MOST PRINCIPAL PERMUTATION CLASSES HAVE NONRATIONAL GENERATING FUNCTIONS

MIKLÓS BÓNA

Abstract. We prove that for any fixed n, and for most permutation patterns q, the number $\mathrm{Av}_{n,\ell}(q)$ of q-avoiding permutations of length n that consist of ℓ skew blocks is a monotone decreasing function of ℓ. We then show that this implies that for most patterns q, the generating function $\sum_{n\geq 0} \mathrm{Av}_n(q)z^n$ of the sequence $\mathrm{Av}_n(q)$ of the numbers of q-avoiding permutations is not rational.

1. Introduction

We say that a permutation p *contains* the pattern $q = q_1q_2 \cdots q_k$ if there is a k-element set of indices $i_1 < i_2 < \cdots < i_k$ so that $p_{i_r} < p_{i_s}$ if and only if $q_r < q_s$. If p does not contain q, then we say that p *avoids* q. For example, $p = 3752416$ contains $q = 2413$, as the first, second, fourth, and seventh entries of p form the subsequence 3726, which is order-isomorphic to $q = 2413$. A recent survey on permutation patterns can be found in [11] and a book on the subject is [4]. Let $\mathrm{Av}_n(q)$ be the number of permutations of length n that avoid the pattern q. In general, it is very difficult to compute, or even describe, the numbers $\mathrm{Av}_n(q)$, or their sequence as n goes to infinity. As far as the generating function $A_q(z) = \sum_{n\geq 0} \mathrm{Av}_n(q)z^n$ goes, there are known examples when it is algebraic, (when q is of length three, or when $q = 1342$), and known examples when it is not algebraic (when q is the monotone pattern $12 \cdots k$, where k is an even integer that is at least four). The question whether $A_q(z)$ is always differentiably finite was raised in 1996 by John Noonan and Doron Zeilberger, and is still open. See Chapter 6 of [10] for an introduction to the theory of differentiably finite generating functions and their importance.

In this paper, we prove that for patterns $q = q_1q_2 \cdots q_k$, where $k > 2$ and $\{q_1, q_k\} \neq \{1, k\}$, the generating function $A_q(z)$ is *never rational*, and for even for a few patterns for which $\{q_1, q_k\} = \{1, k\}$. It is plausible to think that our result holds for the less than $1/[k(k-1)]$ of patterns of length k for which we cannot prove it. On the other hand, the statement obviously fails for the pattern $q = 12$, since for that q, we trivially have that $\mathrm{Av}_n(q) = 1$ for all n, so $A_q(z) = 1/(1-z)$. The set of permutations of any length that avoid a given pattern q is often called a *principal permutation class*, explaining the title of this paper. As rational functions are differentiably finite, this paper

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excludes a small subset of differentiably finite power series from the set of possible generating functions of principal permutation classes.

In proving the result described in the preceding paragraph, our main tool will be a theorem that is interesting on its own right. We say that a permutation p is *skew indecomposable* if it is not possible to cut p into two parts so that each entry before the cut is larger than each entry after the cut. For instance, $p = 3142$ is skew indecomposable, but $r = 346512$ is not as we can cut it into two parts by cutting between entries 5 and 1, to obtain $3465|12$.

If p is not skew indecomposable, then there is a unique way to cut p into **nonempty** skew indecomposable strings $s_1, s_2, \cdots, s_\ell$ of consecutive entries so that each entry of s_i is larger than each entry of s_j if $i > j$. We call these strings s_i the *skew blocks* of p. For instance, $p = 67|435|2|1$ has four skew blocks, while skew indecomposable permutations have one skew block.

The number of skew blocks of a permutation is of central importance for this paper. For permutations with no restriction, it is easy to prove that almost all permutations of length n are skew indecomposable. In this paper, we consider a similar question for pattern avoiding permutations. We prove that if q is a skew indecomposable pattern, and n is any fixed positive integer, then the number $\mathrm{Av}_{n,\ell}(q)$ of q-avoiding permutations of length n that consist of ℓ skew blocks is a monotone decreasing function of ℓ. That is, as the number ℓ of skew blocks increases, the number of q-avoiding permutations with ℓ skew blocks decreases. We will only need a special case of these inequalities (the one relating to $\ell = 1$ and $\ell = 2$) to prove our main result in Section 5.

