2t_+$, reflections from the whole interval $[0, x_+]$ contribute, resulting in a constant asymptotic reflected amplitude.

Although (2.8) does not account for characteristics that have been reflected more than once, it gives a surprisingly good approximation to the reflected wave for this simple example, as seen in Figure 6. This approximation is less accurate at later times because the contributions from the multiply-reflected characteristics begin to make a noticeable difference to the solution. The following section details a method that incorporates as many sets of characteristics as needed for an accurate approximation to both the reflected and transmitted wave.

**3. The general solution by characteristics with reflections.** In this section we construct a series for the solution at the boundaries of the variable region:

$$p(0,t) - p(-c_-t, 0) = \sum_{m=0}^{\infty} R_{2m+1}(t) \tag{3.1a}$$

$$p(x_+, t) = \sum_{m=0}^{\infty} T_{2m}(t). \tag{3.1b}$$

Here $R_n$ and $T_n$ denote contributions from characteristics involving $n$ reflections. We have effectively computed $T_0(t)$ and $R_1(t)$ already in the previous sections; from our derivations of (2.6) and (2.8) it is

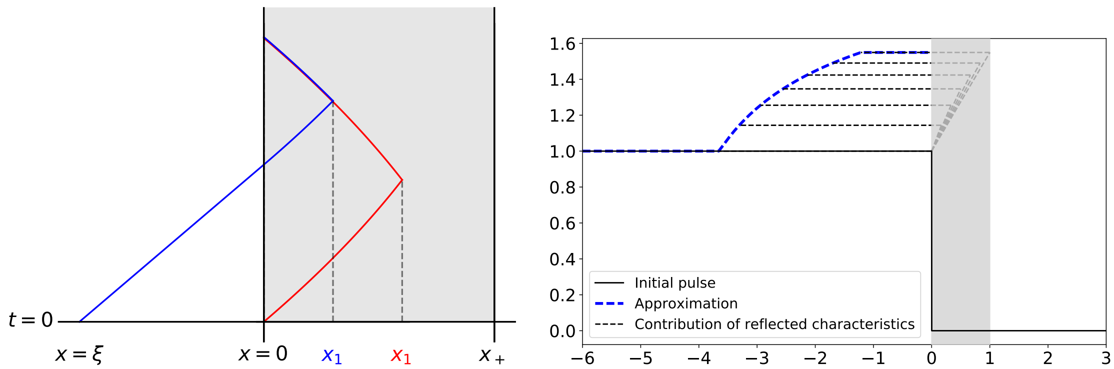



FIG. 5. *The reflected wave exists because of the cumulative effect of characteristics that get reflected within* $[0, x_+]$ *and come out to the left side with some infinitesimal amplitude. This figure only shows characteristics that have been reflected only once which is why the blue curve is an approximation to the reflected wave.*

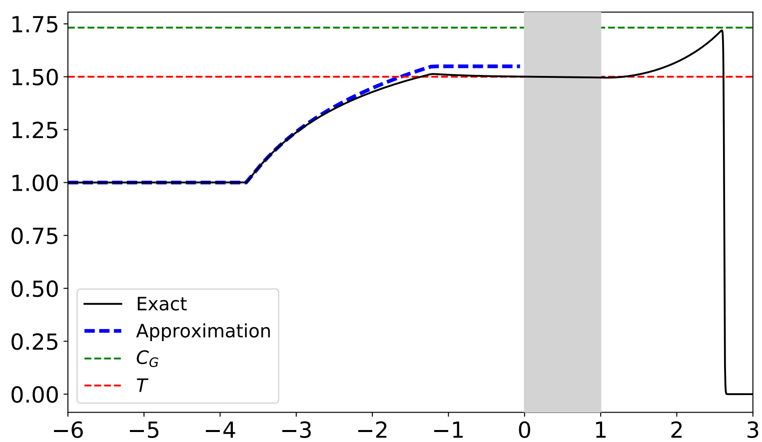



FIG. 6. *Equation* (2.8) *provides a good approximation to the reflected wave by considering only the characteristics that have been reflected once. This overestimates the solution because the next significant set of characteristics are the thrice-reflected ones which have negative amplitudes (in this case).*

straightforward to obtain the more general expressions

$$T_0(t) = C_G p_0(-c_-(t - t_+))$$
$$R_1(t) = \int_0^{X(t/2)} p_0(-c_-(t - 2\tau_1))r(x_1)dx_1$$

which give the part of the transmitted solution due to unreflected characteristics and the part of the reflected solution due to characteristics with a single reflection, respectively. Here and below, $\tau_j$ denotes the time for a characteristic to reach $x_j$ from $x = 0$.

The function $X(t)$ defined in (2.3) gives the path of a characteristic that is not reflected. More generally, consider a characteristic path involving reflection at the sequence of points $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in [0, x_+]$, which we refer to as the reflection point sequence for this path. This path is a union of curves

$X_j(t)$ $(j = 0, 1, 2, \dots)$, each of which is the solution of an initial value problem:

$$X_j'(t) = (-1)^j c(X_j(t)) \qquad X_j(t_j) = x_j \qquad t \in [t_j, t_{j+1}].$$

Here $x_0 = 0$ and $x_{n+1}$ is either zero (for reflected characteristics) or $x_+$ (for transmitted characteristics). The value of $t_j$ is the time at which the path reaches $x_j$. For a given medium, a path is determined completely by the reflection points $\mathbf{x}$ and the initial time $t_0$. Some examples of such paths are given in Figures 7b and 7a. Notice that the shape of the curves $X_j$ depends on the variation of $c(x)$, but all can be obtained by applying a temporal offset to $X(t)$ and (for left-going segments) reflecting the curve $X(t)$ vertically in the $x$–$t$ plane.

