AN IDENTITY FOR VERTICALLY ALIGNED ENTRIES IN PASCAL'S TRIANGLE

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Abstract. The classic way to write down Pascal's triangle leads to entries in alternating rows being vertically aligned. In this paper, we prove a linear dependence on vertically aligned entries in Pascal's triangle. Furthermore, we give an application of this dependence to morphisms between hyperelliptic curves.

1. Introduction

We consider entries in the nth row of Pascal's triangle, where n is any nonnegative integer. It is well known that the ith entry in this row can be computed as $\binom{n}{i}$, where $0 \leq i \leq n$. For example, the 3rd entry in row 11 is $\binom{11}{3} = \frac{11\cdot 10\cdot 9}{3!} = 165$. Figure 1 shows rows 0 through 12 of Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 136 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

Figure 1. Pascal's triangle.

Notice that entries in alternating rows are vertically aligned. For example, in Figure 2 below we have circled the entries that are vertically aligned with the 3rd entry in the 11th row. In Figure 3 we have circled the entries that are vertically aligned with the 6th entry in the 12th row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 136 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

FIGURE 2. Entries vertically aligned with the 3rd entry in the 11th row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 136 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

FIGURE 3. Entries vertically aligned with the 6th entry in the 12th row.

We can describe these entries in the following way. Starting with the ith entry in the nth row, i.e. $\binom{n}{i}$, the entries that are vertically aligned with this entry and above it are all of the form

$$\binom{n-2k}{i-k},$$

where $1 \leq k \leq i$ and $k \leq \lfloor \frac{n}{2} \rfloor$.

For example, when $n = 11$ and $i = 3$, the entries that are above $\binom{11}{3}$ and vertically aligned with it are

$$\binom{9}{2}, \binom{7}{1}, \binom{5}{0}.$$

Observe that

$$\binom{11}{3} - 11\binom{9}{2} + 44\binom{7}{1} - 77\binom{5}{0} = 165 - 11 \cdot 36 + 44 \cdot 7 - 77 \cdot 1 = 0.$$

When $n = 12$ and $i = 6$, we have

$$\begin{aligned}\binom{12}{6} - 12\binom{10}{5} + 54\binom{8}{4} - 112\binom{6}{3} + 105\binom{4}{2} - 36\binom{2}{1} + 2\binom{0}{0} \\ = 924 - 12 \cdot 252 + 54 \cdot 70 - 112 \cdot 20 + 105 \cdot 6 - 36 \cdot 2 + 2 \cdot 1 \\ = 0.\end{aligned}$$

In the next section, we prove a general formula for the linear dependence on vertically aligned entries in Pascal's triangle.

2. General Formula

Theorem 2.1. *Let n be a nonnegative integer and $0 < i < n$. Then*

$$\sum_{k=0}^{i} (-1)^k \frac{n}{n-k}\binom{n-k}{k}\binom{n-2k}{i-k} = 0.$$

Remark 1. Note that the $k = 0$ term is simply $\binom{n}{i}$. If $i > n/2$, there will be some values of k for which $n - 2k < i - k$. For example, if $n = 11$ and $i = 8$, then $k = 4$ has $n - 2k = 3 < 4 = i - k$. But recall that

$$\binom{m}{r} = 0$$

whenever $0 \le m < r$ (see, for example, [3, Section 1.9]). Thus, terms for which $0 \le n-2k < i - k$ do not contribute to the sum in Theorem 2.1.

If $n - 2k < 0$, then $\binom{n-2k}{i-k}$ is no longer 0. However in this case, we have $n < 2k$, which implies, $n - k < k$. Thus, $\binom{n-k}{k} = 0$ instead.

Hence, all terms for which $i > n/2$ do not contribute to the sum in Theorem 2.1.

Remark 2. The expressions $\frac{n}{n-k}\binom{n-k}{k}$ that appear in Theorem 2.1 are referred to as the Triangle of coefficients of Lucas (or Cardan) polynomials, denoted $T(n, k)$, in the On-Line Encyclopedia of Integer Sequences [1].

Proof of Theorem 2.1. The following proof starts with an identity attributed to E.H. Lockwood. For any $n \ge 1$,

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(xy)^k(x+y)^{n-2k}$$

(see, for example, [3, Section 9.8]).

