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CLASSIFICATION OF UNIFORM FLAG TRIANGULATIONS OF THE LEGENDRE POLYTOPE

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Abstract. The Legendre polytope is the convex hull of all pairwise differences of the standard basis vectors. It is also known as the full root polytope of type A. We completely classify all flag triangulations of this polytope that are uniform in the sense that the edges may be described as a function of the relative order of the indices of the four basis vectors involved. These triangulations fall naturally into three classes: the lex class, the revlex class and the Simion class. We also determine that the refined face counts of these triangulations only depend on the class of the triangulations. The refined face generating functions are expressed in terms of the Catalan and Delannoy generating functions and the modified Bessel function of the first kind.

1. Introduction

Triangulations of root polytopes and of products of simplices have been a subject of intense study in recent years [2, 5, 6, 7, 13]. Motivated by an observation made in [9], we recently [11] established that the Simion type B associahedron [19] may be realized as a pulling triangulation of the Legendre polytope, defined as the convex hull of all differences of pairs of standard basis vectors in Euclidean space. These vertices can be thought of as *arrows* between numbered nodes. We also showed that all pulling triangulations are flag. The Legendre polytope is the centrally symmetric variant of the type A root polytope whose lexicographic and revlex triangulations were studied by Gelfand, Graev and Postnikov [13]. A question naturally arises: Are there other reasonably uniform triangulations of the Legendre polytope?

In this paper we fully answer this question. We classify all flag triangulations that are uniform in the sense that the flag condition depends only on the relative order on the numbering of the basis vectors involved. The key tool we use is a characterization of triangulations of a product of simplices given by Oh and Yoo [16, 17] in terms of *matching ensembles*. We determine that there are three classes of triangulations: variants of the lexicographic pulling triangulation, variants of the revlex pulling triangulation, and variants of the triangulation representing the Simion type B associahedron.

It is known that all triangulations of the boundary of the Legendre polytope have the same face numbers. For pulling triangulations, this was shown in [14]. For general triangulations it may be

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shown by generalizing the argument in [14] and using the fact that all triangulations of a product of simplices have the same face numbers [17, Lemma 2.3].

To distinguish between the three major classes of uniform flag triangulations, we introduce a refined face count which keeps track of the number of forward and backward arrows in each face. Surprisingly we find that the refined face count of a triangulation only depends on which class it belongs to, regardless how we fix the number of forward and backward arrows. For the Simion class the generating function for Catalan numbers and weighted generalizations of the Delannoy numbers play a crucial role in the refined face count. For the revlex class different weighted generalizations of the Delannoy numbers and modified Bessel functions of the first kind play the important role. Finally for the lex class, the Catalan numbers play the essential role.

Our paper is structured as follows. In Section 3 we state our main classification theorem and prove the sufficiency part. The necessity part is in Section 4. In Section 5 we outline that all triangulations that we consider are actually pairwise non-isomorphic. Results facilitating the refined face count in all cases are presented in Section 6. The actual refined face count appears in Sections 7, 8 and 9, respectively. We end the paper with concluding remarks.

2. Preliminaries

2.1. **The Legendre polytope.** The *Legendre polytope* P_n is the convex hull of the $n(n+1)$ vertices $e_j - e_i$ where $i \neq j$ and $\{e_1, e_2, \ldots, e_{n+1}\}$ is the orthonormal basis of the Euclidean space $\mathbb{R}^{n+1}$. This polytope was first studied by Cho [8], and it is called the "full" type A root polytope in the work of Ardila, Beck, Hoşten, Pfeifle and Seashore [2]. The name Legendre polytope [14] is motivated by the fact that the polynomial $\sum_{j=0}^{n} f_{j-1} \cdot ((x-1)/2)^j$ is the nth Legendre polynomial, where f_i is the number of i-dimensional faces in any pulling triangulation of the boundary of P_n. Another way to view the Legendre polytope is to intersect the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$ with the $(n+1)$-dimensional cross-polytope formed by the convex hull of the vertices $\pm 2e_1$, $\pm 2e_2$, …, $\pm 2e_{n+1}$.

The Legendre polytope P_n contains the root polytope P_n^+, defined as the convex hull of the origin and the set of points $e_j - e_i$, where $i < j$. The polytope P_n^+ was first studied by Gelfand, Graev and Postnikov [13] and later by Postnikov [18]. Many properties of the Legendre polytope P_n are straightforward generalizations of the properties of the root polytope P_n^+.

We use the shorthand notation (i, j) for the vertex $e_j - e_i$ of the Legendre polytope P_n. We may think of these vertices as the set of all directed nonloop edges on the vertex set $\{1, 2, \ldots, n+1\}$. To avoid confusion between edges and vertices of the Legendre polytope, we will refer to the vertices of P_n as *arrows*. The root polytope P_n^+ is then the convex hull of the origin and of the forward arrows. Suppressing the orientation, we arrive at the edges representing vertices in the work of Gelfand, Graev and Postnikov [13].

A subset of arrows is contained in some face of P_n exactly when there is no $i \in \{1, 2, \ldots, n+1\}$ that is both the head and the tail of an arrow; see [14, Lemmas 4.2 and 4.4]. Equivalently, the faces are products of two simplices [11, Lemma 2.2]: we may write them as $\Delta_I \times \Delta_J$ where $I, J \neq \emptyset$, $I \cap J = \emptyset$

and the symbol Δ_K denotes the convex hull of the set $\{e_i : i \in K\}$ for $K \subseteq \{1, 2, \ldots, n+1\}$. The analogous observations regarding the faces of the root polytope P_n^+ that do not contain the origin may be found in [13] where the sets of edges (forward arrows in our terminology) representing vertices contained in a proper face are called *admissible*. Facets of P_n are exactly the faces $\Delta_I \times \Delta_J$ where the disjoint union of I and J is $\{1, 2, \ldots, n+1\}$. The edges of P_n are of the form $\Delta_I \times \Delta_J$ where $\{|I|, |J|\} = \{1, 2\}$. The two-dimensional faces $\Delta_I \times \Delta_J$ of P_n are either squares when $|I| = |J| = 2$ or triangles when $\{|I|, |J|\} = \{1, 3\}$.

Affine independent subsets of vertices of faces of the Legendre polytope are described as follows. A set $S = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ is a $(k-1)$-dimensional simplex if and only if, disregarding the orientation of the directed edges, the set S contains no cycle, that is, it is a forest [14, Lemma 2.4]. The analogous observations were made for the root polytope P_n^+ in [13] and for products of simplices in [10, Lemma 6.2.8] (see also [5, Lemma 2.1]).

In a recent paper [11], the authors have shown that the Simion type B associahedron [19] is combinatorially equivalent to a pulling triangulation of the boundary of the Legendre polytope. For an exact definition of a pulling triangulation we refer the reader to [11]. Here we only point out the following key observation [11, Theorem 3.1]. Recall that a simplicial complex is *flag* if the minimal non-faces have cardinality 2.

Theorem 2.1. *Every pulling triangulation of the boundary of the Legendre polytope P_n is flag.*

Thus the triangulation giving rise to a combinatorial equivalent of the Simion type B associahedron is completely determined by the rules given in the associated column of Table 1.

The last two columns in Table 1 are the analogous rules for two other pulling triangulations of the boundary of the Legendre polytope P_n, also discussed in [11]. These are the lexicographic (lex) and revlex pulling orders. Their restriction to the root polytope P_n^+ are called the antistandard, respectively, standard triangulations in [13]. The terminology we use in [11] was introduced in [14], where it was observed that these are pulling triangulations. (These pulling triangulations are not to be confused with the terms "lexicographic triangulation" and "reverse lexicographic triangulation" used in [21] where the first is a *placing* triangulation, and only the second is a true pulling triangulation.) The lexicographic pulling order was also studied in [2]. The reader may take Table 1 as a definition of these flag complexes. We obtain an independent verification of the fact that these rules yield triangulations of the boundary ∂P_n of the Legendre polytope. However, our results in this paper also apply to the twelve other variants of these triangulations.

2.2. **Characterizing triangulations of $\Delta_{a-1} \times \Delta_{b-1}$ via matching ensembles.** Our key tool to verify that the flag complexes we define triangulate the boundary of the Legendre polytope P_n is the characterization of the triangulations of the Cartesian product $\Delta_{a-1} \times \Delta_{b-1}$ given by Oh and Yoo [16, 17]. See also [5]. We identify the vertices of $\Delta_{a-1} \times \Delta_{b-1}$ with edges in the complete bipartite graph $K_{a,b}$, whose vertex set is $\{1, 2, \ldots, a\} \uplus \{\overline{1}, \overline{2}, \ldots, \overline{b}\}$, and call this the *bipartite graph representation* of $\Delta_{a-1} \times \Delta_{b-1}$. By [10, Lemma 6.2.8] a set of affine independent vertices of $\Delta_{a-1} \times \Delta_{b-1}$ corresponds to a forest in the bipartite graph representation, and maximal affine independent sets correspond to trees. Facets of a triangulation of $\Delta_{a-1} \times \Delta_{b-1}$ thus correspond to spanning trees. The

Type	Order of nodes	Type B associahedron	Lexicographic pulling	Revlex pulling
$THTH$	$i_1 < j_1 < i_2 < j_2$	$i_1 \; j_1 \; i_2 \; j_2$	$i_1 \; j_1 \; i_2 \; j_2$	$i_1 \; j_1 \; i_2 \; j_2$
$HTHT$	$j_1 < i_1 < j_2 < i_2$	$j_1 \; i_1 \; j_2 \; i_2$	$j_1 \; i_1 \; j_2 \; i_2$	$j_1 \; i_1 \; j_2 \; i_2$
$THHT$	$i_1 < j_1 < j_2 < i_2$	$i_1 \; j_1 \; j_2 \; i_2$	$i_1 \; j_1 \; j_2 \; i_2$	$i_1 \; j_1 \; j_2 \; i_2$
$HTTH$	$j_1 < i_1 < i_2 < j_2$	$j_1 \; i_1 \; i_2 \; j_2$	$j_1 \; i_1 \; i_2 \; j_2$	$j_1 \; i_1 \; i_2 \; j_2$
$TTHH$	$i_1 < i_2 < j_1 < j_2$	$i_1 \; i_2 \; j_1 \; j_2$	$i_1 \; i_2 \; j_1 \; j_2$	$i_1 \; i_2 \; j_1 \; j_2$
$HHTT$	$j_1 < j_2 < i_1 < i_2$	$j_1 \; j_2 \; i_1 \; i_2$	$j_1 \; j_2 \; i_1 \; i_2$	$j_1 \; j_2 \; i_1 \; i_2$

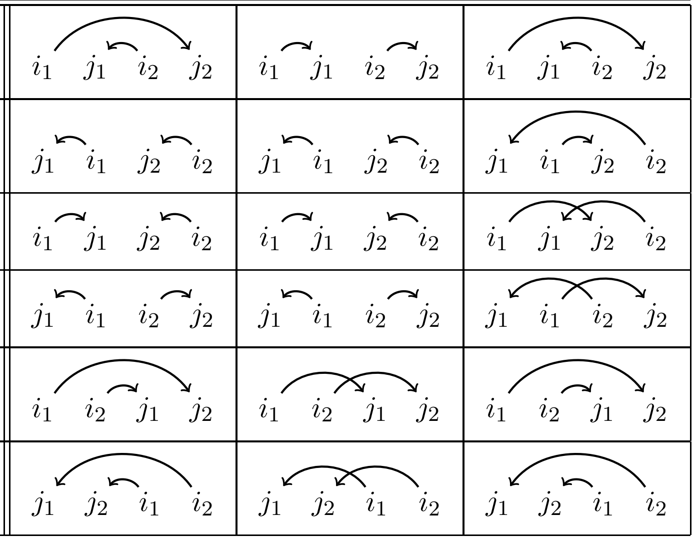


TABLE 1. Pairs of arrows that are edges in three triangulations of the boundary ∂P_n of the Legendre polytope.

results of Oh and Yoo characterize those sets of spanning trees that correspond to a triangulation of $\Delta_{a-1} \times \Delta_{b-1}$.

Definition 2.2. *A family $\mathcal{M}$ of matchings of $K_{a,b}$ is a* matching ensemble *if it satisfies the following three axioms:*

- **Support axiom:** *For $I \subseteq \{1, 2, \ldots, a\}$ and $\overline{J} \subseteq \{\overline{1}, \overline{2}, \ldots, \overline{b}\}$ with $|I| = |\overline{J}|$ there is a unique matching in $\mathcal{M}$ that matches the elements of I with the elements of $\overline{J}$ in the subgraph induced by $I \uplus \overline{J}$ of $K_{a,b}$.*
- **Closure axiom:** *Any subset of a matching in $\mathcal{M}$ is also a matching in $\mathcal{M}$.*
- **Linkage axiom:** *If m is a non-empty matching in $\mathcal{M}$ and v is any vertex of $K_{a,b}$ not incident to any edge of m then there is an edge $e \in m$ and there is an edge $e' \notin m$ incident to v such that the resulting matching $m' = (m - e) \cup e'$ also belongs to $\mathcal{M}$.*

For T a spanning tree of $K_{a,b}$ define $\phi(T)$ to be the set of all matchings contained in the edges of the tree T. Extend this notion to families of spanning trees by defining

$$\Phi(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} \phi(T).$$

S. Oh and H. Yoo proved the following result [17, Theorem 5.4].

Theorem 2.3 (Oh–Yoo)**.** *The function Φ is a bijection between families of spanning trees representing triangulations of $\Delta_{a-1} \times \Delta_{b-1}$ and matching ensembles of the bipartite graph $K_{a,b}$.*

Ceballos, Padrol and Sarmiento [6, Lemma 2.5] explicitly describe the inverse Φ^{-1}.

Lemma 2.4 (Ceballos–Padrol–Sarmiento)**.** *Given a matching ensemble $\mathcal{M}$ on $K_{a,b}$, the spanning tree T of $K_{a,b}$ belongs to $\Phi^{-1}(\mathcal{M})$ if and only if for each matching $m \in \mathcal{M}$, there is no cycle in $T \cup m$ that alternates between T and m.*

Closely related to this result is the following lemma, essentially due to Postnikov; see Lemma 12.6 in [18]. Although Postnikov originally made the statement in the case of spanning trees, the proof carries over with very little modification to the case of forests. Recall that for a forest F in $K_{a,b}$ we denote Δ_F to be simplex in $\Delta_{a-1} \times \Delta_{b-1}$ whose vertices correspond to the edges of the forest.

Lemma 2.5 (Postnikov)**.** *Let F and F' be two forests in the bipartite graph $K_{a,b}$. The intersection of the two simplices $\Delta_F \cap \Delta_{F'}$ is either empty or a simplex represented by a set of edges of $K_{a,b}$ if and only if the graph $F \cup F'$ does not contain a cycle of length greater than or equal to 4 in which the edges alternate between F and F'.*

Proof. The proof of the necessity is exactly the same as for spanning trees. If there is a cycle $(i_1, \overline{j_1}, i_2, \overline{j_2}, \ldots, i_k, \overline{j_k})$ such that

$$\{\{i_1, \overline{j_1}\}, \{i_2, \overline{j_2}\}, \ldots, \{i_k, \overline{j_k}\}\} \subseteq F \quad \text{and} \quad \{\{i_1, \overline{j_k}\}, \{i_2, \overline{j_1}\}, \ldots, \{i_k, \overline{j_{k-1}}\}\} \subseteq F'$$

then the point $1/k \cdot \sum_{s=1}^{k}(e_{i_s} - e_{\overline{j_s}})$ belongs to the intersection $\Delta_F \cap \Delta_{F'}$, but all vertices of the smallest dimensional faces of the two simplices containing the point do not belong to the intersection. To prove the converse, we only need to add one sentence to Postnikov's proof. Direct all edges $\{i, \overline{j}\} \in F \setminus F'$ from i to $\overline{j}$ and direct all edges $\{i, \overline{j}\} \in F' \setminus F$ from $\overline{j}$ to i. The resulting set $U(F, F')$ of directed edges is acyclic. We select a height function that is constant on the connected components of $F \cap F'$ and increases along the directed edges in $U(F, F')$ joining two connected components of $F \cap F'$. Since we started with forests instead of spanning trees, the resulting height function is still undefined on those nodes that are not incident to any edge in the union $F \cup F'$. For these, we select the height to be the same constant selected to be less than any of the already defined values. ☐

3. Classifying uniform flag triangulations of the Legendre polytope

3.1. Uniform triangulations and their classification.

A common property of all three flag complexes described in Table 1 is that the edges are defined in a *uniform fashion*. We make this clear in the following definition. First, let V_n be the vertex set defined by

$$V_n = \{(i, j) \ : \ 1 \le i, j \le n+1, i \ne j\}.$$

Definition 3.1. *A flag simplicial complex $\triangle_n$ on the vertex set V_n is a* uniform flag complex *if determining whether or not a pair of vertices $\{(i_1, j_1), (i_2, j_2)\}$ forms an edge depends only on the equalities and inequalities between the values of i_1, i_2, j_1 and j_2.*

We begin with the necessary conditions for describing uniform flag triangulations. To facilitate making statements, we introduce some new terminology and notation. We use the letter T to mark the tail of each arrow and the letter H to mark the head. For each pair of arrows on four nodes, we will indicate the relative order of the two heads and two tails by writing down the appropriate letters left to right in the order as they occur. We will refer to the resulting word as the *type* of the pair of arrows. After that we will simply state in words the condition that a pair of arrows of a given type must satisfy to be an edge of the triangulation. Examples of this convention are given in Table 1.