In keeping with the method of characteristics, we would like to add up the contributions of all characteristic paths arriving at a given place and time $(x, t)$. One way to do this is to sum over all valid reflection point sequences. Notice that the reflection point sequence cannot be an arbitrary sequence of points in $[0, x_+]$. We need to sum over all characteristics with an *alternating sequence* of reflection points, as defined by:

Definition 3.1. *A sequence* $\mathbf{x} = \{x_1, x_2, \cdots x_n\}$ *is an* alternating sequence *if*

$$\begin{aligned} x_j &\le x_{j-1} && \textit{for } j \textit{ even, and} \\ x_j &\ge x_{j-1} && \textit{for } j \textit{ odd.} \end{aligned}$$

For characteristics with $n$ reflection points, the terms in (3.1) are given by the following iterated integrals. Note that $x_1$ in the outermost integral can be anywhere in $[0, x+]$; then $x_2$ must be chosen in $[0, x_1]$, and $x_3$ in $[x_2, x+]$, etc.

(3.2a)
$$R_{2m+1}(t) := (-1)^m \int_0^{x_+} r(x_1)dx_1 \int_0^{x_1} r(x_2)dx_2 \int_{x_2}^{x_+} r(x_3)dx_3 \cdots \int_{x_{2m}}^{x_+} p_0(\xi_R(\mathbf{x}, t))r(x_{2m+1})dx_{2m+1}$$

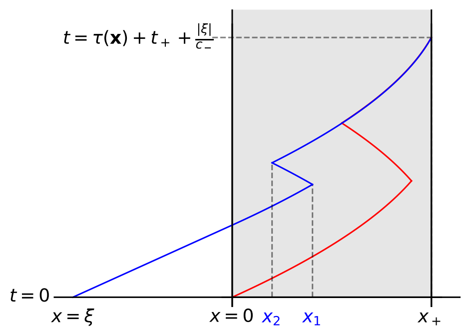
(3.2b)
$$T_{2m}(t) := (-1)^m C_G \int_0^{x_+} r(x_1)dx_1 \int_0^{x_1} r(x_2)dx_2 \int_{x_2}^{x_+} r(x_3)dx_3 \cdots \int_0^{x_{2m-1}} p_0(\xi_T(\mathbf{x}, t))r(x_{2m})dx_{2m}.$$

Here $\xi_R(\mathbf{x}, t)$ is the starting point for the characteristic with reflection points $\mathbf{x}$ arriving eventually at $(x = 0, t)$, while $\xi_T(\mathbf{x}, t)$ is the starting point for the characteristic with reflection points $\mathbf{x}$ arriving eventually at $(x = x_+, t)$. The factor $(-1)^m$ appears because the reflection coefficient for a characteristic initially going left is $-r(x)$, so every even-numbered reflection involves a factor of $-1$. Thus $p_0(\xi)$ gives the initial solution value corresponding to that characteristic path, and the product of reflection characteristics gives the part of that value that eventually contributes to $p(x, t)$. The limits of integration take into account that the reflection points must be an alternating sequence.

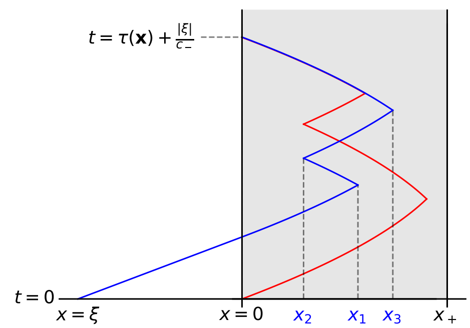
The factor $C_G$ appearing in (3.2b) is due to variation along a characteristic as described by the solution of (2.4). It is absent in (3.2a) because the value of the solution along a characteristic traveling left changes by exactly the reciprocal factor and so there is no net change in amplitude for a characteristic that returns to $x = 0$.

We can compute the full solution (to any desired accuracy) by considering the contributions from all characteristics involving $n = 1, 2, \dots, N$ reflections. To complete this approach we only need to determine how $\xi$ depends on $\mathbf{x}$ and $t$, which we do in the next two subsections.

**3.1. Reflection.** Let us work out the initial location $\xi(\mathbf{x}, t)$ for a characteristic that is eventually reflected. Consider a characteristic passing through $x = 0$ (going to the right) that is subsequently reflected at the points $x_1, x_2, \dots, x_n \in (0, x_+)$ and eventually emerges (going to the left) back at $x = 0$,




(a) Characteristics contributing to $T_2(t)$




(b) Characteristics contributing to $R_3(t)$

FIG. 7. *Characteristic paths starting from different points but arriving simultaneously to contribute to the indicated transmission and reflection terms. For clarity, only the reflection points of the blue trajectories are marked.*

at time $t$ (see Figure 7b). The total time taken for the characteristic to traverse this path is

$$\tau(\mathbf{x}) = 2\sum_{j=1}^{n}(-1)^{j+1}\tau_j, \tag{3.3}$$

where again $\tau_j$ is the travel time from $x = 0$ to $x_j$, and we have used the fact that the travel time from $x_i$ to $x_{i+1}$ is given by $|\tau_{i+1} - \tau_i|$. Thus this characteristic must have first passed through $x = 0$ at time $t - \tau(\mathbf{x})$. Hence it must have originated at time zero from

$$\xi_R(\mathbf{x}, t) = -c_-(t - \tau(\mathbf{x})). \tag{3.4}$$

This holds for all $t \geq \tau(\mathbf{x})$.

