GOLDBACH'S LIKE CONJECTURES ARISING FROM ARITHMETIC PROGRESSIONS WHOSE FIRST TWO TERMS ARE PRIMES

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ABSTRACT. For two odd primes p and q such that $p < q$, let $A(p,q) := (a_k)_{k=1}^{\infty}$ be the arithmetic progression whose kth term is given by $a_k = (k-1)(q-p)+p$ (i.e., with $a_1 = p$ and $a_2 = q$). Here we conjecture that for every positive integer $a > 1$ there exist a positive integer n and two odd primes p and q such that a can be expressed as a sum of the first $2n$ terms of the arithmetic progression $A(p,q)$. Notice that in the case of even a, this conjecture immediately follows from Goldbach's conjecture. We also propose the analogous conjecture for odd positive integers $a > 1$ as well as some related Goldbach's like conjectures arising from the previously mentioned arithmetic progressions.

1. CONJECTURES ON ARITHMETIC PROGRESSIONS WHOSE FIRST TWO TERMS ARE PRIMES

Let p and q be two primes such that $p < q$ and let $A(p,q) := (a_k)_{k=1}^{\infty}$ be the arithmetic progression whose kth term is given by

$$a_k = (k-1)(q-p)+p, \quad k = 1, 2, \ldots$$

In other words, $A(p,q)$ is an arithmetic progression whose first two terms are p and q (i.e., $a_1 = p$ and $a_2 = q$). The sum $S_n(p,q) = S_n$ of the first n terms of the progression $A(p,q)$ is equal to

$$S_n(p,q) = \frac{n}{2}((n-1)q-(n-3)p). \tag{1}$$

From (1) we have that for all $n = 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ the sum $S_{n,m}(p,q) := \sum_{i=m+1}^{n+m} a_i$ of some n consecutive terms of progression $A(p,q)$ is equal to

$$S_{n,m}(p,q) := S_{n+m}(p,q) - S_m(p,q) = \frac{n}{2}((n+2m-1)q-(n+2m-3)p). \tag{2}$$

We start with following example.

Example 1.1 (An extension of a Sylvester's result). Here we examine positive integers a which can be written as a sum $S_{n,m}(2,3)$ (given by (2) with $p = 2$ and $q = 3$) for some $n \geq 2$ and $m \geq 1$. The sum of kth term and $(k+1)$th term of the progression

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$A(2,3) = (k+1)_{k=1}^{\infty}$ is equal to $2k+3$. Therefore, every odd integer greater than 3 is a sum of some two consecutive terms of $A(2,3)$. Furthermore, by (2) we have

$$S_{n,m}(2,3) = \sum_{i=m+1}^{n+m} (i+1) = \frac{n}{2}(n+2m+3). \tag{3}$$

If a is an even positive integer which is not a power of 2, then $a = (2d+1)2^u$ for some positive integers $d \geq 1$ and $u \geq 1$. If $1 \leq d \leq 2^u - 2$ for such a a, we have $S_{2d+1,2^u-d-2} = (2d+1)2^u = a$. (If $d = 0$ then $n = 1$ and $S_{1,m}(2,3) = m+2$ is in fact the $(m+1)$th term of $A(2,3)$). Similarly, if $d \geq 2^u + 1$, then $S_{2^{u+1},d-2^u-1} = (2d+1)2^u = a$. This shows that each even positive integer $a = (2d+1)2^u$ with $1 \leq d \leq 2^u - 2$ or $d \geq 2^u + 1$ can be expressed as a sum of at least two consecutive terms of the arithmetic progression $A(2,3)$.

It remains to consider the cases when a is of the form 2^u, $(2^{u+1}-1)2^u$ or $(2^{u+1}+1)2^u$ with some positive integer u. If $a = 2^u$, then by (3) the equality $S_{n,m}(2,3) = a$ is equivalent to $n(n+2m+3) = 2^{u+1}$, which is impossible in view of the fact that one among numbers n and $n+2m+3$ is an odd integer.

