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ON SOME DETERMINANTS INVOLVING THE TANGENT FUNCTION

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Abstract. Let p be an odd prime and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. In this paper we mainly evaluate

$$T_p^{(\delta)}(a,b):=\det\left[\tan\pi\frac{aj^2+bk^2}{p}\right]_{\delta\leqslant j,k\leqslant (p-1)/2}\quad(\delta=0,1).$$

For example, in the case $p\equiv 3\pmod 4$ we show that $T_p^{(1)}(a,b)=0$ and

$$T_p^{(0)}(a,b)=\begin{cases}2^{(p-1)/2}p^{(p+1)/4} & \text{if } (\frac{ab}{p})=1,\\ p^{(p+1)/4} & \text{if } (\frac{ab}{p})=-1,\end{cases}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. When $(\frac{-ab}{p})=-1$, we also evaluate the determinant $\det[\cot\pi\frac{aj^2+bk^2}{p}]_{1\leqslant j,k\leqslant (p-1)/2}$. We also pose several conjectures one of which states that the class number of the quadratic field $\mathbb{Q}(\sqrt{p^*})$ with $p^*=(-1)^{(p-1)/2}p$ is equal to

$$\left(\frac{-2}{p}\right)2^{-(p-3)/2}p^{-(p-5)/4}\det\left[\cot\pi\frac{jk}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}.$$

1. Introduction

Let p be an odd prime. It is well known that the numbers

$$0^2,\ 1^2,\ \dots,\ \left(\frac{p-1}{2}\right)^2$$

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are pairwise incongruent modulo p. In [S19] the author investigated the determinants

$$S(d,p)=\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant (p-1)/2}$$

$$T(d,p)=\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant (p-1)/2},$$

where d is an integer not divisible by p, and $(\frac{\cdot}{p})$ is the Legendre symbol. In particular, Sun [S19] showed that if $(\frac{d}{p})=1$ then

$$\left(\frac{-S(d,p)}{p}\right)=1 \text{ and } T(d,p)=\frac{p-1}{2}S(d,p).$$

Recall that the tangent function $\tan x$ has period π. For $a,b\in\mathbb{Z}$ we define

$$T_p^{(0)}(a,b):=\det\left[\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2} \tag{1.1}$$

and

$$T_p^{(1)}(a,b):=\det\left[\tan\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}. \tag{1.2}$$

In this paper we aim to evaluate the determinants $T_p^{(0)}(a,b)$ and $T_p^{(1)}(a,b)$.

Now we present our main results.

Theorem 1.1. *Let p be an odd prime and let $a,b\in\mathbb{Z}$ with $p\nmid ab$.*

(i) *Assume that $p\equiv 1 \pmod 4$. Then*

$$T_p^{(0)}(a,b)=0. \tag{1.3}$$

If $(\frac{ab}{p})=1$ and $b\equiv ac^2 \pmod p$ with $c\in\mathbb{Z}$, then

$$T_p^{(1)}(a,b)=\left(\frac{2c}{p}\right)p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})(2-(\frac{2}{p}))h(p)}, \tag{1.4}$$

where ε_p and $h(p)$ are the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$. When $(\frac{ab}{p})=-1$, we have

$$T_p^{(1)}(a,b)=\pm 2^{(p-1)/2}p^{(p-3)/4}. \tag{1.5}$$

(ii) *Suppose that $p\equiv 3 \pmod 4$. Then*

$$T_p^{(1)}(a,b)=0. \tag{1.6}$$

Also,

$$T_p^{(0)}(a,b)=\begin{cases}2^{(p-1)/2}p^{(p+1)/4} & \text{if } (\frac{ab}{p})=1,\\ p^{(p+1)/4} & \text{if } (\frac{ab}{p})=-1.\end{cases} \tag{1.7}$$

Theorem 1.2. *Let $n > 1$ be an odd integer and let a and b be integers with* $\gcd(ab, n) = 1$. *Then*

$$\det\left[\tan\pi\frac{aj+bk}{n}\right]_{0\leqslant j,k\leqslant n-1} = 0 \tag{1.8}$$

and

$$\det\left[\tan\pi\frac{aj+bk}{n}\right]_{1\leqslant j,k\leqslant n-1} = \left(\frac{-ab}{n}\right)n^{n-2}, \tag{1.9}$$

where $(\frac{\cdot}{n})$ is the Jacobi symbol.

Theorem 1.3. *Let $p > 3$ be a prime, and let $a, b \in \mathbb{Z}$ with $(\frac{-ab}{p}) = -1$. Then*

$$\begin{aligned}&\det\left[\cot\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\\ =&\begin{cases}\pm 2^{(p-1)/2}/\sqrt{p} & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{(h(-p)+1)/2}(\frac{a}{p})2^{(p-1)/2}/\sqrt{p} & \text{if } p\equiv 3 \pmod 4,\end{cases}\end{aligned} \tag{1.10}$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Remark 1.1. It is known that $2 \nmid h(-p)$ for each prime $p \equiv 3 \pmod 4$. In 1961 L. J. Mordell [M61] even proved that for any prime $p > 3$ with $p \equiv 3 \pmod 4$ we have

$$\frac{p-1}{2}! \equiv (-1)^{(h(-p)+1)/2} \pmod p.$$

We are going to provide several lemmas in the next section and then prove Theorem 1.1 in Section 3. Theorems 1.2 and 1.3 will be shown in Section 4. In Section 5, we pose some conjectures on determinants involving the tangent function.

2. Some Lemmas

Lemma 2.1. *Let A be the matrix $[a_{jk}]_{0\leqslant j,k\leqslant n}$ with a_{jk} complex numbers for all $i, j = 0, \ldots, n$. Then*

$$\det[x + a_{jk}]_{0\leqslant jk\leqslant n} = \det A + x\det B, \tag{2.1}$$

where $B = [b_{jk}]_{1\leqslant j,k\leqslant n}$ with $b_{jk} = a_{jk} - a_{j0} - a_{0k} + a_{00}$.

