Directed Ramsey and Anti-Ramsey Algebras and the Flexible Atom Conjecture

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Abstract

We generalize the notion of a Ramsey algebra to directed graphs (asymmetric relations). We construct m-color Directed Anti-Ramsey algebras for $2 \le m \le 1000$ (except for m = 8). We also construct m-color Directed Ramsey algebras for various m < 500. As a corollary, we give finite representations for relation algebras 35_{37} , 77_{83} , 78_{83} , 80_{83} , 82_{83} , 1310_{1316} , 1313_{1316} , and 1315_{1316} , none of which were previously known to be finitely representable.

1 Introduction

A Ramsey algebra in m colors is a partition of a set $U \times U$ into disjoint binary relations $\mathrm{Id}, R_0, \ldots, R_{m-1}$ such that

- (I.) $R_i^{-1} = R_i$;
- (II.) $R_i \circ R_i = R_i^c$;
- (III.) for $i \neq j$, $R_i \circ R_j = \mathrm{Id}^c$.

Here, $\mathrm{Id} = \{(x, x) : x \in U\}$ is the identity over U, \circ is relational composition, $^{-1}$ is relational inverse, and c is complementation with respect to $U \times U$.

Ramsey algebras are representations of relation algebras first defined (but not named) in a 1982 paper by Maddux [7]. Kowalski later called the (abstract) relation algebras "Ramsey Relation Algebras" [6].

The usual method of attempting to construct the relations R_0, \ldots, R_{m-1} is a "guess-and-check" prime field method due to Comer [4], as follows: Fix $m \in \mathbb{Z}^+$, and let $X_0 = H$ be a multiplicative subgroup of \mathbb{F}_p of order (p-1)/m, where $p \equiv 1 \pmod{2m}$. Let $X_1, \ldots X_{m-1}$ be its cosets; specifically, let $X_i = g^i X_0 = \{g^{am+i} \pmod{p} : a \in \mathbb{Z}^+\}$, where g is a generator of \mathbb{F}_p^{\times} . Suppose the following conditions obtain:

(i.)
$$-X_i = X_i$$
;

(ii.)
$$X_i + X_i = \mathbb{F}_p \setminus X_i$$
,

(iii.) for
$$i \neq j$$
, $X_i + X_j = \mathbb{F}_p \setminus \{0\}$.

Then define $A_i = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : x - y \in X_i\}$. It is easy to check that (i.)-(iii.) imply (I.)-(III.), and we get a Ramsey algebra. Condition (ii.) implies that all the X_i s are sum-free. The fact that $p \equiv 1 \pmod{2m}$ implies that the order of X_0 is even, which implies $-X_0 = X_0$, i.e., X_0 is symmetric. In [1], Ramsey algebras were constructed for all $m \leq 2000$, except for m = 8, 13, and it was shown that if $p > m^4 + 5$, then X_0 contains a solution to x + y = z, hence \mathbb{F}_p does not give an m-color Ramsey algebra via Comer's construction.

A natural generalization is to consider X_0 of odd order, so that $-X_0 \cap X_0 = \emptyset$; this gives relations R_i that are antisymmetric. This motivates the following definition.

Definition 1. Let k be odd and n be even, with p = nk+1 prime. Let $m = \frac{n}{2}$. Let g be a primitive root modulo p. Let $X_0 = \{g^{\alpha n} : 0 \le \alpha < k\}$ be the multiplicative subgroup of \mathbb{F}_p of index n, and let $X_i = \{g^{\alpha n+i} : 0 \le \alpha < k\}$ be its cosets $(1 \le i < n)$. Suppose the following conditions obtain:

1.
$$\forall i, X_i + X_i = \bigcup_{j \neq i} X_j$$

2.
$$\forall i, X_i + X_{i+m} = \{0\} \cup \bigcup_{i \neq j \neq i+m} X_j$$

3.
$$\forall i, j \text{ with } i \neq j \neq i + m, X_i + X_j = \bigcup_{\ell} X_{\ell} = \mathbb{F}_n^{\times}$$
.

Then the X_i s are called a (prime field) Directed Ramsey Algebra in m colors.