We separate the $k = 0$ term from the summation to get

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}. \tag{1}$$

The Binomial Theorem tells us that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = x^n + y^n + \sum_{i=1}^{n-1} \binom{n}{i} x^{n-i} y^i \tag{2}$$

Substituting Equation 2 into Equation 1 yields

$$x^n + y^n = x^n + y^n + \sum_{i=1}^{n-1} \binom{n}{i} x^{n-i} y^i + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}.$$

Hence,

$$\sum_{i=1}^{n-1} \binom{n}{i} x^{n-i} y^i + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = 0. \tag{3}$$

Thus, when combining the two sums, the coefficient of each $x^{n-i}y^i$ term must equal 0. We expand the second summand in order to identify all terms of the form $x^{n-i}y^i$. The Binomial Theorem tells us that, for each k,

$$(x+y)^{n-2k} = \sum_{j=0}^{n-2k} \binom{n-2k}{j} x^{n-2k-j} y^j.$$

Hence,

$$(xy)^k (x+y)^{n-2k} = \sum_{j=0}^{n-2k} \binom{n-2k}{j} x^{n-k-j} y^{j+k}. \tag{4}$$

The values of j that yield $x^{n-i}y^i$ terms are $j = i - k$. Note that we must have $k \le i$, since otherwise $j \le 0$. Thus, the coefficient of $x^{n-i}y^i$ in Equation 4 is

$$\sum_{k=1}^{i} \binom{n-2k}{i-k}.$$

Hence, the sum of the coefficients of the $x^{n-i}y^i$ terms in Equation 3 is

$$\sum_{k=0}^{i} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{n-2k}{i-k} = 0,$$

where the $k = 0$ term is $\binom{n}{i}$, which comes from the first summation in Equation 3.

□

3. Application to Hyperelliptic Curves

In this section we give an application of the identity in Theorem 2.1. Work on this application in [2, Section 5.1] is what led the author to discover the identity in Theorem 2.1.

Let C be the genus g hyperelliptic curve $y^2 = x^{2g+1} + x$. The map

$$\phi(x,y) = \left(\frac{x^2+1}{x}, \frac{y}{x^a} \right),$$

where $a = \frac{g+1}{2}$, is a nonconstant morphism from C to some curve, denoted C'. Note that the curve C' will also be hyperelliptic. We initially define C' to be of the form

$$y^2 = c_d x^d + \ldots + c_{d-i} x^{d-i} + \ldots + c_0$$

and we will apply the transformation of variables given by ϕ to determine the coefficients c_j. Applying the transformation yields

$$\begin{aligned}
\left(\frac{y}{x^a}\right)^2 &= c_d \left(\frac{x^2+1}{x}\right)^d + \ldots + c_{d-i}\left(\frac{x^2+1}{x}\right)^{d-i} + \ldots + c_0 \\
\frac{y^2}{x^{g+1}} &= c_d x^{-d}(x^2+1)^d + \ldots + c_{d-i} x^{i-d}(x^2+1)^{d-i} + \ldots + c_0 \\
y^2 &= c_d x^{g+1-d}(x^2+1)^d + \ldots + c_{d-i} x^{g+1+i-d}(x^2+1)^{d-i} + \ldots + c_0 x^{g+1}.
\end{aligned}$$

In order for ϕ to be a morphism from C to C', this last equation should, in fact, be the equation for the curve C. Note that the degree of the expression in x will be $g+1-d+2d = g+1+d$. Hence, we need $c_d = 1$ and $g+1+d = 2g+1$, so that $d = g$. We use this to simplify the above equation to

$$y^2 = x(x^2+1)^g + \ldots + c_{g-i} x^{1+i}(x^2+1)^{g-i} + \ldots + c_0 x^{g+1}. \tag{5}$$

In order to determine the coefficients c_i, we need to expand the right-hand side of the equation and match coefficients with those of C. We now work through two examples to better understand what the coefficients of C' will be.

Example 3.1. *Let $g = 5$, so that C is the hyperelliptic curve $y^2 = x^{11} + x$. From our above work, we know that the degree of C' will be 5. Consider the following terms from Equation 5: $A_1 = x(x^2+1)^5$, $A_2 = x^3(x^2+1)^3$, and $A_3 = x^5(x^2+1)^1$. We expand each of these to get*

$$\begin{aligned}
A_1 &= x(x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1) \\
&= x^{11} + 5x^9 + 10x^7 + 10x^5 + 5x^3 + x, \\
A_2 &= x^3(x^6 + 3x^4 + 3x^2 + 1) \\
&= x^9 + 3x^7 + 3x^5 + x^3, \\
A_3 &= x^5(x^2+1) \\
&= x^7 + x^5.
\end{aligned}$$

Note that $A_1 - 5A_2 + 5A_3 = x^{11} + x$. Hence, ϕ is a morphism from C to $y^2 = x^5 - 5x^3 + 5x$.