Proof. As $(x+a_{jk})-(x+a_{0k})=a_{jk}-a_{0k}$ for all $0<j\leqslant n$ and $0\leqslant k\leqslant n$, we have

$$\det[x+a_{jk}]_{0\leqslant j,k\leqslant n}=\begin{vmatrix} x+a_{00} & x+a_{01} & x+a_{02} & \dots & x+a_{0n} \\ a_{10}-a_{00} & a_{11}-a_{01} & a_{12}-a_{02} & \dots & a_{1n}-a_{0n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0}-a_{00} & a_{n1}-a_{01} & a_{n2}-a_{02} & \dots & a_{nn}-a_{0n} \end{vmatrix}$$

$$=\begin{vmatrix} x & x & x & \dots & x \\ a_{10}-a_{00} & a_{11}-a_{01} & a_{12}-a_{02} & \dots & a_{1n}-a_{0n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0}-a_{00} & a_{n1}-a_{01} & a_{n2}-a_{02} & \dots & a_{nn}-a_{0n} \end{vmatrix}$$

$$+\begin{vmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} \\ a_{10}-a_{00} & a_{11}-a_{01} & a_{12}-a_{02} & \dots & a_{1n}-a_{0n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0}-a_{00} & a_{n1}-a_{01} & a_{n2}-a_{02} & \dots & a_{nn}-a_{0n} \end{vmatrix}.$$

and hence $\det[x+a_{jk}]_{0\leqslant j,k\leqslant n}-\det A$ coincides with

$$\begin{vmatrix} x & 0 & \dots & 0 \\ a_{10}-a_{00} & a_{11}-a_{01}-(a_{10}-a_{00}) & \dots & a_{1n}-a_{0n}-(a_{10}-a_{00}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}-a_{00} & a_{n1}-a_{01}-(a_{n0}-a_{00}) & \dots & a_{nn}-a_{0n}-(a_{n0}-a_{00}) \end{vmatrix}=x\det B.$$

This concludes the proof of (2.1). □

Corollary 2.1. *Let m and n be positive integers with $2\nmid n$. Let $f(x):\mathbb{R}\to\mathbb{R}$ be an odd function, where $\mathbb{R}$ is the field of real numbers. Then, for any integer d, the determinant*

$$\det\left[x+f((j+d)^m-(k+d)^m)\right]_{0\leqslant j,k\leqslant n}$$

does not depend on x.

Proof. Let

$$a_{jk}=f((j+d)^m-(k+d)^m)\quad \text{for } j,k=0,\dots,n.$$

For $1\leqslant j,k\leqslant n$ set $b_{jk}=a_{jk}-a_{j0}-a_{0k}+a_{00}$. As f is an odd function, we have

$$b_{jk}=f((j+d)^m-(k+d)^m)-f((j+d)^m-d^m)-f(d^m-(k+d)^m)=-b_{kj}$$

Thus

$$\det[b_{jk}]_{1\leqslant j,k\leqslant n}=(-1)^n\det[b_{kj}]_{1\leqslant j,k\leqslant n}=-\det[b_{jk}]_{1\leqslant j,k\leqslant n}$$

and hence $\det[b_{jk}]_{1\leqslant j,k\leqslant n}=0$. Applying Lemma 2.1, we immediately get the desired result. □

The following lemma is a known result (cf. [K05, (5.5)]).

Lemma 2.2. *We have*

$$\det\left[\frac{1}{x_j+y_k}\right]_{1\leqslant j,k\leqslant n}=\frac{\prod_{1\leqslant j<k\leqslant n}(x_k-x_j)(y_k-y_j)}{\prod_{j=1}^{n}\prod_{k=1}^{n}(x_j+y_k)}. \tag{2.2}$$

Lemma 2.3 (Pan [P06])**.** *Let $n>1$ be an odd integer and let c be any integer relatively prime to n. For each $j=1,\ldots,(n-1)/2$ let $\pi_c(j)$ be the unique $r\in\{1,\ldots,(n-1)/2\}$ with cj congruent to r or $-r$ modulo n. For the permutation π_c on $\{1,\ldots,(n-1)/2\}$, its sign is given by*

$$\mathrm{sign}(\pi_c)=\left(\frac{c}{n}\right)^{(n+1)/2}. \tag{2.3}$$

Lemma 2.4 (Sun [S18])**.** *Let p be an odd prime. Let $\zeta=e^{2\pi i/p}$ and $a\in\mathbb{Z}$ with $p\nmid a$.*

(i) *If $p\equiv 1 \pmod 4$, then*

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}+\zeta^{ak^2})=\pm\varepsilon_p^{(\frac{a}{p})h(p)((\frac{2}{p})-1)/2} \tag{2.4}$$

and

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}-\zeta^{ak^2})^2=(-1)^{(p-1)/4}p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})h(p)}, \tag{2.5}$$

where ε_p and $h(p)$ are the fundamental units of the real quadratic field $\mathbb{Q}(\sqrt{p})$.

