

TREE-DECORATED PLANAR MAPS

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ABSTRACT. We introduce the set of (non-spanning) tree-decorated planar maps, and show that they are in bijection with the Cartesian product between the set of trees and the set of maps with a simple boundary. As a consequence, we count the number of tree decorated triangulations and quadrangulations with a given amount of faces and for a given size of the tree. Finally, we generalise the bijection to study other types of decorated planar maps and obtain explicit counting formulas for them.

1. INTRODUCTION

In this paper, we study the combinatorial properties of tree-decorated maps via the use of a simple bijection. Tree-decorated maps are a couple made of a (planar) map and a subtree. They are one-parameter families interpolating between planar maps, when the subtree has size 1, and the spanning-tree decorated map, when the subtree has the same number of vertices that the whole graph.

Planar maps and spanning-tree decorated maps have been thoroughly studied, both in combinatorics and probability. Planar maps were introduced in [Edm60] and afterwards have been thoroughly studied in many works from both a combinatorial (see for example [Tut62, Sch98, BDFG04]) and a probabilistic perspective (see for example [MM06, LG13, Mie13]). Spanning-tree decorated maps were first studied in [Mul67], where they obtain a simple closed formula to count them, which later was explained in [WL72, CDV86, Ber07] through bijective methods. These bijections became key to the study of planar maps decorated by statistical physics models [She16].

1.1. Motivation. The main motivation for the introduction of this model is to try to understand the difference, as metric spaces, between uniformly chosen planar maps and uniformly chosen spanning-tree decorated planar maps.

The study of planar maps as metric spaces has been an active area of research in combinatorics and probabilities. In the case of a uniformly chosen planar quadrangulation, Cori-Vauquelin-Schaeffer (CVS) bijection [Sch98] has been used to understand many of its asymptotic properties: the distances scale like the number of edges to the power $1/4$ and there is an explicit limiting metric space called Brownian map [MM06, Mie13, LG13]. However, in the world of uniformly chosen spanning-trees decorated planar maps, we do not know much. We can only give bounds on the order of the diameter as a function of the number of edges [GHS17, DG18]. On the optimistic side, Walsh and Lehman's bijection [WL72] shows that the spanning tree decorating the map is chosen uniformly over all trees with a certain size (number of vertices).

The main reason why it is difficult to understand distances in the context of spanning-tree decorated maps becomes clear when one compares it to the case of planar quadrangulations. The main tool used to study distances in these planar maps is the (CVS) bijection, which relates a planar map to a pair of trees. In this bijection, one of the trees encodes the distance to a marked point, thus by knowing this tree, one gets an immediate upper bound on the diameter of the map. On

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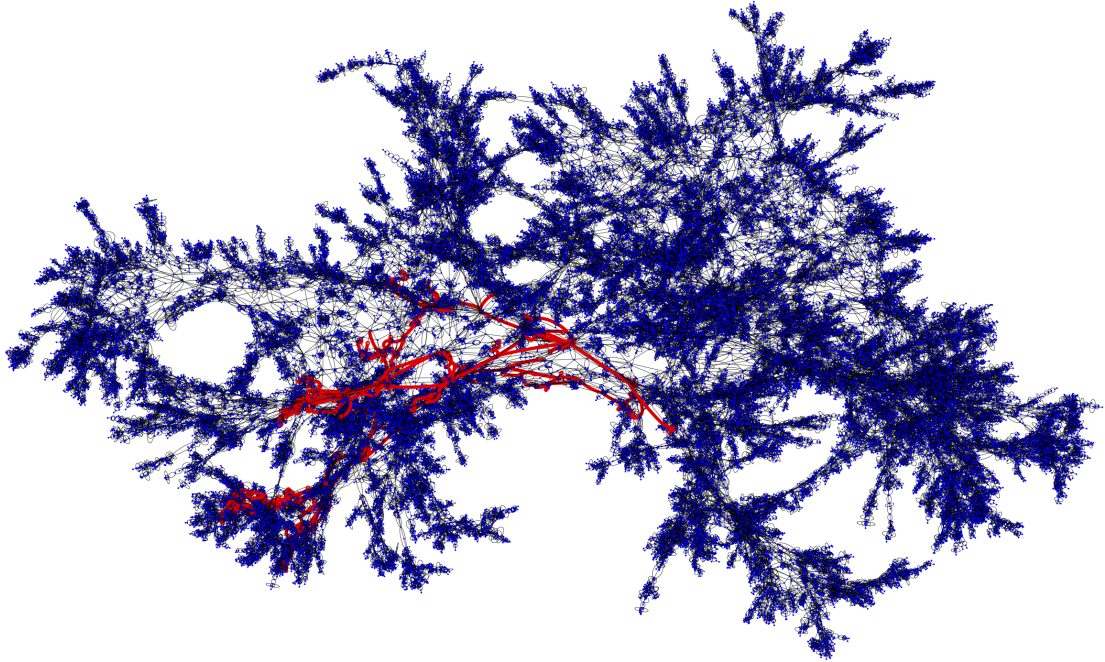


FIGURE 1. A simulation, based in our bijection, of a uniformly chosen tree decorated quadrangulation with 90000 faces and with a tree of 500 edges. The tree is in red and the image is close to an isometrically embedding of the map in \mathbb{R}^3 .

the side of tree-decorated planar maps, we are not that lucky. Even though Walsh and Lehman's bijections [WL72, Ber07] relate them to a pair of trees, it is not possible to extract any information about distances in the original graph from just looking at either of them.

At this point, let us make a remark from the point of view of conformal field theory, where these two models are not expected to look the same. Uniformly chosen planar maps are model associated to central charge equal to 0 ($c = 0$), while spanning-tree decorated maps have an associated central charge of -2 ($c = -2$) [JS04]. Thus, two objects, in the world of tree-decorated maps, have two-different central charges, meaning that theirs conformal properties are interacting. Trying to understand how this interaction works in the limit is the main interest of a follow up paper [FS19].

1.2. Results. Let us, now, present the main results of the paper. For detailed definitions, we refer the reader to Section 2.

The main result of this paper is an explicit bijection between tree-decorated maps and maps with a simple boundary quotiented by a tree.

Theorem 1.1. *The set of tree-decorated maps with f faces, with a tree of m edges and a root-edge on the tree, is in bijection with the Cartesian product of trees with m edges and maps with a simple boundary of size $2m$ and f interior faces.*

The bijection can be summarise as followed: the tree is kept and the map with a boundary is produced by a duplication of the edges of the tree. We call this direction of the bijection ungluing, and its inverse gluing. Note that to gluing consists only in identifying the boundary of the map using the equivalence relationship generated by the tree. Additionally, let us note that as the decorating tree is kept without any changes, in a uniformly chosen tree-decorated map, the law of the tree is uniform among all the trees of a given size (Corollary 3.8). This is a generalisation of the result for spanning-trees decorated maps.

The main contribution of Theorem 1.1 is to connect the study of tree-decorated maps to that of maps with a boundary. In the spirit of the motivation, note that maps with boundary are also an interpolation model, as when the boundary is of size 2, we recover planar maps, and, when the boundary has twice the number of edges, we recover planar trees. This effect was explored, for uniformly chosen maps with a boundary in [Bet15], and we hope that the bijection allows us to transfer the transition from map to a tree to that from a map to a spanning tree-decorated map.