We can compute the contribution of all characteristics that are eventually reflected, for any initial condition $p_0$, using (3.2a) with $\xi_R$ given by (3.4).

**3.2. Transmission.** Consider a characteristic starting at $x = 0$ (going to the right) that is reflected at the points $x_1, x_2, \ldots, x_n \in (0, x_+)$ and eventually emerges (going to the right) at $x = x_+$ (see Figure 7a). The total time taken for the characteristic to traverse this path is $\tau(\mathbf{x}) + t_+$. Hence it must have originated at time zero from

$$\xi_T(\mathbf{x}, t) = -c_-(t - \tau(\mathbf{x}) - t_+). \tag{3.5}$$

Because the characteristic path starts at $x \leq 0$ (with impedance $Z_-$) and ends at $x = x_+$ (with impedance $Z_+$), the net change in the value of the solution along this characteristic due to Green's law is given by the factor $C_G$ defined in (2.6). Hence the contribution to the solution is given by $C_G r(x_1) r(x_2) \cdots r(x_n)$, leading to the integral (3.2b) for the total contribution of all characteristics with $n$ reflections.

We can compute the contribution of all characteristics that are eventually transmitted, for any initial condition $p_0$, using (3.2b) with $\xi$ given by (3.5).

**4. Transmission and reflection of a step.** In this section we apply the approach just outlined to the propagation of an initial condition consisting of a step:

$$p_0(x) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0. \end{cases} \tag{4.1}$$

Since the step function is the integral of a $\delta$-function, the resulting solution gives the integral of the Green's function for the problem, and can be used as a basis to obtain solutions for arbitrary initial data.

Straightforward calculation shows that the required values of $\xi$ in this case are simply

$$p_0(\xi_R) = \begin{cases} 0 & t < \tau(\mathbf{x}) \\ 1 & t \ge \tau(\mathbf{x}) \end{cases} \tag{4.2}$$

for reflected components and

$$p_0(\xi_T) = \begin{cases} 0 & t < \tau(\mathbf{x}) + t_+ \\ 1 & t \ge \tau(\mathbf{x}) + t_+ \end{cases} \tag{4.3}$$

for transmitted components, where $\tau$ is defined in (3.3).

The integrals (3.2b) for the transmitted components can thus be written

(4.4)
$$T_{2m}(t) = (-1)^m C_G \int_0^{x_+} \frac{Z'(x_1)}{2Z(x_1)} dx_1 \int_0^{x_1} \frac{Z'(x_2)}{2Z(x_2)} dx_2 \int_{x_2}^{x_+} \frac{Z'(x_3)}{2Z(x_3)} dx_3 \cdots \int_0^{\alpha_{2m}(\mathbf{x},t)} \frac{Z'(x_{2m})}{2Z(x_{2m})} dx_{2m},$$

where the upper limit $\alpha_{2m}(\mathbf{x}, t)$ imposes the condition that the path must reach $x = x_+$ by time $t$.

**4.1. Relation between Green's coefficient and the transmission/reflection coefficients.** Let us consider what happens for long times; let $T_{2m}^{\infty} = \lim_{t\to\infty} T_{2m}(t)$. Then $\alpha_{2m} = x_+$ and it is straightforward but tedious to evaluate the multiple integral (3.2b); the result depends only on $C_G$ and $m$. For each value of $m$, $T_{2m}^{\infty} = (-1)^m a_{2m} C_G (\log(C_G))^{2m}$, where the constants $a_{2m}$ for $m = 1, 2, 3, \ldots$ are

$$1,\ 1/2,\ 5/24,\ 61/720,\ 277/8064,\ 50521/3628800,\ 540553/95800320, \ldots \tag{4.5}$$

We now explain where this sequence comes from.

Let a sequence of reflection points $\mathbf{x}$ and a time $t$ be given. We call the sequence admissible if it is an alternating sequence and $\tau(\mathbf{x}) < t$; we denote the set of admissible paths by $\mathcal{P}_n(t)$:

$$\mathcal{P}_n(t) = \{\mathbf{x} \in [0, x_+]^n : \mathbf{x} \text{ is an alternating sequence and } \tau(\mathbf{x}) \le t\}. \tag{4.6}$$

The above is valid for reflected characteristics ($n$ odd); for transmitted characteristics ($n$ even) the admissible paths are given by $\mathcal{P}_n(t + t_+)$. Then we can summarize the bounds of the integral (4.4) by saying that the integral is over $\mathcal{P}_{2m}(t + t_+)$.

For $t \ge (n + 1)t_+$, all possible (alternating) sequences of reflection points are admissible and $\alpha = x_+$. The integral for $T_{2m}$ can be simplified using the substitution $y(x) = \log(Z(x))/2$. Also let $y_+ = \log(Z_+)/2, y_- = \log(Z_-)/2$. For simplicity we assume that $Z(x)$ is monotone increasing. Then

$$\frac{T_{2m}^{\infty}}{C_G} = (-1)^m \int_{y_-}^{y_+} dy_1 \int_{y_-}^{y_1} dy_2 \int_{y_2}^{y_+} dy_3 \cdots \int_{y_-}^{y_{2m-1}} dy_{2m}. \tag{4.7}$$

Let $n = 2m$; then (ignoring the sign for the moment) this integral is the volume of some subset of the $n$-dimensional hypercube $[y_-, y_+]^n$. It is does not include the full hypercube because the reflection points are required to be an alternating sequence (this requirement is enforced by the limits of integration). Notice that since $Z(x)$ is monotone increasing, this is equivalent to the condition that the sequence $y_1, y_2, \ldots, y_n$ be alternating. The integral in (4.7) gives the volume of the subset of the hypercube that satisfies this alternating condition. The volume of the whole hypercube is of course $(y_+ - y_-)^n = (\log(C_G))^n$.

