Independent set and matching permutations

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Abstract

Let G be a graph G whose largest independent set has size m. A permutation π of $\{1, \ldots, m\}$ is an *independent set permutation* of G if

$$a_{\pi(1)}(G) \leq a_{\pi(2)}(G) \leq \cdots \leq a_{\pi(m)}(G)$$

where $a_k(G)$ is the number of independent sets of size k in G. In 1987 Alavi, Malde, Schwenk and Erdős proved that every permutation of $\{1, \ldots, m\}$ is an independent set permutation of some graph. They raised the question of determining, for each m, the smallest number $f(m)$ such that every permutation of $\{1, \ldots, m\}$ is an independent set permutation of some graph with at most $f(m)$ vertices, and they gave an upper bound on $f(m)$ of roughly m^{2m}. Here we settle the question, determining $f(m) = m^m$.

Alavi et al. also considered *matching permutations*, defined analogously to independent set permutations. They observed that not every permutation is a matching permutation of some graph, putting an upper bound of 2^{m-1} on the number of matching permutations of $\{1, \ldots, m\}$. Confirming their speculation that this upper bound is not tight, we improve it to $O(2^m/\sqrt{m})$.

We also consider an extension of independent set permutations to weak orders, and extend Alavi et al.'s main result to show that every weak order on $\{1, \ldots, m\}$ can be realized by the independent set sequence of some graph with largest independent set size m, and with at most m^{m+2} vertices.

1 Introduction

To a real sequence $a_1, a_2, \ldots, a_m$ we can associate a permutation π of $[m] := \{1, \ldots, m\}$, which gives information about the shape of the histogram of the sequence, via

$$a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots \leq a_{\pi(m)}. \tag{1}$$

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If there are some repetitions among the a_i then π is not unique. For example, the sequence $(5, 10, 10, 5, 1)$ has associated with it each of the sequences 51423, 54123, 51432 and 54132. (Here and elsewhere we present permutations in one-line notation, so for example 51423 represents the permutation π with $\pi(1) = 5$, $\pi(2) = 1$, et cetera.)

This association was introduced by Alavi, Malde, Schwenk and Erdős in [1], where they proposed using it to investigate sequences associated with graphs. For example, let $\mathcal{I}_m$ denote the set of (simple, finite) graphs G with $\alpha(G) = m$, that is, whose largest independent set (set of mutually non-adjacent vertices) has size m. The *independent set sequence* of $G \in \mathcal{I}_m$ is the sequence $(i_k(G))_{k=1}^m$ where $i_k(G)$ is the number of independent sets of size k in G. Say that π is an *independent set permutation* of G if π is one of the permutations that can be associated to the independent set sequence of G via (1). (We do not consider $i_0(G)$, as it equals 1 for every G.)

The main theorem of [1] is that all $m!$ permutations of $[m]$ are independent set permutations.

Theorem 1.1. *[1] Given $m \geq 1$ and a permutation π of $[m]$, there is a graph G with $\alpha(G) = m$ and with*

$$i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(m)}. \tag{2}$$

In the language of [1] the independent set sequence of a graph is *unconstrained* — it can exhibit arbitrary patterns of rises and falls.

For a permutation π denote by $g(\pi)$ the minimum order (number of vertices) over all graphs G for which π is an independent set permutation of G, and for each m denote by $f(m)$ the maximum, over all permutations π of $[m]$, of $g(\pi)$. Alavi et al. showed that $f(m)$ is at most roughly m^{2m+1} (they did not calculate their upper bound explicitly). They speculated that $f(m) \geq m^m$, and proposed the question of determining $f(m)$.

Problem 1.2. [1, Problem 1] Determine the smallest order large enough to realize every permutation of order m as the sorted indices of the vertex independent set sequence of some graph.

Our first result settles this question exactly.

Theorem 1.3. *(Part 1, $f(m) \leq m^m$) For each $m \geq 1$ there is a graph G_m on m^m vertices with $\alpha(G) = m$ and with*

$$i_1(G_m) = i_2(G_m) = \cdots = i_m(G_m) = m^m \tag{3}$$

(and so for every permutation π of $[m]$, π is an independent set permutation of G_m).

(Part 2, $f(m) \geq m^m$) On the other hand, if $\alpha(G) = m$ and $i_m(G) < m^m$ then $i_m(G) < i_{m-1}(G)$ (and so for a permutation of $[m]$ of the form $\cdots(m-1)\cdots m \cdots 1 \cdots$ to be an independent set permutation of some graph G, G must have at least m^m vertices).

Our proof that $f(m) \geq m^m$ follows fairly quickly from results of Frankl, Füredi and Kalai [6] and Frohmader [7] on Kruskal-Katona type theorems for flag complexes.

Our construction of G_m, to establish $f(m) \leq m^m$, follows the same general scheme introduced in [1]. There, it is shown how to construct a graph G with $\alpha(G) = m$, with $i_k(G)$ being a sum. The first term of the sum is $\pi^{-1}(k)T$ (for some arbitrary constant T), and for T sufficiently large the sum of the remaining terms can be bounded above by T. This puts $i_k(G)$ in the interval $[\pi^{-1}(k)T, \pi^{-1}(k)T + 1)$, and so π is a (actually, the unique) independent set permutation of G. (We describe this construction in more detail in Section 2). We obtain $f(m) \leq m^m$ by carefully carrying out the construction in a way that allows perfect control over the lower order terms in the sum.