(ii) *Suppose that $p\equiv 3 \pmod 4$. Then*

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}+\zeta^{ak^2})=1, \tag{2.6}$$

and

$$\begin{aligned}&\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}-\zeta^{ak^2})\\ =&\begin{cases}(-p)^{(p-3)/8} & \text{if } p\equiv 3 \pmod 8,\\ (-1)^{(p+1)/8+(h(-p)-1)/2}(\frac{a}{p})p^{(p-3)/8}i & \text{if } p\equiv 7 \pmod 8.\end{cases}\end{aligned} \tag{2.7}$$

Also,

$$\prod_{k=1}^{(p-1)/2}(1-\zeta^{ak^2})=(-1)^{(h(-p)+1)/2}\left(\frac{a}{p}\right)\sqrt{p}\,i. \tag{2.8}$$

Lemma 2.5. *Let p be an odd prime and let $a,b\in\mathbb{Z}$ with $(\frac{-ab}{p})=-1$. Then*

$$\begin{aligned}&\prod_{j=1}^{(p-1)/2}\prod_{k=1}^{(p-1)/2}\left(1-e^{2\pi i(aj^2+bk^2)/p}\right)\\ =&p^{(p-1)/4}\times\begin{cases}1&\text{if } p\equiv 1\pmod 4,\\ (-1)^{(h(-p)-1)/2}(\frac{a}{p})i&\text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}\tag{2.9}$$

Proof. For $m\in\mathbb{Z}$ set

$$\begin{aligned}r(m):=&\left|\left\{(j,k):\ 1\leqslant j,k\leqslant\frac{p-1}{2}\text{ and } aj^2+bk^2\equiv m\pmod p\right\}\right|\\ =&\left|\left\{1\leqslant x\leqslant p-1:\ \left(\frac{x}{p}\right)=1\text{ and }\left(\frac{m-ax}{p}\right)=\left(\frac{b}{p}\right)\right\}\right|.\end{aligned}$$

Note that $r(0)=0$ since $(\frac{-ab}{p})\neq 1$.

Let $m\in\{1,\ldots,p-1\}$. Then

$$\begin{aligned}r(m)=&\sum_{\substack{0<x<p\\ p\nmid ax-m}}\frac{(\frac{x}{p})+1}{2}\cdot\frac{(\frac{b(m-ax)}{p})+1}{2}\\ =&\frac14\sum_{x=1}^{p-1}\left(\left(\frac{bx(m-ax)}{p}\right)+\left(\frac{x}{p}\right)+\left(\frac{b(m-ax)}{p}\right)+1\right)-\frac{(\frac{am}{p})+1}{4}\\ =&\frac14\sum_{x=0}^{p-1}\left(\frac{-abx^2+bmx}{p}\right)+\frac14\sum_{x=0}^{p-1}\left(\frac{x}{p}\right)+\frac14\sum_{x=0}^{p-1}\left(\frac{-abx+bm}{p}\right)\\ &-\frac14\left(\frac{bm}{p}\right)+\frac{p-1}{4}-\frac{(\frac{am}{p})+1}{4}\\ =&\frac14\sum_{x=0}^{p-1}\left(\frac{-abx^2+bmx}{p}\right)+\frac{p-1}{4}-\frac{(\frac{am}{p})+(\frac{bm}{p})+1}{4}.\end{aligned}$$

It is well known that for any $a_0,a_1,a_2\in\mathbb{Z}$ with $p\nmid a_0$ we have

$$\sum_{x=0}^{p-1}\left(\frac{a_0x^2+a_1x+a_2}{p}\right)=\begin{cases}-(\frac{a_0}{p})&\text{if } p\nmid a_1^2-4a_0a_2,\\ (p-1)(\frac{a_0}{p})&\text{if } p\mid a_1^2-4a_0a_2.\end{cases}\tag{2.10}$$

(See, e.g., [BEW, p. 58].) Therefore

$$r(m)=-\frac14\left(\frac{-ab}{p}\right)+\frac{p-1}{4}-\frac{(\frac{am}{p})+(\frac{bm}{p})+1}{4}=\frac{p-1}{4}-\frac{1-(\frac{-1}{p})}{4}\left(\frac{am}{p}\right).$$

In view of the above,

$$\prod_{j=1}^{(p-1)/2}\prod_{k=1}^{(p-1)/2}\left(1-e^{2\pi i(aj^2+bk^2)/p}\right)$$
$$=\prod_{m=1}^{p-1}(1-e^{2\pi im/p})^{r(m)}=\frac{\prod_{m=1}^{p-1}(1-e^{2\pi im/p})^{(p-1+(\frac{a}{p})(1-(\frac{-1}{p})))/4}}{\prod_{\substack{0<m<p\\(\frac{m}{p})=1}}(1-e^{2\pi im/p})^{(\frac{a}{p})(1-(\frac{-1}{p}))/2}}.$$

Clearly,

$$\prod_{m=1}^{p-1}(1-e^{2\pi im/p})=\lim_{x\to1}\frac{x^p-1}{x-1}=p.$$

In view of (2.8),

$$\prod_{\substack{0<m<p\\(\frac{m}{p})=1}}(1-e^{2\pi im/p})^{(1-(\frac{-1}{p}))/2}=\begin{cases}1 & \text{if } p\equiv1\pmod 4,\\ (-1)^{(h(-p)+1)/2}\sqrt{p}\,i & \text{if } p\equiv3\pmod 4.\end{cases}$$

Thus the desired (2.9) follows. □

3. Proof of Theorem 1.1

We can easily verify the desired results for $p=3$. Below we assume that $p>3$. For convenience, we set $n=(p-1)/2$ and $\zeta:=e^{2\pi i/p}$. Since

$$\sum_{k=0}^{n}k^2=\frac{n(n+1)(2n+1)}{6}=\frac{p^2-1}{24}p\equiv0\pmod p,$$

we have

$$\prod_{k=0}^{n}\zeta^{k^2}=1. \tag{3.1}$$

As

$$\tan x=\frac{2\sin x}{2\cos x}=\frac{(e^{ix}-e^{-ix})/i}{e^{ix}+e^{-ix}}=\frac{-i(e^{2ix}-1)}{e^{2ix}+1}=-i+\frac{2i}{e^{2ix}+1},$$

we also have

$$i+\tan\pi\frac{aj^2+bk^2}{p}=\frac{2i}{\zeta^{aj^2+bk^2}+1}\quad\text{for all } j,k=0,\dots,n. \tag{3.2}$$

By Lemma 2.3, for each $\delta\in\{0,1\}$ and integer $d\not\equiv0\pmod p$, we have

$$T_p^{(\delta)}(a,\pm ad^2)=\left(\frac{d}{p}\right)^{n+1}T_p^{(\delta)}(a,\pm a). \tag{3.3}$$

Proof of the First Part of Theorem 1.1. As $p\equiv 1 \pmod 4$, we have $2\mid n$. For $q=((p-1)/2)!$ we have $q^2\equiv -1 \pmod p$ by Wilson's theorem, hence

$$\begin{aligned}-T_p^{(0)}(a,b)=&\det\left[-\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}\\ =&\det\left[\tan\pi\frac{a(qj)^2+b(qk)^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}=T_p^{(0)}(a,b)\end{aligned}$$

and thus $\det T_p^{(0)}(a,b)=0$.