Condition 1.-3. make each X_i sum-free, but otherwise all sumsets are as large as possible. One might think that the exclusions in 2. are unnecessary, but one would be mistaken; they are implied by the exclusion $X_i + X_i \not\supseteq X_i$, the "triangle symmetry"

$$X_i + X_j \supseteq X_\ell \iff X_j + X_{\ell+m} \supseteq X_{i+m}$$

and the "rotational symmetry"

$$X_i + X_j \supseteq X_\ell \iff X_{i+a} + X_{j+a} \supseteq X_{\ell+a}$$
.

(Proofs of these symmetries are left to the reader.)

We say m colors rather than n colors since the n cosets come in m pairs, where $X_i = -X_{i+m}$. One can give a directed coloring of the edges of K_p in m colors by labelling the vertices with the elements of \mathbb{F}_p with color classes $C_i = \{(x,y) : x-y \in X_i\}$ for $0 \le i < m$. (So if X_0 is "directed red", then X_m would just be the "backward" red color. The first m cosets X_0, \ldots, X_{m-1} are sufficient to determine the directed edge-coloring of K_p .)

In such a directed coloring, the monochromatic "transitive" triangles (i.e., triples (x, y), (y, z), (x, z)) are forbidden. This suggests a different generalization, where we forbid monochromatic "intransitive" triangles (i.e., triples (x, y), (y, z), (z, x)).

Definition 2. Let k be odd and n be even, with p = nk+1 prime. Let $m = \frac{n}{2}$. Let g be a primitive root modulo p. Let $X_0 = \{g^{\alpha n} : 0 \le \alpha < k\}$ be the multiplicative subgroup of \mathbb{F}_p of index n, and let $X_i = \{g^{\alpha n+i} : 0 \le \alpha < k\}$ be its cosets $(1 \le i < n)$. Suppose the following conditions obtain:

1.
$$\forall i, X_i + X_i = \bigcup_{j \neq i+m} X_j$$

2.
$$\forall i, X_i + X_{i+m} = \{0\} \cup \bigcup_{\ell} X_{\ell} = \mathbb{F}_p$$

3.
$$\forall i, j \text{ with } i \neq j \neq i + m, X_i + X_j = \bigcup_{\ell} X_{\ell} = (\mathbb{F}_p)^{\times}.$$

Then the X_i s are called a (prime field) Directed Anti-Ramsey Algebra in m colors.

So we ask: for which m > 1 is there a prime p so that there exists a Directed Ramsey algebra (resp., a Directed Anti-Ramsey Algebra) over \mathbb{F}_p ?

(Note to experts in relation algebras: we refer to the explicit combinatorial objects of Definitions 1 and 2 as Directed (Anti-)Ramsey algebras, and to the associated abstract algebras as Directed (Anti-)Ramsey Relation algebras, following Kowalski and Maddux. We observe, for example, that the 1-color Directed Anti-Ramsey Relation algebra is the so-called point algebra, and is not finitely representable.)

Lemma 3. Let p = nk + 1 be prime, with k odd and n = 2m. If $p > n^4 + 5$, then \mathbb{F}_p does not give either an m-color Directed Ramsey algebra nor an m-color Directed Anti-Ramsey algebra via Comer's construction.

Proof. For Ramsey algebras, the proof of Theorem 4 from [1] goes through with "m" replaced by "n", showing that X_0 is not sum-free. For Anti-Ramsey algebras, one also replaces the equation x+y=z by x+y=-z; the argument goes through, $mutatis\ mutandis$.

(Lemma 3 also follows from [3], a paper of which the author was unaware when proving Theorem 4 from [1].)