To determine the value of the integral (4.7), let us partition the hypercube into $n!$ equal parts, where each part is defined by a particular ordering of the $y_j$. For instance, with $n = 4$ we would write

$$V_{ijkl} = \{(y_1, y_2, y_3, y_4) : y_i < y_j < y_k < y_l\},$$

where $(i, j, k, l)$ ranges over all permutations of $(1, 2, 3, 4)$. Each of the sets $V_{ijkl}$ must have the same volume since there is nothing to distinguish a particular coordinate direction. Thus each has volume $(\log(C_G))^n/n!$. The value of the integral (4.7) is determined by how many of the $V_{ijkl}$ satisfy the alternating condition. With $n = 4$ there are 5 alternating sequences:

$$(4, 2, 3, 1),\ (4, 1, 3, 2),\ (3, 2, 4, 1),\ (3, 1, 4, 2),\ (2, 1, 4, 3),$$

so the integral yields $(5/24)(\log(C_G))^4$. In general, the number of alternating sequences of length $n$ is known as the $n$th *Euler zigzag number* (or just *zigzag number*; for even $n$ these are also known as secant numbers or simply zig numbers [1, 6]. We have proved formula (4.9a) of the following theorem. Formula (4.9b) can be proved by a similar argument.

THEOREM 4.1. *Let $Z(x)$ be monotone and define*

$$b_n(z) = \frac{A_n}{n!} z^n \tag{4.8}$$

*where $A_n$ is the $n$th zigzag number; i.e., the number of alternating permutations of a sequence of length $n$. Then the asymptotic contributions for the step satisfy*

$$T_n^\infty = C_G b_n(i \log(C_G)) = C_G \frac{A_n}{n!}(i \log(C_G))^n \qquad \text{for } n \text{ even} \tag{4.9a}$$

$$R_n^\infty = i b_n(i \log(C_G)) = i \frac{A_n}{n!}(i \log(C_G))^n \qquad \text{for } n \text{ odd,} \tag{4.9b}$$

*where $i$ denotes the imaginary unit.*

The name *zigzag* seems eminently appropriate for numbers that appear in the context of Figure 7. Nevertheless, it is worth noting that the original meaning of the name was a reference to zigzags in the discrete setting and had nothing to do with space or characteristics. There are many recursive formulas for the zigzag numbers; in the course of this work we rediscovered the following formula by evaluating the multiple integrals (4.4). Let $a_n = A_n/n!$; then the $a_n$ are generated by setting $a_0 = a_1 = 1$ and computing

$$a_{2m} = \sum_{j=1}^{m} \frac{(-1)^{j-1}}{(2j)!} a_{2(m-j)}$$

$$a_{2m+1} = \sum_{j=1}^{m} \frac{(-1)^{j-1}}{(2j-1)!} a_{2(m-j+1)}.$$

These formulas recover the values (4.5) and the corresponding sequence for the reflection terms. The following well-known result is known as André's Theorem (1881).

THEOREM 4.2 (André's Theorem). *Let $b_n(z)$ be defined by (4.8). Then*

$$\sum_{m=0}^{\infty} b_n(z) = \sec(z) + \tan(z).$$

Comparison of this result with our series (4.9a)-(4.9b) leads immediately to

$$\sum_{n=1}^{\infty} T_n^{\infty} + \sum_{n=1}^{\infty} R_n^{\infty} = C_G \sec(i \log(C_G)) + i \tan(i \log(C_G)). \tag{4.10}$$

Further comparing with the expressions for the transmission and reflection coefficients yields

COROLLARY 4.3. *Let $e^{-\pi} < Z_+/Z_- < e^{\pi}$. Then*

$$\sum_{m=0}^{\infty} T_{2m}^{\infty} = C_T(Z_+, Z_-) = C_G \operatorname{sech}(\log(C_G)), \tag{4.11a}$$

$$\sum_{m=0}^{\infty} R_{2m+1}^{\infty} = C_R(Z_+, Z_-) = \tanh(\log(C_G)). \tag{4.11b}$$

*Proof.* We prove the transmission coefficient part; the proof for the reflection coefficient is similar. From (4.9a) we have

$$\sum_{m=0}^{\infty} T_{2m}^{\infty} = C_G \sum_{m=0}^{\infty} \frac{A_{2m}}{(2m)!} (i \log(C_G))^{2m} = C_G \operatorname{sech}(\log(C_G)).$$

This is the Maclaurin series for $\operatorname{sech}(z)$ with $z = \log(C_G)$; the sequence is convergent for $|z| < \pi/2$, which is equivalent to the condition $e^{-\pi} < Z_+/Z_- < e^{\pi}$. Meanwhile, we can express the transmission coefficient in terms of the Green's coefficient as follows:

$$C_T(Z_+, Z_-) = \frac{2Z_+}{Z_+ + Z_-} = \frac{2C_G^2}{C_G^2 + 1}.$$

Substituting $z = -i \log(C_G)$ (so $C_G = e^{iz}$) we find

$$C_T(Z_+, Z_-) = \frac{2e^{2iz}}{e^{2iz} + 1} = e^{iz} \sec(z) = C_G \sec(-i \log(C_G)) = C_G \operatorname{sech}(\log(C_G)). \qquad \square$$