2. Preliminaries

The following proposition shows that in order to prove our monotonicity result announced in the introduction, it suffices to prove the relevant inequality for $\ell = 1$. This proposition does not hold for patterns that are not skew indecomposable. Recall that $\mathrm{Av}_{n,\ell}(q)$ denotes the number of q-avoiding permutations of length n that consist of ℓ skew blocks.

Proposition 2.1. *Let q be any skew indecomposable pattern. If, for all positive integers n, the inequality*

$$\mathrm{Av}_{n,2}(q) \leq \mathrm{Av}_{n,1}(q) \tag{1}$$

holds, then for all positive integers n, and all positive integers ℓ, the inequality

$$\mathrm{Av}_{n,\ell+1}(q) \leq \mathrm{Av}_{n,\ell}(q)$$

holds.

Proof. Let $A_{\ell,q}(z) = \sum_{n\geq 1} \mathrm{Av}_{n,\ell}(q) z^n$ be the ordinary generating function of the sequence of the numbers $\mathrm{Av}_{n,\ell}(q)$. As q is skew indecomposable, a permutation p with ℓ skew blocks is q-avoiding if and only if each of its skew blocks is q-avoiding. This implies that $A_{\ell,q}(z) = A_{1,q}(z)^\ell$, so, for all $\ell \geq 2$, we have the equalities

$$A_{\ell,q}(z) = A_{\ell-1,q}(z) \cdot A_{1,q}(z),$$

and

$$A_{\ell+1,q}(z) = A_{\ell-1,q}(z) \cdot A_{2,q}(z).$$

As the coefficient of each term in $A_{1,q}(z)$ is at least as large as the corresponding coefficient of $A_{2,q}(z)$, and the coefficients of $A_{\ell-1,q}(z)$, $A_{1,q}(z)$, and $A_{2,q}(z)$ are all nonnegative, it follows from the way in which the product of power series is computed that the coefficient of each term in $A_{\ell,q}(z)$ is at least as large as the corresponding coefficient of $A_{\ell+1,q}(z)$. This proves our claim. □

We will also need the following simple fact. If $q = q_1q_2\cdots q_k$ is a pattern, let q^{rev} denote its reverse $q_kq_{k-1}\cdots q_1$, and let q^c denote its complement $(k+1-q_1)(k+1-q_2)\cdots(k+1-q_k)$. For instance, if $q = 25143$, then $q^{rev} = 34152$, and $q^c = 41523$. Recall that $\mathrm{Av}_n(q)$ denotes the number of permutations of length n that avoid q. It is then obvious that for all patterns q, the equalities

$$\mathrm{Av}_n(q) = \mathrm{Av}_n(q^{rev}) = \mathrm{Av}_n(q^c) \tag{2}$$

hold. These equalities, and similar others, will be useful for us because of the following fact.

Proposition 2.2. *Let q and q' be two skew indecomposable patterns so that the equality*

$$\mathrm{Av}_n(q) = \mathrm{Av}_n(q') \tag{3}$$

holds for all $n \geq 1$. Then for all positive integers n, and for all positive integers $\ell \leq n$, the equality

$$\mathrm{Av}_{n,\ell}(q) = \mathrm{Av}_{n,\ell}(q') \tag{4}$$

holds.

In other words, if two skew indecomposable patterns are avoided by the same number or permutations of length n for all n, (in this case they are called *Wilf-equivalent*), then they are avoided by the same number of permutations of the length n that have ℓ skew blocks.

Proof. Recall that $A_q(z) = \sum_{n\geq 0} \mathrm{Av}_n(q)z^n$. Then

$$A_q(z) = \sum_{\ell\geq 0} A_{\ell,q}(z) = \sum_{\ell\geq 0} A_{1,q}^{\ell}(z) = \frac{1}{1-A_{1,q}(z)},$$

where the second equality holds because q is skew indecomposable. (We explained this in the proof of Proposition 2.1.)

Therefore,

$$A_{1,q}(z) = 1 - \frac{1}{A_q(z)}, \tag{5}$$

and similarly,

$$A_{1,q'}(z) = 1 - \frac{1}{A_{q'}(z)}.$$

Therefore, our conditions imply that $A_{1,q}(z) = A_{1,q'}(z)$, and therefore, for all ℓ, the equalities

$$A_{\ell,q}(z) = (A_{1,q}(z))^{\ell} = (A_{1,q'}(z))^{\ell} = A_{\ell,q'}(z)$$

hold. Equating coefficients of z^n completes our proof. □

3. The pattern 132

The pattern 132 will be of particular importance to us because it enables us to illustrate a method that we will later apply in a more general setting. As a byproduct, we will prove a simple, but surprising result in Lemma 3.3.