Important consequences of Theorem 1.1 are counting formulas for some subsets of tree-decorated maps. A close look at the bijection of Theorem 1.1 shows that the ungluing procedure changes only the newly created face. Thus, it is possible to obtain counting formulae for tree-decorated (and spanning-tree decorated) q -angulations. To obtain these results, we need to count the maps with a simple boundary and use a re-rooting argument. Luckily, we can find these countings in [Kri07] for triangulations and in [BG09] for quadrangulations.

Corollary 1.2. *For $m \leq f/2 + 1$, the number of tree-decorated triangulations where the tree has m edges and the map has f faces is*

$$(1.1) \quad 2^{f-2m} \frac{(3f/2 + m - 2)!!}{(f/2 - m + 1)!(f/2 + 3m)!!} \frac{3f}{m + 1} \binom{4m}{2m, m, m},$$

where $n!!$ stands for the double factorial of n .

Furthermore, for $m \leq f + 1$, the amount of tree-decorated quadrangulations, where the tree has size m and the map has f faces

$$(1.2) \quad 3^{f-m} \frac{(2f + m - 1)!}{(f + 2m)!(f - m + 1)!} \frac{4f}{m + 1} \binom{3m}{m, m, m}.$$

Note that triangulations, resp. quadrangulations, with f faces have $f/2 + 2$ vertices, resp. $f + 2$ vertices. Thus, one can also find the number of spanning tree-decorated triangulations and quadrangulations. These formulas appeared first in [BM11], and were based in Walsh and Lehman's bijection [WL72]. Thus, these results give a new bijective proof of them.

Corollary 1.3. *The number of spanning-tree decorated triangulations of f faces and the root-edge could be placed in every possible oriented edge is*

$$(1.3) \quad \frac{12f}{(f + 4)(f + 2)^2} \binom{2f}{f, f/2, f/2}.$$

Furthermore, the cardinal of the set of spanning-tree decorated quadrangulation with f faces and the root-edge could be placed in every possible oriented edge is

$$(1.4) \quad \frac{4f}{(f + 1)^2(f + 2)} \binom{3f}{f, f, f}.$$

Let us remark that the counting formula for general maps with a given boundary and edges size exists and is not closed so we decided not to present it here. Furthermore, note that the the spanning-tree decorated maps with a given number of faces is infinite, so the bijection is not interesting in that case.

The main bijection is not only useful to obtain counting results, but we plan to use it in [FS19] to study distances in uniformly chosen tree decorated maps. The bijection becomes useful because distances in uniformly chosen maps with boundaries (simple or not) have been intensively studied [Bet15, BM17, GM19] and are better understood than that of tree-decorated maps.

Let us, also, mention that the bijection presented in this work is simple enough so that decoration on the vertices can be carried between the two objects. This may allow, in the future, to understand what happens when the tree-decorated map is not chosen uniformly but weighted by a statistical physic law. This probability laws, have been the object of great interest in the statistical physics

community, especially after the introduction of the so-called ‘Hamburger-Cheeseburger’-bijection [She16].

Most of the other results of this article consist in using the main idea of the bijection of Theorem 1.1 to produce bijections between other combinatorial objects. In this flavour, we study forest-decorated planar maps (see Corollary 4.3) and tree-decorated planar maps with a simple boundary (see Corollary 4.2). The latter one allows us to make the gluing procedure in a progressive dynamical way (see Proposition 3.11).

On another note, we also explain what happens when one tries to use our bijection with maps whose boundary is not necessarily simple. In this case, the gluing does not produce maps, but what we call bubble-maps (see Section 3.3), which are maps embedded in a tree-like structure of spheres, and which are decorated by a specific type of circuit (see Proposition 3.11).

As a final remark, we would like to say that gluing maps along the boundary has already been done, but not when the decoration is a tree. Some gluings in the literature are, for example, self avoiding walks [DK88, Bet15, GM16] and loops [BBG11].

1.3. Organisation of the paper. The paper is organised as follows: we start with the preliminaries, where we carefully present the classical objects that are important for the paper together with the maps we are interested in. In Section 3, we present all the bijections and its proofs. Finally, in Section 4, we discuss the counting formulas we obtain from the bijections.

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2. PRELIMINARIES

2.1. Elementary definitions. In this section, we present the elementary concepts that appear in this work. As the main result of the paper is a self-contained bijection, the counting results are only stated when needed. For an introduction to planar maps, we recommend, for example, to see [GJ04, FS09] and [BM11].

A ROOTED PLANAR MAP, or MAP for short, is a finite connected graph embedded in the sphere that has a marked oriented edge. We consider that two maps are the same if there is an homeomorphism between them that preserves the orientation (i.e. respecting a cyclic order around every vertex). We call ROOT EDGE the marked oriented edge and ROOT VERTEX its starting point.

The FACES of a map are the connected components of the complement of the edges in the embedding. The DEGREE of a face is the number of oriented edges for which the face lies at its left. We call ROOT FACE the face that is to the left of the root, following the sense of the root edge. A q-ANGULATION is a map where each of the faces has degree q . We set \mathbf{M}_f to be the set of all maps on f faces. Finally, we call a PLANTED PLANE TREE, or TREE for short, to a map with a single face and we denote the set of planted plane trees with m edges as \mathbf{T}_m .

To understand the properties of \mathbf{T}_m , it is useful to codify trees using walks. In the literature, there are several of these codings, see for example Section 1 of [LG05]. In this paper, we are interested in the most elementary of them: the contour function. We present this bijection briefly, for a more detailed description we refer to Section 1.1 of [LG05] and Section 2 of [Ber07].

To introduce the contour function, let us first note that a tree has an intrinsic way of visiting all of its oriented-edges. This visit can be represented by a walker that starts from the root vertex and

follows the direction of the root edge touching the tree with his left hand¹ as long as it walks. The walker, then, continues until it returns to the root edge. Note that this walk visits every oriented edge only once. Now, we define the contour function $C : \llbracket 0, 2m \rrbracket \rightarrow \mathbb{N}$ as giving, at time n , the distance to the root vertex (height) of the vertex visited at time n by the walker.

The inverse of this bijection given by the contour function is explicit. We say that a function $C : \llbracket 0, 2m \rrbracket \rightarrow \mathbb{N}$ is a contour function if $C(0) = C(2m) = 0$ and its increments are ± 1 , i.e., C is a Dyck path. We can construct a plane tree from a contour function by saying that two points $n_1, n_2 \in \llbracket 0, 2m \rrbracket$ are equivalent if

$$(2.1) \quad C(n_1) = C(n_2) = \inf_{n \in \llbracket n_1, n_2 \rrbracket} C(n).$$

The vertices of the tree can be recovered from the equivalence classes of the relation and the edges can be recovered as follows: two vertices have an edge between them if they are the equivalent class of two elements that are exactly at distance 1 in $\llbracket 0, 2m \rrbracket$. Note that each edge comes exactly from two steps of the walk, one going up and the other one coming down.