This finer control allows us to extend Theorem 1.1. To a real sequence $a_1, a_2, \ldots, a_m$ we can associate a unique weak order (an ordered partition $(B_1, \ldots, B_\ell)$ of $[m]$ into non-empty blocks) via $B_i = \{b_{i1}, b_{i2}, \ldots\}$, where

$$a_{b_{11}} = a_{b_{12}} = \cdots < a_{b_{21}} = a_{b_{22}} = \cdots < \cdots < a_{b_{\ell 1}} = a_{b_{\ell 2}} = \cdots .$$

For example the sequence $(4, 6, 4, 1)$ (the independent set sequence of the edgeless graph on four vertices) induces the weak order $B_1 = \{4\}$, $B_2 = \{1, 3\}$, $B_3 = \{2\}$. Theorem 1.1 says that every weak order in which all blocks are singletons is the weak order induced by some graph, while Part 1 of Theorem 1.3 says the same for every weak order with a single block.

Theorem 1.4. *For $m \geq 1$, for every weak order w on $[m]$ there is a graph G with $\alpha(G) = m$, and with fewer than m^{m+2} vertices, which induces w.*

Note that the number of weak orders on $[m]$ grows like $(1/2)m!(\log_2 e)^{m+1}$ [2], faster than the number of permutations. Note also that by Theorem 1.3, any weak order on $[m]$ that has $m - 1$ and m in the same block, and 1 in a block with a higher index, cannot be induced by a graph with m^m or fewer vertices. The analog of Problem 1.2 for weak orders — where in the range (m^m, m^{m+2}) is the smallest order sufficient to realize every weak order on $[m]$? — remains open.

Alavi et al. also considered the *edge independent set sequence* or *matching sequence* of a graph. Let $\mathcal{M}_n$ denote the set of graphs with $\nu(G) = n$, that is, whose largest matching (set of edges no two sharing a vertex) has n edges. The matching sequence of $G \in \mathcal{M}_n$ is $(m_k(G))_{k=1}^n$ where $m_k(G)$ is the number of matchings in G with k edges. Say that π is a *matching permutation* of G if π is one of the permutations that can be associated to the matching sequence of G via (1).

In contrast to independent set permutations, there are permutations that are not the matching permutation of any graph. Indeed, Schwenk [14] showed that the matching sequence of any graph $G \in \mathcal{M}_n$ is unimodal in the strong sense that for some k,

$$m_1(G) < m_2(G) < \cdots < m_k(G) \geq m_{k+1}(G) > m_{k+2}(G) > \cdots > m_m(G).$$

It follows that the permutations of $[n]$ that can be the matching permutations of a graph in $\mathcal{M}_n$ must have

$$\begin{gathered} \pi^{-1}(1) < \pi^{-1}(2) < \cdots < \pi^{-1}(k-1) \\ \text{and} \\ \pi^{-1}(n) < \pi^{-1}(n-1) < \cdots < \pi^{-1}(k+1), \end{gathered} \tag{4}$$

where $k := \pi(n)$. (This restriction on π can also be deduced from the real-rootedness of the matching polynomial, first established by Heilmann and Lieb [10].) Following Alavi et al., we refer to permutations satisfying (4) as *unimodal* permutations. There are $\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$ unimodal permutations of $[n]$ and so, writing M_n for the set of permutations π that are the matching permutations of some graph in $\mathcal{M}_n$, we have $M_n \leq 2^{n-1}$. This bound was observed in [1], where the following problem was posed.

Problem 1.5. [1, Problem 2] Characterize the permutations realized by the edge independence sequence. In particular, can all 2^{n-1} unimodal permutations of $[n]$ be realized?

We do not address the characterization problem, but our next result answers the particular question: a vanishing proportion of unimodal permutations are the matching permutations of some graph.

Theorem 1.6. *We have $M_n = o(2^n)$. More precisely, there is a constant c such that for $n \geq 1$*

$$M_n \leq \frac{c2^n}{\sqrt{n}}. \tag{5}$$

In the other direction, the perfect matching gives a lower bound on M_n of $2^{\lfloor (n-1)/2 \rfloor}$. We can improve this by an additive term of $\Omega(n)$, but we do not give the details here.

We give the proofs of our results concerning independent set permutations and weak orders in Section 2, and address matching permutations in Section 3. We end with some questions and comments in Section 4.

2 Independent set permutations

We begin with the proof of Part 2 of Theorem 1.3, $f(m) \geq m^m$.

The set $\mathcal{I}(G)$ of independent sets in a graph G forms a flag complex, whose ground set is the vertex set $V(G)$ of G: it is closed under taking subsets, all (singleton) elements of $V(G)$ are independent sets (so $\mathcal{I}(G)$ is a simplicial complex), and, since a subset of vertices that is not an independent set spans at least one edge, minimal non-independent sets have size 2 (so $\mathcal{I}(G)$ is a flag complex).

The dimension of $\mathcal{I}(G)$ is $m-1$, where $m = \alpha(G)$, and its 1-skeleton is $\overline{G}$, the complement of G. The face sequence of $\mathcal{I}(G)$ — the sequence whose kth term is the number of elements of $\mathcal{I}(G)$ of size k — is exactly the independent set sequence of G.

A flag complex of dimension $m-1$ is said to be *balanced* if its 1-skeleton has chromatic number m. The complex $\mathcal{I}(G)$ is not necessarily balanced; consider, for example, the graph $G = C_5$ (the cycle on five vertices), which has dimension 1 but whose 1-skeleton is C_5, which has chromatic number 3. However, Frohmader [7, Theorem 1.1], settling a conjecture of Eckhoff and (independently) Kalai, showed that

$$\text{for any flag complex } \mathcal{S} \text{ there is a balanced simplicial complex } \mathcal{S}' \text{ with the same face sequence as } \mathcal{S}. \tag{6}$$

(For example, the set of independent sets of the graph on vertex set $\{a, b, c, d, e\}$, with edges ab, ac, bc, cd and de, forms a balanced simplicial complex, in fact a flag complex, whose face sequence agrees with that of $\mathcal{I}(C_5)$.)