The main classification result in this paper is the following.

Theorem 3.2. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n for some $n \geq 5$ that satisfies the necessary conditions stated in Proposition 3.4. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope if and only if it satisfies exactly one of the following conditions:*

(1) Both $THTH$ and $HTHT$ types of pairs of arrows do not nest, and both $HTTH$ and $THHT$ types of arrows do not cross.
(2) Both $THTH$ and $HTHT$ types of pairs of arrows nest, and both $HTTH$ and $THHT$ types of arrows cross.
(3) Exactly one of the $THTH$ and $HTHT$ types of pairs of arrows nest. Furthermore, if both $THHT$ and $HTTH$ types of pairs cross then both $TTHH$ and $HHTT$ types of pairs nest.

The three classes in listed in Theorem 3.2 are pairwise mutually exclusive. We name them as follows, and give brief motivations why.

(1) This class contains the triangulation obtained by the lexicographic pulling order of the Legendre polytope and hence is named *the lex class.*
(2) This class contains the triangulation obtained by the revlex pulling order of the Legendre polytope and hence is named *the revlex class.*
(3) This class contains the Simion type B associahedron and hence is called *the Simion class*. Furthermore, we subdivide this class into the three subclasses, the Simion subclass of types a through c, according to:
  (a) Both $THHT$ and $HTTH$ types of pairs do not cross.
  (b) Exactly one of the $THHT$ and $HTTH$ types of pairs cross.
  (c) Both $THHT$ and $HTTH$ types of pairs cross, and both $TTHH$ and $HHTT$ types of pairs nest.

In Subsection 3.2 we prove the necessity part of Theorem 3.2 in Propositions 3.6 through 3.8. The sufficiency part of Theorem 3.2 is proved Section 4. The main tool for proving these results is Theorem 3.5 which gives necessary and sufficient conditions for a simplicial complex on the vertex set V_n to be a triangulation of the boundary ∂P_n of the Legendre polytope. These conditions, called the support and the linkage axioms, are based upon Definition 2.2.

We end this subsection by introducing two commuting operations on triangulations. Let $\triangle_n$ be a uniform flag complex on the vertex set V_n. Let the *dual triangulation* $\triangle_n^*$ be the triangulation obtained by reversing all the arrows. Let the *reflected dual triangulation* $\overline{\triangle_n}$ be the triangulation obtained by reversing all the arrows and replacing node i with $n + 2 - i$.

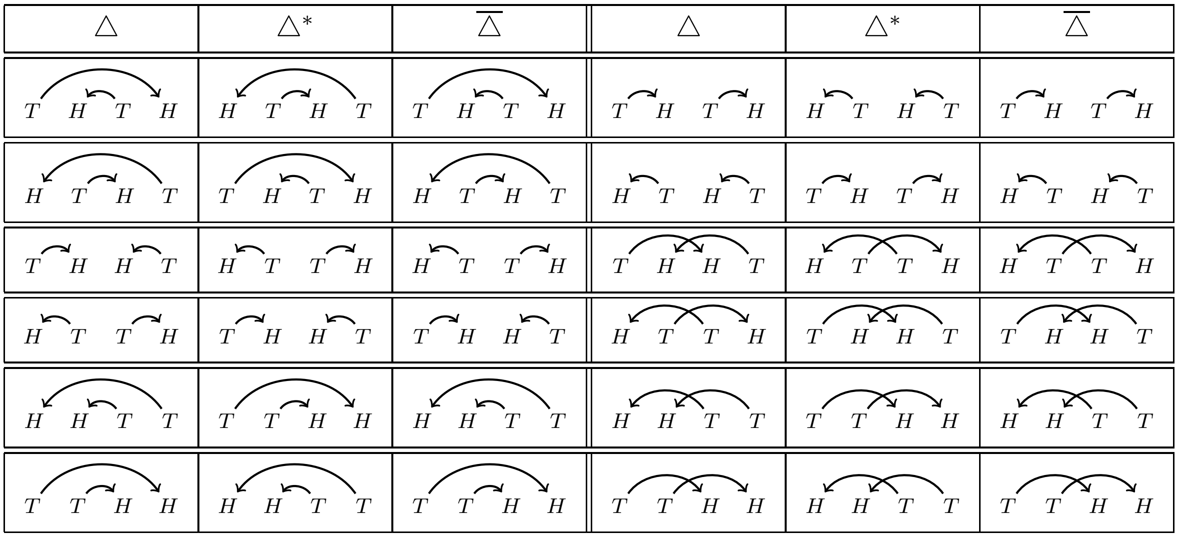


TABLE 2. The action of the two involutions $\triangle \longmapsto \triangle^*$ and $\triangle \longmapsto \overline{\triangle}$.

Lemma 3.3. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n. Then the uniform flag complexes $\triangle_n$ and $\triangle_n^*$ belong to the same (sub)class. Furthermore, the conditions on the types $THTH$, $HTHT$, $TTHH$ and $HHTT$ stay invariant under the involution $\triangle_n \longmapsto \overline{\triangle_n}$, whereas the condition on the types $THHT$ and $HTTH$ switches.*

Proof. See Table 2. □

3.2. **Necessary conditions for uniform flag complex.** We now state necessary conditions for a uniform flag complex to represent a triangulation of the boundary ∂P_n of the Legendre polytope.

Proposition 3.4. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n where each vertex $(i, j) \in V_n$ is identified with the vertex $e_j - e_i$ of the Legendre polytope P_n. If the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope then it satisfies the following criteria:*

(1) There is no edge of the form $\{(i, j), (j, k)\}$ in the complex $\triangle_n$.
(2) For each three-element subset $\{i, j, k\}$ of $\{1, 2, \ldots, n+1\}$, the two sets $\{(i, j), (i, k)\}$ and $\{(j, i), (k, i)\}$ are edges in the complex $\triangle_n$.
(3) For each four-element subset $\{i_1, i_2, j_1, j_2\}$ of $\{1, 2, \ldots, n+1\}$, exactly one of the two sets $\{(i_1, j_1), (i_2, j_2)\}$ and $\{(i_1, j_2), (i_2, j_1)\}$ is an edge in the complex $\triangle_n$.

Proof. Condition (1) is equivalent to requiring that the faces of the flag complex represent affine independent vertex sets. Condition (2) is necessary to make sure that the 2-dimensional triangular faces of P_n belong to the triangulation. Condition (3) is necessary to ensure that each 2-dimensional square face is subdivided into two triangles by a diagonal. □

Condition (2) and the fact that the complex $\triangle_n$ is a flag complex imply that for two disjoint subsets I and J such that $|I| = 3$ and $|J| = 1$ that the set $I \times J$ forms a 2-dimensional face (triangle) of $\triangle_n$. Similarly, $J \times I$ is also a triangular face of the complex $\triangle_n$.

The next result is essential for proving the necessary and sufficient conditions in Theorem 3.2.

Theorem 3.5. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n satisfying the conditions of Proposition 3.4. Let $\mathcal{M}$ be the family of all faces that are matchings, that is, let*

$$\mathcal{M} = \{\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\} \in \triangle_n \, : \, |\{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}| = 2k\}.$$

Identify each vertex (i, j) with the vertex $e_j - e_i$ of the Legendre polytope. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope if and only if the family of matchings $\mathcal{M}$ satisfies the following two properties:

- (SA) *For two disjoint subsets $I, J \subset \{1, 2, \ldots, n + 1\}$ satisfying $|I| = |J|$ there is a unique $\sigma \in \mathcal{M}$ such that $\sigma \subseteq I \times J$ and $|\sigma| = |I|$;*
- (LA) *Assume I and J are disjoint subsets of $\{1, 2, \ldots, n + 1\}$. Let σ be a non-empty matching in $\mathcal{M}$ such that $\sigma \subseteq I \times J$. Then for each $k \notin I \cup J$ there is an edge $(i, j) \in \sigma$ such that $(\sigma - \{(i, j)\}) \cup \{(k, j)\} \in \mathcal{M}$. Also, for each $k \notin I \cup J$ there is an edge $(i, j) \in \sigma$ such that $(\sigma - \{(i, j)\}) \cup \{(i, k)\} \in \mathcal{M}$.*

Proof. The conditions stated in Proposition 3.4 imply that each face of $\triangle_n$ represents a subset of a proper face $\Delta_I \times \Delta_J$ of the Legendre polytope P_n, where I and J are disjoint subsets of $\{1, 2, \ldots, n+1\}$. Under these conditions $\triangle_n$ represents a triangulation of ∂P_n if and only if for each pair (I, J) of disjoint nonempty subsets of $\{1, 2, \ldots, n + 1\}$ the set of faces whose vertices are contained in $I \times J$ represent a triangulation of the face $\Delta_I \times \Delta_J$ of P_n. Property (SA) is equivalent to the support axiom in the definition of a matching ensemble, while property (LA) is equivalent to the linkage axiom. The closure axiom is an immediate consequence of the fact that a subset of a face is a face in a simplicial complex. By Theorem 2.3 the family of matchings $\mathcal{M}$ must satisfy the stated axioms.

To prove the converse, assume that $\mathcal{M}$ satisfies the stated axioms and let I and J be an arbitrary pair of nonempty disjoint subsets of $\{1, 2, \ldots, n + 1\}$. Then the *restriction* $\mathcal{M}|_{I \times J}$ of $\mathcal{M}$ to $I \times J$, i.e., the set of matchings contained in $\mathcal{M}$ whose elements belong to $I \times J$, is a matching ensemble. By Theorem 2.3 there is a triangulation $\triangle(I, J)$ of $\Delta_I \times \Delta_J$ corresponding to this matching ensemble for which the elements of $\mathcal{M}|_{I \times J}$ are the matchings of the complete bipartite graph $K_{I,J}$ contained in the spanning trees representing the facets of $\triangle(I, J)$. It suffices to show that $\triangle(I, J)$ is the family $\triangle_n|_{I \times J}$ of faces of $\triangle_n$ whose vertices are contained in $I \times J$.

Assume, by way of contradiction, that the two simplicial complexes $\triangle(I, J)$ and $\triangle_n|_{I \times J}$ differ. If there is a face σ of $\triangle(I, J)$ that does not belong to $\triangle_n|_{I \times J}$ then (the vertex sets being equal) this face σ also contains an edge $\{(i_1, j_1), (i_2, j_2)\}$ that does not belong to $\triangle_n|_{I \times J}$, since $\triangle_n|_{I \times J}$ is a flag complex which would contain σ if it contained all of its edges. By part (2) of Proposition 3.4 the set $\{i_1, j_1, i_2, j_2\}$ must have four distinct elements, and by part (3) of the same proposition we must have $\{(i_1, j_2), (i_2, j_1)\} \in \triangle_n|_{I \times J}$. By the definition of $\mathcal{M}$ we have $\{(i_1, j_2), (i_2, j_1)\} \in \mathcal{M}|_{I \times J}$ and by our assumption we also have $\{(i_1, j_1), (i_2, j_2)\} \in \mathcal{M}|_{I \times J}$. This violates the uniqueness part of the support

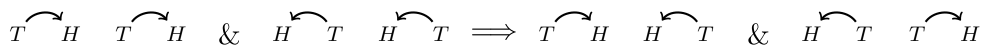


FIGURE 1. A graphical representation of Proposition 3.6.

axiom (SA) for the pair of subsets $(\{i_1, i_2\}, \{j_1, j_2\})$. Hence $\triangle(I, J)$ is contained in $\triangle_n|_{I\times J}$, that is, $\triangle(I, J) \subseteq \triangle_n|_{I\times J}$.

We are left with the possibility of having a face σ in $\triangle_n|_{I\times J}$ that does not belong to $\triangle(I, J)$. After adding a few more vertices, if necessary, we may assume that this face σ is a facet of $\triangle_n|_{I\times J}$. The vertices of the facet σ are arrows from I to J which, disregarding their orientation, must form a forest. If some arrows of σ form a cycle $\{\{i_1, j_1\}, \{i_2, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}, \{i_1, j_k\}\}$ then the uniqueness part of the support axiom (SA) is violated for the pair of sets $(\{i_1, i_2, \ldots, i_k\}, \{j_1, j_2, \ldots, j_k\})$, since both $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ and $\{(i_1, j_k), (i_2, j_1), \ldots, (i_k, j_{k-1})\}$ belong to $\mathcal{M}$. Consider the centroid $1/|\sigma| \cdot \sum_{(i,j)\in\sigma}(e_j - e_i)$ of the face of P_n represented by σ. This point belongs to some facet Δ_T of the triangulation $\triangle(I, J)$. Here T is the spanning tree of $K_{I,J}$ representing the facet. All vertices of σ cannot be represented by edges belonging to T, for otherwise σ belongs to $\triangle(I, J)$. Hence, by Lemma 2.5 there is a cycle $(i_1, j_1, i_2, j_2, \ldots, i_k, j_k)$ in $K_{I,J}$ such that the matching $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ belongs to T and the matching $\{(i_1, j_k), (i_2, j_1), \ldots, (i_k, j_{k-1})\}$ belongs to σ. Both matchings belong to $\mathcal{M}$, and we have reached a contradiction with Lemma 2.4. □

We now begin to obtain the necessary conditions.

Proposition 3.6. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n representing a triangulation of the boundary ∂P_n of the Legendre polytope for $n \geq 4$. Assume that pairs of arrows of types $THTH$ and $HTHT$ do not nest in the triangulation $\triangle_n$. Then pairs of arrows of types $HTTH$ and $THHT$ do not cross in the triangulation $\triangle_n$.*

Proof. Assume by way of contradiction, that $HTTH$ type pairs of arrows cross in $\triangle_n$. Hence the edge $\sigma = \{(2, 5), (4, 1)\}$ is in the triangulation $\triangle_n$. Now let $k = 3$ and apply the linkage axiom (LA) in Theorem 3.5. We obtain either the edge $\{(2, 5), (4, 3)\}$ or $\{(2, 3), (4, 1)\}$, contradicting the assumed condition on $THTH$ or $HTHT$. The second conclusion follows by reversing all the arrows in the proof. □

The next necessary condition is completely analogous to that of Proposition 3.6.

Proposition 3.7. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n representing a triangulation of the boundary ∂P_n of the Legendre polytope for $n \geq 4$. Assume that pairs of arrows of the types $THTH$ and $HTHT$ nest in the triangulation $\triangle_n$. Then pairs of arrows of types $HTTH$ and $THHT$ cross in the triangulation $\triangle_n$.*

Proof. The proof is completely analogous to the proof of Proposition 3.6, but this time use the edge $\sigma = \{(2, 1), (4, 5)\}$ in $\triangle_n$. □

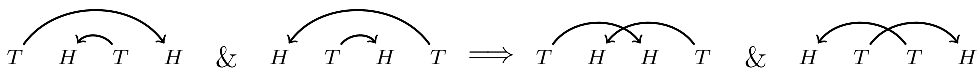


FIGURE 2. A graphical representation of Proposition 3.7.

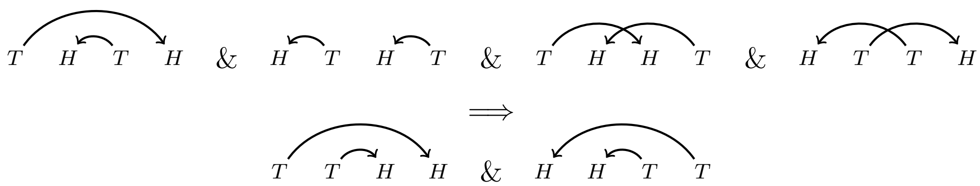


FIGURE 3. A graphical representation of the first case of Proposition 3.8.