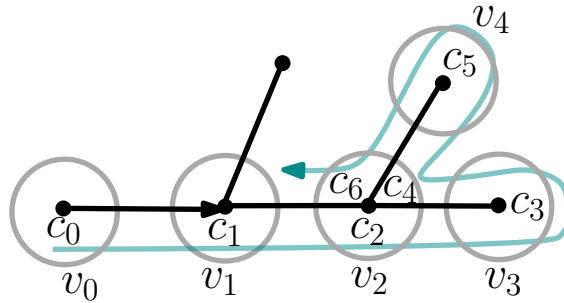


FIGURE 2. Tree with part of the contour in green, the corners are numbered as c_i and are shown in gray. All the corners belonging to the same circle belongs to the same equivalence class associated to a vertex, for example $v_2 = [c_2]_c = [c_4]_c = [c_6]_c$.

We say that \mathfrak{f} is a FOREST WITH r TREES, if $\mathfrak{f} = (\mathfrak{t}_1, \dots, \mathfrak{t}_r)$, where \mathfrak{t}_i is a tree for all $i \in \{1, 2, \dots, r\}$. We define $\mathbb{F}_{m_1, m_2, \dots, m_r}$ as the set of all forest on r trees with prescribed sizes m_1, m_2, \dots, m_r , meaning that for every $i \in \{1, 2, \dots, r\}$, \mathfrak{t}_i has m_i edges.

We define a NON-SELF CROSSING CIRCUIT as a sequence of directed edges $(e_i)_{i=0}^{l-1}$, for some $l \in \mathbb{N}$, embedded in the plane, such that the head of e_i is the tail of $e_{i+1 \bmod l}$ for all $i \in \llbracket 0, l-1 \rrbracket$, and that it does not cross itself. The last part means that for every vertex in the circuit, we do not find the pattern $e_i, e_{j+1}, e_{i+1}, e_j$ in cyclical order, where $i, j \in \llbracket 0, l-1 \rrbracket$ and the sum in the indexes is modulo l .

A MAP WITH A BOUNDARY is a map that has one special face, which has arbitrary degree, however, we will work only with those having arbitrary even degree. The special face is denoted EXTERNAL FACE and all the others are called INTERNAL FACES. The oriented edges incident to the external face form the boundary of the map and the root edge must belong to the boundary. Recall that an oriented edge is incident to a face if the face lies at the left of the oriented edge, this implies that the external face is the root face. We call $\mathbb{B}_{f, m}$ the set of maps with a boundary with f internal faces and with a boundary of size $2m$.

Note that oriented edges in the boundary of a map with a boundary have a canonical label in $\llbracket 0, 2m \rrbracket$. This label is obtained from the amount of steps that a walker, who starts from the root edge and follows the boundary of the root face, takes to arrive to a given edge. An interesting case

¹Note that, in the literature, the walker usually walks following its right hand. In this work, the left hand convention makes some statements easier.

is when the boundary of the map is simple, i.e., it is simple as a curve. In this case, the labels of the edges gives rise to a labelling of the vertices of the boundary. To separate them from the non-simple ones, we denote the set of maps with a simple boundary in $\mathbf{B}_{f,m}$ as $\mathbf{SB}_{f,m}$.

We can also work with more than one boundary. Let $r \in \mathbb{N}$ and define a MAP WITH r -BOUNDARIES as a map with multiple special faces, called the boundary faces, which are pairwise vertex-disjoint and have simple boundaries. Furthermore, this map is multi-rooted, that is to say, every boundary faces has a root and the roots are labelled from 1 to r , where the root labelled 1 coincides with the root of the map. We called $\mathbf{MSB}_{f,m_1,\dots,m_r}$ the set of all multi-rooted maps with boundaries on f internal faces and with boundaries of size $2m_1, \dots, 2m_r$ respectively for the labels of the boundaries.

A map \mathbf{m}_1 is said to be a submap of \mathbf{m}_2 (with the notation $\mathbf{m}_1 \subset_M \mathbf{m}_2$) if \mathbf{m}_1 can be obtained from \mathbf{m}_2 by suppressing edges and vertices. This definition implies that \mathbf{m}_1 respects the cyclic order of \mathbf{m}_2 in the vertices and edges remaining. A decorated map is a map with an especial submap, i.e. it is a pair $(\mathbf{m}, \mathbf{sm}(\mathbf{m}))$ with $\mathbf{m} \in \mathbf{M}_f$ for some $f \in \mathbb{N}$ and $\mathbf{sm}(\mathbf{m}) \subset_M \mathbf{m}$.

2.2. Objects of interest and notation. We present, now, the main objects of study of the paper. In particular, we rigorously define the tree-decorated map.

Take a map $\mathbf{m} \in \mathbf{M}_f$ and define $\mathbf{T}_m(\mathbf{m})$ to be the set of (unrooted) trees $\mathbf{t} \subset_M \mathbf{m}$ with m edges. A TREE-DECORATED MAP is a couple (\mathbf{m}, \mathbf{t}) with $\mathbf{m} \in \mathbf{M}_f$ and $\mathbf{t} \in \mathbf{T}_m(\mathbf{m})$. We denote $\mathbf{M}_f^{\mathbf{T},m}$ the set of all of these pairs. Furthermore, we denote $\mathring{\mathbf{M}}_f^{\mathbf{T},m}$ the set of all tree-decorated maps where the root of the tree is in the map. Note that one can do similar definitions when the map \mathbf{m} is a q -angulation. The set of tree-decorated triangulations and quadrangulations is counted in Corollary 1.2 and that of spanning-tree decorated triangulations and quadrangulations is counted in Corollary 1.3.

Let us now consider a map with a simple boundary $\mathbf{m}^b \in \mathbf{SB}_{f,m_1}$ and define $\mathbf{T}_{m_2}^b(\mathbf{m})$ to be the set of trees $\mathbf{t} \subset_M \mathbf{m}^b$ with m_2 edges containing a unique vertex on the boundary, the root vertex. A TREE-DECORATED MAP WITH A SIMPLE BOUNDARY is a pair $(\mathbf{m}^b, \mathbf{t})$ with $\mathbf{m}^b \in \mathbf{SB}_{f,m_1}$ and $\mathbf{t} \in \mathbf{T}_{m_2}^b(\mathbf{m}^b)$. We define the set of all these pairs as $\mathbf{B}_{f,m_1}^{\mathbf{T},m_2}$. The set of all tree-decorated triangulations with a simple boundary is counted by (4.3) and that of quadrangulations is counted by (4.4).

To finish, we define an r -FOREST-DECORATED MAP as a map decorated on a forest with r non vertex-intersecting (unrooted) trees. The set of all r -forest decorated maps on f faces and with the trees of the forest of size m_1, \dots, m_r respectively is denote by $\mathbf{M}_f^{\mathbf{F},m_1,1,\dots,m_r}$, note that here the order of m_1, \dots, m_r is not important. Furthermore, we denote $\mathring{\mathbf{M}}_f^{\mathbf{F},m_1,m_2,\dots,m_r}$, the set of r -forest-decorated map where each tree is rooted, each root is labelled with a different number in $\llbracket 1, r \rrbracket$ and the root labelled 1 coincides with the root edge of the map. The counting of r -forest decorated triangulations is given in (4.5) and that of quadrangulation is given in (4.6).

