EXPECTED f-VECTOR OF THE POISSON ZERO POLYTOPE AND RANDOM CONVEX HULLS IN THE HALF-SPHERE

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Abstract. We prove an explicit combinatorial formula for the expected number of faces of the zero polytope of the homogeneous and isotropic Poisson hyperplane tessellation in $\mathbb{R}^d$. The expected f-vector is expressed through the coefficients of the polynomial

$$(1+(d-1)^2x^2)(1+(d-3)^2x^2)(1+(d-5)^2x^2)\dots.$$

Also, we compute explicitly the expected f-vector and the expected volume of the spherical convex hull of n random points sampled uniformly and independently from the d-dimensional half-sphere. In the case when $n=d+2$, we compute the probability that this spherical convex hull is a spherical simplex, thus solving an analogue of the Sylvester four-point problem on the half-sphere.

1. Main results

1.1. **Poisson zero polytope.** Poisson hyperplane processes and the corresponding random tessellations of the Euclidean space by polytopes have been extensively studied in stochastic geometry since the works of Miles [21, 18, 19, 20] and Matheron [14, 15, 16]; see Section 4.4 and Chapter 10 of the book by Schneider and Weil [25] for more information and references. Poisson hyperplane tessellations give rise to (at least) two natural random polytopes: *the Poisson zero polytope* (defined as the a.s. unique polytope of the tessellation containing the origin) and the *typical Poisson polytope* (defined essentially as a polytope picked uniformly at random from the set of polytopes of the tessellation contained in some very large observation window). One of the most interesting characteristics of a random polytope is its expected f-vector whose k-th component is, by definition, the expected number of k-dimensional faces of the polytope. While it is well known that the expected f-vector of the typical Poisson polytope coincides with the f-vector of the cube of the same dimension (see, for example, Theorems 10.3.1 and 10.3.2 in [25]), a corresponding result for the zero polytope seems to be missing. In the present paper we close this gap by providing an explicit formula for the expected f-vector of the zero polytope of the isotropic and homogeneous Poisson hyperplane tessellation on $\mathbb{R}^d$.

Let us recall the definitions of the Poisson hyperplane process and the Poisson zero polytope. Denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the Euclidean norm and the standard scalar product on $\mathbb{R}^d$, respectively. Let $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d\colon \|x\|=1\}$ be the unit sphere in $\mathbb{R}^d$. Let $A(d,d-1)$ be the Grassmannian manifold of all affine hyperplanes in $\mathbb{R}^d$. Every affine hyperplane $H\in A(d,d-1)$ can be represented in the form

$$H=H(w,\tau):=\{x\in\mathbb{R}^d\colon \langle x,w\rangle=\tau\}$$

with some "direction" $w\in\mathbb{S}^{d-1}$ and some (possibly negative) "distance" $\tau\in\mathbb{R}$. In fact, $H(w,\tau)=H(-w,-\tau)$, and every $H\in A(d,d-1)$ has exactly two such representations. A

2010 *Mathematics Subject Classification.* Primary: 52A22, 60D05; Secondary: 52B11, 52A20, 51M20, 52A55.

Key words and phrases. Poisson hyperplane tessellation, Poisson zero polytope, Crofton polytope, f-vector, random polytope, random cone, Stirling numbers, conic intrinsic volumes, internal and external angles, beta' polytope, convex hulls on the half-sphere, Sylvester four-point problem.