We now need a result of Frankl, Füredi and Kalai [6, Theorem 5.1], which addresses the question of how the bounds in the Kruskal-Katona theorem change in the presence of information about the chromatic number of the underlying set system. Fix natural numbers $1 \leq \ell \leq k \leq r$. Suppose $\mathcal{F}$ is a family of k-subsets of $\mathbb{N}$ such that for any member of $\mathcal{F}$, no two of its elements are congruent modulo r. The *ℓ-shadow* of $\mathcal{F}$, denoted $\partial_\ell(\mathcal{F})$, is the family of ℓ-subsets of $\mathbb{N}$ that are subsets of some element of $\mathcal{F}$. Frankl, Füredi and Kalai show that if $|\mathcal{F}| = \binom{r}{k}x^k$ for some $x \geq 0$, then

$$|\partial_\ell(\mathcal{F})| \geq \binom{r}{\ell}x^\ell. \tag{7}$$

Proof. (Theorem 1.3, Part 1) Let G be any graph with $\alpha(G) = m$, and with $i_m < m^m$. By (6) there is a balanced simplicial complex $\mathcal{S}'$ whose face sequence is the independent set sequence of G. Because $\mathcal{S}'$ is balanced and has dimension $m - 1$, it can be realized as a set of subsets of $\mathbb{N}$, each of which has the property that no two of its elements are congruent modulo m. Let $\mathcal{F}$ be the set of elements of $\mathcal{S}'$ in this realization, that have size m, and $\mathcal{F}'$ the set of elements of size $m - 1$. We have

$$|\mathcal{F}| = i_m(G) = x^m$$

for some $0 \leq x < m$ (since $i_m(G) < m^m$), so by (7) (in the case $r = k = m$, $\ell = m - 1$) we have

$$|\partial_{m-1}(\mathcal{F})| \geq mx^{m-1} > x^m = i_m(G). \tag{8}$$

But also, because $\mathcal{S}'$ is a simplicial complex, we have $\partial_{m-1}(\mathcal{F}) \subseteq \mathcal{F}'$ and so

$$|\partial_{m-1}(\mathcal{F})| \leq |\mathcal{F}'| = i_{m-1}(G). \tag{9}$$

Combining (8) and (9) we obtain $i_m(G) < i_{m-1}(G)$, as claimed. $\square$

We now move on to the proof of Part 1 of Theorem 1.3, $f(m) \geq m^m$. We begin with an outline of the construction, which is very similar to one described in [1]. Recall that our goal is to construct a graph G_m with $\alpha(G) = m$ that has m^m independent sets of size k for each $k \in [m]$. A key idea that we use throughout is the effect of the join operation on independent set sequences. For a collection $\{G_j : j \in J\}$ of graphs, denote by $\oplus_{j\in J}G_j$ the graph consisting of a union of disjoint copies of the G_j, with every vertex in each G_j adjacent to every vertex in $G_{j'}$ for each $j' \neq j$ — the *mutual join* of the G_j. The effect of $\oplus$ on independent set sequences is additive: if $G = \oplus_{j\in J}G_j$ then for $k \geq 1$,

$$i_k(G) = \sum_{j\in J} i_k(G_j), \tag{10}$$

because no independent set in G can have vertices in two different G_j's.

Given a permutation π of $[m]$, to construct a graph G satisfying (2) (i.e., $i_{\pi(1)}(G) < \cdots < i_{\pi(m)}(G)$) Alavi et al. [1] consider a graph of the form

$$G_\pi := \oplus_{k=1}^{m} kK_{n_k},$$

where $n_k = (\pi^{-1}(k)T)^{1/k}$ for some large integer T, and where kK_{n_k} denotes k vertex disjoint copies of the complete graph K_{n_k} on n_k vertices. By (10) we have

$$i_k(G_\pi) = \pi^{-1}(k)T + \sum_{j=k+1}^{m} \binom{j}{k} (\pi^{-1}(j)T)^{\frac{k}{j}}. \tag{11}$$

Here the term $\pi^{-1}(k)T$ is the count of independent sets of size k in kK_{n_k}, and for $j > k$ the summand $\binom{j}{k}(\pi^{-1}(j)T)^{\frac{k}{j}}$ counts independent sets of size k in jK_{n_j}; there are no independent sets of size k in any jK_{n_j} for $j < k$. For large enough $T = T(m)$ the sum in (11) is strictly smaller than T, so that $\pi^{-1}(k)T \le i_k(G_\pi) < (\pi^{-1}(k) + 1)T$ and (2) holds.

To more carefully control the sum in (11), and allow us to construct a graph G_m with m^m independent sets of all sizes from 1 to m, we modify this construction. Before doing so, we give some intuition.