3. MAIN BIJECTIONS

3.1. The basic bijection.

3.1.1. From tree decorated maps to maps with simple boundary and trees. In the following paragraphs, we construct the first part of the bijection: the ungluing procedure. This procedure is represented by a function ϕ that takes a tree-decorated map in $\mathring{\mathbf{M}}_f^{\mathbf{T},m}$ and gives a tree in \mathbf{T}_m together with a map with a simple boundary in $\mathbf{SB}_{f,m}$. Basically, in this procedure, the resulting tree is equal to the decorating tree and the map with a boundary is obtained by a duplication of the oriented edges of the tree in such a way that the newly appeared oriented edges all belong to the same face, see Figure 3.

Consider a tree decorated map $(\mathbf{m}, \mathbf{t}) \in \mathring{\mathbf{M}}_f^{\mathbf{T},m}$, and denote $(\mathbf{m}^b, \mathbf{t}')$ as the (soon-to-be-constructed) image of (\mathbf{m}, \mathbf{t}) under ϕ . The tree \mathbf{t}' is taken equal to the tree \mathbf{t} , in particular that the root edge of \mathbf{t}' is the same as that of \mathbf{t} , i.e., the one of the tree decorated map.

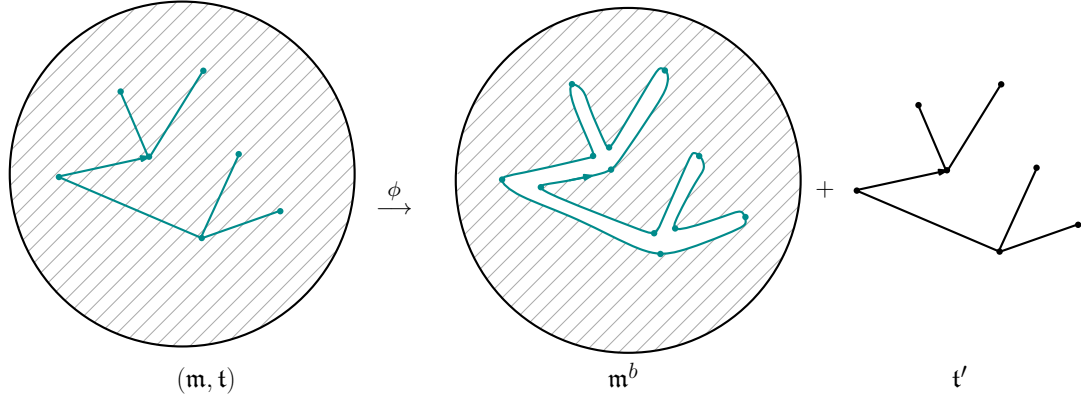


FIGURE 3. A simple sketch of the ungluing procedure.

The construction of the map with a boundary \mathbf{m}^b is not as straight-forward. We start by defining the notion of a corner of \mathfrak{t} at a vertex x as a pair of two consecutive (for the clockwise order) oriented edges, where the first one finishes at x and the second one starts at x . Define K as the set of corners of \mathfrak{t} . Note that an oriented edge that starts or ends in x can be said to go to a corner (e_1, e_2) of x if it is between e_1 and e_2 for the clockwise order.

We can now define \mathbf{m}^b . The set of vertices is the union between K and the vertices of \mathbf{m} that are not in the tree \mathfrak{t} , i.e., $V(\mathbf{m}) \setminus V(\mathfrak{t})$. To define the edges we just need to come back to the original ones. Each oriented edge e of \mathbf{m} whose vertices do not lie in \mathfrak{t} is also an edge of \mathbf{m}^b . Each oriented edge e of \mathbf{m} that have at least one vertex in \mathfrak{t} , becomes an oriented edge that instead of the vertex in the tree has the corner in which the edge is incident. Finally, we add the oriented edges (c_1, c_2) between two elements of K if (c_2, c_1) was already added before.

To finish the definition of \mathbf{m}^b we are only missing the root edge and the embedding. The root edge of \mathbf{m}^b is the edge created from the root edge of \mathbf{m} by the procedure described. The embedding, is characterised by the cyclic orientation of the edges which is the same as that of \mathbf{m} . In other words, in a corner the cyclic orientation is kept the same as in \mathbf{m} , and this can be done because all edges in \mathbf{m}^b that belong to a corner can be identify with a unique edge in \mathbf{m} .

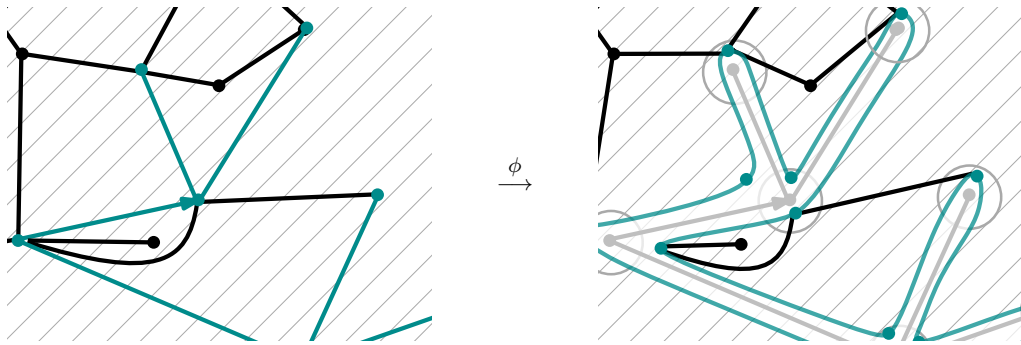


FIGURE 4. Local image of ϕ

Let us, now, show that the constructed map actually belongs to $\text{SB}_{f,m}$.

Lemma 3.1. *The constructed map \mathbf{m}^b belongs to $\text{SB}_{f,m}$.*

Proof. We first need to note that the boundary of the root face has size $2m$. To do that we just note that the boundary of the root face is made by all the oriented edges that went between two

corners, thus it has exactly size $2m$. Furthermore, the boundary of the map \mathbf{m}^b is simple, otherwise the tree would have at least one identification of vertices generating a cycle.

To finish, we just need to show that \mathbf{m}^b is a (planar) map, thus that it can be embedded in the sphere, \mathbf{S}^2 . This comes directly from the fact that there is an orientation preserving isomorphism that goes from $\mathbf{S}^2 \setminus \mathfrak{t}$ to $\mathbf{S}^2 \cap \{(x, y, z) : x > 0\}$, and that can be extended to the boundary in a way that it sends prime ends of \mathfrak{t} in $\mathbf{S}^2 \setminus \mathfrak{t}$ to points in $\mathbf{S}^2 \cap \{x = 0\}$ keeping the cyclical order between them². Thus, one can embed \mathbf{m}^b in \mathbf{S}^2 using any of these functions. To finish, it is enough to see that any of this isomorphisms produce an embedding of \mathbf{m}^b with boundary contained in $\mathbf{S}^2 \cap \{x = 0\}$. \square

Finally, let us note that the ungluing procedure gives \mathbf{m}^b with exactly one more face than \mathbf{m} : the external one. Furthermore, the internal faces of \mathbf{m}^b are in one to one correspondence with the faces of \mathbf{m} . Thus, we can conclude the following lemma.