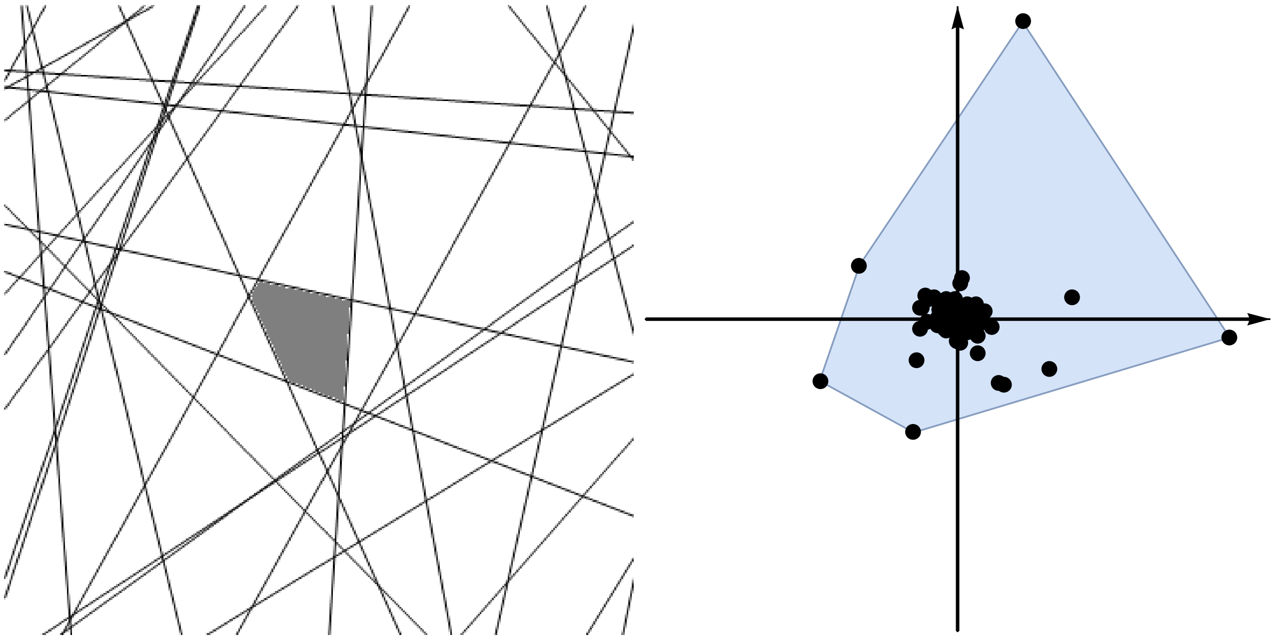

Figure 1.1. Left: The Poisson line tessellation in the plane, together with the zero polygone. Right: The dual Poisson point process $\Pi_{2,1}$ on $\mathbb{R}^2$ with intensity $\|x\|^{-3}$, together with its convex hull. The lines of the tessellation correspond to the points of the Poisson process via projective duality.

homogeneous *Poisson hyperplane process* with intensity $\gamma > 0$ is a random, countable collection of affine hyperplanes $X = \{H(W_i, T_i)\}_{i\in\mathbb{Z}}$, where

(a) $\{T_i\}_{i\in\mathbb{Z}}$ are the arrivals of a homogeneous, intensity γ Poisson process on the real line;
(b) $\{W_i\}_{i\in\mathbb{Z}}$ are independent, identically distributed random vectors with certain centrally symmetric probability distribution μ on the unit sphere $\mathbb{S}^{d-1}$;
(c) $\{T_i\}_{i\in\mathbb{Z}}$ is independent of $\{W_i\}_{i\in\mathbb{Z}}$.

Equivalently, we can view the Poisson hyperplane process X as a Poisson point process on $A(d, d-1)$ whose intensity measure Θ is given by

$$\Theta(A) := \gamma \int_{\mathbb{S}^{d-1}} \left(\int_{-\infty}^{+\infty} \mathbb{1}_{\{H(w,\tau)\in A\}} \mathrm{d}\tau \right) \mu(\mathrm{d}w),$$

for all Borel sets $A \subset A(d, d-1)$; see [25, Section 4.4]. In the present paper, we restrict our attention to the *isotropic* case meaning that the direction measure μ is chosen to be the uniform probability distribution on $\mathbb{S}^{d-1}$. Without restriction of generality, we may choose $\gamma := 1$. It is known that X consists of countably many random affine hyperplanes whose probability law is invariant with respect to the natural action of the isometry group of $\mathbb{R}^d$ on the set of hyperplanes $A(d, d-1)$. The hyperplanes of the Poisson hyperplane process X dissect $\mathbb{R}^d$ into countably many polytopes; see the left panel of Figure 1.1 for a realization when $d = 2$. The *Poisson zero polytope* or the *Crofton polytope* is the a.s. unique polytope of this tessellation that contains the origin.

1.2. **Statement of the main result.** Given a polytope P, the number of its k-dimensional faces is denoted by $f_k(P)$, for $k \in \{0, 1, \ldots, d\}$. For example, $f_0(P)$ is the number of vertices, $f_1(P)$ is the number of edges, $f_{d-1}(P)$ is the number of $(d-1)$-dimensional faces (called *facets*), and $f_d(P) = 1$. The *f-vector* of the polytope P is defined by $\mathbf{f}(P) := (f_0(P), \ldots, f_{d-1}(P))$.

The main result of the present paper is the following formula for the expected f-vector of the Poisson zero polytope.