The graph $G_0 := mK_m$ has $\alpha(G_0) = m$, $i_m(G_0) = i_{m-1}(G_0) = m^m$, and $i_k(G_0) = \binom{m}{k}m^k < m^m$ for $k < m-1$. We need to increase the count of independent sets of size $m-2$ by

$$m^m - \binom{m}{2}m^{m-2} = m^{m-2}\left(m^2 - \binom{m}{2}\right) := a_2 m^{m-2},$$

without changing the number of independent sets of sizes m or $m-1$. By (10), the graph $G_2 := \oplus_{i=1}^{a_2}(m-2)K_m$ (the mutual join of a_2 copies of $(m-2)K_m$) has $i_{m-2}(G_2) = a_2 m^{m-2}$, and also has $i_m(G_2) = i_{m-1}(G_2) = 0$. Hence, again by (10), $\alpha(G_0 \oplus G_2) = m$, $i_m(G_0 \oplus G_2) = i_{m-1}(G_0 \oplus G_2) = i_{m-2}(G_0 \oplus G_2) = m^m$, and $i_{m-3}(G_0 \oplus G_2) = \binom{m}{3}m^{m-3} + a_2(m-2)m^{m-3}$. We need to add

$$m^{m-3}\left(m^3 - \binom{m}{3} - a_2(m-2)\right) := a_3 m^{m-3}$$

independent sets of size $m-3$ (without adding any independent sets of sizes $m, m-1$ or $m-2$). We achieve this by setting

$$G_3 := \oplus_{i=1}^{a_3}(m-3)K_m$$

and considering $G_0 \oplus G_2 \oplus G_3$. (Note that $a_3 \ge 0$, being a cubic in m with non-negative coefficients.)

We continue in this manner until we reach a graph which satisfies (3), which we declare to be G_m. We have to check that at no point, while fixing the number of independent sets of size k to be m^m, do we cause the number of independent sets of size j to be greater than m^m, for some $1 \le j < k$. This check is the main point of the formal proof of Theorem 1.3, Part 1.

Proof. (Theorem 1.3, Part 1) For $m \geq 1$, define a sequence $(a_0, a_1, \ldots, a_{m-1})$ via

$$m^k = a_0\binom{m}{k} + a_1\binom{m-1}{k-1} + \cdots + a_{k-1}\binom{m-(k-1)}{1} + a_k\binom{m-k}{0} \qquad (12)$$

for $k = 0, \ldots, m-1$ (so $a_0 = 1$, $a_1 = 0$, $a_2 = m^2 - \binom{m}{2}$, et cetera). The motivation behind this definition as follows: we will go through an iterative procedure (the one described above) to set the number of independent sets of each size to be m^m, starting with independent sets of size m, and working down. When we come to fix the number of independent sets of size $m-k$ to be m^m, it will turn out that we need to add $a_k m^{m-k}$ such, which we will achieve by successively joining a_k copies of $(m-k)K_m$ to what has thus far been constructed. Evidently each a_i is an integer; but in fact $a_i \geq 0$, as we now show by induction.

For $m = 1$ the sequence $(a_0, a_1, \ldots, a_{m-1})$ consists of the single term $a_0 = 1$, and for $m = 2$ the sequence is $(1, 0)$. So consider $m \geq 3$. We have $a_0 = 1$. Now suppose $a_0, \ldots, a_k$ are all non-negative, for some k, $0 \leq k \leq m-2$. We have

$$\begin{aligned} m^{k+1} &= a_0 m\binom{m}{k} + a_1 m\binom{m-1}{k-1} + \cdots + a_{k-1} m\binom{m-(k-1)}{1} + a_k m\binom{m-k}{0} \\ &\geq a_0\binom{m}{k+1} + a_1\binom{m-1}{k} + \cdots + a_{k-1}\binom{m-(k-1)}{2} + a_k\binom{m-k}{1} \\ &= m^{k+1} - a_{k+1}, \end{aligned}$$

so $a_{k+1} \geq 0$. The first equality here uses (12), the inequality uses

$$m\binom{m-j}{k-j} \geq \binom{m-j}{k-j+1},$$

valid for $m \geq 3$, $k \in \{0, \ldots, m-2\}$ and $j \in \{0, \ldots, k\}$, and the second equality uses (12) again (this time with k replaced by $k+1$).

Now consider the graph $G_m = \oplus_{k=0}^{m-1} G_k$ where $G_k = \oplus_{j=1}^{a_k}(m-k)K_m$. We have $\alpha(G_m) = m$ and, for each $k \in \{0 \ldots, m-1\}$

$$\begin{aligned} i_{m-k}(G_m) &= a_0\binom{m}{k}m^{m-k} + a_1\binom{m-1}{k-1}m^{m-k} + \cdots + a_k\binom{m-k}{0}m^{m-k} \\ &= m^{m-k}\left(a_0\binom{m}{k} + a_1\binom{m-1}{k-1} + \cdots + a_k\binom{m-k}{0}\right) \\ &= m^m, \end{aligned}$$

the last inequality by (12). The main points of the calculation above are that the only parts of G_m that contribute to $i_{m-k}(G_m)$ are those of the form aK_m for $a \geq m-k$, and that

$$i_{m-k}(aK_m) = \binom{a}{m-k}m^{m-k} = \binom{m-(m-a)}{k-(m-a)}m^{m-k}.$$

$\square$

We now turn to the proof of Theorem 1.4, concerning weak orders. The case $m = 1$ is trivial, and $m = 2$ is easy: the three weak orders on $[2]$ are achieved by K_2, $2K_2$ and $K_3 \cup K_2$. So from here on we assume $m \geq 3$.

We will construct

- a graph H_1 with $m^m + m^{m-1}$ vertices, with m^m independent sets of each size in $\{2, \ldots, m\}$, $m^m + m^{m-1}$ independent sets of size 1, and with $\alpha(H_1) = m$;
- a graph H_m with $2m^m - m^{m-1}$ vertices, with $2m^m - m^{m-1}$ independent sets of each size in $\{1, \ldots, m-1\}$, $2m^m$ independent sets of size m, and with $\alpha(H_m) = m$;
- and for each $k \in \{2, \ldots, m-1\}$, we will construct a graph H_k with m^m vertices, with m^m independent sets of each size in $\{1, \ldots, m\} \setminus \{k\}$, with $m^m + m^{m-1}$ independent sets of size k, and with $\alpha(H_m) = m$.