Lemma 3.2. *For any $q \in \mathbb{N}$, the un-gluing function ϕ takes m -tree decorated q -angulations in to a pair of an m -tree and a q -angulation with boundary of size $2m$.*

Remark 3.3. *Let us also note that as vertex and faces are kept the same, except for the newly created external face, any face or vertex decoration commutes with the bijection.*

3.1.2. *From maps with simple boundary and trees to tree decorated maps.* It is time to define the gluing function ψ that takes a map with a boundary in $\mathbf{SB}_{f,m}$ and a tree in \mathbf{T}_m and gives an m -tree decorated map and is the inverse of the ungluing function ϕ . Informally, ψ should identify two oriented edges incident to the external face of the map using the relation given by the oriented edges in the tree.

Let $(\mathbf{m}^b, \mathfrak{t}')$ be an element of $\mathbf{SB}_{f,m} \times \mathbf{T}_m$ and construct $(\mathbf{m}, \mathfrak{t}) \in \mathring{\mathbf{M}}_f^{\mathbf{T},m}$, the value of $\psi((\mathbf{m}^b, \mathfrak{t}'))$ as follows. Recall that the vertices of the external face of \mathbf{m}^b are indexed from 0 to $2m - 1$, and call C the contour function of \mathfrak{t}' . Recall that C induces an equivalent relationship on vertices via equation (2.1), and define V' as the set of equivalence classes.

Let us now construct \mathbf{m} . The vertex set is made by the union of V' with all vertices of \mathbf{m}^b that do not belong to the exterior face. The edge set of \mathbf{m} is constructed from those of \mathbf{m}^b in the following way. Let (x, y) be an oriented edge of \mathbf{m}^b , then the edge $(G(x), G(y))$ is in \mathbf{m}^b , where

$$(3.1) \quad G(x) := \begin{cases} [l] & \text{if } x \in V', \\ x & \text{else,} \end{cases}$$

where l is the label of x , and $[l]$ is the equivalence class of l under the equivalence relationship defined by C .

Before defining the embedding, let us give some properties of the graph built.

- If (x, y) is an oriented edge that belongs to the boundary of \mathbf{m}^b , its reverse (y, x) is associated exactly to one other edge (y', x') such that (x', y') belongs to the boundary of \mathbf{m}^b . In other words, C induces a perfect matching of the edges in the boundary
- As the boundary of \mathbf{m}^b is simple, the image of the edges in the boundary of \mathbf{m}^b has the same tree structure as \mathfrak{t}' . We define \mathfrak{t} as this image.

To finish the construction of \mathbf{m} , we need to set the cyclical order of the edges around each vertex in the map. If v is a vertex of \mathbf{m} that does not belong to V' , we set the order of the edges surrounding it as the order of \mathbf{m}^b . In the case where $v \in V'$, we consider the order as the gluing of orders, following the corner identification around v (see Figure (5)). Note that this also makes that the cycle order of \mathfrak{t} is the same that the one of \mathfrak{t}' .

²In this case it follows from the fact that $\mathbf{S}^2 \setminus \mathfrak{t}$ is simply connected (so one can use Riemann's theorem), together with results of the behaviour of conformal functions close to the boundary. See Chapter 2 of [Pom13] for definitions of prime ends and the main results in the boundary behaviour of conformal maps.

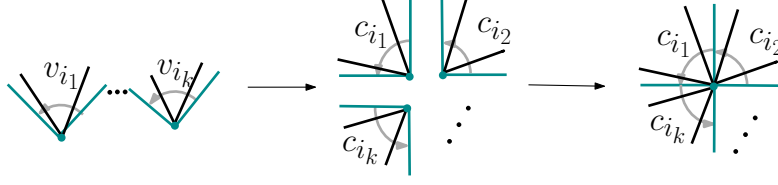


FIGURE 5. Gluing of corners and edge order around a vertex. For vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ with corners in the same class, we associate their order of edges and we glue these orders following the corner sequence of the contour function.

To finish, let us actually prove that there exist an embedding that satisfy the given properties.

Lemma 3.4. (\mathbf{m}, \mathbf{t}) belongs to $\mathring{M}_f^{\mathbf{T}, m}$

Proof. As \mathbf{t} is a subset of \mathbf{m} , is a tree and has m edges, we just need to prove that there exists an embedding that satisfies the order we imposed. This follows like in Lemma 3.1, because of the existence of a orientation preserving isomorphism that goes from $\mathbf{S}^2 \cap \{(x, y, z) : x > 0\}$ to $\mathbf{S}^2 \setminus \mathbf{t}$, and that takes any cyclically ordered prime ends of t in $\mathbf{S}^2 \setminus \mathbf{t}$ to elements of $\mathbf{S}^2 \cap \{x = 0\}$ keeping the same cyclical order. \square

3.1.3. *The gluing is actually a bijection.* Let us now prove that the gluing and ungluing are inverse of each other.

Proposition 3.5. For $f, m \in \mathbb{N}$, $\psi \circ \phi = Id$ in $\mathring{M}_f^{\mathbf{T}, m}$ and $\phi \circ \psi = Id$ in $\mathbf{SB}_{f, m} \times \mathbf{T}_m$.

Proof. Consider $\mathbf{m} \in \mathring{M}_f^{\mathbf{T}, m}$, and let us show that $\psi(\phi(\mathbf{m})) = \mathbf{m}$. First note that the composition of both functions preserves the number of vertices. This is because, every vertex that does not belong to the decoration does not change by the transformations and the vertices on the decoration are separated by ϕ in corners and gathered by ψ in exactly the same vertices of \mathbf{m} .

Let us now note that the edges are kept the same. This is because every edge without endpoints in the decoration remains after the transformations and edges with endpoints in the decoration again are unglued from the decoration by ϕ and glued after by ψ .

Finally, we see that the orientation of the edges is kept. The reason for this is that for vertices that are not in the tree the orientation is kept the same, and for edges that hit in the tree, the orientation is carried by the tree that is the same in both cases. \square

Remark 3.6. Proposition 3.5 can also be proven using that the isomorphisms taken in Lemma 3.1 and 3.4 may be chosen to be the inverse of each other.

In fact, Lemma 3.2 together with Proposition 3.5 imply that the gluing-ungluing procedure is also a bijection when restricted to q -angulations.

Proposition 3.7. ϕ is a bijection between m -tree decorated q -angulations with trees of m -edges and q -angulations with boundary of size $2m$.

Furthermore, as the ungluing keeps the tree without any changes we obtain the following probabilistic result.

Corollary 3.8. The tree of a uniformly chosen m -tree decorated map (or q -angulation) with $e + m$ edges is uniformly chosen between all of the trees of size m .

3.2. **Extensions.** In the following section, we discuss some extensions of the gluing procedure which allows us to make bijections between other combinatorial objects.

3.2.1. *Tree-decorated map with a simple boundary.* In this subsection, we discuss what happens when gluing a tree to a map with a boundary that is bigger than the contour of the tree, and how this allows to create a dynamic gluing of the boundary.

Let us consider the gluing procedure described in Section 3.1.2, but now, let us glue a tree with a contour smaller than the boundary of the exterior face. Note that in this case the result is that the exterior face has shrunk but not disappeared. Furthermore, the glued part becomes a tree that shares only one vertex with the exterior face. This allows us to conclude the following proposition.

Proposition 3.9. *Let $f, m_1, m_2 \in \mathbb{N}$. The set of tree-decorated map with a simple boundary, $\mathbf{B}_{f, m_1}^{\mathbb{T}, m_2}$, is in bijection with the Cartesian product between trees and maps with a simple boundary, $\mathbb{T}_{m_2} \times \mathbf{SB}_{f, m_1 + 2m_2}$.*

In this moment, let us present a fact about gluing that we believe may become useful in the future, especially when working with local limits. This fact is going to be presented in probabilistic terms, as it may be its main use.