Skew blocks of 132-avoiding permutations have a simple property that we state and prove below.

Proposition 3.1. *Let $p \in \mathcal{A}v_n(132)$ have ℓ skew blocks, of sizes $a_1, a_2, \cdots, a_\ell$ when listed* from the right. *Then for $1 \leq i \leq \ell$, the rightmost entry of the ith skew block from the right is $a_1 + a_2 + \cdots + a_\ell$.*

Example 3.2. *If $p = 6758|4|123$, then $a_1 = 3$, $a_2 = 1$, and $a_3 = 4$, and indeed, the rightmost skew block ends in $a_1 = 3$, the second skew block from the right ends in $a_1 + a_2 = 4$, and the leftmost skew block ends in $a_1 + a_2 + a_3 = 8$.*

Proposition 3.1. We prove our statement for each skew block *moving from left to right*. For the leftmost skew block, the statement claims that its rightmost entry will be $a_1 + a_2 + \cdots + a_\ell = n$. Indeed, n is always a member of the leftmost skew block (in any permutation of length n, regardless of pattern avoidance), since it is the largest entry. On the other hand, in a 132-avoiding permutation, all entries on the left of n must be larger than all entries on the right of n. Therefore n always *ends* the first (leftmost) skew block.

In order to prove the statement for the second skew block from the left, simply remove all entries weakly on the left of n from our permutation p. We obtain a 132-avoiding permutation of length $a_1 + a_2 + \cdots + a_{\ell-1}$, and then we can repeat the same argument for the leftmost skew block of this permutation (that skew block was the second skew block of p). Then repeat the same argument for each skew block. □

Next we show the interesting fact that when $q = 132$, then in (1), equality holds if $n > 1$.

Lemma 3.3. *Let $n \geq 2$. Then the equality*

$$\mathrm{Av}_{n,2}(132) = \mathrm{Av}_{n,1}(132)$$

holds.

Proof. We define a map $f : \mathcal{A}v_{n,2}(132) \to \mathcal{A}v_{n,1}(132)$, and show that it is a bijection. The special case of $\ell = 1$ of Proposition 3.1 shows that a 132-avoiding permutation is skew-indecomposable if and only if it ends in its maximum entry.

Let $p \in \mathcal{A}v_{n,2}(132)$, and let us define $f(p)$ by moving the maximum entry n of p into the last position of p. It follows from the characterization of $\mathcal{A}v_{n,1}(132)$ given above that $f(p) \in \mathcal{A}v_{n,1}(132)$.

Note that Proposition 3.1 implies that if the rightmost skew block R of p was of length a_1, then the next-to-last entry of $f(p)$ is a_1. Therefore, if $w = w_1 w_2 \cdots w_n$ is a permutation in $\mathcal{A}v_{n,1}(132)$, then we obtain $f^{-1}(w)$ by moving its last entry n to the immediate left of the skew block R, that is, in position $n - a_1$. This always results in a 132-avoiding permutation, since we placed n between two skew blocks, and the obtained permutation will always have two skew blocks, namely R and the rest of $f^{-1}(w)$, ending in n. So f has an inverse function, and hence it is a bijection. $\square$

Example 3.4. *If $p = 534612$, then $f(p) = 534126$.*

Now Proposition 2.1 and Lemma 3.3, and the fact that $1 = \mathrm{Av}_{1,1}(132) > \mathrm{Av}_{1,2}(132) = 0$ together immediately imply the following.

Theorem 3.5. *For all positive integers n, and all positive integers $\ell \leq n-1$, the inequality*

$$\mathrm{Av}_{n,\ell+1}(132) \leq \mathrm{Av}_{n,\ell}(132)$$

holds.

Note that the fact that $\mathrm{Av}_{1,1}(132) > \mathrm{Av}_{1,2}(132)$ implies that the inequality in Theorem 3.5 is strict unless $\ell = 1$ and $n > 1$.