The main point here is that for each k there is $s(k)$ such that H_k has $s(k)$ independent sets of all sizes except k, and has $s(k) + m^{m-1}$ independent sets of size k (specifically $s(k) = m^m$ for $k \neq m$ and $s(m) = 2m^m - m^{m-1}$).

Let $w = (B_1, \ldots, B_\ell)$ be a weak order on $[m]$. Construct a graph $H(w)$ as follows: $H(w)$ is the mutual join of one copy of G_m for each $k \in B_1$ (here and later, G_m is the graph from Theorem 1.3, Part 1); one copy of H_k for each $k \in B_2$, and in general $j - 1$ copies of H_k for each $k \in B_j$. For $k \in B_j$, for any $1 \leq j \leq \ell$, we have

$$i_k(H_w) = \left(m^m|B_1| + \sum_{i \in B_2} s(i) + 2\sum_{i \in B_3} s(i) + \cdots + \sum_{i \in B_\ell} (\ell - 1)s(i) \right) + (j-1)m^{m-1}.$$

Noting that the term in parenthesis above is independent of j and k, we see that the weak order induced by $H(w)$ is indeed w.

Among the H_k none has more than $2m^m - m^{m-1}$ vertices, so the order of $H(w)$ is at most

$$m^m|B_1| + (|B_2| + 2|B_3| + \cdots + (\ell - 1)|B_\ell|)(2m^m - m^{m-1}). \tag{13}$$

If any of the B_i's has size at least 2, then the quantity in (13) can be increased by replacing $|B_i|$ with $|B_i| - 1$ and $|B_{i+1}|$ with $|B_{i+1}| + 1$ (creating a new, $(\ell+1)$st, block if $i = \ell$). It follows that subject to the constraints $\sum_i |B_i| = m$ and $|B_i| \geq 1$, the quantity in (13) is maximized by

$$m^m + (1 + 2 + \ldots + (m-1))(2m^m + m^{m-1}) < m^{m+2}.$$

This gives Theorem 1.4; so our goal (which occupies the rest of the section) is to construct H_k, for $k \in \{1, \ldots, m\}$.

In the proof of Theorem 1.3, we required $a_k \geq 0$. To construct H_k, we need a better bound.

Lemma 2.1. *For $k \geq 2$ (and $m \geq 3$), $a_k \geq m^{k-1}$.*

Proof. We will use an explicit expression for the a_k. It will be convenient in what follows to extend the sequence $(a_0, \ldots, a_{m-1})$ to $(a_0, \ldots, a_m)$, by using (12) to also define a_m.

Let $\vec{a}$ be the column vector with a_j in the jth position (with the positions indexed from 0 to m), and $\vec{m}$ the column vector with m^j in the jth position. We have $M\vec{a} = \vec{m}$ where M is the $(m+1)$ by $(m+1)$ matrix with $\binom{m-j}{i-j}$ in the (i,j) position (rows and columns indexed from 0). Here we understand $\binom{n}{c}$ to be 0 for negative c. M is lower triangular, with 1's down the diagonal, so invertible. We claim that M^{-1} has $(-1)^{i-j}\binom{m-j}{i-j}$ in the (i,j) position.

Indeed, consider the matrix $M\overline{M}$, where $\overline{M}$ has $(-1)^{i-j}\binom{m-j}{i-j}$ in the (i,j) position. The (k,ℓ) entry of $M\overline{M}$ is clearly 0 for $k < \ell$, and 1 for $k = \ell$. For $\ell < k$ the (k,ℓ) entry is

$$\sum_{t=\ell}^{k}(-1)^{t-\ell}\binom{m-t}{k-t}\binom{m-\ell}{t-\ell} = (-1)^{\ell-k}\binom{m-\ell}{m-k}\sum_{t=\ell}^{k}(-1)^{k-t}\binom{k-\ell}{k-t} = 0,$$

the first inequality via some elementary rearrangements, and the second via the standard fact that the alternating sum of binomial coefficients is 0. This shows that $M\overline{M}$ is the identity, and so the inverse of M is as claimed.

Since $\vec{a} = M^{-1}\vec{m}$ we have

$$a_k = m^k - m^{k-1}\binom{m-(k-1)}{1} + m^{k-2}\binom{m-(k-2)}{2} - \cdots + (-1)^k\binom{m}{k}. \qquad (14)$$

For $m \geq 3$ and $k \geq 2$, it is easily checked that the sequence

$$m^k,\ m^{k-1}\binom{m-(k-1)}{1},\ m^{k-2}\binom{m-(k-2)}{2}, \ldots,\ \binom{m}{k}$$

is strictly decreasing. Lower bounding a_k by the sum of the first two terms of the decreasing alternating sum on the right-hand side of (14) we get

$$a_k > m^k - m^{k-1}\binom{m-(k-1)}{1} = (k-1)m^{k-1} \geq m^{k-1},$$

as claimed. □

Another tool we will need in the construction of the H_k is the following easy observation.

Lemma 2.2. *If $k \leq n$ (with k, n natural numbers), then the sequence*

$$n^k,\ \binom{k}{1}n^{k-1},\ \binom{k}{2}n^{k-2}, \ldots,\ \binom{k}{k-1}n$$

is non-increasing. In fact it is strictly decreasing, except that when $k = n$ the first two terms are equal.

Lemma 2.2 gives an alternate proof that the procedure described in the proof of Theorem 1.3 (the construction of G_m) is valid. The construction starts with mKm, or $G_0 \oplus G_1$, which has m^m independent sets of size m, and m^m of size $m-1$. By Lemma 2.2 the sequence $(i_{m-2}(mK_m), \dots, i_1(mK_m))$ is strictly decreasing, with first term at most m^m, and with $i_{m-k}(mK_m)$ a multiple of m^{m-k}.