Proposition 3.8. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n representing a triangulation of the boundary ∂P_n of the Legendre polytope for $n \geq 5$. Assume exactly one of the $THTH$ and $HTHT$ type of pairs of arrows nest, and the other type does not nest. Also assume that both $THHT$ and $HTTH$ type of pairs of arrows cross. Then both $TTHH$ and $HHTT$ type of pairs nest.*

Proof. Without loss of generality we may assume that $THTH$ type of arrows nest and $HTHT$ type of pairs do not nest; the opposite case may be dealt with by reversing all arrows. Assume that $HHTT$ type of arrows cross and observe that both $\{(2,1),(4,3),(6,5)\}$ and $\{(2,5),(4,1),(6,3)\}$ form faces in the triangulation $\triangle_n$. Note that this contradicts the support axiom (SA) in Theorem 3.5 and thus $HHTT$ type of arrows must nest. A similar contradiction may be reached when $TTHH$ type of arrows cross, by considering the faces $\{(1,6),(3,2),(5,4)\}$ and $\{(1,4),(3,6),(5,2)\}$. □

We conclude this section by an observation that we will frequently use in our proofs in the situations when we need to consider only forward arrows or only backward arrows.

Theorem 3.9. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4, and the condition that $THTH$ type of pairs of arrows do not nest. Then the restriction $\triangle_n^+$ of $\triangle_n$ to the set of forward arrows represents a triangulation of the union of those boundary facets of the polytope P_n^+ that do not contain the origin.*

Proof. Since all the arrows are forward arrows and any non-incident pair of such arrows is of the type $TTHH$ or $THTH$, the face structure of $\triangle_n^+$ depends on the rules associated with $TTHH$ and $THTH$. If such arrow pairs cross then the faces of any $\triangle_n^+|_{I\times J}$ are all sets of forward arrows $\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\}$ where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ hold. If such pairs of arrows nest then the faces of $\triangle_n|_{I\times J}$ are all sets of forward arrows $\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\}$ such that $i_1 < i_2 < \cdots < i_k$ and $j_1 > j_2 > \cdots > j_k$ hold. In other words, we obtain the well-known lexicographic and revlex pulling triangulations of P_n^+. □

4. Verifying the sufficiency part of the classification

In this section we show that any uniform flag complex $\triangle_n$ on V_n that satisfies one of the sets of criteria listed in Theorem 3.2 represents a triangulation of the boundary ∂P_n of the Legendre polytope. We do this by verifying in each case that the support and linkage axioms of Theorem 3.5 are satisfied.

To verify the support axiom (SA), we will list the elements of the disjoint union $I \cup J$ in order, marking each element of I with a T (tail) and each element of J with a H (head). Thus we obtain a word containing the same number of letters T and H. We also associate to each tail an up $(1,1)$ step, and to each head a down $(1,-1)$ step. These steps yield a lattice path starting at the origin and ending on the x-axis. We will describe the unique matching contained in $\triangle_n$ whose tail set is I and head set is J in terms of this *associated TH-word* and *associated lattice path*.

During the verification of the support axiom (SA) we will often treat forward and backward arrows separately. This separation leads to considering special associated TH-words and lattice paths.

Definition 4.1. *We call a lattice path consisting of up $(1,1)$ steps and down $(1,-1)$ steps a* lower (upper) Dyck path *if it starts and ends on the x-axis and never goes above (below) the x-axis. We call a TH-word a lower (upper) Dyck word if replacing each T with an up step and each H with a down step results in a lower (upper) Dyck path.*

The following lemma is straightforward.

Lemma 4.2. *For any uniform flag complex on V_n and any face $\sigma \subset V_n$ that is a matching and consists of backward arrows only the lattice path associated to the tail set I and head set J is a lower Dyck path.*

Indeed, at any stage the number of tails listed cannot exceed the number of heads listed. The next lemma is a partial converse of Lemma 4.2.

Lemma 4.3. *Let $\triangle_n$ be a uniform flag complex on V_n that satisfies the necessary conditions stated in Theorem 3.2 and has the property that $HTHT$ type of pairs of arrows do not nest. Let I and J be two sets satisfying $I, J \subset \{1, 2, \ldots, n+1\}$, $I \cap J = \emptyset$ and $|I| = |J| \neq 0$. Assume that the TH-word w associated with the two sets I and J is a lower Dyck word. Then there is a unique matching contained in $\triangle_n$ that matches I to J. Furthermore, this matching consists of backward arrows only.*

Proof. First we show that no matching from I to J contains a forward arrow. Assume by way of contradiction that there is a smallest counterexample to this statement, and let (i, j) be the forward arrow with the smallest tail i in such an example. The associated lattice path must start with a down step, hence the least element of $I \cup J$ is a head j_1, the head of a backward arrow (i_1, j_1). If $i_1 < i$ holds then the removal of the arrow (i_1, j_1) yields a smaller counterexample, in contradiction with our assumption of minimality. We cannot have $i_1 > j$ either as $HTHT$ type of pairs of arrows do not nest. Hence we have $j_1 < i < i_1 < j$ and $HTTH$ type of pairs of arrows cross. The same reasoning may be repeated for any backward arrow whose head is to the left of i: the tails of these arrows are all between i and j, in particular no tail of a backward arrow is to the left of i. By the choice of i, there is no tail of a forward arrow to the left of i either: the first up step in the associated lattice path is contributed by the tail i. Hence i is preceded by $k \geq 1$ heads: $j_1, j_2, \ldots, j_k$, and the tails $i_1, i_2, \ldots, i_k$ of these backward arrows all occur before j. The associated lattice path goes above the x-axis before the down step associated to j unless there is a backward arrow (i', j') whose head j' occurs before j, while i' occurs only after it. The pair $\{(i,j),(i',j')\}$ is a crossing $THHT$ type of pair of arrows. Since both $THHT$ and $HTTH$ type of pairs cross, the complex $\triangle_n$ cannot belong to the lex class. It cannot belong to the revlex class either because $HTHT$ type of pairs of arrows do not nest by our assumption. We are left with the possibility that $HTHT$ type of pairs do not nest and $THTH$ type of pairs nest. By Proposition 3.8 $HHTT$ type of pairs nest. As a consequence no backward arrow (i', j') satisfying $j' < j < i'$ can cross any arrow (i_s, j_s) satisfying $j_s < i < i_s < j$. But then the associated lattice path goes above the x-axis at the step associated to $\max(i_1, i_2, \ldots, i_k)$ and we obtain a contradiction.

Having established that no matching can contain a forward arrow, we may show the existence of a unique matching by induction. Regardless on the condition on the $HHTT$ type of pairs of arrows, the first step of the associated lattice path is a down step, corresponding to a head j_1. We only need to show that there is a unique way to identify the tail i_1 of this arrow, and that the removal of the steps associated to j_1 and i_1 results in a lattice path that does not go above the horizontal axis. In the case when $HHTT$ type of pairs nest, i_1 must be the tail marking the first return to the horizontal axis, because all arrows whose head is between j_1 and i_1 must also have their tail in the same interval as backward arrows cannot cross. The removal of the first down step and the first return to the x-axis yields a lattice path that does not go above the x-axis. Finally, in the case when $HHTT$ type of pairs cross, i_1 must be the least tail, marking the first up step, as any back arrow whose tail precedes i_1 would form a $HHTT$ type of nesting pair with (i_1, j_1). The removal of the first up step and the first down step yields once again a lattice path that does not go above the x-axis. □

Remark 4.4. The proof of Lemma 4.3 defines the matchings induced by the associated lattice paths in a recursive fashion, but it is not difficult to prove the following explicit rules by induction:

(1) If $HHTT$ type of arrows nest then each head j, representing a down step, is matched to the tail i representing the first return to the same level.
(2) If $HHTT$ type of arrows cross then the kth head in the left to right order is matched to the kth tail in the left to right order.
(3) In either case, each head j is matched to a tail i in such a way that there is no return to the x-axis strictly between the down step associated to j and the up step associated to i.

Lemma 4.5. *Assume the same conditions hold as in Lemma 4.3, but with the extra condition that the associated word w factors as a product of two lower Dyck words w_1 and w_2. Then the matching as a graph is a disjoint union of the two matchings for w_1, respectively w_2.*

Proof. The matching obtained by taking the union of the two matchings for w_1, respectively w_2, satisfies all the conditions. Hence by uniqueness the result follows. □

Remark 4.6. We will use Lemmas 4.2 and 4.3 for backward arrows most of the time, but the reader should note that the dual statements, for forward arrows and associated lattice paths being upper Dyck paths, also hold.

These lemmas were about the case when $THTH$ type of arrows do not nest. We will consider the case when they nest in the dual form below.

Lemma 4.7. *Let $\triangle_n$ be a uniform flag complex on V_n that satisfies the necessary conditions stated in Theorem 3.2 and has the property that $HTHT$ type of pairs of arrows nest. Let I and J be two sets satisfying $I, J \subset \{1, 2, \ldots, n+1\}$, $I \cap J = \emptyset$ and $|I| = |J| \neq 0$. There is a matching in $\triangle_n$ consisting of forward arrows only that matches I to J if and only if every tail precedes every head, that is, $i < j$ holds for all $i \in I$ and $j \in J$. Furthermore, if every tail precedes every head in $I \cup J$ then the matching contained in $\triangle_n$ that matches I to J is unique.*

Proof. It is a direct consequence of the condition on $THTH$ type of pairs of arrows that all tails must precede all heads in every face that consists of forward arrows only. Conversely assume that $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ satisfy $i_1 < i_2 < \cdots < i_k$ and $i_k < j$ for all $j \in J$. Clearly, any arrow whose tail is in I and whose head is in J is a forward arrow. Any pair of such arrows is a $TTHH$ type of pair. Hence the only matching contained in $\triangle_n$ is $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ where $j_1 < j_2 < \cdots < j_k$ holds if $TTHH$ type of pairs cross and $j_1 > j_2 > \cdots > j_k$ holds when $TTHH$ type of pairs nest. □

The verification of the linkage axiom (LA) is facilitated by the following observations.

Lemma 4.8. *Let $\triangle_n$ be any uniform flag complex on V_n satisfying the conditions stated in Proposition 3.4. Then the complex $\triangle_n$ satisfies the relevant part of the linkage axiom (LA) when k is inserted as a tail and there exist $i \in I$ such that there is no element of $I \cup J$ strictly between k and i. Similarly, when k is inserted as a head and there exist $j \in J$ such that there is no element of $I \cup J$ strictly between k and j, the linkage axiom is satisfied.*

In other words, if we insert a new tail in position k next to a tail, then we may extend the arrow containing the old tail to contain the new tail instead. A similar observation can be made about inserting a new head. We only need to verify the linkage axiom (LA) in the cases when a new head is inserted between two tails (or as the least or largest node, next to a tail), and when a new tail is inserted between two heads (or as the least or largest node, next to a head).

Lemma 4.9. *Let $\triangle_n$ be any uniform flag complex on V_n satisfying the necessary conditions stated in Theorem 3.2, and that $HTHT$ type of arrows do not nest. Let I and J be two sets satisfying $I, J \subset \{1, 2, \ldots, n+1\}$, $I \cap J = \emptyset$, and $|I| = |J| \neq 0$. Assume that TH-word associated with the two sets I and J is a lower Dyck word. If $k > \min(I \cup J)$ is inserted as a tail or $k < \max(I \cup J)$ and k is inserted as a head, then the relevant part of the linkage axiom is verified in such a way that the new arrow is also backward arrow.*

Proof. By symmetry, it is enough to consider the case when k is inserted as a tail. By Lemma 4.8 we can assume that k is inserted after head j of an arrow (i, j). If $HHTT$ type of pairs nest then we match j to k and we remove the tail i. Note that the new arrow does not introduce any crossings. If $HHTT$ type of pairs cross then consider the least tail $i' > k$, which is the tail of an arrow (i', j'). Note that $j' \leq j$ since otherwise there would be two nesting backward arrows. Match k to j' and remove the tail i'. By Remark 4.4 part (2), it is straightforward to see that we obtain a matching belonging to $\triangle_n$. Note that in both cases, all the new arrows are backward arrows. □

4.1. **The lex class.** In this subsection we show that all four triangulations in the lex class are possible.

Theorem 4.10. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the property that both $THTH$ and $HTHT$ types of arrows do not nest. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

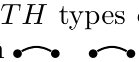
Proof. We begin by observing that Proposition 3.6 implies that $THHT$ and $HTTH$ types of arrows do not cross. Next we verify the support and linkage axioms of Theorem 3.5.

To verify the support axiom (SA), mark maximal runs of the associated lattice path that are above the x-axis with forward arrows. Similarly, mark the maximal runs of the lattice path that are below the x-axis with backward arrows. The fact that no end of a forward arrow can occur between the head and tail of a backward arrow follows from the fact that arrows of opposite direction neither cross nor nest. By the same reason no tail of a backward arrow can occur between the head and tail of a forward arrow. Within each maximal run, match the heads H and the tails T according to the $TTHH$ and $HHTT$ rules. By Lemmas 4.2, 4.3 and their duals, this can be done in a unique way. Furthermore, note that arrows of opposite direction do not cross and do not nest. This shows that there is a unique way to obtain a matching between the set of tails I and the set of heads J.

To verify the linkage axiom (LA), we may by symmetry assume that we are inserting a new tail at position k. Denote this node by T'. If the new tail is adjacent to an old tail, we are done by Lemma 4.8. If a new tail T' is inserted in between two heads we have three possible cases. If the two heads are part of a $THHT$ pattern then we are between two maximal runs as described above. Take any of the two heads adjacent to the inserted element, unlink it from its pair and link it to the inserted tail; see the first line of Figure 4. If the two heads are part of a $HHTT$ pattern, there are two subcases depending upon whether the two arrows nest or cross, and these two cases are explained in the second and third line of Figure 4. In each of these two subcases, one can verify that the new matching is in fact a face. Finally, if the two heads are part of a $TTHH$ pattern, it is mirror symmetric to the previous case. □

4.2. **The revlex class.** We now turn our attention to the revlex class and show that all four triangulations are possible.

Theorem 4.11. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the properties that both $THTH$ and $HTHT$ types of arrows nest. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

Proof. By Proposition 3.7 we conclude that $THHT$ and $HTTH$ types of arrows cross. Hence if we disregard the direction of the arrows, we have that the pattern  cannot occur. Thus all pairs of arrows must either nest or cross.

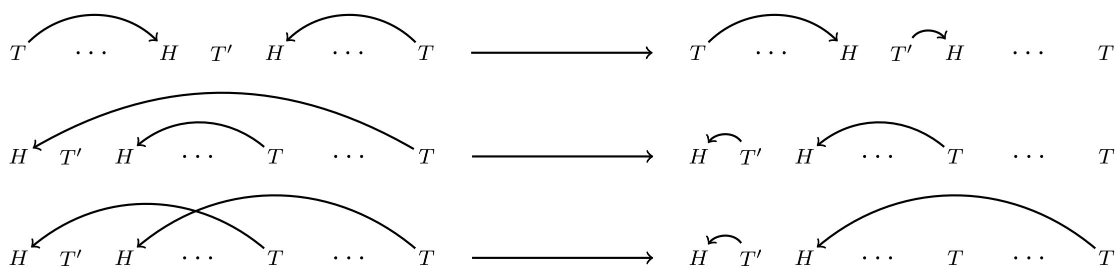


Figure 4. Verifying the linkage axiom in the lex class of triangulations.

The fact that every pair of arrows nests or crosses implies that all arrows must arch over the *midpoint* of the set $I \cup J$, that is, the point that has the same number of elements of $I \cup J$ to the left of it as the number of such elements to the right. For example, for the word $TTHT|HHTH$, the midpoint is marked with a vertical bar. Hence the number of tails to the left of the midpoint must equal the number of heads to the right of it. The tails to the left of the midpoint are tails of the forward arrows. They must be matched with the heads to the right of the midpoint. The analogous statement is true for the heads and tails of the backward arrows. Now match the left of midpoint tails with the right of the midpoint heads according to the $TTHH$ rule. Similarly, match the right of midpoint tails with the left of the midpoint heads according to the $HHTT$ rule. Both of these matchings are unique, proving the support axiom.

To verify the linkage axiom (LA), by symmetry it is enough to verify the linkage axiom after inserting a new tail at position k, denoted by T'. Assume that there another tail T of an arrow (i, j) in the set of tails I adjacent to T' and that the position i is on the same side of the midpoint as position k. Then we can replace the arrow (i, j) with (k, j). If there is no such arrow (i, j), there is no tail on the same side as T', that is, the situation is $T \cdots T|HH \cdots HT'H \cdots H$ or $H \cdots HT'H \cdots HH|T \cdots T$. These two possibilities are symmetric, so it enough to consider the first one. Note that all the arrows are forward arrows. Let (i, j) be the arrow with the smallest value of j, that is, the arrow attached to the first H. Replace the arrow (i, j) with the new arrow (k, j). Note that this yields the new word $T \cdots TH|H \cdots HT'H \cdots H$, where there is now an H on the left of the new midpoint. This completes the verification of the linkage axiom. □

4.3. **The Simion class of triangulations.** In this section we conclude the proof of Theorem 3.2. The remaining case is the Simion class of triangulations.