Take $x \in \llbracket 0, m-1 \rrbracket$ and a contour function C , i.e., a Dyck path. Then, for every level $l \in \mathbb{N}$ smaller than the value of $C(x)$ there is a unique connected subset $\llbracket \underline{m}(x, l), \overline{m}(x, l) \rrbracket \ni x$ where $C(\cdot) - l$ is a Dyck path and

$$(3.2) \quad l = C(\underline{m}(x, l)) = C(\overline{m}(x, l)).$$

Note that the set of contour functions where this set is exactly $[\underline{m}, \overline{m}]$ can just be reconstructed by the Cartesian product between a positive path going from 0 to $l-1$ in $\underline{m}-1$ steps and a positive path going from $l-1$ to 0 in $\overline{m}-1$ steps.

Looking from a tree point of view, this decomposition allows us to progressively recover all the vertices of the tree whose common ancestors with x is at distance at least l from the root. This allow us to define a progressive procedure of gluing trees, where at time $t \in \llbracket 0, l-1 \rrbracket$ we glue the sub-tree whose vertices live in $\llbracket \underline{m}(x, l-t), \overline{m}(x, l-t) \rrbracket$ with a given map having a simple boundary of size $2m$.

From a probabilistic perspective these remarks imply the following result.

Proposition 3.10. *Let \mathbf{m}^b be a map with a boundary of size $2m$ and n -vertices, chosen uniformly at random in $\mathbf{SB}_{f, m}$ and \mathbf{t} be a tree of size m , also chosen uniformly at random in \mathbb{T}_m . Additionally, take $x, l, \underline{m}, \overline{m} \in \mathbb{N}$ with $x \in \llbracket 0, m-1 \rrbracket$. Call \mathbf{m}_t the map generated by gluing the sub-tree generated by the vertices $\llbracket \underline{m}(x, l-t), \overline{m}(x, l-t) \rrbracket$ with \mathbf{m}^b . Conditionally on the event*

$$\{\llbracket \underline{m}(x, l-t), \overline{m}(x, l-t) \rrbracket = \llbracket \underline{m}, \overline{m} \rrbracket\},$$

the law \mathbf{m}_t is uniform over the $(\underline{m} - \overline{m})/2$ -tree decorated maps with a simple boundary of size $m - (\underline{m} - \overline{m})$ and whose root is at cyclic distance $\min\{\underline{m}, m - \overline{m}\}$ of the tree. Furthermore, under this conditioning, the interior of \mathbf{m}_l , the boundary of \mathbf{m}_t , and the boundary \mathbf{m}_l minus the tree generated at time t are all independent.

In more probabilistic words, the maps \mathbf{m}_t works as a renewal submap for the gluing process of the tree.

3.3. Gluing with a non-simple boundary. To finish this section, we would like to discuss what happens when the boundary of the map we are gluing is not simple. The combinatorial objects that appear do not look that canonical so we discuss with more emphasis the complication that make them arise.

First note, that when one glues a non-simple boundary with a tree, the resulting glued “map” has a decoration that is not necessarily a tree (see Figure 6). This generates two problems. The first one comes from the fact that two different gluing may give the same glued map with the same

submap. This is not a central problem as it can be fixed by, instead of decorating by a subset, decorating by a non-self crossing circuit, defined in Section 2.1.

The second problem that arises in the gluing is the main one. The truth is that there are, in fact two types of cycles that may appear. The first ones are made by vertices where the boundary is not simple, but that are not identified in the tree. This cycles are harmless. The wicked cycles are those made by vertices that are both points where the boundary is not simple, and points identified by the tree (vertices in the boundary associated to corners belonging to the same equivalent class in the tree). This point pinches down the sphere, thus it should be embedded in a topological space with bubbles.

Let us explain better what happens with the wicked points, for that it is useful to assume that the map with a boundary is already embedded in the sphere. Let x be a wicked point. We know that if we retire x , the map with a boundary is left with two or more connected components of interior faces. Note that two faces, f_1, f_2 made by edges in two different connected components can only be connected through a continuous path in the sphere that either pass through x or pass through the exterior face of the map. After the gluing has been done, the exterior face disappears. Thus, any continuous path that goes from the image of f_1 to that of f_2 has to pass by x , as the tree gives no useful identification between these edges. Thus, *the resulting glued "map", is not a map* since it cannot be embedded in the sphere: removing the point x results in a disconnection of the faces, which never happens in \mathbf{S}^2 .

Additionally note that the bubbles are connected in an arborescent way, as they do not form cycles. This is because, each bubble is associated with a starting point and a subtree of the original tree, where two subtrees can only intersect in at most one vertex (See Figure 6). Let us call bubble-maps any rooted graph that can be embedded in such a tree like topology.

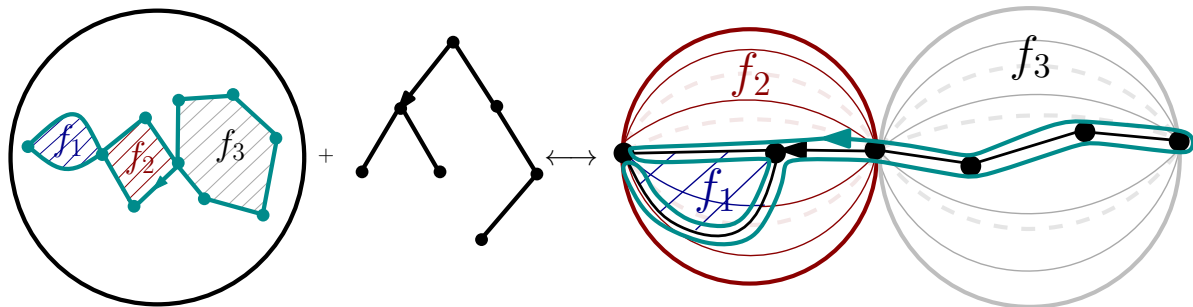


FIGURE 6. Left: Map with a non-simple boundary (the interior faces are filled) and a tree.

Right: Bubbles (3D plot) form by the gluing of a map with non-simple boundary and a tree. We chose to leave the scarf generated by the tree after the gluing (black), even though the decoration is the green circuit (oriented edges following the sense of the root edge from the root edge).

Furthermore, let us note that the image of the path given by the walker in the tree, generates a non-self crossing circuit that passes exactly two times by each edge. From this circuit, it is possible to recover the original tree structure. Let us describe an algorithm that allows to recover the contour function, C , of the tree by using the circuit. Start at $n = 0$ and $C(0) = 0$ from the root-vertex and following the root-edge, and iterate for each new edge of the circuit:

- If the edge has been visited, set $C(n + 1) = C(n) - 1$.
- If this is the first time the edge has been visited and it visits a new vertex, set $C(n + 1) = C(n) + 1$.
- If this is the first time the edge has been visited and it visits an already visited vertex, set $C(n + 1) = C(n) + 1$ and create a new non-self crossing circuit, where the visited vertex is

duplicated. All edges visited before time n (not including the one in this step), goes with the vertex going to the right, and all the others go with the vertex going to the left. At distance $\varepsilon > 0$ from this point, the graph is embedded homeomorphically as before. This can be done because the circuit is not self-crossing.