Example 3.2. *Now let $g = 6$, so that C is the hyperelliptic curve $y^2 = x^{13} + x$. From our above work, we know that the degree of C' will be 6. Consider the following terms from Equation 5: $B_1 = x(x^2+1)^6$, $B_2 = x^3(x^2+1)^4$, $B_3 = x^5(x^2+1)^2$, and $B_4 = x^7(x^2+1)^0$.*

We expand each of these to get

$$\begin{aligned} B_1 &= x(x^{12} + 6x^{10} + 15x^8 + 2 - x^6 + 15x^4 + 6x^2 + 1) \\ &= x^{13} + 6x^{11} + 15x^9 + 2 - x^7 + 15x^5 + 6x^3 + x, \\ B_2 &= x^3(x^8 + 4x^6 + 6x^4 + 4x^2 + 1) \\ &= x^{11} + 4x^9 + 6x^7 + 4x^5 + x^3, \\ B_3 &= x^5(x^4 + 2x^2 + 1) \\ &= x^9 + 2x^7 + x^5 \\ B_4 &= x^7. \end{aligned}$$

One can easily show that $B_1 - 6B_2 + 9B_3 - 2B_4 = x^{13} + x$*, which tells us that* ϕ *is a morphism from* C *to* $y^2 = x^6 - 6x^4 + 9x^2 - 2$*.*

While working on [2, Section 5.1], the author determined (by hand) the curve C' for $g = 11$. The coefficients she found were 1, 11, 44, 77, 55, and 11, with alternating signs (see Table 1 below). The author entered this sequence of numbers into the On-line Encyclopedia of Integer Sequences [1] and found that these numbers are the Triangle of coefficients of Lucas (or Cardan) polynomials, $T(n, k)$. The coefficients that appear in Examples 3.1 and 3.2 are also of the form $T(n, k)$. As noted in Remark 2,

$$T(n, k) = \frac{n}{n-k}\binom{n-k}{k}.$$

This leads us to the following theorems.

Theorem 3.3. *Let* C *be the hyperelliptic curve* $y^2 = x^{2g+1} + x$ *and let* C' *be the hyperelliptic curve*

$$y^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k} x^{g-2k}.$$

Then the map

$$\phi(x, y) = \left(\frac{x^2+1}{x}, \frac{y}{x^a}\right),$$

where $a = \frac{g+1}{2}$*, is a nonconstant morphism from* C *to* C'*.*

We can generalize Theorem 3.3. Let $c \in \mathbb{Q}^*$ be constant and ζ be a primitive g-th root of unity. In the following theorem we work over the field $\mathbb{F} = \mathbb{Q}(\zeta, c^{1/g})$.

Theorem 3.4. *Let* C *be the hyperelliptic curve* $y^2 = x^{2g+1} + cx$ *and let* C_i *be the hyperelliptic curve*

$$y^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k} \zeta^{ik} c^{k/g} x^{g-2k}$$

for $i = 0, 1$*. Then the map*

$$\phi_i(x, y) = \left(\frac{x^2 + \zeta^i c^{1/g}}{x}, \frac{y}{x^a}\right),$$

where $a = \frac{g+1}{2}$*, is a nonconstant morphism from* C *to* C_i*.*

Since

$$\frac{g}{g-k}\binom{g-k}{k} = \left[\binom{g-k}{k} + \binom{g-k-1}{k-1}\right]$$

(see, for example, [3, Section 9.9]), Theorem 3.4 also generalizes Lemma 5.1 in [2] because we are no longer restricting g to be odd. The proofs of Theorems 3.3 and 3.4 are nearly identical to the proof of Lemma 5.1 in [2], and so we omit them.