Theorem 4.12. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the properties that exactly one of the types $THTH$ and $HTHT$ do nest. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

Using the involution $\triangle_n \longmapsto \triangle_n^*$ it is enough to consider the cases where $THTH$ type of arrows nest and $HTHT$ type of arrows do not nest in the following propositions. We begin by considering all four flag complexes in the Simion subclass of type a.

Proposition 4.13. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the properties that the $THTH$ type of arrows nest, the $HTHT$ type of arrows do not nest, and both $THHT$ and $HTTH$ types of arrows do not cross. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

Proof. Let w be the associated TH-word to the two sets I and J. Factor the word w as follows

$$w = w_1 T \cdots T w_h T w_{h+1} H w_{h+2} H \cdots H w_{2h+1}$$

where the factors $w_1, w_2, \ldots, w_{2h+1}$ are lower Dyck words. Note that such a factorization exists and is unique since the h Ts in the expression correspond to left-to-right maxima of the lattice path. Similarly, the Hs correspond to right-to-left maxima. Create a matching from I to J by making h forward arrows from the h Ts to the h Hs, according to the $TTHH$ rule. Finally, apply Lemma 4.3 to each factor w_r to create a matching consisting of only backward arrows. It is straightforward to see that there is no crossing between a forward and a backward arrow, that is, the $THHT$ and $HTTH$ conditions hold. Furthermore, no backward arrow nests a forward arrow, the $HTHT$ condition follows and finally no two forward arrows follow each other, so the $THTH$ condition is also true.

Next we show the uniqueness part of the support axiom. Assume a matching from I to J contains h forward arrows. Since $THTH$ type of pairs nest, the tails of all forward arrows precede all heads. The heads and tails of the forward arrows partition the number line into $2h+1$ segments. Since $THHT$ and $HTTH$ types of pairs do not cross and $HTHT$ do not nest, backward arrows that have one end in one of these line segments have their other end in the same line segment. By Lemma 4.2 the part of the lattice path associated to all backward arrows in one of these line segments represent a lattice path starting and ending at the same level and never going above the level where it started. Hence the tails of the forward arrows mark the first ascents to level 1, 2, …, h and the heads of the forward arrows mark the last descents to levels $h-1$, $h-2$, …, 0. This observation shows the unique determination of the endpoints of the forward arrows. By Lemma 4.7 there is a unique way to match the heads and tails of the forward arrows, and by Lemma 4.3 there is a unique way to match the heads and tails of the backward arrows within each segment created by the endpoints of the forward arrows.

To verify the linkage axiom (LA), note that Lemma 4.9 is applicable unless k is inserted as a tail at the beginning of a run of backward arrows or as a head at the end of such a run. Assume k is inserted as a tail; the case when k is inserted as a head is completely analogous. If k is inserted right after a tail, then we are done by Lemma 4.8. If k is inserted right after a head j, then this head is necessarily the head of a forward arrow (i, j). In this case insert the backward arrow (k, j) and remove the arrow (i, j). Note that this results in a matching in $\triangle_n$ in which there is one less forward arrow and the arrow (k, j) becomes part of the run of backward arrows immediately succeeding it. We are left with the case when the inserted tail k satisfies $k < \min(I \cup J)$. If the current matching contains at least one forward arrow, then we select the forward arrow (i, j) with the smallest i. We remove the arrow (i, j) and add the arrow (k, j). This move does not change the crossing or nesting properties of arrows of the same direction, nor does it create crossing arrows of opposite directions. Finally, if the current matching on $I \cup J$ consists of backward arrows only, then we associate an initial NE step to k, and continue with the lattice path associated to $I \cup J$ which now starts and ends at level 1 and never goes above that level. Let $j \in J$ be the head associated to the last descent from level 1 to level 0. This is the head of a backward arrow (i, j). Removing (i, j) and adding (k, j) results in a valid matching because of part (3) of Remark 4.4. □

We next turn our attention to the Simion subclass of type c.

Proposition 4.14. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the properties that the $THTH$ type of arrows nest, the $HTHT$ type of arrows do not nest, both $THHT$ and $HTTH$ types of arrows cross and both $TTHH$ and $HHTT$ type of arrows nest. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

Proof. Let w be the associated TH-word to the two sets I and J and assume that the associated lattice path reaches height h. That is, there are at least h ascents (T) before the path reaches height h, and at least h descents (H) since it leaves height h the last time. Hence we factor the word w uniquely as

$$w = H^{m_1}TH^{m_2}T \cdots TH^{m_h}T \cdot u \cdot HT^{n_1}HT^{n_2}H \cdots HT^{n_h}$$

Using the rule for $TTHH$ (nest), match the first h tails in this expression with the last h heads to create h pairwise nesting forward arrows. The remaining nodes contribute the subword $w' = H^{m_1+m_2+\cdots+m_h} \cdot u \cdot T^{n_1+n_2+\cdots+n_h}$. Note that this is a lower Dyck word. Thus apply Lemma 4.3 to this word to create the remaining arrows, which are all backward arrows. Directly by the construction rules hold for types $THTH$, $TTHH$ and $HHTT$. Next notice that $HTTH$ must cross, since all tails of backward arrows are after the tails of forward arrows. By the symmetric argument $THHT$ must also cross. The final condition for $HTHT$ is that no backward arrow can nest a forward arrow. Recall that the word w reaches its maximum height somewhere in the factor u. Hence the word w' reaches the maximum of 0 also in the factor u. Thus w' can be factored into two lower Dyck words $w' = (H^{m_1+m_2+\cdots+m_h} \cdot u_1) \cdot (u_2 \cdot T^{n_1+n_2+\cdots+n_h})$. By Lemma 4.5 each backward arrow has either its tail or head (or both) in the word $u_1 \cdot u_2$. Hence it lies between the tail and and head of every forward arrow and last condition is proved.

Next we prove uniqueness. Assume that the associated lattice path reaches height h. Observe that the conditions imply that all the forward arrows must nest. Let (i', j') be the innermost forward arrow. Since no backward arrow nests, whether in front or behind this shortest forward arrow, each backward arrow must have either its tail or head (or both) in the interval from i' to j'. Let h' be the number of forward arrows. Hence the first h' Ts in the associated TH-word correspond to the tails of the forward arrows. Similarly, the last h' Hs correspond to the head of the forward arrows. Since the backward arrows nest. Consider the backward arrow (i'', j'') with the smallest head j''. If $j'' < i'$ then there is no backward arrow above the tail i'', only h' forward arrows. Similarly, if there is no such backward arrow then there is no backward arrow above i', and only h' forward arrows. In both cases we obtain that the maximal height h is h'. This agrees with the construction in the previous paragraph. The uniqueness in Lemma 4.3 implies that the matching from I to J is unique.

Before proving the linkage axiom recall that the TH-word w has the following factorization

$$w = \underbrace{H^{m_1}TH^{m_2}T \cdots TH^{m_h}T}_{\text{1st factor}} \cdot z \cdot \underbrace{HT^{n_1}HT^{n_2}H \cdots HT^{n_h}}_{\text{5th factor}},$$

where

$$z = \underbrace{u_1 T u_2 T \cdots u_m T}_{\text{2nd factor}} \cdot \underbrace{x}_{\text{3rd factor}} \cdot \underbrace{H v_1 H v_2 \cdots H v_n}_{\text{4th factor}},$$

where $m = \sum_{i=1}^{h} m_i$, $n = \sum_{i=1}^{h} n_i$ and $u_1, u_2, \ldots, u_m, x, v_1, v_2, \ldots, v_n$ are all lower Dyck words. Consider the case when we insert a new tail T'. By Lemma 4.9 it can be inserted in the one of the lower Dyck words $u_1, \ldots, u_m, x, v_1, \ldots, v_n$ or immediately after one of these words. If inserted in the 1st factor, or immediate afterward, we can move one of the tails of the forward arrows. The case when inserted in the 2nd or 3rd factor is already taken care of. In the 4th factor, in front of a v_i then it is immediately after a head, which can be used for the switch. Finally, in the 5th factor when $n > 0$ we can move one of tails of the closest backward edge from the 5th factor to the 4th factor. When $n = 0$, remove the innermost forward arrow and use its head. Observe that this is only case that changes the number of forward arrows. This completes the proof of the linkage axiom (LA). □

Finally, we consider half of the cases in the Simion subclass of type b. Observe that the proof is a mixture of the two previous proofs.

Proposition 4.15. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the properties that the $THTH$ type of arrows nest, the $HTHT$ type of arrows do not nest, the $HTTH$ type of arrows cross and the $THHT$ type of arrows do not cross. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

Proof. Let w be the associated TH-word to the two sets I and J and assume that the associated lattice path reaches height h. Factor the word w as follows $z_1 \cdot z_2 = z_1 \cdot H w_{h+2} H \cdots H w_{2h+1}$ where the factors $w_{h+2}, w_{h+3}, \ldots, w_{2h+1}$ are lower Dyck words. Note that such a factorization exists and is unique since the h Hs are right-to-left maxima. Note that the word z_1 contains at least h Ts. Factor z_1 as $H^{m_1} T H^{m_2} T \cdots T H^{m_h} T w_{h+1}$. That is, we have

$$w = z_1 \cdot z_2 = H^{m_1} T H^{m_2} T \cdots T H^{m_h} T w_{h+1} \cdot H w_{h+2} H \cdots H w_{2h+1}.$$

Create a matching from I to J by making h forward arrows from the first h Ts in z_1 to the h Hs in z_2, according to the $TTHH$ rule. Apply Lemma 4.3 to each factor w_{h+2} through w_{2h+1} to create matchings consisting of only backward arrows. Finally, on the remaining letters H^{m_1}, H^{m_2}, …, H^{m_h}, w_{h+1} apply Lemma 4.3 again to create the remaining backward arrows. It is straightforward to see that this matching satisfies all the conditions and hence the existence part of the support axiom (SA) holds.

We now prove the uniqueness part of the support axiom (SA). Let us denote the number of forward arrows in the matching between I and J by h. The endpoints of these arrows partition the number line into $2h+1$ segments. Since $THHT$ pairs do not cross, just as in the proof of Proposition 4.13, backward arrows having one end in one of the last h segments have both ends in the same segment. The endpoints of these arrows within the same segment are associated to a lattice path starting and ending at the same level. Before the leftmost head of a forward arrow the numbers of tails exceeds the number of heads by h and this is the largest height reached by the associated lattice path. The heads of the forward arrows mark the last descents to level $h-1$, $h-2$, …, 0, respectively. Since $HTTH$

type of pairs cross, no tail of a backward arrow may occur before the tail of the forward arrow: the leftmost h tails are tails of forward arrows, and they mark the first h ascents in the associated lattice path. The rest of the proof of the uniqueness and existence parts of the support axiom is very similar to the one in the proof of Proposition 4.13, thus we omit the details. We only underscore the key difference: we treat all backward arrows whose tail is to the right of the heads of the forward arrows as a single set: after removing the tails of the forward arrow this yields a lattice path from level 0 to level 0 that never goes above the x-axis, and the tails of the forward arrows are correctly reinserted if and only if they are to the left of the tails of all backward arrows. Hence we consider $h+1$ runs of backward arrows: the last h runs are just like in the proof of Proposition 4.13 and the first run is different.

To verify the linkage axiom (LA) we observe that Lemma 4.9 is applicable in all cases except when a new head is inserted at the end or a new tail is inserted at the beginning of any of the $h+1$ runs of backward arrows. If a new tail is inserted at the beginning of one of the last h runs of backward arrows then we may proceed exactly as in the proof of Proposition 4.13 when k is inserted right after the head of a forward arrow. If $k < \min(I \cup J)$ is inserted as a tail, again, we may proceed as in the proof of Proposition 4.13 (note that there is nothing to the left of this inserted new tail). If k is inserted as a head at the end of one of the first h runs of backward arrows then k is immediately followed by the head j of a forward arrow (i, j) and Lemma 4.8 is applicable. We are left with the case when $k > \max(I \cup J)$ and k is inserted as a head. If the matching on $I \cup J$ contains at least one forward arrow, we proceed as in the proof of Proposition 4.13 (note that the dual case when k is inserted as a tail at the beginning was explained). Finally, when the matching on $I \cup J$ consists of backward arrows only, then let (i, j) be the backward arrow whose tail i is $\min(I)$. Replacing (i, j) with (i, k) yields a matching in $\triangle_n$. □

Proposition 4.16. *Let $\triangle_n$ be a uniform flag complex on the vertex set V_n that satisfies the necessary conditions stated in Proposition 3.4 and has the properties that the $THTH$ type of arrows nest, the $HTHT$ type of arrows do not nest, the $HTTH$ type of arrows do not cross and the $THHT$ type of arrows cross. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope.*

Proof. Follows from Proposition 4.15 and Lemma 3.3 by applying the involution $\triangle_n \longmapsto \overline{\triangle_n}$. □

Proof of Theorem 4.12. The result in the case $THTH$ nests and $HTHT$ do not nest follows by combining Propositions 4.13, 4.14, 4.15 and 4.16. The case when $THTH$ do not nest and $HTHT$ nests follows by the involution $\triangle_n \longmapsto \triangle_n^*$. □

5. Fifteen distinct triangulations

The classification given in Theorem 3.2 was accompanied by the observation that some triangulations are (reflected) duals of each other. Taking the dual of a uniform flag triangulation amounts to taking a centrally symmetric copy, taking the reflected dual amounts to composing this reflection about the origin with a reflection about the subspace defined as the intersection of the $\lceil n/2 \rceil$ hyperplanes defined by the equations $x_i = x_{n+2-i}$ for $1 \le i \le \lceil n/2 \rceil$. In this section we outline how to prove that there are no other isomorphisms between the uniform flag triangulations of the Legendre polytope.

Theorem 5.1. *For $n \geq 4$ there are 15 non-isomorphic uniform flag triangulations of the boundary of the Legendre polytope P_n. They are distributed as follows: the lex class and the revlex class each contain 3 triangulations; the two Simion subclasses a and b each contain 4 triangulations; and finally the Simion subclass c only contains 1 triangulation.*

The upper bound of 15 is a direct consequence of Lemma 3.3. In the rest of this section we show how the triangulations differ already when $n = 4$.

In every flag triangulation of the boundary ∂P_n of the Legendre polytope each arrow (i, j) has at least $2n - 2$ neighbors: the set $\{(i, k) : k \notin \{i, j\}\} \cup \{(k, j) : k \notin \{i, j\}\}$ is a subset of cardinality $2n - 2$ of the set of neighbors of (i, j).

Definition 5.2. *The* excess degree $\varepsilon(i, j)$ *of the arrow $(i, j) \in V_n$ in a uniform flag complex $\triangle_n$ on the vertex set V_n is the number of arrows (i', j') such that $|\{i, i', j, j'\}| = 4$ and $\{(i, j), (i', j')\} \in \triangle_n$.*

In other words, the excess degree $\varepsilon(i, j)$ is the amount by which the degree of (i, j) exceeds $2n - 2$. Table 3 shows how to compute the excess degrees. For example, whenever $THTH$ pairs of arrows nest and $i < j$, then for each arrow (i, j) there are $\binom{q}{2}$ ways to select (i', j') satisfying $i < j' < i' < j$, where $p = i - 1$, $q = j - i - 1$ and $r = n + 1 - j$.

In Table 4 we show the sorted lists of the excess degrees for the 15 triangulations when $n = 4$. It is straightforward to observe that these lists are distinct, showing that the triangulations are non-isomorphic.

6. Tools for refined face enumeration

In this section we introduce some terminology and results that we will use to prove theorems regarding the refined face counting in uniform flag triangulations of the boundary ∂P_n of the Legendre polytope. Our triangulations are defined by a set of rules, independent of the dimension. After fixing such a set of rules, we will simultaneously consider each triangulation $\triangle_n$ determined on the vertex set V_n defined by these rules, for each $n \geq 0$. Note that the set V_0 is the empty set, and the only face contained in $\triangle_0$ is the empty set.