If $a = (2^{u+1}-1)2^u$ for a positive integer u, then the equality $S_{n,m}(2,3) = a$ is equivalent to

$$n(n+2m+3) = (2^{u+1}-1)2^{u+1}.$$

If $2^{u+1}-1$ is a composite number, then it can be written as a product $2^{u+1}-1 = tv$ with odd integers $t \geq 3$ and $v \geq 3$. Then the above equality holds for $n = v \geq 3$ and $m = (t(tv+1)) - v - 3)/2 = ((t^2-1)v + t - 3)/2 \geq 12$. If $2^{u+1}-1$ is a prime number, then easily follows that the above equality holds only for $n = 1$ and $m = (2^{u+1}-1)2^u - 2$.

Now consider the last case, i.e., when $a = (2^{u+1}+1)2^u$ for a positive integer u. Then the equality $S_{n,m}(2,3) = a$ is equivalent to

$$n(n+2m+3) = (2^{u+1}+1)2^{u+1}.$$

If $2^{u+1}+1$ is a composite number, then it can be written as a product $2^{u+1}+1 = tv$ with odd integers $t \geq 3$ and $v \geq 3$. Then the above equality holds for $n = v \geq 3$ and $m = (t(tv-1)) - v - 3)/2 = ((t^2-1)v - t - 3)/2 \geq 9$. If $2^{u+1}+1$ is a prime number, then easily follows that the above equality holds only for $n = 1$ and $m = (2^{u+1}+1)2^u - 2$.

In view ot the above considerations, we have shown that every integer $a \geq 4$ is equal to $S_{n,m}(2,3)$ for some integers $n \geq 2$ and $m \geq 1$ in all the cases excluding the following ones:

1) a is not a power of 2;
2) a is not of the form $(2^{u+1}-1)2^u$, where $2^{u+1}-1$ is a prime number and
3) a is not of the form $(2^{u+1}+1)2^u$, where $2^{u+1}+1$ is a prime number.

Remark 1.2. Notice that if $a = 2^u(2^{u+1}+1)$ for an integer $u \geq 1$, then $a = \sum_{i=1}^{2^{u+1}} i$, while if $a = 2^u(2^{u+1}-1)$ for an integer $u \geq 1$, then $a = \sum_{i=1}^{2^{u+1}-1} i$. These two identities together with Example 1.1 imply the well known fact that every integer $a > 1$ which is not a power of 2, is a sum of two or more consecutive integers (see, e.g., Dickson's History [1, 1, Ch. III, p. 139], where this result was attributed to Sylvester).

Remark 1.3. Note that it is well known (see, e.g., [4, Subsections 2.2 and 2.3]) that in order to the so-called a *Mersenne number* $M_{u+1} := 2^{u+1} - 1$ to be prime, $u + 1$ must itself be prime. A Mersenne number which is prime is called *Mersenne prime* (this is Sloane's sequence A000668 in [6] corresponding to indices given by Sloane's sequence A000043). Moreover, it is easy to show that in order to $2^{u+1} + 1$ to be prime, $u + 1$ must be a power of 2. Such numbers are in fact *Fermat numbers* $F_s := 2^{2^s} + 1$ ($s = 0, 1, 2, \ldots$; this is Sloane's sequence A000215 in [6]). Fermat conjectured in 1650 that every Fermat number is prime and Eisenstein proposed as a problem in 1844 the proof that there are an infinite number of *Fermat primes* (i.e., Fermat numbers which are primes) (see [5, p. 88]). However, the only known Fermat primes are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ and $F_4 = 65537$ (Sloane's sequence A019434 in [6]). For more information on classical and alternative approaches to the Mersenne and Fermat numbers, see [3].

Note that the conclusion at the end of Example 1.1 immediately yields the following interesting assertion.

Proposition 1.4. *The following two statements are equivalent:*

(i) *There are infinitely many Fermat primes or there are infinitely many Mersenne primes*;

(ii) *The set* $\{S_{n,m}(2,3) : n = 2, 3, \ldots; m = 1, 2, \ldots\}$ *omits infinitely many positive integer values which are not powers of 2.*

Example 1.5. For the progression $A(3,5) = (2k+1)_{k=1}^{\infty}$ we have $S_{n,m}(3,5) = 2(2m+4)$. From this it can be easily seen that a positive integer $a \geq 8$ is equal to some sum $S_{n,m}(3,5)$ with $n \geq 2$ if and only if a is divisible by 4 or a is an odd composite integer greater than 14 which is not a square of a prime.