4. The case containing most patterns

In the last section, we discussed a map that took a permutation with two skew blocks and moved its largest entry in its last position. For 132-avoiding permutations, this led to a bijection between two sets in which we were interested. In this section, we will replace 132 by a pattern q coming from a very large set of patterns. Furthermore, instead of moving the largest entry to the back, we will move the *last entry of the first skew block* to the end of the whole permutation. (In the special case of $q = 132$, that entry happens to be the largest entry as well.) We will be able to show that this map is an *injection* from $\mathcal{A}v_{n,2}(q)$ to $\mathcal{A}v_{n,1}(q)$.

For the rest of this section, the pattern q is assumed to be skew indecomposable. Let us call a pattern $q = q_1 q_2 \cdots q_k$ *good* if there does not exist a positive integer $i \leq k-1$ so that $\{q_{k-i}, q_{k-i+1}, \cdots, q_{k-1}\} = \{1, 2, \cdots, i\}$. That is, q is good if there is no proper segment immediately preceding its last entry whose entries would be the smallest entries of q. For instance, $q = 132$ and $q = 3142$ are good, but $q = 1324$ and $q = 35124$ are not, because of the choices of $i = 3$ in the former, and $i = 2$ in the latter. In particular, q is never good if $q_k = k$, because then we can choose $i = k-1$.

Lemma 4.1. *Let q be a good pattern. Then for all positive integers n, the inequality*

$$\mathrm{Av}_{n,2}(q) \leq \mathrm{Av}_{n,1}(q)$$

holds.

Proof. We define a map $g : \mathcal{A}v_{n,2}(q) \rightarrow \mathcal{A}v_{n,1}(q)$, and show that it is an injection.

Let $p \in \mathcal{A}v_{n,2}(q)$. That means p has two skew blocks; let us call the entries of the first skew block the *big* entries, and the entries of the second skew block the *small* entries. Let us define $g(p)$ by moving the *rightmost big entry* x of p into the last position of p. The obtained permutation $g(p)$ still avoids q. Indeed, as p avoids q, the only way $g(p)$ could possibly contain a copy C of q would be if C contained the recently moved entry x that is at the end of $g(p)$. However, C could not consist entirely of big entries, since then p would contain q as well. Therefore, C must start with a (possibly empty) string of big entries, followed by a non-empty string of small entries, and end by its maximal entry x, which is a large entry. This contradicts our assumption that q is a good pattern.

Now we prove that $g : \mathcal{A}v_{n,2}(q) \rightarrow \mathcal{A}v_{n,1}(q)$ is an injection. Let $p*$ be the (partial) permutation we obtain from p when we remove x. The structure of $p*$ is as follows.

(1) The small entries of p form a skew block at the end, and
(2) the big entries of p can form
   (a) zero skew blocks, if there was only one large entry in p, or
   (b) one block, if the removal of x did not split the first skew block of p into more blocks, or
   (c) more than one block, if the removal of x did split the first skew block of p into more blocks.

   In each case, $g(p)$ without its last entry will have one additional skew block, namely the skew block of the small entries.

An example for the first case is $p = 53124$ and $f(p) = 31245$, an example for the second case is $p = 536412$ and $f(p) = 536124$, and an example for the last case is $p = 6534712$, and $f(p) = 6534127$. In each case, the pattern q can be any skew indecomposable pattern that p avoids.

Let $w = w_1 w_2 \cdots w_n$ be a permutation in $\mathcal{A}v_{n,1}(q)$. Consider its segment $w* = w_1 w_2 \cdots w_{n-1}$, and count how many skew blocks it has.

(a) If $w*$ has one skew block, then $g(p) = w$ implies that p has only one large entry, the entry n, and that is in the first position of p. So we obtain the unique permutation $g^{-1}(w) = p$ by placing the last entry w_n of w into the first position.
(b) If $w*$ has two skew blocks, and the one on the right contains entries larger than w_n, then w does not have a preimage under g, since the rightmost block of $w*$ must coincide with the small entries of its purported preimage, and those entries must be smaller than the big entry $w_n = x$. If $w*$ has two skew blocks, and the one on the right

consists of entries smaller than w_n, then the unique permutation $g^{-1}(w) = p$ is obtained by placing $w_n = x$ just in front of that block. Indeed, this is the only way to assure that $w_n = x$ is the last entry of the first block of $g^{-1}(w) = p$.