If $(\frac{ab}{p})=1$ and $b\equiv ac^2 \pmod p$ with $c\in\mathbb{Z}$, then $b\equiv -a(qc)^2 \pmod p$ and hence

$$T_p^{(1)}(a,b)=\left(\frac{2c}{p}\right)T_p^{(1)}(a,-a)$$

by (3.3) and the equality $(\frac{q}{p})=(\frac{2}{p})$ (cf. [S19, Lemma 2.3]).

By Corollary 2.1,

$$\begin{aligned}&\det\left[x+\tan\pi\frac{aj^2-ak^2}{p}\right]_{1\leqslant j,k\leqslant n}\\ =&\det\left[x+\tan\pi\frac{a(j+1)^2-a(k+1)^2}{p}\right]_{0\leqslant j,k\leqslant n-1}\end{aligned}$$

does not depend on x. So, with the help of (3.2), we get

$$\begin{aligned}T_p^{(1)}(a,-a)=&\det\left[i+\tan\pi\frac{aj^2-ak^2}{p}\right]_{1\leqslant j,k\leqslant n}\\ =&\det\left[\frac{2i}{e^{2\pi ia(j^2-k^2)/p}+1}\right]_{1\leqslant j,k\leqslant n}\\ =&\prod_{k=1}^n(2i\zeta^{ak^2})\times\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{1\leqslant j,k\leqslant n}.\end{aligned}$$

In light of Lemma 2.2,

$$\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{1\leqslant j,k\leqslant n}=\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{k=1}^n(\zeta^{ak^2}+\zeta^{ak^2})\times\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}.$$

Therefore,

$$\begin{aligned}T_p^{(1)}(a,-a)=&i^n\prod_{1\leqslant j<k\leqslant n}\left(\frac{\zeta^{ak^2}-\zeta^{aj^2}}{\zeta^{ak^2}+\zeta^{aj^2}}\right)^2\\ =&(-1)^{(p-1)/4}\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}=p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})h(p)(2-(\frac{2}{p}))}\end{aligned}$$

with the help of Lemma 2.4(i).

Now suppose that $(\frac{ab}{p}) = -1$. Clearly, $T_p^{(1)}(a,b) = \det[c_{jk}]_{1\leqslant j,k\leqslant n}$ with $c_{jk} = \tan\pi(aj^2+bk^2)/p$. By Lemma 2.1,

$$\det[x+c_{jk}]_{1\leqslant j,k\leqslant n} = T_p^{(1)}(a,b) + x\det[d_{jk}]_{1<j,k\leqslant n}, \tag{3.4}$$

where $d_{jk} = c_{jk} - c_{j1} - c_{1k} + c_{11}$. In light of (3.1),

$$\det\left[\frac{2i}{\zeta^{aj^2+bk^2}+1}\right]_{1\leqslant j,k\leqslant n} = \det[i+c_{jk}]_{1\leqslant j,k\leqslant n} = T_p^{(1)}(a,b) + i\det[d_{jk}]_{1<j,k\leqslant n}$$

and hence (1.5) is implied by

$$D_p(a,b) := \det\left[\frac{2i}{\zeta^{aj^2+bk^2}+1}\right]_{1\leqslant j,k\leqslant n} = \pm 2^{(p-1)/2}p^{(p-3)/4}. \tag{3.5}$$

(Note that both $T_p^{(1)}(a,b)$ and $\det[d_{jk}]_{1<j,k\leqslant n}$ are real numbers.)

With the help of Lemma 2.2,

$$\begin{aligned}
D_p(a,b) &= \prod_{k=1}^n\left(\frac{2i}{\zeta^{bk^2}}\right)\times\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n}\\
&=\frac{(2i)^n}{\prod_{k=1}^n\zeta^{bk^2}}\cdot\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2}+\zeta^{-bk^2})}\\
&=(-1)^{(p-1)/4}2^{(p-1)/2}\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}+1)}.
\end{aligned}$$

Note that

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2(\zeta^{-bk^2}-\zeta^{-bj^2})^2 = p^{(p-3)/2}\varepsilon_p^{((\frac{a}{p})+(\frac{-b}{p}))h(p)} = p^{(p-3)/2}$$

by (2.5), and

$$\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}+1) = \prod_{j=1}^n\prod_{k=1}^n\frac{1-\zeta^{2aj^2+2bk^2}}{1-\zeta^{aj^2+bk^2}} = 1$$

by Lemma 2.5. Therefore (3.5) holds and hence so does (1.5).

In view of the above, we have completed the proof of part (i) of Theorem 1.1. □

Proof of the Second Part of Theorem 1.1. As $p\equiv 3 \pmod 4$, we have $2\nmid n$.