More generally, if $q = p + 2$, then $S_{n,m}(p,q) = n(n + 2m + p - 1)$. From this it follows that a positive integer a is equal to some sum $S_{n,m}(3,5)$ with $n \geq 2$ if and only if $a = 4s$ with $s \geq (p+1)/2$ or a is an odd composite integer which can be expressed as a product $n = ab$ with odd integers a and b such that $a \geq 3$ and $b \geq a + p - 1$.

From Examples 1.1 and 1.5 it follows that every integer greater than 10 can be expressed as a sum of two or more consecutive terms of the progression $A(2,3)$ or $A(3,5)$. Accordingly, it can be of interest to consider a problem of representation of a positive integer as a sum of two or more first consecutive integers in some progression $A(p,q)$. Notice that

$$S_2(p,q) = p + q,$$

and even Goldbach's conjecture states that every even positive integer greater than 2 can be expressed as a sum of two primes. This famous conjecture was proposed on 7 June 1742 by the German mathematician Christian Goldbach in a letter to Leonhard

Euler [2] (cf. [1]). This conjecture has been shown to hold for all integers less than 4×10^{18}, but remains unproven despite considerable effort.

In view of the above equality, this conjecture is equivalent with the following set equality:

$$\{S_2(p,q) : p \text{ and } q \text{ are odd primes}\} = \{2n : n \in \mathbb{N} \setminus \{1,2\}\}.$$

This fact suggests the investigations of the values of $S_n(p,q)$ given by (1). Namely, for each positive integer n, we will consider the values

$$S_{2n}(p,q) = n((2n-1)q - (2n-3)p), \tag{4}$$

where p and q are odd primes.

Using some heuristic arguments and computational results, we propose the following "weak even Goldbach conjecture".

Conjecture 1.6 ("weak even Goldbach conjecture")**.** *For each even positive integer a greater than 2 there exist a positive integer n and odd primes p and q such that $a = S_{2n}(p,q)$; or equivalently, that*

$$a = n((2n-1)q - (2n-3)p). \tag{5}$$

Clearly, the following conjecture is stronger than Conjecture 1.6.

Conjecture 1.7. *For any positive integer $n > 1$ there exist odd primes p and q such that*

$$(2n-1)q - (2n-3)p = 2. \tag{6}$$

Note that the equality (6) can be written as

$$q = p - \frac{2(p-1)}{2n-1},$$

whence it follows that $p = 2k(2n-1)+1$ and $q = 2k(2n-3)+1$ for a positive integer k. Hence, Conjecture 1.7 is equivalent to the following one.

Conjecture 1.7'. *For any integer $n > 1$ there exists a positive integer k such that both numbers $p = 2k(2n-1)+1$ and $q = 2k(2n-3)+1$ are primes.*

If p and q are odd primes, then from the expression (1) we see that $S_n(p,q)$ is odd if and only if n is even. The following conjecture is the odd analogue of Conjecture 1.6.

Conjecture 1.8 ("weak odd Goldbach conjecture")**.** *For each odd positive integer a greater than 2 there exist a positive integer n and odd primes p and q such that $a = S_{2n+1}(p,q)$; or equivalently, that*

$$a = (2n+1)(nq - (n-1)p). \tag{7}$$

Clearly, the following conjecture is stronger than Conjecture 1.8.

Conjecture 1.9. *For any positive integer $n > 1$ there exist odd primes p and q such that*

$$nq - (n-1)p = 1. \tag{8}$$

From the equality (8) we have

$$q = p - \frac{p-1}{n},$$

whence we conclude that $p = nk+1$ and $q = (n-1)k+1$ for a positive integer k. This together with the fact that $k = p - q$ is even shows that Conjecture 1.9 is equivalent to the following one.

Conjecture 1.9'. *For any integer $n > 1$ there exists a positive integer k such that both numbers $p = 2kn + 1$ and $q = 2k(n-1) + 1$ are primes.*

Finally, notice that Conjectures 1.6 and 1.8 can be joined into the following conjecture.

Conjecture 1.10 ("weak Goldbach conjecture"). *Conjectures 1.6 and 1.8 are true if and only if the following statement holds true*:

For each positive integer a greater than 2 there exist a positive integer n and odd primes p and q such that

$$a = \frac{n}{2}((n-1)q - (n-3)p). \tag{9}$$

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