(c) If $w*$ has more than two skew blocks, then just in the previous case, w_n must be larger than all entries in the rightmost of those skew blocks. Next, we claim that even in this case, the unique permutation $g^{-1}(w) = p$ is obtained by placing w_n immediately to the left of the skew block of small entries.

Indeed, let us assume that $w_n = x$ is somewhere else. The structure of w is as follows.

$$w = B_1|B_2|\cdots|B_t|S|x,$$

with $t \geq 2$, where B_1 is the first skew block of w, and S is the skew block of the small entries of p.

Then in $g^{-1}(w)$, the entry x cannot be between B_i and B_{i+1}, because then $g^{-1}(w) \notin \mathcal{A}v_{n,2}(q)$. Indeed, either x is larger than everything in B_{i+1}, and then there is at least one skew block on the left of B_{i+1}, and B_{i+1} is a skew block, and S is a skew block, yielding that $g^{-1}(w)$ has at least three skew blocks, or x is smaller than everything in B_i, and then similarly, B_i is a skew block, the string starting with x and ending immediately before S is a union of skew blocks, and S is a skew block, leading to at least three skew blocks again.

It is also impossible that in $g^{-1}(w)$, the entry $w_n = x$ is at the very front, because x is supposed to be the rightmost entry of the first skew block of $g^{-1}(w)$, and the only way for the rightmost entry of a skew block to be also the leftmost one is by that skew block being of size 1. However, that would mean that $g^{-1}(w) = p$ starts with its largest entry, so the rest of p, and therefore, $w*$, has only one skew block, and so we are in case (a), not case (c).

So we have seen that if $w \in \mathcal{A}v_{n,1}(q)$, then w has at most one preimage under g, proving that $g : \mathcal{A}v_{n,2}(q) \to \mathcal{A}v_{n,1}(q)$ is an injection, and hence proving our lemma. □

Now we are going to extend the reach of Lemma 4.1 to other patterns.

Lemma 4.2. *Let $q = q_1 \cdots q_k$ be a skew indecomposable pattern so that $q_1 \neq 1$ or $q_k \neq k$ or both. The the inequality*

$$\mathrm{Av}_{n,2}(q) \leq \mathrm{Av}_{n,1}(q)$$

holds.

Proof. Let $q = q_1 q_2 \cdots q_k$ be a pattern that is not a good pattern and does not end in its largest entry. That means that there exists an $i < k-1$ so that $\{q_{k-i}, q_{k-i+1}, \cdots, q_{k-1}\} = \{1, 2, \cdots, i\}$, and $q_k = y \neq k$. Therefore, in the reverse q^{rev} of q, the entry $y \neq k$ is in the first position, and the entries in positions $2, 3, \cdots, i+1$ are the entries $1, 2, \cdots, i$ in some order. In particular, the entry 1 precedes the entry k, so q^{rev} is skew indecomposable. Furthermore, q^{rev} is a good pattern, since again, the entry 1 precedes the entry k, so all ending segments that contain 1 also contain k, so the only way for q^{rev} to be not good would be by ending in k. However, that would imply that q starts in k, contradicting the assumption that q is indecomposable.

If q is a skew indecomposable pattern that is not good and ends in its largest entry, but does not start in the entry 1, then the reverse complement $(q^c)^{rev} := q^{rc}$ of q is a skew indecomposable pattern that does not end in its largest entry. So, by the previous paragraph, either q^{rc} or its reverse q^c is a good pattern. In either case, we finish our proof by applying Lemma 4.1 to either q^{rc} or to q^c, and then applying Proposition 2.2 to conclude that our statement holds for q as well. □

Lemma 4.2 does not cover patterns that start with their minimal element and end with their largest element, like 1324. However, if q is such a pattern, we can still prove the statement of Lemma 4.2 for q if q is Wilf-equivalent to a pattern q' that *is* covered by Lemma 4.2. Indeed, this is an immediate consequence of Proposition 2.2. So, for instance, the statement of Lemma 4.2 also holds for all monotone patterns $12\cdots k$, since it is well-known [1] that $12\cdots k$ is Wilf-equivalent to the pattern $12\cdots(k-2)k(k-1)$.

The proof of the monotonicity result announced in the introduction is now immediate.