If $(\frac{ab}{p})=-1$, then $b\equiv -ad^2 \pmod p$ for some integer $d\not\equiv 0 \pmod p$ and hence by (2.9) we have $T_p^{(0)}(a,b)=T_p^{(0)}(a,-a)$ and $T_p^{(1)}(a,b)=T_p^{(1)}(a,-a)$. Note that $T_p^{(1)}(a,-a)=0$ since

$$T_p^{(1)}(a,-a)=\det\left[\tan\pi\frac{k^2-j^2}{p}\right]_{1\leqslant j,k\leqslant n}=(-1)^nT_p^{(1)}(a,-a)=-T_p^{(1)}(a,-a).$$

Now we determine $T_p^{(0)}(a,-a)$. In view of Corollary 2.1 and (3.2), we have

$$\begin{aligned}T_p^{(0)}(a,-a)&=\det\left[i+\tan\pi\frac{aj^2-ak^2}{p}\right]_{0\leqslant j,k\leqslant n}\\&=\det\left[\frac{2i}{e^{2\pi ia(j^2-k^2)/p}+1}\right]_{0\leqslant j,k\leqslant n}\\&=\prod_{k=0}^n(2i\zeta^{ak^2})\times\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{0\leqslant j,k\leqslant n}.\end{aligned}$$

By Lemma 2.2,

$$\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{0\leqslant j,k\leqslant n}=\frac{\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{k=0}^n(\zeta^{ak^2}+\zeta^{ak^2})\times\prod_{0\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}.$$

Therefore,

$$\begin{aligned}T_p^{(0)}(a,-a)=&i^{n+1}\prod_{0\leqslant j<k\leqslant n}\left(\frac{\zeta^{ak^2}-\zeta^{aj^2}}{\zeta^{ak^2}+\zeta^{aj^2}}\right)^2\\=&(-1)^{(p+1)/4}\prod_{k=1}^n\left(\frac{\zeta^{ak^2}-1}{\zeta^{ak^2}+1}\right)^2\times\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}.\end{aligned}$$

By Lemma 2.4,

$$\prod_{k=1}^n(\zeta^{ak^2}-1)^2=-p\quad\text{and}\quad\prod_{k=1}^n(\zeta^{ak^2}+1)^2=\prod_{k=1}^n\frac{(\zeta^{2ak^2}-1)^2}{(\zeta^{ak^2}-1)^2}=\frac{-p}{-p}=1,$$

and

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2=(-p)^{(p-3)/4}\ \text{and}\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}+\zeta^{aj^2})^2=1.$$

Therefore

$$\det T_p^{(0)}(a,-a)=(-1)^{(p+1)/4}(-p)(-p)^{(p-3)/4}=p^{(p+1)/4}.$$

If $(\frac{ab}{p}) = 1$, then $b \equiv ac^2 \pmod p$ for some $c \in \mathbb{Z}$ with $p \nmid c$, and hence by Lemma 2.3 we have $T_p^{(0)}(a,b) = T_p^{(0)}(a,a)$ and $T_p^{(1)}(a,b) = T_p^{(1)}(a,a)$ since $(p+1)/2$ is even.

Clearly $T_p^{(0)}(a,a) = \det[a_{jk}]_{0\leqslant j,k\leqslant n}$ with $a_{jk} = \tan\pi(aj^2+ak^2)/p$. By Lemma 2.1,

$$\det[x+a_{jk}]_{0\leqslant j,k\leqslant n} = T_p^{(0)}(a,a) + x\det[b_{jk}]_{1\leqslant j,k\leqslant n} \tag{3.6}$$

where

$$b_{jk} := a_{jk} - a_{j0} - a_{0k} + a_{00} = \tan\pi\frac{aj^2+ak^2}{p} - \tan\pi\frac{aj^2}{p} - \tan\pi\frac{ak^2}{p}.$$

Recall that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}.$$

So we have

$$b_{jk} = \tan\pi\frac{aj^2}{p} \times \tan\pi\frac{ak^2}{p} \times \tan\pi\frac{aj^2+ak^2}{p}$$

and hence

$$\det[b_{jk}]_{1\leqslant j,k\leqslant n} = T_p^{(1)}(a,a)\prod_{j=1}^n \tan^2\pi\frac{aj^2}{p}. \tag{3.7}$$

In view of (3.2), (3.6) and (3.7),

$$\begin{aligned}\det\left[\frac{2i}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n} =& \det[i+a_{jk}]_{0\leqslant j,k\leqslant n} \\ =& T_p^{(0)}(a,a) + iT_p^{(1)}(a,a)\prod_{j=1}^n \tan^2\pi\frac{aj^2}{p}.\end{aligned}$$

Thus

$$T_p^{(0)}(a,a) = 2^{(p-1)/2}p^{(p+1)/4} \text{ and } T_p^{(1)}(a,a) = 0 \tag{3.8}$$

if and only if

$$\det\left[\frac{2i}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n} = 2^{(p-1)/2}p^{(p+1)/4}. \tag{3.9}$$

With the help of Lemma 2.2,

$$\begin{aligned}\det\left[\frac{2i}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n} =& \prod_{k=0}^n\left(\frac{2i}{\zeta^{ak^2}}\right) \times \det\left[\frac{1}{\zeta^{aj^2}+\zeta^{-ak^2}}\right]_{0\leqslant j,k\leqslant n} \\ =& \frac{(2i)^{n+1}}{\prod_{k=0}^n \zeta^{ak^2}} \cdot \frac{\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-ak^2}-\zeta^{-aj^2})}{\prod_{j=0}^n\prod_{k=0}^n(\zeta^{aj^2}+\zeta^{-ak^2})}.\end{aligned}$$

Therefore

$$\det\left[\frac{2i}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n} =(-1)^{(p+1)/4}2^{(p+1)/2}\frac{\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-ak^2}-\zeta^{-aj^2})}{\prod_{j=0}^n\prod_{k=0}^n(\zeta^{a(j^2+k^2)}+1)}. \tag{3.10}$$

By Lemma 2.4(ii),

$$\begin{aligned}&\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-ak^2}-\zeta^{-aj^2})\\ =&\prod_{k=1}^n(\zeta^{ak^2}-1)(\zeta^{-ak^2}-1)\times\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-aj^2}-\zeta^{-ak^2})\\ =&p\times p^{(p-3)/4}=p^{(p+1)/4}.\end{aligned}$$

In view of Lemma 2.4(ii) and Lemma 2.5,

$$\begin{aligned}\prod_{j=0}^n\prod_{k=0}^n(\zeta^{a(j^2+k^2)}+1)=&(\zeta^0+1)\prod_{j=1}^n\left(\frac{1-\zeta^{2aj^2}}{1-\zeta^{aj^2}}\right)^2\times\prod_{j=1}^n\prod_{k=1}^n\frac{1-\zeta^{2a(j^2+k^2)/p}}{1-\zeta^{a(j^2+k^2)/p}}\\ =&2\left(\frac{2}{p}\right)^2\left(\frac{2}{p}\right)=2(-1)^{(p+1)/4}.\end{aligned}$$

Combining these with (3.10) we get (3.9) and hence (3.8) holds.