2 Data

We found m-color Directed Ramsey algebras for 37 values of m < 500; see Table 1. We found m-color Directed Anti-Ramsey algebras for all values of $1 < m \le 1000$ except for m = 8. All data collected for this paper made use of the fast algorithm for computing cycle structures of Comer's algebras over \mathbb{F}_p given in [2]. The data for Directed Anti-Ramsey algebras are available at https://oeis.org/A294615. See Figure 1.

m	p	m	p	m	p
1	3	106	497141	198	2192653
10	3221	111	559219	206	2020861
15	4231	116	679993	210	2728741
17	11527	122	814717	213	2420959
23	15319	129	764971	295	4017311
35	38011	132	1118569	311	4618351
48	91873	141	1043683	394	8881549
59	135347	176	1946209	419	9071351
66	209221	177	1470871	447	11279599
67	228203	179	1521859	474	12190333
74	309173	186	1514413	482	11383877
89	476863	188	1968361	483	12390883
				495	15468751

Table 1: Smallest modulus p for m-color Directed Ramsey algebras

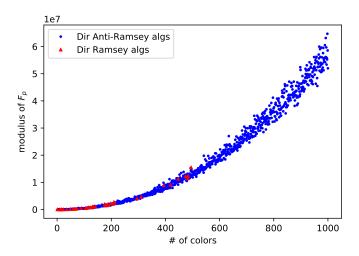


Figure 1: Smallest modulus p for m-color Directed (Anti-)Ramsey algebras

2.1 Asymmetry between Directed Ramsey and Anti-Ramsey Algebras

The difference in ease in finding Directed Ramsey and Anti-Ramsey algebras was at first surprising to the author. But there is a fundamental asymmetry that is Ramsey-theoretic in nature. For consider the subgraph of the coloring described in section 1 induced by any one color. With Directed Anti-Ramsey algebras, this subgraph will consist of a bunch of transitive triangles pasted together, and can contain large cliques. With Directed Ramsey algebras, however, any such color class contains only intransitive triangles. This forces the color class to be sparse in the sense that the *only* cliques are triangles, since any tournament (directed clique) on 4 or more vertices contains a transitive triple (a, b), (b, c), (a, c). This is a plausible explanation for the greater difficulty in finding Directed Ramsey algebras.

3 Some finite Relation algebras

This section is directed at experts in relation algebras. We will give some explicit finite representations of relation algebras. Many of these algebras contain flexible atoms. Thus we eliminate several potential small counterexamples to the Flexible Atom Conjecture:

Conjecture 4. Every finite integral relation algebra with a flexible atom, i.e., an atom that does not participate in any forbidden diversity cycles, is representable over a finite set.

Since all representations considered here are group representations, throughout we let ρ be a representation that maps into the powerset of \mathbb{F}_p (although the base set for the representation is actually $\mathbb{F}_p \times \mathbb{F}_p$), where p is given by context. We use the numbering system and notation given in Maddux's book [8]. Unless otherwise noted, all algebras in this section were not previously known to be finitely representable. In most cases, we use the fact that the algebra in question embeds in some Directed Ramsey or Anti-Ramsey algebra.

3.1 The class a, r, \ddot{r}

3.1.1 33₃₇

Relation algebra 33₃₇ has atoms 1', a, r, \check{r} . The forbidden cycle is $rr\check{r}$. (See Figure 2.) Let p=29 and m=2. Then

$$\rho(a) = X_1 \cup X_3$$

$$\rho(r) = X_0$$

$$\rho(\check{r}) = X_2$$

is a representation over \mathbb{F}_p . Notice that 33_{37} is a subalgebra of the 2-color Directed Anti-Ramsey algebra.

Relation algebra 33_{37} was previously shown to be finitely representable by J. Manske and the author, but the representation given here is smaller.

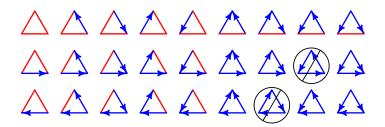


Figure 2: Cycle structure of 33_{37}

3.1.2 35₃₇

Relation algebra 35_{37} has atoms 1', a, r, \check{r} . The forbidden cycle is rrr. (See Figure 3.) Let p=3221 and m=10. Then

$$\rho(a) = \bigcup_{0 \neq i \neq 10} X_i$$

$$\rho(r) = X_0$$

$$\rho(\breve{r}) = X_{10}$$

is a representation over \mathbb{F}_p . Notice that 35_{37} is a subalgebra of the 10-color Directed Ramsey algebra.