Corollary 4.3 gives simple expressions for the transmission and reflection coefficients in terms of the Green's coefficient. It also says that if we add up all the long-time asymptotic contributions from paths with any even number of reflections, we obtain the same value given by the transmission coefficient. Similarly, if we add up all contributions from paths with any odd number of reflections, we obtain the same value as the reflection coefficient. Thus the asymptotic state near $x = 0$ for the reflection of the step is just the middle state resulting from the Riemann problem. In fact, Corollary 4.3 could instead be proven directly, using PDE-based arguments to show that the net effect of all terms asymptotically depends only on $Z_+, Z_-$ and so must sum to the traditional transmission and reflection coefficients.

It may seem unnatural to involve the imaginary unit in (4.11) since $T_n^{\infty}$ and $R_n^{\infty}$ are real numbers.

**4.2. Convergence.** It is evident that if our series approximation converges, it converges to the solution of the PDE. The crucial question is whether the series converges at all. We have already seen that in the long-time limit, the series may diverge for large impedance ratios. Nevertheless, in most situations the series seems to give good accuracy with only a small number of terms. In this section we give two convergence results.

The first result shows that the series always converges for any finite time. However, in the worst case the error may grow exponentially in time.

THEOREM 4.4. *Consider problem (1.2) with coefficients (2.1) and initial data (2.2). Let $R_{2m+1}(t)$ and $T_{2m}(t)$ be defined as in (3.2). Then for any time $0 \le t < \infty$,*

$$\lim_{N\to\infty} \sum_{m=0}^{N} R_{2m+1}(t) = p(0,t) - p(-c_- t, 0) \tag{4.12}$$

$$\lim_{N\to\infty} \sum_{m=0}^{N} T_{2m}(t) = p(x_+, t). \tag{4.13}$$

*Furthermore, we have*

$$\left|\sum_{n=0}^{N} R_{2m+1}(t) - (p(0,t) - p(-c_- t, 0))\right| \le M \frac{(\zeta C t)^{2N+2}}{(2N+2)!} \sinh(\zeta C t_*)$$

$$\left|\sum_{n=0}^{N} T_{2m}(t) - p(x_+, t)\right| \le M \frac{(\zeta C t)^{2N+2}}{(2N+2)!} \cosh(\zeta C t_*)$$

*for some $t_* \in [0, t]$, where*

$$C = \max_x |c(x)|$$

$$\zeta = \max_x \frac{|Z'(x)|}{2Z(x)}$$

$$M = \max_x |p_0(x)|.$$

*Proof.* In the limit $n \to \infty$, the series above represent the contribution of all characteristics. Hence, if the series converge then they must converge to the true solution. We now prove that the reflection series converges; the proof for the transmission series is similar.

First, for simplicity take $c(x) = 1$ so that the travel time between two points is just the distance between them, step function initial data (4.1), and $Z(x) = e^{2x}$ so that $r(x) = Z'(x)/(2Z(x)) = 1$. Then for odd $n$

$$|R_n(t)| = \int \cdots \int_{\mathcal{P}_n(t)} d\mathbf{x} = \mathrm{Vol}(\mathcal{P}_n(t)),$$

Because $c(x) = 1$, in this case the set $\mathcal{P}_n(t)$ (defined in (4.6)) is just

$$\mathcal{X}_n(t) := \left\{ \mathbf{x} \in [0, x_+]^n : \mathbf{x} \text{ is an alternating sequence and } 2\sum_j (-1)^{j+1} x_j \le t \right\},$$

the set of alternating sequences with path length at most $t$. In other words, $|R_n(t)|$ is given by the volume of the set $\mathcal{X}_n(t)$. The idea of the proof is that this volume is smaller than $t^n/n!$. Since $\sum_n t^n/n!$ converges, our series converges also.

Define the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_i(\mathbf{x}) = \begin{cases} x_1 & i = 1 \\ x_{i-1} - x_i & \text{for } i \text{ even} \\ x_i - x_{i-1} & \text{for } i > 1 \text{ odd.} \end{cases}$$

This mapping can be represented by a lower-triangular matrix whose diagonal entries are $\pm 1$, so it preserves volume. Note also that

$$\|f(\mathbf{x})\|_1 = \tau(\mathbf{x}) - x_n,$$

and for any alternating sequence $\mathbf{x} \geq 0$ we have $f(\mathbf{x}) \geq 0$. Let $\mathcal{B}^1_{n+}(t)$ denote the intersection of the $n$-dimensional $L_1$ ball of radius $t$ with the positive orthant:

$$\mathcal{B}^1_{n+}(t) = \{x \in [0,\infty)^n : \|x\|_1 \leq t\}.$$

For any $\mathbf{x} \in \mathcal{X}_n(t)$, we have $f(\mathbf{x}) \in \mathcal{B}^1_{n+}(t)$, so

$$|R_n(t)| = \text{Vol}(\mathcal{X}_n(t)) = \text{Vol}(f(\mathcal{X}_n(t))) \leq \text{Vol}(\mathcal{B}^1_{n+}(t)) = \frac{t^n}{n!}.$$