In order to compute the associated generating function, we introduce the following more general notion. Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$ be a sequence of collections of arrows, where $\mathcal{F}_n$ is a subset of the power set of V_n, that is, $\mathcal{F}_n \subseteq 2^{V_n}$. We define the associated generating function

$$F(\mathcal{F}, x, y, t) = \sum_{n,i,j \geq 0} f(\mathcal{F}_n, i, j) \cdot x^i y^j t^n$$

where $f(\mathcal{F}_n, i, j)$ is the number of sets in $\mathcal{F}_n$ consisting of i forward arrows and j backward arrows. Our interest is to compute this generating function when $\mathcal{F}$ is the collection $\mathcal{F} = (\triangle_0, \triangle_1, \ldots)$. In this

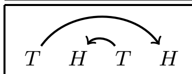
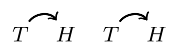
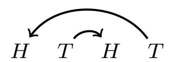
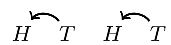
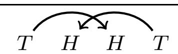
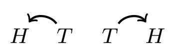
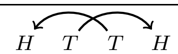
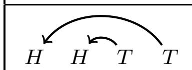
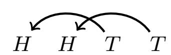
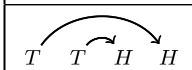
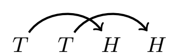
Condition	Contribution to $\varepsilon(i,j)$	Contribution to $\varepsilon(j,i)$	Condition	Contribution to $\varepsilon(i,j)$	Contribution to $\varepsilon(j,i)$
	$\binom{q}{2}$	pr		$\binom{p}{2}+\binom{r}{2}$	0
	pr	$\binom{q}{2}$		0	$\binom{p}{2}+\binom{r}{2}$
	$\binom{r}{2}$	$\binom{p}{2}$		qr	pq
	$\binom{p}{2}$	$\binom{r}{2}$		pq	qr
	0	$pr+\binom{q}{2}$		0	$pq+qr$
	$pr+\binom{q}{2}$	0		$pq+qr$	0

TABLE 3. Table to compute the excess degrees $\varepsilon(i,j)$ and $\varepsilon(j,i)$ for $1 \le i < j \le n+1$, where $p = i-1$, $q = j-i-1$ and $r = n+1-j$.

case $f(\triangle_n, i, j)$ counts the faces of the simplicial complex $\triangle_n$ with i forward arrows and j backward arrows, and we call the polynomial $\sum_{i,j\ge 0} f(\triangle_n, i, j)\cdot x^i y^j$ the *face polynomial*. We note that the power of t in a term of $F(\triangle_n, x, y, t)$ is the same as the number of vertices in an associated facet of the triangulation. The number of cases to be considered can be reduced by extending the notion of the dual and reflected dual triangulations to all families $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$.

The following lemma is a straightforward consequence of the definitions.

Lemma 6.1. *Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$ be a sequence of collections of arrows, where $\mathcal{F}_n \subseteq 2^{V_n}$. Let $\mathcal{F}^* = (\mathcal{F}_0^*, \mathcal{F}_1^*, \ldots)$ be the dual family obtained by reversing all arrows in all sets, and let $\overline{\mathcal{F}} = (\overline{\mathcal{F}_0}, \overline{\mathcal{F}_1}, \ldots)$ be the reflected dual family obtained by reversing each arrow, and replacing each node i in V_n with $n+2-i$. Then the following two equalities hold:*

$$F(\mathcal{F}^*, x, y, t) = F(\mathcal{F}, y, x, t) \quad \text{and} \quad F(\overline{\mathcal{F}}, x, y, t) = F(\mathcal{F}, x, y, t).$$

The families of uniform flag triangulations defined by a set of rules are *coherent* in the sense that they are closed under the insertion and removal of isolated nodes. To make this informal observation precise, consider the map $\pi_k : \mathbb{N} - \{k\} \to \mathbb{N}$ given by

$$\pi_k(m) = \begin{cases} m & \text{if } m < k, \\ m-1 & \text{if } m > k. \end{cases}$$

Definition 6.2. *Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$ be a collection of families of sets such that for each n the family $\mathcal{F}_n$ consists of subsets of V_n. We call such a collection* coherent *if for each subset σ of V_n and*

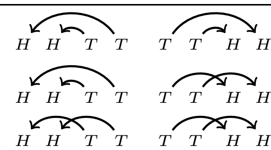
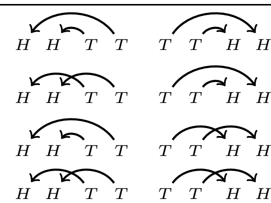
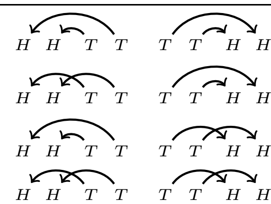
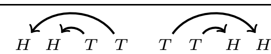
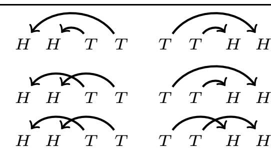
Class/ subclass	$HHTT$ & $TTHH$ conditions	Sorted list of excess degrees
Lex		$1^6, 2^4, 3^2, 4^4, 6^4$
		$0^1, 1^3, 2^7, 3^1, 4^4, 6^4$
		$0^2, 2^{10}, 4^4, 6^4$
Simion a		$1^5, 2^5, 3^5, 6^5$
		$0^1, 1^3, 2^4, 3^5, 4^4, 6^3$
		$1^4, 2^4, 3^8, 6^4$
		$0^1, 1^2, 2^3, 3^8, 4^4, 6^2$
Simion b		$0^1, 1^2, 2^5, 3^7, 4^1, 5^1, 6^3$
		$0^2, 1^1, 2^5, 3^4, 4^5, 5^1, 6^2$
		$0^2, 1^2, 2^1, 3^{10}, 4^1, 5^2, 6^2$
		$0^3, 1^1, 2^1, 3^7, 4^5, 5^2, 6^1$
Simion c		$0^2, 2^4, 3^8, 4^3, 5^2, 6^1$
Revlex		$0^4, 2^4, 4^{10}, 6^2$
		$0^4, 2^4, 3^1, 4^7, 5^3, 6^1$
		$0^4, 2^4, 3^2, 4^4, 5^6$

TABLE 4. The sorted lists of the excess degrees for $n = 4$, where superscripts denotes multiplicities.

each $k \in \{1, 2, \ldots, n+1\}$ that is not incident to any arrow in σ, the set $\pi_k(\sigma) = \{(\pi_k(i), \pi_k(j)) : (i, j) \in \sigma\}$ belongs to $\mathcal{F}_{n-1}$ if and only if $\sigma \in \mathcal{F}_n$. In particular, for a coherent collection $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$, the empty set either belongs to all $\mathcal{F}_n$ or it belongs to none of them.

By abuse of terminology we will say that $\mathcal{F}$ *contains the empty set* if all families $\mathcal{F}_n$ in it contain it. Sets of arrows in coherent collections may be enumerated by counting only the *saturated* sets in the families, which we now define.

Definition 6.3. *For $n \geq 1$, a subset σ of V_n is* saturated *if the set of endpoints of its arrows is the set $\{1, 2, \ldots, n+1\}$. We also consider the empty set to be a saturated subset of $V_0 = \emptyset$. Given a coherent collection $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots)$, we denote the family of saturated sets in $\mathcal{F}_n$ by $\widehat{\mathcal{F}}_n$.*

Note that $\widehat{\mathcal{F}_0} = \{\emptyset\}$ exactly when $\mathcal{F}$ contains the empty set. One of our key enumeration tools is the following observation.

Lemma 6.4. *Given a coherent collection of families $\mathcal{F}$ of arrows, the face generating function satisfies*

$$F(\mathcal{F},x,y,t) = -\frac{t}{(1-t)^2}\cdot \delta_{\mathcal{F}_0,\{\emptyset\}} + \frac{1}{(1-t)^2}\cdot F\left(\widehat{\mathcal{F}},x,y,\frac{t}{1-t}\right), \tag{6.1}$$

$$F(\widehat{\mathcal{F}},x,y,z) = \frac{z}{1+z}\cdot \delta_{\widehat{\mathcal{F}}_0,\{\emptyset\}} + \frac{1}{(1+z)^2}\cdot F\left(\mathcal{F},x,y,\frac{z}{1+z}\right), \tag{6.2}$$

where δ denotes the Kronecker delta function.

Proof. Consider the second term of equation (6.1). The power n in a term $x^i y^j t^n$ in $F(\widehat{\mathcal{F}},x,y,t)$ counts the number of spaces between nodes in the digraph. The substitution $t \longmapsto t/(1-t)$ corresponds to subdividing each space into more spaces, that is, inserting isolated nodes into these spaces. Finally, the factor of $1/(1-t)^2$ corresponds to inserting isolated nodes before and after the full subset. This completes the proof of the first equation in the case when $\widehat{\mathcal{F}}_0 = \emptyset$. In the case when $\widehat{\mathcal{F}}_0 = \{\emptyset\}$ we need to correct the right-hand side of the first equation by subtracting $1/(1-t)^2$ contributed by z^0 in $F(\widehat{\mathcal{F}},x,y,z)$ and we need to add $1/(1-t)$ to account for the empty set belonging to all families $\mathcal{F}_n$. Equation (6.1) is equivalent to (6.2) by noting that $z = t/(1-t)$ is equivalent to $t = z/(1+z)$. ◻

An interesting special case is counting all facets with a given number of forward and backward arrows. The following statement is straightforward.

Lemma 6.5. *Let $\triangle_n$ be any uniform flag triangulation of the boundary ∂P_n of the Legendre polytope. Then a face $\sigma \in \triangle_n$ is a facet if and only if it is saturated and contains no isolated nodes, that is, every $i \in \{1,2,\ldots,n+1\}$ is incident to some element of σ.*

In other words, as a subset of V_n, a facet is the set of edges of a forest with no isolated nodes. Such a forest has $n+1$ nodes and n arrows. If the number of forward arrows is i then the number of backward arrows is $n-i$, contributing a term $x^i y^{n-i} t^n$ to the generating function of all faces and the term $x^i y^{n-i} z^n$ to the generating function of all saturated faces.

Corollary 6.6. *Let $\mathcal{F} = (\triangle_0, \triangle_1, \ldots)$ be a coherent family of uniform flag triangulations. Then the facet generating function $\sum_{n\geq 0}\sum_{i=0}^{n} f(\triangle_n, i, n-i)x^i y^{n-i} z^n$ may be obtained by substituting x/w into x, y/w into y and wz into t in $F(\mathcal{F},x,y,t)$ and then evaluating the resulting expression at $w = 0$. Alternatively it may also be obtained by substituting x/w into x, y/w into y and wz into z in $F(\widehat{\mathcal{F}},x,y,z)$ and then evaluating the resulting expression at $w = 0$.*

For any uniform flag triangulation of the boundary ∂P_n of the Legendre polytope, the vertex sets consisting only of forward (backward) arrows form subcomplexes whose face numbers are easier to count. We first review the rephrasing of a known result.

A useful way to express our results is in terms of the generating function for the Catalan numbers:

$$C(u) = \sum_{n\geq 0} C_n \cdot u^n = \frac{1-\sqrt{1-4u}}{2u}.$$

Proposition 6.7. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the rule that $HTHT$ types of pairs of arrows do not nest. Then the following two identities hold:*

$$F(\mathcal{F}, 0, y, t) = \frac{1 - t - \sqrt{1 - (4y + 2)t + t^2}}{2yt}, \tag{6.3}$$

$$F(\widehat{\mathcal{F}}, 0, y, z) = \frac{1}{1 + z} \cdot (C(yz(z + 1)) + z). \tag{6.4}$$

Proof. Setting $x = 0$ implies that we are only interested in digraphs with backward arrows. By the dual of Theorem 3.9 the subcomplex of faces formed by backward arrows represents the lexicographic or revlex pulling triangulation of the faces of P_n^+ not containing the origin. (The choice depends on the rule for the $HHTT$ pairs of arrows.) Both triangulations have the same face numbers. The proof of Theorem 5.4 in [14, Theorem 5.4] implies the quadratic equation

$$F(\mathcal{F}, 0, y, t) = 1 + t \cdot F(\mathcal{F}, 0, y, t) + yt \cdot F(\mathcal{F}, 0, y, t)^2, \tag{6.5}$$

and solving it yields (6.3). Identity (6.4) follows by applying (6.2) in Lemma 6.4 and (6.3). □

Remark 6.8. Another way to prove (6.3) is to use Theorem 5.4 and Corollary 5.6 in [14], which states

$$\sum_{j=0}^{n} f(\mathcal{F}_n, 0, j) \cdot \left(\frac{u-1}{2}\right)^j = \sum_{j=0}^{n} \frac{1}{j+1} \cdot \binom{n+j}{j} \cdot \binom{n}{j} \cdot \left(\frac{u-1}{2}\right)^j = \frac{P_n^{(-1,1)}(u)}{n+1},$$

where $P_n^{(-1,1)}(u)$ is a Jacobi polynomial. The stated equation follows by integrating the well-known generating function [1, 22.9.1] of the Jacobi polynomials $P_n^{(-1,1)}(u)$.

We will later use the following corollary of Proposition 6.7. It has a direct bijective proof; see [12].

Corollary 6.9. *Let $\mathcal{F}$ be a family of uniform flag triangulations such that $HTHT$ types of pairs of arrows do not nest. Then the sum over all forests F consisting of $k \geq 1$ backward arrows, no forward arrows and no isolated nodes is*

$$G_k(z) = \sum_F z^{\#\text{nodes of } F} = C_k \cdot z^{k+1} \cdot (z+1)^{k-1}. \tag{6.6}$$

Similarly, if the uniform flag triangulations $\mathcal{F}$ satisfies the requirement that $THTH$ types of pairs of arrows do not nest, then the sum over all forests F consisting of $k \geq 1$ forward arrows, no backward arrows and no isolated nodes also yields the identity (6.6)*.*

Proof. Equation (6.6) follows by considering the coefficient of y^k in equation (6.4). Observe that there is an extra factor of z since we are counting the number of nodes. The second statement follows by reversing the first statement. □

Note that the lower extreme cases of Corollary 6.9 enumerate the anti-standard trees and the noncrossing alternating trees of Gelfand, Graev and Postnikov [13]. We will also use the following refined variant of Proposition 6.7.

Proposition 6.10. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the rule that $HTHT$ type of pairs of arrows do not nest. For each $n \geq 0$, let $\mathcal{F}_n^{(i)}$ denote the set of all faces where the sequence of heads and tails, listed in increasing order, satisfies the condition that the i smallest nodes are heads and the next node is a tail. Then the two resulting collections $\mathcal{F}^{(i)} = \left(\mathcal{F}_0^{(i)}, \mathcal{F}_1^{(i)}, \ldots\right)$ and $\widehat{\mathcal{F}^{(i)}} = \left(\widehat{\mathcal{F}_0^{(i)}}, \widehat{\mathcal{F}_1^{(i)}}, \ldots\right)$ satisfy*

$$F(\mathcal{F}^{(i)}, 0, y, t) = \frac{y^i t^i F(\mathcal{F}, 0, y, t)^i}{(1-t)^{i+1}}, \tag{6.7}$$

$$F(\widehat{\mathcal{F}^{(i)}}, 0, y, z) = \frac{1}{1+z} \cdot (yz(1+z)C(yz(z+1)))^i. \tag{6.8}$$

Proof. Without loss of generality we may assume that $HHTT$ type pairs of arrows nest. Indeed, if we fix the head-tail pattern of the nodes then we are counting the faces in a triangulation of the convex hull of all vertices represented by all backward arrows whose heads and tails are selected from the prescribed set of heads and tails. For example, if we fix $n = 3$ and the pattern $THTH$ (contributing to the collection $\mathcal{F}^{(1)}$) then we have to count the faces in the triangulation of the convex hull of $e_2 - e_1$, $e_4 - e_1$ and $e_4 - e_3$. We obtain the same face numbers for the lexicographic and revlex triangulations.