Theorem 1.1. *Let $Z^{(d)}$ be the d-dimensional Poisson zero polytope, $d \in \mathbb{N}$. For all $\ell \in \{0, \ldots, d\}$ such that the codimension $d - \ell$ is* ***even****, we have*

$$\mathbb{E} f_\ell(Z^{(d)}) = \frac{\pi^{d-\ell}}{(d-\ell)!} A[d, d-l], \tag{1.1}$$

where[1]

$$A[n,k] = [x^k]\Big((1+(n-1)^2x^2)(1+(n-3)^2x^2)(1+(n-5)^2x^2)\ldots\Big) \tag{1.2}$$

and $[x^k]Q(x)$ denotes the coefficient of x^k in the polynomial $Q(x)$.

A simple recursive algorithm computing the values of $\mathbb{E} f_\ell(Z^{(d)})$ without restriction on the parity of $d-\ell$ will be provided in Section 1.6. At this place, let us only point out that the values of $\mathbb{E} f_\ell(Z^{(d)})$ with **odd** $d - \ell$ are defined uniquely by the values with even $d - \ell$ (see Theorem 1.1) and the Dehn-Sommerville relations. Indeed, the random polytope $Z^{(d)}$ is *simple* with probability 1 (that is, each vertex of this polytope is adjacent to exactly d edges (and also exactly d facets). Equivalently, the dual polytope of $Z^{(d)}$ (which will be described explicitly in Section 1.5) is *simplicial* with probability 1, that is all of its facets (and, hence, all faces) are simplices; see [7, Section 4.5] for a discussion of these classes of polytopes. For a simple d-dimensional polytope P, the *Dehn-Sommerville relations* (see, e.g., [7, Section 9.2]) state that for all $\ell \in \{0, \ldots, d\}$ we have

$$f_\ell(P) = \sum_{i=0}^{\ell} (-1)^i \binom{d-i}{d-\ell} f_i(P). \tag{1.3}$$

Applying this to $P := Z^{(d)}$ and taking the expectation, we arrive at the equations

$$\mathbb{E} f_\ell(Z^{(d)}) = \frac{1}{2}\sum_{i=0}^{\ell-1} (-1)^i \binom{d-i}{d-\ell} \mathbb{E} f_i(Z^{(d)}), \tag{1.4}$$

for all **odd** $\ell \in \{1, \ldots, d\}$. Let us show that these equations, together with Theorem 1.1, determine $\mathbb{E} f_\ell(Z^{(d)})$ uniquely for every $\ell \in \{0, \ldots, d\}$ irrespective of its parity.

Case 1: d is even. Then, the values of $\mathbb{E} f_\ell(Z^{(d)})$ with even ℓ are given by Theorem 1.1. For odd ℓ, relation (1.4) expresses $\mathbb{E} f_\ell(Z^{(d)})$ through the values $\mathbb{E} f_i(Z^{(d)})$ with $i < \ell$.

Case 2: d is odd. Then, the values of $\mathbb{E} f_\ell(Z^{(d)})$ with odd ℓ are given by Theorem 1.1. Let ℓ be even. Using (1.4) with ℓ replaced by $\ell + 1$, we obtain

$$\mathbb{E} f_{\ell+1}(Z^{(d)}) = \frac{1}{2}\sum_{i=0}^{\ell} (-1)^i \binom{d-i}{d-\ell-1} \mathbb{E} f_i(Z^{(d)}),$$

for all even $\ell \in \{0, \ldots, d-1\}$. Separating the term with $i = \ell$, we arrive at

$$\mathbb{E} f_\ell(Z^{(d)}) = \frac{2}{d-\ell} \mathbb{E} f_{\ell+1}(Z^{(d)}) - \frac{1}{d-\ell}\sum_{i=0}^{\ell-1} (-1)^i \binom{d-i}{d-\ell-1} \mathbb{E} f_i(Z^{(d)}). \tag{1.5}$$

[1]By convention, the last factor in the product defining $A[n,k]$ is $1 + x^2$ or $1 + 2^2x^2$ depending on whether n is even or odd. The product has finitely many terms, and all terms are non-zero.

This gives an expression for $\mathbb{E}f_\ell(Z^{(d)})$ in terms of $\mathbb{E}f_{\ell+1}(Z^{(d)})$ (which is given by Theorem 1.1) and $\mathbb{E}f_i(Z^{(d)})$ with $i < \ell$.