The value of the last integral is a classical result due to Dirichlet [3, p. 168]. So

$$\sum_{m=0}^{\infty} |R_{2m+1}(t)| \leq \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(2m+1)!} = \sinh(t).$$

To extend the proof to arbitrary $c(x)$, let $C = \max_x |c(x)|$. Then the length of a characteristic path emerging at time $t$ is no greater than $Ct$, so $\mathcal{P}_n(t) \subset \mathcal{X}_n(Ct)$. Thus

$$|R_n(t)| = \text{Vol}(\mathcal{P}_n(t)) \leq \text{Vol}(\mathcal{X}_n(Ct)) \leq \text{Vol}(\mathcal{B}^1_{n+}(Ct)) = \frac{(Ct)^n}{n!}.$$

To extend the proof to arbitrary $Z(x)$ and initial data $p_0(x)$, let $\zeta = \max_x |r(x)|$ and $M = \max_x |p_0(x)|$. Then

$$\begin{aligned}
|R_n(t)| &= \left| \int \cdots \int_{\mathcal{P}_n(t)} p_0(\xi_R(\mathbf{x},t)) \prod_j r(x_j) d\mathbf{x} \right| \\
&\leq \int \cdots \int_{\mathcal{P}_n(t)} |p_0(\xi_R(\mathbf{x},t))| \prod_j |r(x_j)| d\mathbf{x} \\
&\leq M \int \cdots \int_{\mathcal{P}_n(t)} \prod_j \zeta d\mathbf{x} \\
&= M\zeta^n \,\text{Vol}(\mathcal{P}_n(t)) \\
&\leq M\zeta^n \,\text{Vol}(\mathcal{B}^1_{n+}(Ct)) = M\frac{(\zeta C t)^n}{n!}.
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} |R_{2m+1}(t)| \leq M \sinh(\zeta C t).$$

The error bounds in the theorem then follow from Taylor's theorem. ∎

The error estimate given in the theorem above is typically too large to be useful. As we will see in the examples of Section 4.3, the series often converges much faster. The next theorem gives an example of conditions under which more rapid convergence can be guaranteed.

THEOREM 4.5. *Consider problem* (1.2) *with coefficients* (2.1) *and unit step function initial data* (4.1). *Let* $R_n(t)$ *and* $T_n(t)$ *be defined as in* (3.2). *Let* $Z(x)$ *be monotone with* $e^{-2\sqrt{2}} < Z_+/Z_- < e^{2\sqrt{2}}$. *Then for any time* $0 \leq t < \infty$ *we have*

$$\lim_{N\to\infty} \sum_{m=1}^{N} R_{2m+1}(t) = p(0,t) - p(-c_- t, 0) \tag{4.14}$$

$$\lim_{N\to\infty} \sum_{m=1}^{N} T_{2m}(t) = p(x_+, t). \tag{4.15}$$

*Furthermore, the terms $|R_{2m+1}(t)|$ and $|T_{2m}(t)|$ decrease monotonically with $m$ and the approximation error can be bounded as follows:*

$$\left|\sum_{m=0}^{N-1} R_{2m+1}(t) - (p(0,t) - p(-c_-t, 0))\right| \leq |R_{2N+1}(t)| \leq \left(\frac{C_G^2}{2}\right)^N |R_1(t)| \tag{4.16}$$

$$\left|\sum_{m=0}^{N-1} T_{2m}(t) - p(x_+, t)\right| \leq |T_{2N}(t)| \leq \left(\frac{C_G^2}{2}\right)^N |T_0(t)|. \tag{4.17}$$

*Proof.* Since in the limit $n \to \infty$ we are accounting for all characteristics, then if the limit exists it must be equal to the solution value. From (3.2) we see that if $Z(x)$ is monotone then the series $R_{2m+1}$ and $T_{2m}$ are alternating series (i.e., successive terms in each series have opposite sign). It is sufficient to prove that the terms $|R_{2m+1}(t)|$ and $|T_{2m}(t)|$ decrease monotonically with $m$; then the rest of the theorem follows from standard results for alternating series. We prove convergence of the transmission series $T_{2m}(t)$. The proof for the reflection series is similar. For simplicity, we consider the case in which $Z(x)$ is increasing.

Let $m$ and $t$ be fixed and let $Z(x)$ be as stated in the Theorem. As discussed already $T_{2m}(t)$ is given by integrating over $\mathcal{P}_{2m}(t + t_+)$. For clarity, in the remainder of the proof we write $\mathcal{P}_{2m}$ with no argument; it is implicitly $t + t_+$.

$$\begin{aligned}
|T_{2m}(t)| &= \left|C_G \int\int \cdots \int_{\mathbf{x}\in\mathcal{P}_{2m}} \prod_{j=1}^{2m} r(x_j)dx_j\right| \\
&= |C_G| \int\int \cdots \int_{\mathbf{x}\in\mathcal{P}_{2m}} \prod_{j=1}^{2m} |r(x_j)|dx_j.
\end{aligned}$$

The second equality holds because, since $Z(x)$ is monotone, the integrand has the same sign for all paths. This also means that if $\mathcal{P}_{2m}$ is replaced by a larger set of paths, the resulting integral provides an upper bound on $|T_{2m}(t)|$.