- Set $n = n + 1$.

Let us, now, define the image set of the gluing of a tree with a map with a boundary. For $f, m \in \mathbb{N}$, we say that $(\mathbf{m}, \mathbf{c}) \in \mathring{M}_{f,m}^{\mathbf{C}}$ if: \mathbf{m} is a bubble-map of f faces, \mathbf{c} is a non-crossing circuit with length $2m$, going through each edge twice, passing by every pinched point of the bubble-map and containing the root edge.

Let us summarize the discussion.

Proposition 3.11. *When one glues a tree $\mathfrak{t} \in \mathbb{T}_m$ with a map with a (non-simple) boundary $\mathbf{m}^b \in \mathbb{B}_{f,m}$, one only obtains a map if there is no vertex where the boundary of \mathbf{m}^b is not simple and that is identified by \mathfrak{t} .*

More importantly, the gluing is a bijection between $\mathbb{T}_m \times \mathbb{B}_{f,m}$ and $\mathring{M}_{f,m}^{\mathbf{C}}$.

Proof. Given the above discussion, we only need to show that we can perform the ungluing. Let us note that the tree can be recovered from the algorithm described above. To recover the map with a boundary, one just needs to duplicate the edges crossed as described in Figure 4. Now, we only have to explain what needs to be done close to the pinch points. In those points, one just needs to locally modify the underlying space so that the point have thickness and all the newly added points belong to the internal face. \square

Remark 3.12. *Again, the bijection does not modify the degree of inner faces, so the same statement is valid when we restrict our attention to bubble q -angulations.*

4. COUNTINGS

In this section, we discuss how the bijections presented are translated in to counting formulae. Before stating the results, we need to introduce a re-rooting procedure, which allows us to obtain formulas for decorated maps whose root is not necessarily in the decoration.

For convenience, in this section, we write (q) for a family of maps to consider the case where all the inner faces are q -angulations.

4.1. Re-rooting procedure. In this section, we explain how the re-rooting procedure works for r -forest q -angulations. The main result of this section is the fact that

$$(4.1) \quad |\mathring{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q)| m(f, q)! = |M_f^{\mathbf{F}, m_1, \dots, m_r}(q)| r! \left(\prod_{i=1}^r m_i \right) \left(\prod_k c_k! \right)^{-1},$$

where $m(f, q)$ is the amount of edges of a q -angulation with f faces, and c_k counts the multiplicity of the elements of identical m_i , i.e.,

$$c_k := |\{i \in \llbracket 1, r \rrbracket : m_i = k\}|.$$

Let us now deduce Equation (4.1). It just follows from counting a slightly bigger set. Let us call

$$\tilde{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q) := \left\{ (\mathbf{m}, \mathbf{t}_1, \dots, \mathbf{t}_r, \vec{e}) : (\mathbf{m}, \mathbf{t}_1, \dots, \mathbf{t}_r) \in \mathring{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q) \text{ and } \vec{e} \text{ is an oriented edge of } \mathbf{m} \right\}.$$

In other words, an element of $\tilde{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q)$ is made of an element of $\mathring{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q)$ together with an additional marked edge. As any automorphism of a map that fix one oriented edge, fixes all of them, we have that we can compute the cardinal of $\tilde{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q)$ in two ways. The first one is by multiplying the amount of possible values of \vec{e} by the cardinal of $\mathring{M}_f^{\mathbf{F}, m_1, \dots, m_r}(q)$, this gives the

left-hand side of (4.1). The other one, is to multiply the cardinal of $\mathbf{M}_f^{\mathbf{F},m_1,\dots,m_r}(q)$ by the possible position of the root of each tree, and then by all possible orders of these r trees, this gives the right-hand side of (4.1).

Remark 4.1. *It is important to remark that we can do this procedure as soon as we work with two types of rootings of the same family of maps. The justification is subtle since if one unroots a given map, certain symmetries may appear, breaking down the argument. Instead, when distinguishing more than one edge symmetries do not appear and this type of identities follow.*

4.2. Counting results. Before presenting the results, let us start by recalling that the number of (rooted) trees on n edges are counted by Catalan numbers $\{C_n\}_{n \in \mathbb{N}}$ (see, for example, Section 1 in [Ber07])

$$(4.2) \quad |\mathbb{T}_n| = C_n := \frac{1}{n+1} \binom{2n}{n}.$$

4.2.1. Tree and forest decorated maps. Let us now obtain the formulas that directly come from the bijection for tree-decorated maps. The counting formulas obtained for maps come from [Kri07] for the case of triangulations with multiples simple boundaries, and from [BF18] on the case of quadrangulations.

Corollary 4.2. *Set $m = m_1 + m_2$. The number of tree-decorated triangulations with boundary of size m_1 and a tree of size m_2 is*

$$(4.3) \quad |\mathbb{B}_{f,m_1}^{\mathbb{T},m_2}(3)| = 2^{f-2m} \frac{(3f/2 + m - 2)!!}{(f/2 - m + 1)!(f/2 + 3m)!!} \frac{2m}{2m_2 + 1} \binom{4m}{2m} \binom{2m_2}{m_2},$$

where $n!!$ stands from double factorial (or semifactorial). The cardinal of tree-decorated quadrangulations with boundary of size m_1 and a tree of size m_2 is

$$(4.4) \quad |\mathbb{B}_{f,m_1}^{\mathbb{T},m_2}(4)| = 3^{f-m} \frac{(2f + m - 1)!}{(f + 2m)!(f - m + 1)!} \frac{2m}{m_2 + 1} \binom{3m}{m} \binom{2m_2}{m_2}$$

Proof. This is obtained from Proposition 3.9, and the comment above Lemma 3.2. The formula for the number of trees with m_2 edges and Theorem 1 of [Kri07] and Section 2.2 of [BG09] respectively. \square

We can now discuss the results for trees.

Corollary 4.3. *The cardinal of r -forest decorated triangulations, with trees of size m_1, m_2, \dots, m_r , is given by*

$$(4.5) \quad |\mathbb{M}_f^{\mathbf{F},m_1,\dots,m_r}(3)| = 2^{f-2m} 3^f \frac{(3f/2 + m - 2)!!}{(f/2 - m + 2 - r)!(f/2 + 3m)!!} \frac{r!}{\prod_{i \in \mathbb{N}} c_k!} \prod_{i=1}^r \frac{1}{m_i + 1} \binom{4m_i}{2m_i, m_i, m_i}$$

and r -forest decorated quadrangulations, with trees of size m_1, m_2, \dots, m_r , is given by:

$$(4.6) \quad |\mathbb{M}_f^{\mathbf{F},m_1,\dots,m_r}(4)| = 3^{f-m} 4^f \frac{(2f + m - 1)!}{(f + 2m)!(f - m + 2 - r)!} \frac{r!}{\prod_{i \in \mathbb{N}} c_k!} \prod_{i=1}^r \frac{1}{m_i + 1} \binom{3m_i}{m_i, m_i, m_i}$$