Theorem 4.3. *Let $q = q_1 \cdots q_k$ be a skew indecomposable pattern so that at least one of the following conditions hold*

(1) $q_1 \neq 1$, *or*
(2) $q_k \neq k$, *or*
(3) $q_1 = 1$ *and* $q_k = k$, *but q is Wilf-equivalent to a skew-indecomposable pattern in which the first entry is not 1 or the last entry is not k.*

Then the inequality

$$\mathrm{Av}_{n,\ell+1}(q) \leq \mathrm{Av}_{n,\ell}(q)$$

holds for all nonnegative integers n and all positive integers ℓ.

Proof. Proposition 2.2 implies that we can assume that q does not start with 1, or does not end in k. Then the proof of our claim is immediate from Lemma 4.2 and Proposition 1. □

5. Why $A_q(z)$ is not rational

We can now prove the result mentioned in the title of the paper.

Theorem 5.1. *Let $q = q_1 q_2 \cdots q_k$ be a pattern so that either $\{1, k\} \neq \{q_1, q_k\}$, or q is Wilf-equivalent to a pattern $v = v_1 v_2 \cdots v_k$ so that $\{1, k\} \neq \{v_1, v_k\}$ Then the generating function $A_q(z)$ is not rational.*

Proof. First, note that we can assume that q is skew indecomposable. Indeed, if q is not, then q^{rev} is, and clearly, $A_q(z) = A_{q^{rev}}(z)$.

So let q be skew indecomposable, and let us assume that $A_q(z)$ is rational. Then by (5), the power series $A_{1,q}(z)$ is also rational. Let $r > 0$ be the radius of convergence of $A_{1,q(z)}$. We know that $r > 0$, since we know [6] that $\mathrm{A}v_{n,1}(q) \leq \mathrm{A}v_n(q) \leq c_q^n$ for some constant c_q. As the coefficients of $A_{1,q(z)}$ are all nonnegative real numbers, it follows from Pringsheim's theorem [5] that the positive real number r is a singularity of $A_{1,q(z)}$. As $A_{1,q}(z)$ is rational, r is a pole of $A_{1,q}(z)$, so $\lim_{z\to r} A_{1,q}(z) = \infty$. Therefore, there exists a positive real number $z_0 < r$ so that $A_{1,q}(z_0) > 1$. Therefore,

$$\sum_{n\geq 1} \mathrm{Av}_{n,1}(q) z_0^n = A_{1,q}(z_0) < A_{1,q}(z_0)^2 = A_{2,q}(z_0) = \sum_{n\geq 2} \mathrm{Av}_{n,2}(q) z_0^n,$$

contradicting the fact, proved in Theorem 4.3, that for each n, the coefficient of z^n in the leftmost powers series is at least as large as it is in the rightmost power series. □

The elegant argument in the previous paragraph is due to Robin Pemantle [9]. It shows that the square of a rational power series with nonnegative coefficients and a positive convergence radius will have at least one coefficient that is larger than the corresponding coefficient of the power series itself. A significantly more complicated argument proves a stronger statement. The interested reader should consult [2] for details.

6. Further directions

It goes without saying that it is an intriguing problem to prove Lemma 4.2 for the remaining patterns. Of course, Theorem 5.1 could possibly be proved by other means, but numerical evidence seems to suggest that Theorem 5.1 will hold even for patterns that start with their minimum entry and end in their largest entry. Interestingly, the shortest patterns for which we cannot prove Theorem 5.1 are 1324 and 4231, which also happen to be the shortest patterns for which no exact formula is known for $\mathrm{Av}_n(q)$.

It is important to point out that our results do not hold at all for permutation classes that are generated by more than one pattern. For instance, let $\mathrm{Av}_n(123, 132)$ denote the number of permutations of length n that avoid both 123 and 132. It is then easy to prove that $\mathrm{Av}_n(123, 132) = 2^{n-1}$, so $A_{123,132}(z) = (1 - z)/(1 - 2z)$, a rational function. Note that in this case, $\mathrm{Av}_{n,1}(123, 132) = 1$, since the only such permutation is $(n-1)(n-2)\cdots 1n$, while $\mathrm{Av}_{n,2}(123, 132) = n - 1$, so Lemma 4.2 does not hold.

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Department of Mathematics, University of Florida, 358 Little Hall, PO Box 118105, Gainesville, FL, 32611 − 8105 (USA)

E-mail address: `bona@ufl.edu`