Notice that every path in $\mathcal{P}_{2m+2}$ can be obtained in exactly one way by taking a particular path in $\mathcal{P}_{2m}$ and appending two (admissible) reflection points $x_{2m+1}, x_{2m+2}$. Admissibility of the resulting path involves a restriction in the total path length (travel time $\tau(\mathbf{x}) \leq t$) and the condition that $x_{2m+1} \geq \max(x_{2m}, x_{2m+2})$. Let us consider the larger set $\widehat{\mathcal{P}}_{2m+2}$ obtained by omitting the path length restriction and requiring only that $x_{2m+1} \geq x_{2m+2}$. In other words, $\widehat{\mathcal{P}}_{2m+2}$ is obtained by appending, for each path in $\mathcal{P}_{2m}$, all pairs $(x_{2m+1}, x_{2m+2})$ such that $0 \leq x_{2m+1} \leq x_{2m+2} \leq x_+$. Clearly $\mathcal{P}_{2m+2} \subset \widehat{\mathcal{P}}_{2m+2}$, so we have

$$\begin{aligned}
|T_{2m+2}(t)| &= \left|C_G \int\int \cdots \int_{\mathbf{x}\in\mathcal{P}_{2m+2}} \prod_{j=1}^{2m+2} r(x_j)dx_j\right| \\
&\leq \left|C_G \int\int \cdots \int_{\mathbf{x}\in\widehat{\mathcal{P}}_{2m+2}} \prod_{j=1}^{2m+2} r(x_j)dx_j\right| \\
&= |C_G| \int\int \cdots \int_{\mathbf{x}\in\widehat{\mathcal{P}}_{2m+2}} \prod_{j=1}^{2m+2} |r(x_j)|dx_j \\
&= |T_{2m}(t)| \int_0^{x_+} \int_{x_{m+2}}^{x_+} r(x_{2m+2})r(x_{2m+1})dx_{2m+1}dx_{2m+2} \\
&= |T_{2m}(t)| \cdot \frac{1}{2}C_G^2.
\end{aligned}$$

Since $|\log(Z_+/Z_-)| < 2\sqrt{2}$, we have $\frac{1}{2}C_G^2 < 1$, so $|T_{2m+2}(t)| < |T_{2m}(t)|$, so the alternating series is convergent. □

**4.3. Examples.** In this section we illustrate, through numerical examples, the method just proposed. We take $x_+ = 1$ in all examples. In the first three examples we take the functions $c(x), Z(x)$ to be linear in the interval $(0,1)$:

$$(c(x), Z(x)) = \begin{cases} (c_-, Z_-) & x < 0 \\ ((1-x)c_- + xc_+, (1-x)Z_- + xZ_+) & 0 \le x \le x_+ \\ (c_+, Z_+) & x > x_+. \end{cases} \tag{4.18}$$

Let $s = (c_+ - c_-)/x_+$. Then a right-going characteristic starting from $x = 0$ at $t = 0$ satisfies the ODE

$$X'(t) = c(x) = \frac{(x_+ - x)c_- + xc_+}{x_+} \qquad X(0) = 0, \tag{4.19}$$

with solution

$$X(t) = \frac{c_-}{s}(e^{st} - 1). \tag{4.20}$$

The total time for an unreflected characteristic to cross from $x = 0$ to $x = x_+$ is thus

$$t_+ = \frac{1}{s}\log\left(\frac{s}{c_-}x_+ + 1\right).$$

For each example, we show the solution corresponding to an initial step function ($p_0(x) = 1$ for all $x < 0$) and a square wave ($p_0(x) = 1$ for $-1 < x < 0$). A first example, with very mild variation in $Z$, is shown in Figure 8. The solution involving only terms up to $T_2$ is already highly accurate. In the second example, shown in Figure 9, $Z$ varies by a factor of 8. In this case it can be seen that the approximation using terms up to $T_4$ gives a significant improvement.

Both of the previous examples satisfy the conditions given in Theorem 4.4. The next two examples do not. In the third example, we take $Z_- = 1$ and $Z_+ = 20$. It can be seen that in this case the convergence for large times is much slower and the series including terms up to $T_4$ is a good approximation only for short times.

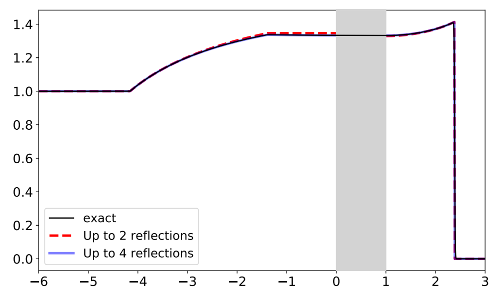
In the final example, $Z(x)$ is non-monotone:

$$Z(x) = 0.25 + 0.75x + \sin(10\pi x)/10.$$

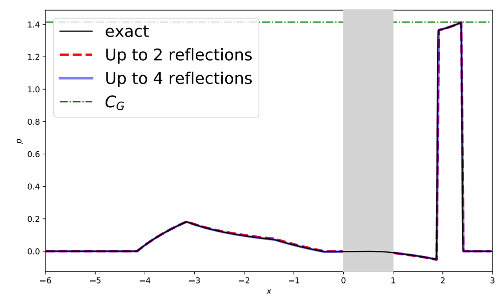
The solution given by including terms up to $T_4$ captures the oscillating solution well. This example also illustrates that when $Z(x)$ is a non-monotone function, the transmitted wave amplitude can exceed $C_G$ at some points.

**5. Conclusions.** We have shown how the method of characteristics can be used to solve the initial value problem for the wave equation in one space dimension in the presence of a region of continuously-varying coefficients. This can be extended in a straightforward way to other linear hyperbolic systems in one dimension.