The construction continues by successively joining a_2 copies of $(m-2)K_m$ to what has currently been constructed, to obtain $G_0 \oplus G_1 \oplus G_2$ where $a_2 \geq 0$ is defined by $a_2 m^{m-2} = m^m - \binom{m}{2} m^{m-k}$ $(= i_{m-2}(mK_m))$. This brings the number of independent sets of sizes $m-2$ up to m^m, and by Lemma 2.2 the sequence $(i_{m-3}(G_0 \oplus G_1 \oplus G_2), \dots, i_1(G_0 \oplus G_1 \oplus G_2))$ is still strictly decreasing, with first term at most m^m, and with $i_{m-k}(G_0 \oplus G_1 \oplus G_2)$ a multiple of m^{m-k}.

Lemma 2.2 now allows this construction to be inductively continued until G_m is reached. We modify things slightly to obtain H_k.

Case 1, $k = 1$: Set $H_1 = G_m \oplus K_{m^{m-1}}$. Note that this requires neither Lemma 2.1 nor Lemma 2.2.

Case 2, $k \neq m, 1$: At the moment when the number of independent sets of size k has reached m^m, there are m^m independent sets of all sizes at least k, while the sequence $(i_{k-1}(G), \dots, i_1(G))$ (where G is the graph constructed so far) is strictly decreasing, with $i_{k-1}(G) = m^m - a_{m-(k-1)} m^{k-1} \leq m^m - m^{m-1}$ (the equality coming from the proof of Theorem 1.3, Part 1, and the inequality using Lemma 2.1), and with $i_j(G)$ a multiple of m^j.

Successively join m^{m-k-1} copies of kK_m to G. This brings the number of independent sets of size k up to $m^m + m^{m-1}$, and it adds

$$km^{k-1} m^{m-k-1} \leq m^{m-1}$$

independent sets of size $k-1$. The result is a graph G' with $i_m(G') = \cdots = i_{k+1}(G') = m^m$, $i_k(G') = m^m + m^{m-1}$, with $(i_{k-1}(G'), \dots, i_1(G'))$ strictly decreasing, with $i_{k-1}(G') \leq (m^m - m^{m-1}) + m^{m-1} = m^m$, and with $i_j(G)$ a multiple of m^j. The inductive procedure described earlier (for the construction of G_m) can now be continued to obtain H_k.

Case 3, $k = m$: Instead of starting the construction with mK_m, we start with $K_{2m} \cup (m-1)K_m$. This has $2m^m$ independent sets of size m, and in general

$$\binom{m-1}{k} m^k + 2m \binom{m-1}{k-1} m^{k-1}$$

independent sets of size k. Two applications of Lemma 2.2 give that the sequence $(i_{m-k}(K_{2m} \cup (m-1)K_m))_{k=1}^{m-1}$ is strictly decreasing, with $i_{m-k}(K_{2m} \cup (m-1)K_m)$ a multiple of m^{m-k}, and with $i_{m-1}(K_{2m} \cup (m-1)K_m) = 2m^m - m^{m-1}$. The inductive procedure described earlier can now be implemented to obtain H_m.

3 Matching permutations

We begin by observing quickly that not all 2^{n-1} unimodal permutations of $\{1, \ldots, n\}$ are realizable as the permutation associated to a graph with largest matching n. Indeed, the following lemma shows that $m_1(G)$ cannot be the largest entry of a matching sequence of any graph whose largest matching has size at least 4, so that for $n \geq 4$ the permutation $n(n-1)\cdots 321$ is not realizable.

Lemma 3.1. *If $\nu(G) \geq 4$ then $m_2(G) > m_1(G)$.*

Proof. We proceed by induction on $e(G)$, the number of edges of G. In the base case, $e(G) = 4$, G must consist of four vertex disjoint edges, and we have $m_2(G) = 6 > 4 = m_1(G)$. For the induction step, let G be a graph on more than four edges with $\nu(G) \geq 4$ and let uv be an arbitrary edge in G, joining vertices u and v. Let G_1 be obtained from G by deleting the edge uv, and G_2 by deleting the vertices u and v. We have $m_2(G) = m_2(G_1)+m_1(G_2)$ (the set of matchings of size 2 in G partitions into those that do not include uv — $m_2(G_1)$ many — and those that do — $m_1(G_2)$ many). Also, $m_1(G) = m_1(G_1) + 1$. Now by induction $m_2(G_1) > m_1(G_1)$, and also $m_1(G_2) \geq 2 > 1$, because on deleting u and v from G at least two of the edges of any matching of size 4 remain. Combining we get $m_2(G) = m_2(G_1) + m_1(G_2) > m_1(G_1) + 1 = m_1(G)$. ☐

We make an incidental observation at this point. The matching polynomial of a graph with maximum matching size n can be expressed in the form $(1 + r_1x)(1 + r_2x)\cdots(1 + r_nx)$ where the r_i's are real and non-negative; this is a consequence of a theorem of Heilmann and Lieb [10]. To a sequence that arises as the coefficient sequence of a polynomial of the form $(1+r_1x)(1+r_2x)\cdots(1+r_nx)$ with r_i real and non-negative, we can associate permutations via (1). Because real-rooted polynomials have unimodal coefficient sequences, at most only the 2^{n-1} unimodal permutations of $[n]$ can arise in this context. The permutation $n(n-1)(n-2)\cdots 321$ can arise: let all r_i be equal, say equal to r, so the polynomial becomes

$$1 + \binom{n}{1}rx + \binom{n}{2}r^2x^2 + \cdots + \binom{n}{n-1}r^{n-1}x^{n-1} + r^nx^n.$$

It's easy to check that if r is sufficiently small,

$$r^n < r^{n-1}\binom{n}{n-1} < \cdots < \binom{n}{2}r^2 < \binom{n}{1}r$$

so that this polynomial has associated with it the unique permutation $n(n-1)(n-2)\cdots 321$. This shows that our observations about restrictions on the matching sequence are not just restrictions coming in disguise from the real-rooted property of the matching polynomial.