By the above, we have finished the proof of part (ii) of Theorem 1.1. □

4. Proofs of Theorems 1.2-1.3

The following lemma is Frobenius' extension (cf. [BC15]) of the Zolotarev lemma [Z].

Lemma 4.1. *Let n be a positive odd integer and let $a\in\mathbb{Z}$ be relatively prime to n. For $j=0,\ldots,n-1$ let $\sigma_a(j)$ be the least nonnegative residue of aj modulo n. Then the permutation σ_a on $\{0,\ldots,n-1\}$ has the sign* $\mathrm{sign}(\sigma_a)=(\frac{a}{n})$.

We also need another lemma.

Lemma 4.2. *Let $n>1$ be an odd number and let $a\in\mathbb{Z}$ with $\gcd(a,n)=1$. Then*

$$\prod_{1\leqslant j<k\leqslant n-1}\left(e^{2\pi iak/n}-e^{2\pi iaj/n}\right)^2=(-1)^{(n-1)/2}n^{n-2}. \tag{4.1}$$

Proof. Let $\zeta=e^{2\pi ia/n}$. Clearly,

$$\prod_{r=1}^{n-1}(1-\zeta^r)=\lim_{x\to1}\frac{x^n-1}{x-1}=n \tag{4.2}$$

and hence

$$\begin{aligned}(-1)^{\binom{n-1}{2}}\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k-\zeta^j)^2 &= \prod_{j=1}^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n-1}(\zeta^j-\zeta^k) = \prod_{j=1}^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n-1}\zeta^j(1-\zeta^{k-j})\\ &= \prod_{j=1}^{n-1}\left(\frac{(\zeta^j)^{n-2}}{1-\zeta^{-j}}\prod_{\substack{k=0\\k\neq j}}^{n-1}(1-\zeta^{k-j})\right)\\ &= \frac{\zeta^{(n-1)\sum_{j=0}^{n-1}j}}{\prod_{j=1}^{n-1}(\zeta^j-1)}\prod_{j=1}^{n-1}\prod_{r=1}^{n-1}(1-\zeta^r)=n^{n-2}.\end{aligned}$$

So (4.1) holds. □

Proof of Theorem 1.2. In view of Lemma 4.1, for each $\delta=0,1$ we have

$$\det\left[\pi\frac{aj+bk}{n}\right]_{\delta\leqslant j,k\leqslant n-1} = \left(\frac{a}{n}\right)\det\left[\pi\frac{j+bk}{n}\right]_{\delta\leqslant j,k\leqslant n-1} = \left(\frac{-ab}{n}\right)D_n^{(\delta)},$$

where

$$D_n^{(\delta)} := \det\left[\tan\pi\frac{j-k}{n}\right]_{\delta\leqslant j,k\leqslant n-1}.$$

Since

$$\begin{aligned}D_n^{(0)} &= \det\left[\tan\pi\frac{k-j}{n}\right]_{0\leqslant j,k\leqslant n-1} = \det\left[-\tan\pi\frac{j-k}{n}\right]_{0\leqslant j,k\leqslant n-1}\\ &=(-1)^n\det\left[\tan\pi\frac{j-k}{n}\right]_{0\leqslant j,k\leqslant n-1} = -D_n^{(0)},\end{aligned}$$

we have $D_n^{(0)}=0$.

Now it remains to show that $D_n^{(1)}=n^{n-2}$. Write $\zeta=e^{2\pi i/n}$. Similar to (3.2), we have

$$i+\tan\pi\frac{j-k}{p} = \frac{2i}{\zeta^{j-k}+1}\quad \text{for all } j,k=1,\dots,n-1.$$

Combining this with Lemma 2.1, we see that $D_n^{(1)}$ is the real part of the determinant

$$D := \det\left[\frac{2i}{\zeta^{j-k}+1}\right]_{1\leqslant j,k\leqslant n-1} = \prod_{k=1}^{n-1}(2i\zeta^k)\times\det\left[\frac{1}{\zeta^j+\zeta^k}\right]_{1\leqslant j,k\leqslant n-1}.$$

By Lemma 2.2,

$$\det\left[\frac{1}{\zeta^j+\zeta^k}\right]_{1\leqslant j,k\leqslant n-1} = \frac{\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k-\zeta^j)^2}{\prod_{j=1}^{n-1}\prod_{k=1}^{n-1}(\zeta^j+\zeta^k)} = \frac{\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k-\zeta^j)^2}{\prod_{k=1}^{n-1}(2\zeta^k)\times\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k+\zeta^j)^2}.$$

Therefore

$$D = i^{n-1}\prod_{1\leqslant j<k\leqslant n-1}\frac{(\zeta^k-\zeta^j)^4}{(\zeta^{2k}-\zeta^{2j})^2} = (-1)^{(n-1)/2}\prod_{1\leqslant j<k\leqslant n-1}\frac{(\zeta^k-\zeta^j)^4}{(\zeta^{2k}-\zeta^{2j})^2}.$$

Combining this with Lemma 4.2, we immediately get $D=(n^{n-2})^2/n^{n-2}=n^{n-2}$. Thus $D_n^{(1)}=\mathrm{Re}(D)=n^{n-2}$ as desired.