It is natural to ask whether the method developed in this work is a practical computational tool. It seems that in most situations, it is less computationally expensive to obtain an accurate solution through numerical discretization of the wave equation than through the approach outlined here, since the evaluation of $R_n(t)$ or $T_n(t)$ requires an $n$-dimensional quadrature. The method developed here has the advantage that the solution at any desired time can be computed directly, without requiring the computation of solutions at earlier times; this might make it advantageous in some circumstances. We

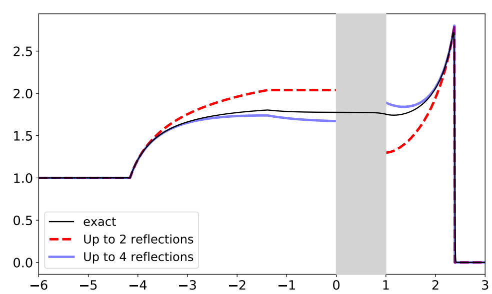



(a) Step function initial condition.

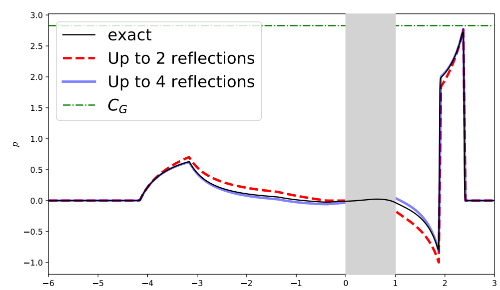



(b) Square wave initial condition.

FIG. 8. *Solution at $t = 3t_+$. Here $x_+ = 1$, $c_- = 2$, $c_+ = 1$, $Z_- = 1/2$, and $Z_+ = 1$. The solution is captured well by considering only two reflections.*

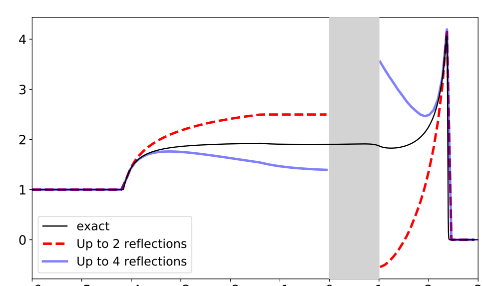



(a) Step function initial condition.

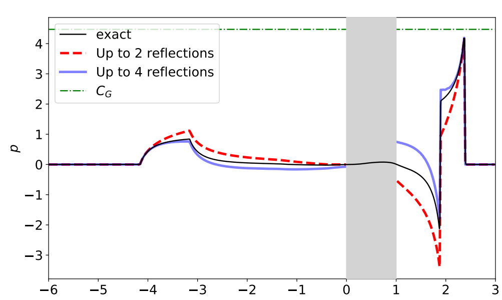



(b) Square wave initial condition.

FIG. 9. *Solution at $t = 3t_+$. Here $x_+ = 1$, $c_- = 2$, $c_+ = 1$, $Z_- = 1/8$, and $Z_+ = 1$. Using more reflections improves the accuracy of both the transmitted and reflected approximations.*

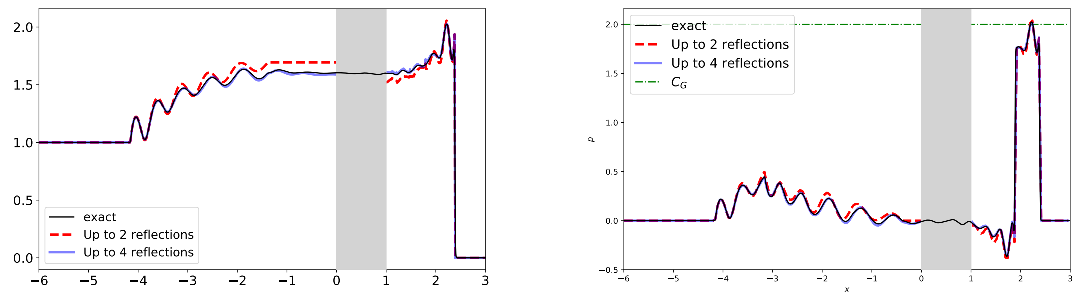



(a) Step function initial condition.




(b) Square wave initial condition.

FIG. 10. *Solution at $t = 3t_+$. Here $x_+ = 1$, $c_- = 2$, $c_+ = 1$, $Z_- = 1$, and $Z_+ = 20$.*




(a) Step function initial condition. (b) Square wave initial condition.

FIG. 11. *Solution at $t = 3t_+$. Here $x_+ = 1$, $c_- = 2$, $c_+ = 1$, $Z_- = 1$, and $Z_+ = 1/4$. In the shaded region, $Z(x) = 0.25 + 0.75x + \sin(10\pi x)/10$.*

have not investigated techniques for reducing the computational cost or made any careful comparisons. Moreover, these results elucidate the relation between transmission and reflection coefficients expected in the limiting case of a sharp interface, with the Greens law behavior expected for sufficiently smooth transitions in material properties.

It is natural to expect that the series (3.1) may converge because characteristics that undergo many reflections contribute in successively smaller amounts to the solution. Examining (3.2), this viewpoint makes sense only if $|r(x)| < 1$. However, our examples and analysis show that (3.1) converges quite independently of any such condition. Theorem 4.4 indicates that in general (3.1) converges for a completely different reason: the number of contributing characteristics (more precisely, the volume they occupy in an appropriate space) becomes vanishingly small as $n \to \infty$.

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