The proof of Lemma 3.1 generalizes considerably. We state and prove the generalization first, and then consider the consequences for matching permutations.

Theorem 3.2. *For each $n \geq 1$, and for each $k = 0, \ldots, \lfloor n/2 \rfloor - 1$, if $\nu(G) \geq n$ then $m_k(G) < m_\ell(G)$ for each ℓ satisfying $k < \ell < n-k$.*

Proof. We begin by dealing with some easy boundary cases. The result is vacuously true for $n = 1$. For $n \geq 2$ and $k = 0$, the claim is that if $\nu(G) \geq n$ then $m_0(G) < m_\ell(G)$ for $1 \leq \ell \leq n-1$. But $m_0(G) = 1$, while $m_\ell(G) \geq \binom{n}{\ell} > 1$ (just consider matchings of size ℓ that are subsets of any particular matching of size n in G), so the claim is valid in this case.

This deals completely with the cases $n = 2, 3$, as well as $n = 4$, $k = 0$. For $n = 4$, $k = 1$, the claim is that if $\nu(G) \geq 4$ then $m_1(G) < m_2(G)$, which is exactly Lemma 3.1. This completes the case $n = 4$.

We now proceed by induction on n. For a particular $n > 4$, assume that we already have the result for all $1 \leq n' < n$. Fix k, $1 \leq k \leq \lfloor n/2 \rfloor - 1$. We will prove, by induction on number $e(G)$ of edges of G, that if $\nu(G) \geq n$ then $m_k(G) < m_\ell(G)$ for any ℓ strictly between k and $n-k$.

In the base case, $e(G) = n$, G must consist of n vertex disjoint edges, and we have $m_\ell(G) = \binom{n}{\ell} > \binom{n}{k} = m_k(G)$.

For the induction step, let G be a graph on more than n edges, with $\nu(G) \geq n$, and let uv be an arbitrary edge in G, joining vertices u and v. As in the proof of Lemma 3.1, let G_1 be obtained from G by deleting the edge uv, and G_2 by deleting the vertices u and v. As in the proof of Lemma 3.1 we have

$$m_\ell(G) = m_\ell(G_1) + m_{\ell-1}(G_2) \text{ and } m_k(G) = m_k(G_1) + m_{k-1}(G_2). \tag{15}$$

Now by the induction hypothesis on $e(G)$, we have

$$m_\ell(G_1) > m_k(G_1). \tag{16}$$

But also,

$$m_{\ell-1}(G_2) > m_{k-1}(G_2). \tag{17}$$

This follows from the $n-2$ case of the of the main induction. Indeed, $\nu(G_2) \geq n-2$ (removing u, v can delete at most two of the edges from any matching of size n). Set $n' = n-2$, $k' = k-1$ and $\ell' = \ell - 1$. We have $1 \leq k \leq \lfloor n/2 \rfloor - 1$ and $k < \ell < n-k$, so $0 \leq k-1 \leq \lfloor n/2 \rfloor - 2$ and $k-1 < \ell - 1 < n-k-1$, or $0 \leq k' \leq \lfloor n'/2 \rfloor - 1$ and $k' < \ell' < n' - k'$, and so the appeal to the earlier case of the main induction is valid.

Combining (16) and (17) with (15) yields $m_\ell(G) > m_k(G)$, as required. ∎

An immediate consequence of Theorem 3.2 is that for any graph G with $\nu(G) \geq n$ we have $m_{\lfloor n/2 \rfloor - 1}(G) < m_{\lfloor n/2 \rfloor}(G)$, which says that the mode of the matching sequence must occur at $\lfloor n/2 \rfloor$ or later. This means that M_n, the number of permutations of $[n]$ that can arise as the permutation associated with a graph with largest matching having size n, satisfies $M_n \leq \sum_{k=\lfloor n/2 \rfloor - 1}^{n-1} \binom{n-1}{k}$. This is asymptotically 2^{n-2} as n goes to infinity; a factor of 2 smaller than the upper bound observed in [1].

A finer analysis of Theorem 3.2 yields the substantially smaller bound (5) on M_n. Let $(m_1, \ldots, m_n)$ be a matching sequence, with mode m_t (perhaps obtained after breaking a tie). Any associated permutation (in one-line notation) puts $\{1, \ldots, t-1\}$ in increasing order and $\{t+1, \ldots, n\}$ in decreasing order in the first $n-1$ spots, and puts t at the end.

This permutation can be encoded by an U-D sequence of length $n-1$ — each time one sees a U, one enters the first as-yet-unused number from $\{1, \ldots, t-1\}$ (remembering that these numbers should be used in increasing order); each time one sees a D, one enters the first as-yet-unused number from $\{t+1, \ldots, n\}$ (remembering that these numbers should be used in decreasing order). For example,

$$UUDDDUUDUDDUU$$

would correspond to $n = 14, t = 8$, and would yield the permutation

$$1\ 2\ 14\ 13\ 12\ 3\ 4\ 11\ 5\ 10\ 9\ 6\ 7\ 8.$$

Notice that this is a bijective encoding — a unique permutation can be read from a sequence. Notice also that in the U-D sequence one is never allowed to have an initial substring that has three more D's than U's, because the first time we see such an initial string, say after j U's and $(j+3)$ D's, we would have seen 1 through j, but not $j+1$, and we would have seen n through $n-(j+2)$, in particular including $n-(j+2)$, so we would have $m_{j+1} > m_{n-(j+2)}$, violating Theorem 3.2. It follows that M_n is bounded above by the number of U-D sequences of length $n-1$ having no initial substring with three more D's than U's.