The proof of Theorem 1.2 is now complete. □

Proof of Theorem 1.3. For any nonzero real number x, we obviously have

$$\cot x = \frac{\cos x}{\sin x} = \frac{(e^{ix}+e^{-ix})/2}{(e^{ix}-e^{-ix})/(2i)} = i + \frac{2i}{e^{2ix}-1}.$$

Thus

$$-i+\cot\pi\frac{aj^2+bk^2}{p} = \frac{2i}{\zeta^{aj^2+bk^2}-1}\quad\text{for all } j,k=1,\ldots,n,$$

where $n=(p-1)/2$ and $\zeta=e^{2\pi i/p}$. Combining this with Lemma 2.1, we see that

$$C := \det\left[\cot\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant n}$$

is the real part of

$$\det\left[\frac{2i}{\zeta^{aj^2+bk^2}-1}\right]_{1\leqslant j,k\leqslant n} = \prod_{k=1}^{n}(2i\zeta^{-bk^2})\times\det\left[\frac{1}{\zeta^{aj^2}-\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n}.$$

By Lemma 2.2,

$$\det\left[\frac{1}{\zeta^{aj^2}-\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n} = \frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(-\zeta^{-bk^2}-(-\zeta^{-bj^2}))}{\prod_{j=1}^{n}\prod_{k=1}^{n}(\zeta^{aj^2}-\zeta^{-bk^2})} = (-1)^{\binom n2}\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{(\prod_{k=1}^{n}\zeta^{-bk^2})^n\prod_{j=1}^{n}\prod_{k=1}^{n}(\zeta^{aj^2+bk^2}-1)}.$$

Note that $\prod_{k=1}^{n}\zeta^{k^2}=1$ by (3.1).

Case 1. $p\equiv 1 \pmod 4$, i.e., $2\mid n$.

As $(\frac{a}{p})=-(\frac{-b}{p})$, in view of (2.5) we have

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2(\zeta^{-bk^2}-\zeta^{-bj^2})^2=p^{(p-3)/2}$$

and hence

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})=\pm p^{(p-3)/4}.$$

By Lemma 2.5,

$$\prod_{j=1}^{n}\prod_{k=1}^{n}(1-\zeta^{aj^2+bk^2})=p^{(p-1)/4}.$$

Therefore

$$\det\left[\frac{2i}{\zeta^{aj^2+bk^2}-1}\right]_{1\leqslant j,k\leqslant n}=\pm(2i)^n\frac{p^{(p-3)/4}}{p^{(p-1)/4}}$$

and hence

$$C=\mathrm{Re}\Bigg(\det\left[\frac{2i}{\zeta^{aj^2+bk^2}-1}\right]_{1\leqslant j,k\leqslant n}\Bigg)=\pm\frac{2^{(p-1)/2}}{\sqrt{p}}.$$

Case 2. $p\equiv 3 \pmod 4$, i.e., $2\nmid n$.

In light of (2.7),

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})=p^{(p-3)/4}.$$

Therefore, with the help of (2.9) we have

$$\begin{aligned}\det\left[\frac{2i}{\zeta^{aj^2+bk^2}-1}\right]_{1\leqslant j,k\leqslant n}=&(2i)^n(-1)^{\binom{n}{2}}\frac{p^{(p-3)/4}}{(-1)^n(-1)^{(h(-p)-1)/2}(\frac{a}{p})p^{(p-1)/4}i}\\ =&\frac{2^n i(i^2)^{(n-1)/2}(-1)^{(n-1)/2}}{(-1)^{(h(-p)+1)/2}(\frac{a}{p})\sqrt{p}\,i}\end{aligned}$$

and hence

$$C=\mathrm{Re}\Bigg(\det\left[\frac{2i}{\zeta^{aj^2+bk^2}-1}\right]_{1\leqslant j,k\leqslant n}\Bigg)=(-1)^{(h(-p)+1)/2}\left(\frac{a}{p}\right)\frac{2^{(p-1)/2}}{\sqrt{p}}.$$

In view of the above, we have completed the proof of Theorem 1.3. □

5. Some open conjectures

Conjecture 5.1. *For any odd prime p, we have*

$$\det\left[\cot\pi\frac{jk}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}=\left(\frac{-2}{p}\right)2^{(p-3)/2}p^{(p-5)/4}h((-1)^{(p-1)/2}p). \quad (5.1)$$

Remark 5.1. This conjecture is interesting since it gives a new formula of the class number of the quadratic field $\mathbb{Q}(\sqrt{p^*})$, where $p^*=(-1)^{(p-1)/2}p$ with p an odd prime. We have verified (5.1) for all odd primes $p<30$.

Conjecture 5.2. *Let n be a positive integer.*

(i) *The number*

$$s_n:=(2n+1)^{-n/2}\det\left[\tan\pi\frac{jk}{2n+1}\right]_{1\leqslant j,k\leqslant n} \quad (5.2)$$

is always an integer.

(ii) *We have*

$$\det\left[\tan^2\pi\frac{jk}{2n+1}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in(2n+1)^{(n+1)/2}4^{n-1}\mathbb{Z}. \quad (5.3)$$

Remark 5.2. Via `Mathematica` we find that

$$s_1=1,\ s_2=-2,\ s_3=s_4=4,\ s_5=48,\ s_6=-160,$$
$$s_7=32,\ s_8=2176,\ s_9=6912,\ s_{10}=0,\ s_{11}=273408.$$

Let t_n denote the nth term of the sequence [I16, A277445]. We guess that $s_n=-t_n$ if $n\equiv 3\pmod 4$, and $s_n=t_n$ otherwise.