Proof. This is obtained from Proposition 3.9, the comment above Lemma 3.2, the formula for the number of trees with m_i edges and the results of [Kri07, Theorem 1] and [BF18, Theorem 1.2] respectively, giving that

$$(4.7) \quad |\mathring{\mathbb{M}}_f^{\mathbf{F},m_1,\dots,m_r}(3)| = \frac{2^{f-2m} (3f/2 + m - 2)!!}{(f/2 - m + 2 - r)!(f/2 + 3m)!!} \prod_{i=1}^r \frac{2m_i}{m_i + 1} \binom{4m_i}{2m_i, m_i, m_i}$$

and

$$(4.8) \quad |\mathring{M}_f^{\mathbf{F}, m_1, \dots, m_r}(4)| = 3^{f-m} \frac{(2f+m-1)!}{(f+2m)!(f-m+2-r)!} \prod_{i=1}^r \frac{2m_i}{m_i+1} \binom{3m_i}{m_i, m_i, m_i}.$$

We conclude using the re-rooting procedure condensed in (4.1). \square

Remark 4.4. *Let us note that Corollary 1.2 can be obtained from Corollary 4.3 using $r = 1$.*

Similar counting formulas can be obtained for triangulations of girth bigger than 2 (loopless triangulations) and 3; and loopless quadrangulations (see [BF18]). In Corollary 4.3, one could also consider a generalization for “tree-decorated maps with boundaries” an analog of the tree-decorated maps with a boundary for multiples boundaries. We hope that the interested reader will have no problem in finding the formula.

For general maps with a simple boundary or with multiple simple boundaries, it is not enough to give the number of faces f , since as soon as f is fixed an infinite number of maps with f faces exist. Therefore, the counting has to be done by specifying the number of edges and the size of the boundary. This can be done, but the formula is not closed, thus we not present it here. To do this one can use the generating function of Section 3.1 in [BM11] and after use the technique to count general maps with a simple boundary (Section 5, [BG09]) from the latter generating function.

Bubble-maps counting is a corollary from Proposition 3.11.

Corollary 4.5. *The cardinal of non-crossing circuit decorated bubble-quadrangulations (\mathbf{m}, \mathbf{c}) with f faces decorated by a circuit of size $2m$ and with root in an oriented edge of the map \mathbf{m} is*

$$(4.9) \quad |\mathbf{M}_{f,m}^{\mathbf{C}}(4)| = 3^f 4^f \frac{(2f+m-1)!}{f!(f+m+1)!} \frac{1}{m+1} \binom{2m}{m}^2$$

Proof. This is obtained from Proposition 3.11, Remark 3.12 using the formula for the number of trees with m_i edges, the results of [BG09, Section 2.2] and a type of re-rooting procedure, similar to the one in Section 4.1. \square

The formula for bubble-triangulations can be stated from Proposition 3.11, but the formula for triangulations with a boundary (non-simple) is complicated [MR72]. Again, the formula for general maps with a boundary is not close but it is known, to recover it use the generating function of Section 3.1 in [BM11], we left this computation to the interesting reader.

4.2.2. Spanning tree-decorated maps. In this subsection, we discuss the consequence of our result for spanning tree-decorated maps. In this case, the counting formula of spanning tree-decorated maps for general face degree was given by Mullin [Mul67], where the root does not necessarily belongs to the tree. It states that spanning tree-decorated maps with e edges are in correspondence with a pair of trees with e and $e+1$ edges, i.e. is counted by $C_e C_{e+1}$. Later Bernardi [WL72, CDV86, Ber07] gave a bijective proof of this result.

We denote by $\mathring{M}_f^{\mathbf{ST}}$, resp. $\mathbf{M}_f^{\mathbf{ST}}$, the set of spanning tree-decorated maps on f faces with one root-edge in the tree, resp. in the map. Note that in the case of triangulations we obtain that for f faces the number of vertices is $2 + f/2$ and the number of edges is $3/2f$, therefore, the number of spanning-tree decorated triangulations are

$$(4.10) \quad |\mathbf{M}_f^{\mathbf{ST}}(3)| = |\mathring{M}_f^{\mathbf{T}, f/2+1}(3)| = \frac{12f}{(f+4)(f+2)^2} \binom{2f}{f, f/2, f/2}.$$

In the case of quadrangulations, the condition of having f faces implies that it has $f+2$ vertices and $2f$ edges. Therefore, we obtain the counting formula for spanning-tree decorated quadrangulations.

$$(4.11) \quad |\mathring{M}_f^{\text{ST}}(4)| = |\mathring{M}_f^{\text{T},f+1}(4)| = \frac{2}{(f+1)(f+2)} \binom{3f}{f, f, f}$$

$$(4.12) \quad |\mathring{M}_f^{\text{ST}}(4)| = |\mathring{M}_f^{\text{T},f+1}(4)| = \frac{4f}{(f+1)^2(f+2)} \binom{3f}{f, f, f}.$$

Here we recover the results obtained from the Walsh and Lehman's Bijection [WL72]. In fact, the dual of $\mathring{M}_f^{\text{ST}}(3)$ are the spanning tree-decorated 3-regular maps with f vertices, and that of $\mathring{M}_f^{\text{ST}}(4)$ are the spanning tree-decorated 4-regular maps with f vertices. These can be counted as appears in Section 6.2 of [BM11], where they explicit the formula for more general families of maps, given that (dual) trees with prescribed degree distribution are easily counted.

Remark 4.6. Notice that we kept the expressions for these spanning-tree decorated quadrangulations, with root in the tree and in the map. We want to point out that the right side of (4.11) looks like a possible generalization of the Catalan numbers. More rigorously, for $n, m \in \mathbb{N}$, $m \geq 1$ define:

$$C_{m,n} = m \left(\prod_{i=1}^m \frac{1}{n+i} \right) \underbrace{\binom{(m+1)n}{n, n, \dots, n}}_{m+1 \text{ times}}$$

When $m = 1$, we recover the Catalan numbers and, for this definition, $C_{2,f}$ counts $|\mathring{M}_f^{\text{ST}}(4)|$. To our knowledge, this extension has not been defined so far and it does not appear in the OEIS.

4.2.3. *Spanning r -forest decorated maps.* The case of forest-decorated maps is obtained from Corollary 4.3. As before, a triangulation with f faces has $2 + f/2$ vertices, which has to be equal to $m + r = \sum_{i=1}^r m_i + r$, the number of vertices covered by the forest. Thus,

$$(4.13) \quad |\mathring{M}_f^{\text{SF},m_1,\dots,m_r}(3)| = 4^{r-2} 3f \frac{(2f+2-r)!!}{(2f+6-3r)!!} \frac{r!}{\prod_{i \in \mathbb{N}} c_k!} \prod_{i=1}^r \frac{1}{m_i+1} \binom{4m_i}{2m_i, m_i, m_i}$$

Again as before, a quadrangulation with f faces has $f+2$ vertices, which has to be equal to $m + r$, the number of vertices cover by the forest. Thus,

$$(4.14) \quad |\mathring{M}_f^{\text{SF},m_1,\dots,m_r}(4)| = 3^{r-2} 4f \frac{(3f-r+1)!}{(3f-2r+4)!} \frac{r!}{\prod_{i \in \mathbb{N}} c_k!} \prod_{i=1}^r \frac{1}{m_i+1} \binom{3m_i}{m_i, m_i, m_i}$$

Remark 4.7. From this formula it is also possible to deduce Corollary 1.3.

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