First we show that for $i > 1$ the formal power series $F(\mathcal{F}^{(i)}, 0, y, t)$ satisfies the relation

$$\begin{aligned} F(\mathcal{F}^{(i)}, 0, y, t) &= t \cdot F(\mathcal{F}^{(i)}, 0, y, t) \\ &\quad + \sum_{d \geq 1} y^d t^d \cdot F(\mathcal{F}^{(i-1)}, 0, y, t) \cdot F(\mathcal{F}, 0, y, t)^{d-1} \cdot (1 + tF(\mathcal{F}, 0, y, t)). \end{aligned} \tag{6.9}$$

The term $t \cdot F(\mathcal{F}^{(i)}, 0, y, t)$ corresponds to the possibility that the node 1 is not incident to any arrow in a face, and the dth term in the next sum accounts for the possibility that the node 1 is the head of exactly d arrows. Assume the set of tails of these arrows is $\{i_1, i_2, \ldots, i_d\}$, where $1 < i_1 < i_2 < \cdots < i_d$. These arrows contribute a factor $y^d t^d$. Since no pairs of arrows are allowed to cross, the d arrows whose tail is 1 partition the set of arrows into $d+1$ classes. Each of these classes may be empty, except for the set of arrows whose head precedes i_1: these arrows form a set where $i-1$ tails are followed by the first head when we list their endpoints in increasing order. These arrows contribute a factor of $F(\mathcal{F}^{(i-1)}, 0, y, t)$. For each j in $\{1, 2, \ldots, d-1\}$, the set of arrows whose head belongs to the set $\{i_j + 1, i_j + 2, \ldots, i_{j+1}\}$ contribute a factor $F(\mathcal{F}, 0, y, t)^{d-1}$. (Note that the factor of t contributed by i_j is already counted.) Finally the set of arrows whose tail is $i_d + 1$ or a larger element contributes a factor of $1 + tF(\mathcal{F}, 0, y, t)$. Rearranging (6.9) yields the recurrence

$$F(\mathcal{F}^{(i)}, 0, y, t) = F(\mathcal{F}^{(i-1)}, 0, y, t) \cdot \frac{yt(1 + tF(\mathcal{F}, 0, y, t))}{(1-t)(1 - ytF(\mathcal{F}, 0, y, t))}.$$

By (6.5) we may replace the factor $1 + tF(\mathcal{F}, 0, y, t)$ with $F(\mathcal{F}, 0, y, t) \cdot (1 - ytF(\mathcal{F}, 0, y, t))$. Thus we obtain

$$F(\mathcal{F}^{(i)}, 0, y, t) = F(\mathcal{F}^{(i-1)}, 0, y, t) \cdot \frac{ytF(\mathcal{F}, 0, y, t)}{1-t}.$$

Combining this recurrence with the expression

$$F(\mathcal{F}^{(1)}, 0, y, t) = \frac{ytF(\mathcal{F}, 0, y, t)}{(1-t)^2},$$

which may be shown in a completely analogous fashion, equation (6.7) follows by induction on i.

Combining equations (6.2), (6.3) and (6.7) yields the identity (6.8). □

When $HTHT$ pairs of arrows do not nest, we obtain a very different expression for enumerating faces containing backward arrows only. We employ its *dual form*, obtained after reversing all arrows, using a generalization of the Delannoy numbers. Recall a *Delannoy path* from $(0,0)$ to (a,b) is a lattice path consisting of North steps $(0,1)$, East steps $(1,0)$ and NE steps $(1,1)$. The number of Delannoy paths from $(0,0)$ to (a,b) is the Delannoy number $D_{a,b}$; see [3] for more on Delannoy numbers.

Definition 6.11. *Given two non-negative integers a and b, the* Delannoy polynomial $D_{a,b}(x)$ *is the total weight of all Delannoy paths from $(0,0)$ to (a,b), where each step contributes a factor of x. Thus the coefficient of x^j in $D_{a,b}(x)$ is the number of Delannoy paths from $(0,0)$ to (a,b) having j steps.*

The bivariate ordinary generating function of the Delannoy polynomials is given by

$$D(u, v, x) = \sum_{a,b \geq 0} u^a v^b \cdot D_{a,b}(x) = \frac{1}{1 - x(u + v + uv)}. \tag{6.10}$$

Proposition 6.12. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules requiring that $THTH$ type of pairs of arrows nest. Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies*

$$F(\widehat{\mathcal{F}}, x, 0, z) = 1 + xz \cdot \sum_{a,b \geq 0} D_{a,b}(x) \cdot z^{a+b}. \tag{6.11}$$

In particular, the contribution to $F(\widehat{\mathcal{F}}, x, 0, z)$ of all saturated faces having $a+1$ tails and $b+1$ heads is $D_{a,b}(x) \cdot xz^{a+b+1}$.

Proof. The constant 1 on the right-hand side of (6.11) accounts for the empty face. The condition on the $THTH$ type of pairs of arrows implies that for any face consisting of forward arrows only, all tails precede all heads. Assume that the set of tails of forward arrows representing a saturated face has cardinality $a+1$ and that the set of heads has cardinality $b+1$. Let the set of tails be $\{i_1 < i_2 < \cdots < i_{a+1}\}$ and the set of heads be $\{j_1, j_2, \ldots, j_{b+1}\}$. If $TTHH$ type of pairs of arrows nest, order the set of heads decreasingly, that is, $j_{b+1} < \cdots < j_2 < j_1$. Otherwise, if $TTHH$ type of pairs of arrows cross, order the set of heads increasingly, that is, $j_1 < j_2 < \cdots < j_{b+1}$. Order the arrows in lexicographic order. Associate a North step to each instance when the tail remains unchanged from the next arrow in the list, an East step to each instance when the head remains unchanged and a NE step to each instance when both head and tail change. We obtain a Delannoy path from $(0,0)$ to (a,b) in which the number of steps is one less than the number of arrows on the list. The correspondence is a bijection between all saturated faces on the given set of heads and tails and all Delannoy paths from $(0,0)$ to (a,b). □

FIGURE 5. The rules for pairs of arrows in the Simion type a subclass.

7. Face enumeration in the Simion class

In this section we compute generating functions for the class of uniform triangulations of the boundaries of the Legendre polytope in the Simion class, that is, exactly one of the $THTH$ and $HTHT$ types of pairs of arrows nest. Our main result on enumerating faces is the following result:

Theorem 7.1. *Let $\mathcal{F}$ be a collection of uniform flag triangulations belonging to the Simion class and let $\widehat{\mathcal{F}}$ be the collection of families of saturated faces. If $THTH$ types of pairs of arrows do not nest and $HTHT$ types of pairs of arrows nest then the following identity holds:*

$$F(\widehat{\mathcal{F}}, x, y, z) = \frac{C(yz(z+1)) + z}{1+z} + \frac{xz \cdot (1 + zC(yz(z+1))) \cdot C(yz(z+1))^2}{(1+z) \cdot (1 - 2C(yz(z+1))xz - C(yz(z+1))^2 xz^2)}. \tag{7.1}$$

If $THTH$ types of pairs of arrows nest and $HTHT$ types of pairs of arrows do not nest, then the symmetric identity holds:

$$F(\widehat{\mathcal{F}}, x, y, z) = \frac{C(xz(z+1)) + z}{1+z} + \frac{yz \cdot (1 + zC(xz(z+1))) \cdot C(xz(z+1))^2}{(1+z) \cdot (1 - 2C(xz(z+1))yz - C(xz(z+1))^2 yz^2)}. \tag{7.2}$$

It suffices to prove the first half of Theorem 7.1. The second half is a direct consequence of Lemmas 3.3 and 6.1, using the duality $\triangle \longmapsto \triangle^*$. We prove the first half for each subclass of the Simion class separately, in Propositions 7.2, 7.3 and 7.4, respectively.

We begin by studying the type a subclass of the Simion class; see Figure 5.

Proposition 7.2. *Let $\mathcal{F}$ be a collection of uniform flag triangulations defined by a set of rules that contain the following rules:*

(1) $THTH$ type of pairs of arrows nest.
(2) $HTHT$ type of pairs of arrows do not nest.
(3) Both $THHT$ and $HTTH$ types of pairs of arrows do not cross.

Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies:

$$\begin{aligned} F(\widehat{\mathcal{F}}, x, y, z) &= \frac{C(yz(z+1)) + z}{1+z} \\ &+ \frac{xz \cdot (1 + zC(yz(z+1))) \cdot C(yz(z+1))^2}{1+z} \cdot D(z \cdot C(yz(z+1)), z \cdot C(yz(z+1)), x). \end{aligned}$$

Proof. First we show that

$$F(\widehat{\mathcal{F}}, x, y, z) = \frac{1}{1+z} \cdot (C(yz(z+1)) + z) + \sum_{a,b \geq 0} xz^{a+b+1} D_{a,b}(x) \cdot \frac{1 + zC(yz(z+1))}{1+z} \cdot C(yz(z+1))^{a+b+2}. \tag{7.3}$$

By equation (6.4) the term $\frac{1}{1+z} \cdot (C(yz(z+1)) + z)$ accounts for the possibility of a face containing backward arrows only. In all the other cases, the face also contains forward arrows. If these arrows are incident to $a+1$ tails and $b+1$ heads, then by Proposition 6.12 these contribute $D_{a,b}(x) \cdot xz^{a+b+1}$. The $a+1$ tails and $b+1$ heads of forward arrows partition the number line into $a+b+3$ segments. Since $THHT$ and $HTTH$ type of pairs of arrows do not cross, backward arrows that have at least one end in one of these segments must have both ends in the same segment. Hence the contribution of the backward arrows may be written as a product of $a+b+3$ independent factors. Tails of backward arrows whose endpoints are contained in one of the leftmost $a+1$ segments may coincide with the tail of a forward arrow. On these segments, the total weight of nonempty sets of backward arrows must be multiplied by $(1+z)$ to account for the possibility of (not) identifying the rightmost tail of a backward arrow with the tail of a forward arrow. By equation (6.4) the total weight of nonempty sets of arrows is $\frac{1}{1+z} \cdot (C(yz(z+1)) + z) - 1$. Keeping in mind also the possibility of not inserting any backward arrow between the tails of two forward arrows, or to the left of all forward arrows, we obtain that the backward arrows discussed so far contribute a factor of

$$\left(1 + (1+z) \cdot \left(\frac{1}{1+z} \cdot (C(yz(z+1)) + z) - 1\right)\right)^{a+1} = C(yz(z+1))^{a+1}.$$

Similarly, backward arrows inserted between the $b+1$ heads of forward arrows or to the right of the heads of all forward arrows contribute a factor of $C(yz(z+1))^{b+1}$. The only remaining possibility is to insert backward arrows between the rightmost tail of a forward arrow and the leftmost head of a forward arrow. Heads and tails of backward arrows inserted on this segment cannot coincide with the head or tail of a backward arrow. They contribute a factor of

$$1 + z \cdot \left(\frac{1}{1+z} \cdot (C(yz(z+1)) + z) - 1\right) = \frac{1 + zC(yz(z+1))}{1+z}.$$

The statement is a direct consequence of (7.3) and the definition of $D(u, v, x)$ given in (6.10). □

We now consider the uniform triangulations which belong to the Simion subclass of type b. Since there is still no restriction on the rules for the $TTHH$ and $HHTT$ types of pairs, at a first glance this subclass appears to be the largest one. This appearance is misleading, as it is closed under taking the reflected dual triangulations. By Lemma 3.3 this operation takes any uniform flag triangulation in which $THHT$ type pairs cross and $HTTH$ type pairs do not cross into a uniform flag triangulation in which $THHT$ type pairs do not cross and $HTTH$ type pairs cross. As a direct consequence of Lemma 6.1, the generating function $F(\widehat{\mathcal{F}}, x, y, z)$ does not change if we take the reverse of $\mathcal{F}$.

Proposition 7.3. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the following requirements:*

(1) $THTH$ type of pairs of arrows nest.

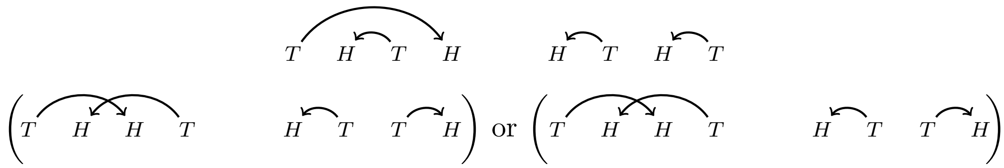


FIGURE 6. The rules for pairs of arrows in the Simion type b subclass.

(2) $HTHT$ type of pairs of arrows do not nest.
(3) Exactly one of the $THHT$ and $HTTH$ types of pairs of arrows cross.

Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies

$$\begin{aligned}F(\widehat{\mathcal{F}}, x, y, z) &= \frac{1}{1+z} \cdot (C(yz(z+1)) + z) \\ &+ D\left(C(yz(z+1))z, \frac{z}{1 - yz(z+1)C(yz(z+1))}, x\right) \\ &\cdot \frac{xz \cdot C(yz(z+1)) \cdot (1 - yzC(yz(z+1)))}{(1 - yz(z+1) \cdot C(yz(z+1)))^2}.\end{aligned}$$

Proof. The proof is similar to the proof of Proposition 7.2. As a consequence of Lemmas 3.3 and 6.1, without loss of generality we may assume that $THHT$ type of pairs do not cross and $HTTH$ type of pairs cross. We first prove that

$$\begin{aligned}F(\widehat{\mathcal{F}}, x, y, z) &= \frac{1}{1+z} \cdot (C(yz(z+1)) + z) \\ &+ \sum_{a,b\geq 0} xz^{a+b+1} D_{a,b}(x) \cdot C(yz(z+1))^{b+1} \\ &\cdot \left(1 + \sum_{i\geq 1} \frac{(yz(z+1) \cdot C(yz(z+1)))^i}{z+1} \cdot \left(\binom{i+a+1}{a+1} z + \binom{i+a}{a}\right)\right).\end{aligned} \tag{7.4}$$

Just as in the proof of Proposition 7.2, the term $\frac{1}{1+z} \cdot (C(yz(z+1)) + z)$ accounts for the possibility of a face containing backward arrows only. In all the other cases, forward arrows that are incident to $a+1$ tails and $b+1$ heads contribute a factor of $D_{a,b}(x) \cdot xz^{a+b+1}$. The $b+1$ heads of the forward arrows partition the number line into $b+2$ segments. Since $THHT$ type of pairs of arrows do not cross, backward arrows that have at least one end between the heads of two forward arrows or to the right of the largest tail of a forward arrow must have both ends in the same position. Backward arrows contained in the right $b+1$ segments contribute a factor of $C(yz(z+1))^{b+1}$, just as in the proof of Proposition 7.2.

It remains the possibility of having backward arrows that are entirely to the left of the head of any forward arrow. Let us list the endpoints of these backward arrows in increasing order. This list

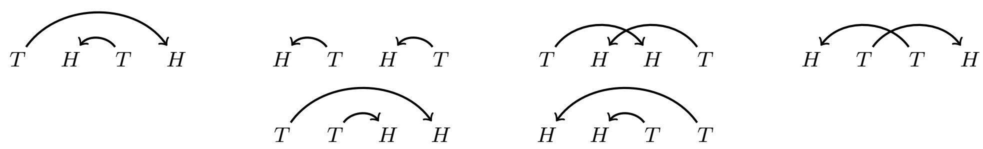


FIGURE 7. The rules for pairs of arrows in the Simion type c subclass.

must begin with a positive number of heads, followed by a tail. Let i be the number of heads of backward arrows preceding all tails. By equation (6.8), the total weight of these backward arrows is $\frac{1}{1+z} \cdot (yz(1+z)C(yz(z+1)))^i$. Since $HTTH$ type of pairs of arrows cross, the $a+1$ tails of the forward arrows must all appear before the first tail of a backward arrow, only the rightmost of them may coincide with the leftmost tail of a backward arrow. There are $\binom{i+a+1}{a+1}$ ways to insert the tails of the forward arrows strictly in front of the leftmost tail of a backward arrow, and there are $\binom{i+a}{a}$ to perform this insertion if the rightmost tail of a forward arrow is equal to the leftmost head of a backward arrow. The contribution of these arrows is the sum after 1 on the last line of (7.4).

Observe that by applying the identity $\sum_{i \geq 1} \binom{i+m}{m} \cdot t^i = 1/(1-t)^{m+1} - 1$ twice to the factor appearing on the last line of (7.4), we can rewrite this factor as

$$1 + \frac{z}{z+1} \cdot \left(\frac{1}{(1 - yz(z+1)C(yz(z+1)))^{a+2}} - 1 \right) + \frac{1}{z+1} \left(\frac{1}{(1 - yz(z+1)C(yz(z+1)))^{a+1}} - 1 \right).$$

Simplifying this expression, including canceling a factor of $z+1$ in the numerator and the denominator, yields

$$\frac{1}{(1 - yz(z+1)C(yz(z+1))^a} \cdot \frac{(1 - yzC(yz(z+1)))}{(1 - yz(z+1)C(yz(z+1)))^2}.$$

Finally, using (6.10) equation (7.4) simplifies to the desired expression in the proposition. □

We now examine the Simion subclass of type c. This is the smallest subclass, as by Proposition 3.8 the $TTHH$ and $HHTT$ types of pairs of arrows must nest.

Proposition 7.4. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by the following rules:*

(1) $THTH$ type of pairs of arrows nest.
(2) $HTHT$ type of pairs of arrows do not nest.
(3) Both $THHT$ and $HTTH$ types of pairs of arrows cross.
(4) Both $TTHH$ and $HHTT$ types of pairs of arrows nest.

Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies:

$$\begin{aligned} F(\widehat{\mathcal{F}}, x, y, z) = & \frac{1}{1+z} \cdot (C(yz(z+1)) + z) \\ & + D\left(\frac{z}{1 - yz(z+1) \cdot C(yz(z+1))}, \frac{z}{1 - yz(z+1) \cdot C(yz(z+1))}, x\right) \\ & \cdot \frac{xz(z+1-yz(z+1) \cdot C(yz(z+1)))^2}{(1+z)(1+z \cdot C(yz(z+1))) \cdot (1-yz(z+1) \cdot C(yz(z+1)))^4}. \end{aligned}$$

Proof. The proof is similar to the proof of Proposition 7.3 in many details. We will highlight the substantial differences. First we show the following equality:

$$\begin{aligned} F(\widehat{\mathcal{F}}, x, y, z) = & \frac{1}{1+z} \cdot (C(yz(z+1)) + z) \\ & + \sum_{a,b \geq 0} xz^{a+b+1} D_{a,b}(x) \cdot \frac{1}{1 + z((C(yz(z+1)) + z)/(z+1) - 1)} \\ & \cdot \left(1 + \sum_{i \geq 1} \frac{(yz(z+1) \cdot C(yz(z+1)))^i}{1+z} \cdot \left(\binom{i+a+1}{a+1} z + \binom{i+a}{a}\right)\right) \\ & \cdot \left(1 + \sum_{j \geq 1} \frac{(yz(z+1) \cdot C(yz(z+1)))^j}{1+z} \cdot \left(\binom{j+b+1}{b+1} z + \binom{j+b}{b}\right)\right). \end{aligned} \tag{7.5}$$

Just as in equation (7.4), the term $\frac{1}{1+z} \cdot (C(yz(z+1)) + z)$ is contributed by the faces containing backward arrows only. The second sum is contributed by faces that contain forward arrows as well; these forward arrows are incident to $a+1$ tails and $b+1$ heads. The total contribution of the forward arrows is $xz^{a+b+1}D_{a,b}(x)$. For the precise count of the contribution of the backward arrows, we use the fact that no pair of backward arrows crosses. We call a saturated face $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ of backward arrows *connected* if it is empty or the arrow $(\max(i_1, i_2, \ldots, i_k), \min(j_1, j_2, \ldots, j_k))$ belongs to the set. Clearly each saturated face of backward arrows is uniquely the disjoint union of maximal connected sets. Introducing $\widehat{\mathcal{G}}$ as the collection of families of connected saturated sets of backward arrows, we have the equality

$$1 + z \cdot (F(\widehat{\mathcal{F}}, 0, y, z) - 1) = \sum_{k \geq 0} F(\widehat{\mathcal{G}}, 0, y, z)^k \cdot z^k = \frac{1}{1 - z \cdot F(\widehat{\mathcal{G}}, 0, y, z)}$$

where k stands for the number of maximal connected sets. Indeed, for $k = 0$ the empty set is saturated and connected. For all nonempty saturated sets, the first component contributes an unnecessary additional factor of z on the right-hand side. Substituting (6.4) yields

$$1 + z \cdot \left(\frac{1}{1+z} \cdot (C(yz(z+1)) + z) - 1\right) = \frac{1}{1 - z \cdot F(\widehat{\mathcal{G}}, 0, y, z)}. \tag{7.6}$$

We partition the backward arrows of a saturated face into three classes. The first class is formed by all backward arrows whose head is weakly to the left of the head of some forward arrow. Since $HTTH$ type of pairs of arrow cross, the tail of such a backward arrow is to the right of all backward arrows. The same condition on the $HTTH$ type of pairs of arrows also guarantees that to the left of any tail of a forward arrow we can only have a head of a backward arrow belonging to the first class. Since $HTHT$ type of pairs do not nest, the tail of a backward arrow in the first class is between the heads and tails of all forward arrows. All such backward arrows form a connected component: they all contain or arch over the rightmost tail of a forward arrow, and any backward arrow that does not contain or arch over this rightmost tail has its head to the right of all backward arrows in the first class. The same reasoning also shows that all heads of backward arrows in the first class are to the left of the tails of these arrows. Introducing i as the number of heads of backward arrows in the first class, the contribution of all backward arrows in the first class is

$$\left(1+\sum_{i\geq 1}\frac{(yz(z+1)\cdot C(yz(z+1)))^i}{1+z}\cdot\left(\binom{i+a+1}{a+1}z+\binom{i+a}{a}\right)\right)\cdot(1-zF(\widehat{\mathcal{G}},0,y,z)).$$

Just as in the proof of Proposition 7.3, each summand in the first factor of the above is the total weight of all saturated faces of backward arrows in which i heads are followed by a tail in the left-to-right order, and the factor of $1-zF(\widehat{\mathcal{G}},0,y,z)$ represents dividing by $1/(1-zF(\widehat{\mathcal{G}},0,y,z))$, i.e., removing the contribution of the additional connected components. Hence the above expression represents the total weight of connected saturated faces.

The second class is formed by all backward arrows whose head is weakly to the right of the head of some forward arrow. A completely analogous reasoning shows that the total weight of these arrows is

$$\left(1+\sum_{j\geq 1}\frac{(yz(z+1)\cdot C(yz(z+1)))^j}{1+z}\cdot\left(\binom{j+b+1}{b+1}z+\binom{j+b}{b}\right)\right)\cdot(1-zF(\widehat{\mathcal{G}},0,y,z)).$$

The remaining arrows form the third class: the heads and tails of these arrows are to the right to the tails of the arrows in the first class and to the left of the heads of the arrows in the second class. They contribute a factor of

$$1+z\cdot(F(\widehat{\mathcal{F}},0,y,z)-1)=1+z\cdot\left(\frac{1}{1+z}\cdot(C(yz(z+1))+z)-1\right).$$

Equation (7.5) is now a consequence of (7.6). The algebraic manipulations used to derive the statement from (7.5) are very similar to the proof of Proposition 7.3, and are therefore omitted. □

Proof of Theorem 7.1. We begin with the proof of (7.1). We have proved three variants of this formula in Propositions 7.2, 7.3 and 7.4. It remains to show that the three generating functions in these propositions are equal to the generating function in equation (7.1). This is straightforward by expanding $D(u,v,x)$ using equation (6.10) and that the Catalan generating function satisfies the quadratic relation $C(u)=1+u\cdot C(u)^2$, especially in the form $1/(1-u\cdot C(u))=C(u)$. Equation (7.2) follows now from equation (7.1) by applying the involution $\triangle\longmapsto\triangle^*$ and Lemmas 3.3 and 6.1. □

By combining Theorem 7.1 and Lemma 6.4 it is possible to give the generating function of all faces. We now explicitly count the facets using Corollary 6.6 and equation (7.1). Since

$$F\left(\widehat{\mathcal{F}}, \frac{x}{w}, \frac{y}{w}, wz\right) = \frac{C(yz(wz+1)) + wz}{1+wz} + \frac{xz \cdot (1 + wzC(yz(wz+1))) \cdot C(yz(wz+1))^2}{(1+wz)(1 - 2C(yz(wz+1))xz - C(yz(wz+1))^2 xwz^2)},$$

we obtain that the facet generating function is given by

$$\sum_{n \geq 0} \sum_{i=0}^{n} f(\triangle_n, i, n-i) x^i y^{n-i} z^n = C(yz) + \frac{xz \cdot C(yz)^2}{1 - 2xzC(yz)} \tag{7.7}$$
$$= C(yz) + \sum_{i \geq 1} 2^{i-1} \cdot (xz)^i \cdot C(yz)^{i+1}.$$

Theorem 7.5. *Let $\triangle_n$ be a uniform flag triangulation of the boundary ∂P_n of the Legendre polytope belonging to the Simion class satisfying the property that $THTH$ type of pairs of arrows nest and $HTHT$ type of arrows do not nest. Then for $i \geq 1$, the number of facets of $\triangle_n$ consisting of i forward arrows and $n-i$ backward arrows is given by*

$$f(\triangle_n, i, n-i) = 2^{i-1} \cdot \frac{(i+1) \cdot (2n-i)!}{(n-i)! \cdot (n+1)!}. \tag{7.8}$$

The number of facets of the triangulation $\triangle_n$ consisting of no forward arrows and n backward arrows is given by the Catalan number C_n.

Proof. Observe that the coefficient of $y^n z^n$ in equation (7.7) is the nth Catalan number C_n. For $i \geq 1$, the coefficient of $x^i y^{n-i} z^n = (xz)^i \cdot (yz)^{n-i}$ is 2^{i-1} times the coefficient of $(yz)^{n-i}$ in $C(yz)^{i+1}$, which is $2^{i-1} \cdot \frac{i+1}{2n-i+1} \cdot \binom{2n-i+1}{n+1}$ by an identity due to Catalan [4]. ☐

Remark 7.6. The formula (7.8) given in Theorem 7.5 may be restated as

$$f(\triangle_n, i, n-i) = 2^{i-1} \cdot C(n, n-i) \tag{7.9}$$

where the numbers

$$C(n,k) = \frac{(n+k)! \cdot (n-k+1)}{k! \cdot (n+1)!}$$

are the entries in the *Catalan triangle*. See OEIS sequence A009766 [15].

We end with two observations. First, it is amusing how the expression in (7.8) is off by a factor of $1/2$ in the case when $i = 0$. Second, when $\triangle_n$ is the Simion type B associahedron triangulation of the boundary of the Legendre polytope, it is possible to give a more constructive proof of Theorem 7.5 by analyzing the tree structure of the digraphs corresponding to facets.

8. Face enumeration in the revlex class

In this section we study the revlex class, that is, the class containing the revlex pulling triangulation. Our first result is similar to equations (7.3), (7.4) and (7.5), and it is perfectly suitable to compute the face numbers with a prescribed number of forward and backward arrows. Unfortunately, it does not seem feasible to produce a closed form formula without infinite sums, that is similar to Propositions 7.2, 7.3 and 7.4.

Theorem 8.1. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the following rules:*

(1) Both $THTH$ and $HTHT$ types of pairs of arrows nest.
(2) Both $HTTH$ and $THHT$ types of pairs of arrows cross.

Then the collection $\widehat{\mathcal{F}}$ of families of saturated faces satisfies

$$F(\widehat{\mathcal{F}}, x, y, z) = 1 + \sum_{a,b \geq 0} (x \cdot D_{a,b}(x) + y \cdot D_{a,b}(y)) \cdot z^{a+b+1} + xy \cdot \sum_{a',b',a'',b'' \geq 0} D_{a',b'}(x) \cdot D_{a'',b''}(y) \cdot z^{a'+b'+a''+b''+2} \cdot C(a', b', a'', b'', z) \tag{8.1}$$

where

$$C(a', b', a'', b'', z) = \binom{a'+b''+2}{a'+1} \cdot \binom{a''+b'+2}{b'+1} \cdot z + \binom{a'+b''+1}{b''} \cdot \binom{a''+b'+1}{b'} + \binom{a'+b''+1}{a'} \cdot \binom{a''+b'+1}{a''}. \tag{8.2}$$

Proof. By Proposition 6.12 the first three terms on the right-hand side of (8.1) are the total weights of all faces that do not contain arrows in both directions. The last sum is the total weight of all faces containing arrows in both directions: forward arrows on $a'+1$ tails and $b'+1$ heads and backward arrows on $a''+1$ tails and $b''+1$ heads. Again, by Proposition 6.12 the subfaces of forward and backward arrows, respectively, contribute factors of $xD_{a',b'}(x)z^{a'+b'+1}$ and $yD_{a'',b''}(y)z^{a''+b''+1}$ respectively. We may collect the contribution of all faces that contain only forward or only backward arrows by identifying a' and a'' with a, and b' and b'' with b. For the remaining faces there is an additional factor of z when the set of endpoints of forward arrows is disjoint from the set of endpoints of backward arrows. By Proposition 3.7 $THHT$ and $HTTH$ type of pairs of arrows cross. As a consequence, heads of backward arrows are to the left of the heads of forward arrows, and tails of backward arrows are to the right of the tails of the backward arrows. These conditions also ensure that the set of endpoints of the backward arrows cannot have two or more nodes in common with the set of endpoints of the forward arrows. The first term factor $C(a', b', a'', b'', z)$ accounts for the number of ways we may line up $a'+1$ tails of forward arrows with $b''+1$ heads of backward arrows on one side and, independently, $a''+1$ tails of backward arrows with $b'+1$ heads of forward arrows on the other side. The remaining terms correspond to the cases when the forward arrows and the backward arrows share one head or one tail, respectively. □

We obtain a more compact expression using the proof of Theorem 8.1 by introducing the following generating function.

Definition 8.2. *Let $\widehat{\mathcal{F}} = (\widehat{\mathcal{F}}_0, \widehat{\mathcal{F}}_1, \ldots)$ be a collection of families of arrows such that for each n the family $\widehat{\mathcal{F}_n}$ consists of saturated subsets of V_n. We define the* node-enriched exponential generating function *of $\widehat{\mathcal{F}}$ as follows:*

(1) The empty set (if it belongs to $\widehat{\mathcal{F}}_0$) contributes a factor of 1.
(2) Each nonempty $\sigma \in \widehat{\mathcal{F}}_n$ contributes a term

$$x^i y^j \cdot \frac{u^{a+1} \cdot v^{b+1}}{(a+1)! \cdot (b+1)!} \cdot t^n,$$

where i is the number of forward arrows, j is the number of backward arrows, $a+1$ is the number of nodes that are left ends of arrows and $b+1$ is the number of nodes that are right ends of arrows.

It should be noted that the numbers $a+1$ and $b+1$ respectively count the left and right ends of arrows and not their heads or tails: a left end is the tail of a forward arrow or the head of a backward arrow. A common tail of a forward and a backward arrow is counted twice: once as a left end and once as a right end. It is easy to derive from the requirements on the $THHT$ and $HTTH$ type of pairs of arrows that for the triangulations in the revlex class, there is at most one node that is simultaneously the left end and the right end of some arrow.

The node-enriched exponential generating function of the saturated faces in a triangulation in the revlex class has a compact expression in terms of the following exponential generating function of the Delannoy polynomials:

$$\widetilde{D}(u, v, x) = \sum_{a,b \geq 0} \frac{D_{a,b}(x) \cdot u^{a+1} \cdot v^{b+1}}{(a+1)! \cdot (b+1)!}. \tag{8.3}$$

Theorem 8.3. *Let $\mathcal{F}$ be a family of uniform flag triangulations defined by a set of rules that contains the following rules:*

(1) Both $THTH$ and $HTHT$ types of pairs of arrows nest.
(2) Both $HTTH$ and $THHT$ types of pairs of arrows cross.

Then the node-enriched exponential generating function of the collection $\widehat{\mathcal{F}}$ of families of saturated faces is given by

$$\begin{aligned} &1 + \frac{1}{z} \cdot \widetilde{D}(uz, vz, x) + \frac{1}{z} \cdot \widetilde{D}(vz, uz, y) + \frac{1}{z} \cdot \widetilde{D}(uz, vz, x) \cdot \widetilde{D}(vz, uz, y) \\ &+ \frac{1}{z^2} \cdot \frac{\partial}{\partial u}\widetilde{D}(uz, vz, x) \cdot \frac{\partial}{\partial v}\widetilde{D}(vz, uz, y) + \frac{1}{z^2} \cdot \frac{\partial}{\partial v}\widetilde{D}(uz, vz, x) \cdot \frac{\partial}{\partial u}\widetilde{D}(vz, uz, y). \end{aligned}$$

Proof. The proof is essentially the same as that of Theorem 8.1 and omitted. □

Theorem 8.3 motivates computing $\widetilde{D}(u, v, x)$ explicitly.

Theorem 8.4. *The exponential generating function* $\widetilde{D}(u,v,x)$ *is given by*

$$\widetilde{D}(u,v,x) = \sum_{k \geq 0} \frac{(uv \cdot (x^2+x))^k}{k!^2} \cdot \psi_{k+1}(ux) \cdot \psi_{k+1}(vx)$$

where $\psi_{k+1}(z) = \frac{d^k}{dz^k}\left(\frac{e^z-1}{z}\right)$.

Proof. We use the identity

$$D_{a,b}(x) = \sum_k \binom{a}{k} \cdot \binom{b}{k} \cdot (x^2+x)^k \cdot x^{a+b-2k}.$$

Here k counts the total number of NE steps and "northwest corners" (i.e., N steps immediately followed by E steps) in a Delannoy path from $(0,0)$ to (a,b). There are $\binom{a}{k}\binom{b}{k}$ ways to select the positions of these steps and corners in the plane, and each such place contributes a factor of x^2+x as the weight of a N step followed by an E step is x^2, whereas the weight of a NE step is x. Using the above expression for $D_{a,b}(x)$, the definition of $\widetilde{D}(u,v,x)$ may be rearranged as follows:

$$\widetilde{D}(u,v,x) = \sum_{k \geq 0} \frac{(uv \cdot (x^2+x))^k}{k!^2} \cdot \sum_{a,b \geq k} \frac{(ux)^{a-k}}{(a-k)!} \cdot \frac{(vx)^{b-k}}{(b-k)!} \cdot \frac{1}{a+1} \cdot \frac{1}{b+1}.$$

The statement follows after noticing that

$$\psi_{k+1}(z) = \sum_{n \geq 0} \frac{1}{n+k+1} \cdot \frac{z^n}{n!}$$

which can easily be shown by induction on k. $\square$

Remark 8.5. It is a direct consequence of Theorems 8.3 and 8.4 that

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\widetilde{D}(u,v,x) = \exp(x \cdot (u+v)) \cdot I_0\left(2\sqrt{(x^2+x) \cdot uv}\right)$$

where $I_0(z)$ is the modified Bessel function of the first kind.