This sequence begins $(1, 2, 4, 7, 14, 25, 50, \ldots)$ [15, A001405], and satisfies the formula

$$a_n = \begin{cases} \frac{3n/2+1}{2n+2}\binom{n+1}{n/2+1} & \text{for even } n, \\ 2a_{n-1} & \text{for odd } n, \end{cases}$$

so that $a_n \sim c2^n/\sqrt{n}$ (with the constant c depending on the parity of n). This verifies (5).

An alternate approach is to say that M_n is bounded above by the number of U-D sequences of length $n+1$ that start with UU and have no initial substring with more D's than U's. This in turn is upper bounded by the number of U-D sequences of length $n+1$ having no initial substring with more D's than U's (with no restriction on how the strings start). These sequences are also known as *left factors* of Dyck words, and it is well-known (see, for example, [15, A001405]) that there are $\binom{n+1}{\lfloor (n+1)/2 \rfloor}$ such. This is asymptotically $c2^n/\sqrt{n}$ (the constant c again depending on the parity of n).

4 Questions and problems

A number of interesting problems remain concerning the behavior of the independent set sequence of a graph. We begin with the natural refinement of our determination of $f(m)$.

Problem 4.1. For each permutation π, determine $g(\pi)$, the minimum order over all graphs G for which π is an independent set permutation of G.

We have shown that at most m^m vertices is enough to induce the constant weak order on $[m]$ from an independent set sequence, but this is definitely not enough to realize all weak orders; for example, the weak order $m-1 < m < m-2 < m-3 < \cdots < 2 < 1$ requires at least $m^m + m - 1$ vertices. In the other direction, we have shown that fewer than m^{m+2} vertices are sufficient to induce any weak order on m.

Problem 4.2. Determine the smallest order large enough to realize every weak order on $[m]$ as the weak order induced by the independent set sequence of some graph.

Problem 4.3. Do the same for weak orders consisting of singleton blocks; equivalently, answer Problem 1.2 with the additional constraint that the permutations associated with independent set sequences are required to be unique.

In [1] the comment is made that Problem 1.2 "is likely to remain exceeding difficult". Given the surrounding discussion in [1], it seems that the authors were implicitly thinking about Problem 4.3 when they made this comment.

A fascinating question is raised in [1], that has attracted some attention, but has remained mostly open. Although the independent set sequence of a graph is unconstrained, if we restrict to special classes of graphs, then it can become constrained. For example the independent set sequence of a claw-free graph is unimodal [9], and so at most only the 2^{m-1} unimodal permutations of $[m]$ can arise as the independent set permutation of a claw-free graph with largest independent set size m. Alavi et al. observed that the independent set sequences of stars and paths are both unimodal, and asked:

Question 4.4. [1, Problem 3] Is the independent set sequence of every tree unimodal?

It is for all trees on 24 or fewer vertices [13]. See, for example, [8] for recent work and other references.

It had been conjectured by Levit and Mandrescu [11] that every bipartite graph has unimodal independent sequence, and they obtained a partial result: if G is a bipartite graph with $\alpha(G) = m \geq 1$, then the final third of the independent set sequence is weakly decreasing, i.e.,

$$i_{\lceil (2m-1)/3 \rceil}(G) \geq \cdots \geq i_{m-1}(G) \geq i_m(G).$$

The unimodality conjecture was, however, disproved by Bhattacharyya and Kahn [3].

Problem 4.5. Characterize the permutations that can occur as the independent set permutations of a bipartite graph.

There is an interesting parallel to the case of well covered graphs. A graph is *well covered* if all its maximal independent sets have the same size. It had been conjectured by Brown, Dilcher, and Nowakowski [4] that every well covered graph has unimodal independent sequence, but this was disproved by Michael and Traves [12], who also showed that the first half of the independent set sequence of a well covered graph is increasing, i.e.,

$$i_1(G) < i_2(G) < \cdots < i_{\lceil m/2 \rceil}(G).$$

They formulated the *roller-coaster* conjecture, that for any $m \geq 1$ and any permutation π of $[\lceil m/2 \rceil, m]$ there is a well covered graph G with $\alpha(G) = m$ and with

$$i_{\pi([\lceil m/2 \rceil])}(G) < i_{\pi([\lceil m/2 \rceil])+1}(G) < \cdots < i_{\pi(m)}(G).$$

This was subsequently proved by Cutler and Pebody [5]. The analog of the roller-coaster conjecture does not hold for Problem 4.5; for example, it is easy to see that for $n \geq 7$, any bipartite graph G on n vertices has $i_2(G) > i_1(G)$.

Turning to matching permutations, the incidental observation made after the proof of Lemma 3.1 raises the following (perhaps easy) question.

Question 4.6. Which unimodal permutations of $[n]$ can arises via (1) from the coefficient sequence of a polynomial of the form $(1 + r_1x)(1 + r_2x) \cdots (1 + r_nx)$ with r_i real and non-negative?

Finally, the greater part of Problem 1.5 remains open.

Problem 4.7. Characterize the permutations that can occur as the matching permutation of a graph, and determine the growth rate of M_n, the number of permutations of $[n]$ that are matching permutations of some graph.

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