We now expand Example 3.1 to show how our work in this section relates to our work in Theorem 2.1. Let $n = g = 5$, and k range from 0 to $\lfloor n/2 \rfloor = 2$. We evaluate

$$(-1)^k \frac{n}{n-k}\binom{n-k}{k}$$

for each of these values of k to get

$$\begin{aligned} k = 0:\ & (-1)^0 \frac{5}{5-0}\binom{5-0}{0} = 1 \\ k = 1:\ & (-1)^1 \frac{5}{5-1}\binom{5-1}{1} = -5 \\ k = 2:\ & (-1)^2 \frac{5}{5-2}\binom{5-2}{2} = 5, \end{aligned}$$

which are the coefficients in the equation for C', i.e. those of A_1, A_2, and A_3, respectively. These coefficients help us cancel certain powers of x in the expansion of Equation 5. For example, in the sum $A_1 - 5A_2 + 5A_3$, the coefficient of x^5 is

$$\begin{aligned} 0 &= 10 - 5 \cdot 3 + 5 \cdot 1 \\ &= \binom{5}{2} - 5\binom{3}{1} + 5\binom{1}{0} \\ &= \frac{5}{5-0}\binom{5-0}{0}\binom{5}{2} - \frac{5}{5-1}\binom{5-1}{1}\binom{3}{1} + \frac{5}{5-2}\binom{5-2}{2}\binom{1}{0}, \end{aligned}$$

which matches the statement of Theorem 2.1 for $n = 5$ and $i = 2$.

3.1. **Higher Genus Examples.** Table 1 below gives C_i for values of g up to 11 and for $c = 1$. Note that this table expands on the table that appears in [2, Section 5.1].

g	curve C_i
5	$y^2 = x^5 - 5\zeta^i x^3 + 5\zeta^{2i} x$
6	$y^2 = x^6 - 6\zeta^i x^4 + 9\zeta^{2i} x^2 - 2\zeta^{3i}$
7	$y^2 = x^7 - 7\zeta^i x^5 + 14\zeta^{2i} x^3 - 7\zeta^{3i} x$
8	$y^2 = x^8 - 8\zeta^i x^6 + 20\zeta^{2i} x^4 - 16\zeta^{3i} x^2 + 2\zeta^{4i}$
9	$y^2 = x^9 - 9\zeta^i x^7 + 27\zeta^{2i} x^5 - 30\zeta^{3i} x^3 + 9\zeta^{4i} x$
10	$y^2 = x^{10} - 10\zeta^i x^8 + 35\zeta^{2i} x^6 - 50\zeta^{3i} x^4 + 25\zeta^{4i} x^2 - 2\zeta^{5i}$
11	$y^2 = x^{11} - 11\zeta^i x^9 + 44\zeta^{2i} x^7 - 77\zeta^{3i} x^5 + 55\zeta^{4i} x^3 - 11\zeta^{5i} x$

TABLE 1.

Note that for all g, the coefficient of second term of the expression in x will always be $-g$ (times a power of ζ). The reason this is the case is that this coefficient corresponds to $k = 1$, and

$$\begin{aligned}(-1)^k \frac{g}{g-k}\binom{g-k}{k} &= -\frac{g}{g-1}\binom{g-1}{1} \\ &= -\frac{g}{g-1}\cdot(g-1) \\ &= -g.\end{aligned}$$

Note that when g is even, the final term corresponds to $k = g/2$, which yields x^0. We compute the coefficient to be

$$\begin{aligned}(-1)^k \frac{g}{g-k}\binom{g-k}{k} &= (-1)^{g/2}\frac{g}{g-g/2}\binom{g-g/2}{g/2} \\ &= (-1)^{g/2}\frac{g}{g/2}\binom{g/2}{g/2} \\ &= (-1)^{g/2}2.\end{aligned}$$

Hence, when g is even, the final term of the expression in x will always be $(-1)^{g/2}2$ (times a power of ζ).

On the other hand, when g is odd, the final term corresponds to $k = (g-1)/2$, which yields x^1. We compute the coefficient to be

$$\begin{aligned}(-1)^k \frac{g}{g-k}\binom{g-k}{k} &= (-1)^{(g-1)/2}\frac{g}{g-(g-1)/2}\binom{g-(g-1)/2}{(g-1)/2} \\ &= (-1)^{(g-1)/2}\frac{g}{(g+1)/2}\binom{(g+1)/2}{(g-1)/2} \\ &= (-1)^{(g-1)/2}\frac{g}{(g+1)/2}\binom{(g-1)/2+1}{(g-1)/2} \\ &= (-1)^{(g-1)/2}\frac{g}{(g+1)/2}\cdot((g-1)/2+1) \\ &= (-1)^{(g-1)/2}g.\end{aligned}$$

Hence, when g is odd, the final term of the expression in x will always be $(-1)^{(g-1)/2}gx$ (times a power of ζ).

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