Remark 8.6. It can be shown by induction that

$$\psi_k(z) = \frac{\left(\sum_{i=0}^{k-1} (-1)^i \cdot \frac{(k-1)!}{(k-1-i)!} \cdot z^{k-1-i}\right) \cdot e^z + (-1)^k \cdot (k-1)!}{z^k}.$$

We conclude this section by counting the facets using Corollary 6.6.

Theorem 8.7. *Let* $\triangle_n$ *be a uniform flag triangulation of the boundary* ∂P_n *of the Legendre polytope that belongs to the revlex class. For* $1 \leq k \leq n-1$*, the number of facets consisting of* k *forward arrows and* $n-k$ *backward arrows, that is,* $f(\triangle_n, k, n-k)$*, is given by*

$$\sum_{i=1}^{k} \sum_{j=1}^{n-k} \binom{k-1}{i-1} \cdot \binom{n-k-1}{j-1} \cdot \left[\binom{n-k+i-j}{i} \cdot \binom{k-i+j}{j} + \binom{n-k+i-j}{i-1} \cdot \binom{k-i+j}{j-1}\right].$$

The number of facets with n *forward arrows and no backward arrows; and the number of facets with no forward arrows and* n *backward arrows are both equal to* 2^{n-1}*.*

Proof. Inspecting (8.1) we see that the total degree of x and y is strictly less than the degree of z, except for the contributions, in which the following rules are observed:

(1) In the expansion of $D_{a',b'}(x)$ only the contribution of those Delannoy paths are kept which contain no NE steps. Hence, to compute the contribution of the facets only, we must replace each appearance of $D_{a',b'}(x)$ in (8.1) with $\binom{a'+b'}{a'} \cdot x^{a'+b'}$.
(2) Similarly, we must replace each appearance of $D_{a'',b''}(y)$ in (8.1) with $\binom{a''+b''}{a''} \cdot y^{a''+b''}$.
(3) Only the z-free part of the factor $C(a', b', a'', b'', z)$ contributes to the calculation of the contribution of the facets.

Therefore we obtain

(8.4)
$$F\left(\widehat{\mathcal{F}}, \frac{x}{w}, \frac{y}{w}, wz\right)\Big|_{w=0} = 1 + \sum_{a',b'\geq 0} \binom{a'+b'}{a'} \cdot (xz)^{a'+b'+1} + \sum_{a'',b''\geq 0} \binom{a''+b''}{a''} \cdot (yz)^{a''+b''+1}$$
$$+ \sum_{\substack{a',b'\geq 0\\ a'',b''\geq 0}} C_0(a', b', a'', b'') \cdot \binom{a'+b'}{a'} \cdot (xz)^{a'+b'+1} \cdot \binom{a''+b''}{a''} \cdot (yz)^{a''+b''+1}$$

where

$$C_0(a', b', a'', b'') = \binom{a'+b''+1}{b''} \cdot \binom{a''+b'+1}{b'} + \binom{a'+b''+1}{a'} \cdot \binom{a''+b'+1}{a''}.$$

The contribution of all facets consisting of forward arrows only is $\sum_{a',b'\geq 0} \binom{a'+b'}{a'} \cdot (xz)^{a'+b'+1}$ on the right-hand side of (8.4). The part of the statement regarding these facets is a direct consequence of the binomial theorem. Similarly, the part of the statement on facets consisting entirely of backward arrows follows from inspecting the next sum on the right-hand side of (8.4). The contribution of all other facets is collected in the last sum. The contribution of all facets consisting of k forward and $n-k$ backward arrows is the sum of all terms satisfying $a' + b' + 1 = k$ and $a'' + b'' + 1 = n - k$. The statement now follows after substituting $i = a' + 1$ and $j = a'' + 1$. □

9. Face enumeration in the lex class

We now turn our attention to face enumeration in the lex class, consisting of the four triangulations studied in Subsection 4.1. Among them are the lexicographic pulling triangulation. So far, the lex class and revlex class have been similar to each other; see Propositions 3.6 and 3.7, Subsections 4.1 and 4.2. In this section this similarity breaks down. This section differs from the previous ones in the simplicity and uniformity of its main result. As the attentive reader might suspect, there is also a purely combinatorial way to prove it. For space considerations, we will present this combinatorial proof in an upcoming paper [12], and here we depend on the tools developed in Section 6.

Theorem 9.1. *Let $\triangle_n$ be a uniform flag triangulation of the boundary ∂P_n of the Legendre polytope in the lex class, that is, $\triangle_n$ satisfies the rules:*

(1) Both $THTH$ and $HTHT$ types of pairs of arrows do not nest.
(2) Both $THHT$ and $HTTH$ types of pairs of arrows do not cross.

Then the number of $(k-1)$-dimensional faces in the triangulation $\triangle_n$ consisting of i forward arrows and $k-i$ backward arrows is given by

$$f(\triangle_n, i, k-i) = \frac{f_{k-1}(\triangle_n)}{k+1} = \frac{1}{k+1} \cdot \binom{n+k}{k} \cdot \binom{n}{k}.$$

Furthermore, this quantity is independent of the parameter i.

We begin with a lemma about Catalan numbers. For a subset S of the integers, call a *run* a maximal interval in S.

Lemma 9.2. *For k a nonnegative integer, let i be an integer satisfying $0 \leq i \leq k$. Then the kth Catalan number is given by the sum of products*

$$C_k = \sum_{\substack{S \subseteq [k] \\ |S|=i}} \prod_{\substack{R \text{ run in } S \\ \text{or in } [k]-S}} C_{|R|}. \tag{9.1}$$

Note that when $i = 0$ or $i = k$, the lemma does not give anything new. When $i = 1$ or $i = k-1$, the lemma yields the classical recursion for the Catalan numbers.

Proof of Lemma 9.2. Recall that $C(x) = \sum_{k \geq 0} C_k \cdot x^k$ is the generating function for the Catalan numbers, which satisfies the quadratic relation $C(x) = 1 + x \cdot C(x)^2$. Let $f(x)$ be $C(x)$ without the constant term, that is, $f(x) = C(x) - 1$. Multiply the right-hand side of (9.1) with $x^i y^{k-i}$ and notice that $x^i y^{k-i} = \prod_{R \text{ run in } S} x^{|R|} \cdot \prod_{R \text{ run in } [k]-S} y^{|R|}$. Sum over all i and k such that $0 \leq i \leq k$. The resulting generating function is given by

$$\sum_{\substack{k \geq 0 \\ 0 \leq i \leq k}} \sum_{\substack{S \subseteq [k] \\ |S|=i}} \prod_{\substack{R \text{ run in } S \\ \text{or in } [k]-S}} C_{|R|} \cdot x^i y^{k-i} = 1 + \frac{f(x)}{1 - f(y)f(x)} + \frac{f(y)}{1 - f(x)f(y)} + 2 \cdot \frac{f(x)f(y)}{1 - f(x)f(y)} \tag{9.2}$$

$$= \frac{(1 + f(x)) \cdot (1 + f(y))}{1 - f(x)f(y)}. \tag{9.3}$$

Note that the constant term 1 in (9.2) corresponds to $k = 0$, the second term to subsets $S \subseteq [k]$ with $1, k \in S$, the third to subsets $S \subseteq [k]$ with $1, k \notin S$, and finally, the fourth term to subsets S such that $|S \cap \{1, k\}| = 1$. Next observe that

$$(1 - f(x)f(y)) \cdot (x \cdot C(x) - y \cdot C(y)) = (x - y) \cdot C(x) \cdot C(y), \tag{9.4}$$

by expanding the product on the left-hand side of (9.4) and simplifying using the quadratic relation $C(x) = 1 + x \cdot C(x)^2$ four times. Using (9.4) the generating function in (9.3) simplifies to

$$\frac{x \cdot C(x) - y \cdot C(y)}{x - y} = \sum_{k \geq 0} \sum_{0 \leq i \leq k} C_k \cdot x^i y^{k-i},$$

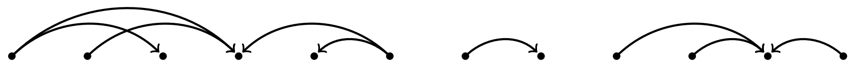


FIGURE 8. A forest with 6 forward arrows and 3 backward arrows on 12 nodes, counted by the term $G_3(z) \cdot z^{-1} \cdot G_2(z) \cdot 1 \cdot G_3(z) \cdot z^{-1} \cdot G_1(z)$ in the proof of Proposition 9.4.

which is the generating function for the left-hand side of (9.1). □

Remark 9.3. Lemma 9.2 is equivalent to the following statement about lattice paths from the origin to $(2n, 0)$ taking up steps $(1, 1)$ and down steps $(1, -1)$. For such a lattice path p, considered as a piecewise linear function, let $b(p)$ be $1/2$ times the sum of the lengths of the intervals where the lattice path is below the x-axis. Then for an integer $0 \leq i \leq n$, the number of lattice paths p such that $b(p) = i$ is given by the Catalan number C_n.

Observe that Corollary 6.9 applies to the case when we only have backward arrows or only forward arrows. Now we turn our attention to the case when we have both forward and backward arrows.

Proposition 9.4. *Consider digraphs such that both $THTH$ and $HTHT$ types of pairs of arrows do not nest and both $THHT$ and $HTTH$ types of pairs of arrows do not cross. The sum over all forests F consisting of i forward arrows, $k - i$ backward arrows and no isolated nodes, where $k \geq 1$, is*

$$\sum_{F} z^{\#nodes\ of\ F} = C_k \cdot z^{k+1} \cdot (z+1)^{k-1}.$$

Proof. Given that we have k undirected arrows, pick a subset S of them. Let S be the set of the forward arrows, and let the complement be the backward arrows. Hence the generating function can be expressed as

$$\sum_{F} z^{\#\text{nodes of } F} = \sum_{\substack{S \subseteq [k] \\ |S|=i}} (1 + z^{-1})^{r(S)-1} \cdot \prod_{\substack{R \text{ run in } S \\ \text{or in } [k]-S}} G_{|R|}(z),$$

where $r(S)$ is the sum of the number of runs in the subset S and the number of runs in the complement subset $[k] - S$ and $G_{|R|}(z)$ is the polynomial appearing in equation (6.6). The factor $1 + z^{-1}$ appears since when we switch the direction of the arrows either the node set is disjoint, yielding the factor 1, or they share an a vertex, yielding the factor z^{-1}; see Figure 8. Expanding $G_{|R|}(z)$ using Corollary 6.9 we have

$$\begin{aligned} \sum_{F} z^{\#\text{nodes of } F} &= \sum_{\substack{S \subseteq [k] \\ |S|=i}} z^{-r(S)+1} \cdot (z+1)^{r(S)-1} \cdot \prod_{\substack{R \text{ run in } S \\ \text{or in } [k]-S}} C_{|R|} \cdot z^{|R|+1} \cdot (z+1)^{|R|-1} \\ &= z^{k+1} \cdot (z+1)^{k-1} \cdot \sum_{\substack{S \subseteq [k] \\ |S|=i}} \prod_{\substack{R \text{ run in } S \\ \text{or in } [k]-S}} C_{|R|}, \end{aligned}$$

where we used $\sum_R |R| = k$. Now by Lemma 9.2 the result follows. □

Proof of Theorem 9.1. Observe that the enumeration in Proposition 9.4 is independent of i, that is, the distribution is uniform. Inserting isolated vertices will not change this fact. Since the total number of k-dimensional faces is given by $f_{k-1}(\triangle_n) = \binom{n+k}{k} \cdot \binom{n}{k}$, the number of faces with exactly i forward arrows is $f_{k-1}(\triangle_n)/(k+1)$. □

10. Concluding Remarks

The result of Oh and Yoo [17, Theorem 5.4] characterizing triangulations of products of simplices is deep but not very direct. It is a straightforward exercise, left to the reader, to show that most of the fifteen uniform flag triangulations of the boundary ∂P_n of the Legendre polytope are pulling triangulations. Since all pulling triangulations of ∂P_n are flag, it suffices to come up with a pulling order that satisfies the given flag conditions. Three of the fifteen uniform flag triangulations do not seem to arise in such an easy manner: the triangulation in the lex class, where both $TTHH$ and $HHTT$ types of pairs of arrows nest, the triangulation in the revlex class where both $TTHH$ and $HHTT$ types of pairs of arrows cross, and the triangulation (up to taking the dual) in the type c subclass of the Simion class. The geometry of these three triangulations is worth a closer look.

All triangulations of the boundary of the Legendre polytope discussed in this paper have the same face numbers. Setting the variables x and y equal in our results yields many equalities linking the Legendre polynomials, the Catalan numbers, the Delannoy numbers and their weighted generalizations. Exploring these identities, relating them to known results, and proving them combinatorially are all subjects of future investigation.

Stanley gives a condition for a lattice polytope so that the f-vector of a triangulation of the polytope would be independent of the triangulation; see [20, Example 2.4 and Corollary 2.7] or [11, Theorem 2.6]. Is there an extension of this condition to explain the invariance of the refined face count, as displayed in Theorems 7.1, 8.1 and 9.1? For instance, the condition "if an arrow is forward or backward" can be replaced with the more geometric condition "if the inner product between the vector $(1, 2, \ldots, n+1)$ and the lattice point v is positive or negative".

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References

[1] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," National Bureau of Standards, Washington, D.C., issued 1964, Tenth Printing, 1972, with corrections.

[2] F. Ardila, M. Beck, S. Hoşten, J. Pfeifle and K. Seashore, Root polytopes and growth series of root lattices, *SIAM J. Discrete Math.* **25** (2011), 360–378.

[3] C. Banderier and S. Schwer, Why Delannoy numbers?, *J. Statist. Plan. Inference* **135** (2005), 40–54.
[4] E. Catalan, Sur les nombres de Segner, *Rend. Circ. Mat. Palermo* **1** (1887), 190–201.
[5] C. Ceballos, A. Padrol and C. Sarmiento, Dyck path triangulations and extendability, *J. Combin. Theory Ser. A* **131** (2015), 187–208.
[6] C. Ceballos, A. Padrol and C. Sarmiento, Geometry of ν-Tamari lattices in types A and B, *Sém. Lothar. Combin.* **78B** (2017), Art. 68, 12 pp.
[7] P. Cellini and M. Marietti, Root polytopes and Abelian ideals, *J. Algebr. Comb.* **2014** (39), 607–645.
[8] S. Cho, Polytopes of roots of type A_n, *Bull. Austral. Math. Soc.* **59** (1999), 391–402.
[9] R. Cori and G. Hetyei, Counting genus one partitions and permutations, *Sém. Lothar. Combin.* **70** (2013), Art. B70e, 29 pp.
[10] J. A. de Loera, J. Rambau and F. Santos, "Triangulations: Structures for Algorithms and Applications," Algorithms Comput. Math. vol. 25, Springer-Verlag, Berlin, 2010.
[11] R. Ehrenborg, G. Hetyei and M. Readdy, Simion's type B associahedron is a pulling triangulation of the Legendre polytope, *Discrete Comput. Geom.* **60** (2018), 98–114.
[12] R. Ehrenborg, G. Hetyei and M. Readdy, Very pure monoids and Catalan combinatorics, in preparation.
[13] I. M. Gelfand, M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, in Arnold-Gelfand Mathematical Seminars: Geometry and Singularity Theory, Birkhäuser, Boston, 1996, 205–221.
[14] G. Hetyei, Delannoy orthants of Legendre polytopes, *Discrete Comput. Geom.* **42** (2009), 705–721.
[15] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, preprint (`http://oeis.org`).
[16] S. Oh and H. Yoo, Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and tropical oriented matroids, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), 717–728, Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011.
[17] S. Oh and H. Yoo, Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and tropical oriented matroids, `arXiv:1311. 6772v1 [math.CO]`.
[18] A. Postnikov, Permutohedra, associahedra, and beyond, *Int. Math. Res. Not. IMRN* (2009), 1026–1106.
[19] R. Simion, A type-B associahedron, *Adv. in Appl. Math.* **30** (2003), 2–25.
[20] R. P. Stanley, Decompositions of rational convex polytopes, *Ann. Discrete Math.* **6** (1980), 333–342.
[21] B. Sturmfels, "Gröbner bases and convex polytopes," University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996.

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