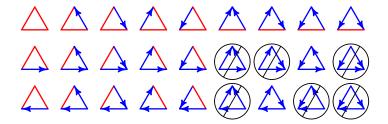


Figure 3: Cycle structure of 35_{37}

3.2 The class $r, \ \breve{r}, \ s, \ \breve{s}$

3.2.1 77₈₃

Relation algebra 77_{83} is the 2-color Directed Anti-Ramsey algebra. The forbidden cycles are $rr\ddot{r}$ and $ss\ddot{s}$. Let p=29 and m=2. Then

$$\rho(r) = X_0$$

$$\rho(\breve{r}) = X_2$$

$$\rho(s) = X_1$$

$$\rho(\breve{s}) = X_3$$

is a representation over \mathbb{F}_p .

3.2.2 78₈₃

Relation algebra 78_{83} has forbidden cycles sss and $ss\breve{s}$. The atoms r and \breve{r} are flexible. Let p=33791 and m=31. Then

$$\rho(r) = \bigcup_{0 < i < 31} X_i$$

$$\rho(\check{r}) = \bigcup_{31 < i < 62} X_i$$

$$\rho(s) = X_0$$

$$\rho(\check{s}) = X_{31}$$

is a representation over \mathbb{F}_p . In this case, \mathbb{F}_p admits a 31-color (symmetric) Ramsey algebra, but the symmetric colors are splittable into asymmetric pairs.

3.2.3 80₈₃

Relation algebra 80_{83} has forbidden cycle $ss\breve{s}$. The atoms r and \breve{r} are flexible. Let p=67 and m=3. Then

$$\rho(r) = X_1 \cup X_2$$

$$\rho(\check{r}) = X_4 \cup X_5$$

$$\rho(s) = X_0$$

$$\rho(\check{s}) = X_3$$

is a representation over \mathbb{F}_p . 80_{83} embeds into the 3-color Directed Anti-Ramsey algebra.

3.2.4 82₈₃

Relation algebra 82_{83} has forbidden cycle sss. The atoms r and \breve{r} are flexible. Let p=3221 and m=10. Then

$$\rho(r) = \bigcup_{0 < i < 10} X_i$$

$$\rho(\breve{r}) = \bigcup_{10 < i < 20} X_i$$

$$\rho(s) = X_0$$

$$\rho(\breve{s}) = X_{10}$$

is a representation over \mathbb{F}_p . 82_{83} embeds into the 10-color Directed Ramsey algebra.

3.2.5 83₈₃

Relation algebra 83_{83} has no forbidden cycles; all diversity atoms are flexible. 82_{83} embeds in the 4-color Directed Anti-Ramsey algebra, hence is representable over \mathbb{F}_p for p=233. There is a smaller "direct" representation over \mathbb{F}_p , p=37 and m=2, where the images of the four diversity atoms are just the four cosets of the multiplicative subgroup of index 4. This algebra was previously known to be finitely representable, by a slight generalization of the probabilistic argument in [5]. The representation given here is probably the smallest known.

3.3 The class $a, b r, \ddot{r}$

3.3.1 1310₁₃₁₆

Relation algebra 1310_{1316} has forbidden cycle $rr\breve{r}$. The atoms a and b are flexible. Let p=67 and m=3. Then

$$\rho(a) = X_1 \cup X_4$$

$$\rho(b) = X_2 \cup X_5$$

$$\rho(r) = X_0$$

$$\rho(\breve{r}) = X_3$$

is a representation over \mathbb{F}_p . 1310₁₃₁₆ embeds into the 3-color Directed Anti-Ramsey algebra.

3.3.2 1313₁₃₁₆

Relation algebra 1313_{1316} has forbidden cycle rrr. The atoms a and b are flexible. Let p=3221 and m=10. Then

$$\rho(a) = (\bigcup_{0 < i < 6} X_i) \cup (\bigcup_{11 < i < 16} X_i)
\rho(b) = (\bigcup_{5 < i < 10} X_i) \cup (\bigcup_{15 < i < 20} X_i)
\rho(r) = X_0
\rho(\breve{r}) = X_{10}$$

is a representation over \mathbb{F}_p . 1313₁₃₁₆ embeds into the 10-color Directed Ramsey algebra.