Let $p=2n+1$ be a prime with n even. Then, for $q=n!$ we have $q^2\equiv-1\pmod p$ by Wilson's theorem, and $(\frac{q}{p})=(\frac{2}{p})$ by [S19, Lemma 2.3]. For any integer $a\not\equiv 0\pmod p$, by Lemma 2.3 we have

$$\det\left[\tan\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant n}=\det\left[\tan\pi\frac{a(qj)^2k^2}{p}\right]_{1\leqslant j,k\leqslant n}$$
$$=\left(\frac{q}{p}\right)\det\left[\tan\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant n}=\left(\frac{2}{p}\right)\det\left[\tan\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant n}.$$

Thus

$$\det\left[\tan\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}=0\ \text{ if } p\equiv 5\pmod 8. \quad (5.4)$$

Similarly,

$$\det\left[\cot\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}=0\ \text{ if } p\equiv 5\pmod 8. \quad (5.5)$$

Conjecture 5.3. *For any prime $p \equiv 1 \pmod 8$ and integer $a \not\equiv 0 \pmod p$, we have*

$$\det\left[\tan\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}=0 \text{ and } \det\left[\cot\pi\frac{aj^2k^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}=0. \tag{5.6}$$

Remark 5.3. We have verified this for $p = 17$. We also conjecture that $\det[\sec 2\pi\frac{jk}{p}]_{0\leqslant j,k\leqslant (p-1)/2}=0$ for any prime $p\equiv 1 \pmod 4$.

Conjecture 5.4. *For any odd integer $n > 1$, we have*

$$\det\left[\tan^2\pi\frac{j-k}{n}\right]_{1\leqslant j,k\leqslant n-1}\in\mathbb{Z} \text{ and } \det\left[\tan^2\pi\frac{j+k}{n}\right]_{1\leqslant j,k\leqslant n-1}\in n^{n-2}\mathbb{Z}. \tag{5.7}$$

Conjecture 5.5. *Let $p \equiv 3 \pmod 4$ be a prime, and let $a, b \in \mathbb{Z}$ with $p \nmid ab$. Then*

$$\det\left[\tan^2\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in p^{(p-3)/4}\mathbb{Z} \tag{5.8}$$

and

$$\det\left[\tan^2\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}\in p^{(p+1)/4}\mathbb{Z}. \tag{5.9}$$

If $(\frac{ab}{p}) = 1$, then

$$\det\left[\cot^2\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in \frac{2^{p-3}}{p}\mathbb{Z}. \tag{5.10}$$

Let $p \equiv 1 \pmod 4$ be a prime, and let $a, b \in \mathbb{Z}$ with $p \nmid ab$. Choose $q \in \mathbb{Z}$ with $q^2 \equiv -1 \pmod p$. Then

$$\begin{aligned}&\det\left[\left(\frac{a(qj)^2+b(qk)^2}{p}\right)\tan\pi\frac{a(qj)^2+b(qk)^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}\\ =&(-1)^{(p+1)/2}\det\left[\left(\frac{aj^2+bk^2}{p}\right)\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}\end{aligned}$$

and hence

$$\det\left[\left(\frac{aj^2+bk^2}{p}\right)\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}=0. \tag{5.11}$$

Conjecture 5.6. *Let $p \equiv 3 \pmod 4$ be a prime and let $a, b \in \mathbb{Z}$ with $p \nmid ab$. Then*

$$\det\left[\left(\frac{aj^2+bk^2}{p}\right)\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2} \in p\mathbb{Z}. \tag{5.12}$$

If $(\frac{ab}{p}) = 1$, then

$$\sqrt{p}\det\left[\left(\frac{aj^2+bk^2}{p}\right)\cot\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2} \in \mathbb{Z}. \tag{5.13}$$

Remark 5.4. For any prime $p \equiv 3 \pmod 4$ set

$$a_p^{\pm} = \frac{1}{p}\det\left[\left(\frac{j^2\pm k^2}{p}\right)\tan\pi\frac{j^2\pm k^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}.$$

Via `Mathematica` we find that

$$a_3^+ = a_3^- = -1,\ a_7^+ = 60,\ a_7^- = 3,\ a_{11}^+ = 2^6\times 3^3,\ a_{11}^- = -373,$$
$$a_{19}^+ = 2^{12}\times 3\times 5^2\times 7\times 11\times 17 \text{ and } a_{19}^- = -5\times 7\times 89\times 3803.$$

Conjecture 5.7. *Let n be a positive integer.*

(i) *We have*

$$\frac{1}{2n}\det\left[\cos\pi\frac{jk}{n}\right]_{0\leqslant j,k\leqslant n} = \det\left[\cos\pi\frac{jk}{n}\right]_{1\leqslant j,k\leqslant n} = (-1)^{\lfloor\frac{n+1}{2}\rfloor}\frac{n^{(n-1)/2}}{2^{(n-1)/2}}. \tag{5.14}$$

If n is odd, then for each $\delta = 0, 1$ we have

$$\det\left[\cos 2\pi\frac{jk}{n}\right]_{\delta\leqslant j,k\leqslant (n-1)/2} = \left(\frac{2}{n}\right)\frac{n^{(n+1)/4-\delta}}{2^{(n-1)/2}}. \tag{5.15}$$

(ii) *If $n > 1$ then*

$$\det\left[\sin\pi\frac{jk}{n}\right]_{1\leqslant j,k\leqslant n-1} = (-1)^{\lfloor\frac{n-1}{2}\rfloor}\frac{n^{(n-1)/2}}{2^{(n-1)/2}}. \tag{5.16}$$

If n is odd, then

$$\det\left[\sin 2\pi\frac{jk}{n}\right]_{1\leqslant j,k\leqslant (n-1)/2} = \left(\frac{-2}{n}\right)\frac{n^{(n-1)/4}}{2^{(n-1)/2}}. \tag{5.17}$$

Remark 5.5. We have checked this conjecture via `Mathematica`; for example, we have verified (5.14) for all $n = 1, \ldots, 12$.

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