3.3.3 1315₁₃₁₆

Relation algebra 1315₁₃₁₆ has forbidden cycle bbb. The atoms a, r, and \check{r} are flexible. Let p=33791 and m=31. Then

$$\rho(a) = (\bigcup_{2 < i < 31} X_i) \cup (\bigcup_{33 < i < 62} X_i)$$

$$\rho(b) = X_0 \cup X_{31}$$

$$\rho(r) = X_1 \cup X_2$$

$$\rho(\check{r}) = X_{32} \cup X_{33}$$

is a representation over \mathbb{F}_p .

3.3.4 1316₁₃₁₆

Relation algebra 1316_{1316} has no forbidden cycles; all diversity atoms are flexible. 1316_{1316} Let p = 73 and m = 4. Then

$$\rho(a) = X_2 \cup X_6$$

$$\rho(b) = X_3 \cup X_7$$

$$\rho(r) = X_0 \cup X_1$$

$$\rho(\breve{r}) = X_4 \cup X_5$$

is a representation over \mathbb{F}_p . This algebra was previously known to be finitely representable, by a slight generalization of the probabilistic argument in [5].

4 Problems

Problem 1. Consider the finite integral algebra with diversity atoms $a, r, \check{r}, s, \check{s}$, with only rrr and $ss\check{s}$ forbidden. Then a is flexible, so a representation exists over a countable set. Can this algebra be finitely represented?

Problem 2. The (abstract) 2-color Directed Ramsey Relation algebra is 81_{83} , and is not representable; it fails to satisfy the equation that Maddux calls (J). We showed above that the 10-color Directed Ramsey algebra is representable. What is the smallest m > 2 such that the m-color Directed Ramsey algebra is representable? (The 1-color Directed Ramsey algebra is representable over $\mathbb{Z}/3\mathbb{Z}$.)

Problem 3. Are all sufficiently large Directed Ramsey and Directed Anti-Ramsey algebras constructible (i.e., are the associated abstract algebras representable)? Clearly the computational approach taken here is of no avail.

Problem 4 (due to Peter Jipsen). Is the smallest set over which relation algebra 1316_{1316} has a representation smaller than the corresponding smallest set for 83_{83} ? Both have four flexible diversity atoms; 83_{83} has four asymmetric diversity atoms, while 1316_{1316} has two symmetric and two asymmetric.

The author's guess is that the answer to Problem 4 is "No". 83_{83} has only two "colors" — in this case, two pairs of asymmetric diversity atoms — while 1316_{1316} has three colors.

References

- [1] Jeremy F. Alm. 401 and beyond: improved bounds and algorithms for the Ramsey algebra search. J. Integer Seq., 20(8):Art. 17.8.4, 10, 2017.
- [2] Jeremy F. Alm and A. Ylvisaker. A fast coset-translation algorithm for computing the cycle structure of Comer relation algebras over $\mathbb{Z}/p\mathbb{Z}$. To Appear.
- [3] Noga Alon and Jean Bourgain. Additive patterns in multiplicative subgroups. *Geom. Funct. Anal.*, 24(3):721–739, 2014.
- [4] S. D. Comer. Color schemes forbidding monochrome triangles. In *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983)*, volume 39, pages 231–236, 1983.
- [5] P. Jipsen, R. D. Maddux, and Z. Tuza. Small representations of the relation algebra $\mathcal{E}_{n+1}(1,2,3)$. Algebra Universalis, 33(1):136–139, 1995.
- [6] Tomasz Kowalski. Representability of Ramsey relation algebras. *Algebra Universalis*, 74(3-4):265–275, 2015.
- [7] R. Maddux. Some varieties containing relation algebras. *Trans. Amer. Math. Soc.*, 272(2):501–526, 1982.
- [8] R. Maddux. Relation algebras, volume 150 of Studies in Logic and the Foundations of Mathematics. Elsevier B. V., Amsterdam, 2006.