In low dimensions, the expected f-vectors (as computed with the help of Mathematica) are given in Table 1. The reader may observe that the values with even codimension $d-\ell$ are "nice" (rational multiples of powers of π^2), whereas the values with odd $d-\ell$ are "ugly" (polynomials in π^2 with rational coefficients computed by means of Dehn-Sommerville relations).

1.3. **Related results.** It has been known [25, Theorem 10.4.9] that

$$\mathbb{E}f_0(Z^{(d)}) = \frac{d!}{2^d}\kappa_d^2, \tag{1.6}$$

where $\kappa_d := \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ is the volume of the d-dimensional unit ball. This also yields a formula for $\mathbb{E}f_1(Z^{(d)})$ via the a.s. relation $2f_1(Z^{(d)}) = df_0(Z^{(d)})$ that is valid for every simple polytope. Finally, it has been known that

$$\mathbb{E}f_{d-2}(Z^{(d)}) = \frac{1}{2}\binom{d+1}{3}\pi^2, \tag{1.7}$$

see [12, Equation (1.15)], where it is explained how this can be derived from a result of [3]. All these results are consistent with Theorem 1.1. It seems that no formulae for $\mathbb{E}f_k(Z^{(d)})$ have been known for $k \notin \{0, 1, d-2, d\}$. Asymptotic properties of expected f-vectors of the Poisson zero polytope (and some more general random polytopes), as $d \to \infty$, have been studied in [8]. Some refinements of these results were obtained in [12, Section 1.7], and it should be possible to obtain even more refined results using the exact formula stated in Theorem 1.1.

Let us also mention that the expected intrinsic volumes of the Poisson zero polytope can be expressed through its expected face numbers [24, p. 693]. Thus, Theorem 1.1 also leads to an explicit formula for the expected intrinsic volumes of $Z^{(d)}$.

1.4. **Recurrence relations.** The triangular array $A[n, 2k]$ is the row-reverse of Entry A121408 and the unsigned version of Entry A182971 in [26], see Table 2 at the end of this paper for a table of these numbers. The numbers $A[n, 2k]$ can be expressed through the central factorial numbers defined as the coefficients in the expansion of $x^{[n]} := x(x + \frac{n}{2} - 1)\dots(x - \frac{n}{2} + 1)$. It is easy to check that the numbers $A[n, k]$ satisfy the recurrence relation

$$A[n+2, k] - A[n, k] = (n+1)^2 A[n, k-2], \tag{1.8}$$

see Lemma 2.5, below. In the proof of Theorem 1.1, an important role will be played by the numbers

$$B\{n, k\} := \frac{1}{(k-1)!(n-k)!}\int_0^\pi (\sin x)^{k-1} x^{n-k}\,\mathrm{d}x, \quad k \in \{1, \dots, n\}, \tag{1.9}$$

which will be shown to satisfy the "dual" relation

$$B\{n, k-2\} - B\{n, k\} = (k-1)^2 B\{n+2, k\}. \tag{1.10}$$

Formally, these relations transform into each other under the substitution $(n, k) \mapsto (-k, -n)$. These properties bear some similarity to the well-known properties of the Stirling numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix}, \qquad \begin{Bmatrix} n+1 \\ k \end{Bmatrix} - \begin{Bmatrix} n \\ k-1 \end{Bmatrix} = k\begin{Bmatrix} n \\ k \end{Bmatrix}, \qquad \begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{bmatrix} -k \\ -n \end{bmatrix}, \tag{1.11}$$

which explains our notation. Using Theorem 1.1, we can write relation (1.8) as

$$k(k-1)\left(\mathbb{E}f_{n+2-k}(Z^{(n+2)}) - \mathbb{E}f_{n-k}(Z^{(n)})\right) = \pi^2(n+1)^2\mathbb{E}f_{n+2-k}(Z^{(n)}) \tag{1.12}$$

for all $n \in \mathbb{N}$ and all even $k \in \{0, \dots, n\}$. The next proposition extends this to odd values of k (for which the corresponding values of $\mathbb{E} f_{n-k}(Z^{(n)})$ in Table 1 are "ugly").

Proposition 1.2. *Relation* (1.12) *holds for all* $n \in \mathbb{N}$ *and all* $k \in \{0, \dots, n+1\}$ *irrespective of the parity, provided we define* $\mathbb{E} f_{-1}(Z^{(d)}) := 0$ *in the case* $k = n+1$.

1.5. **Convex hulls on the half-sphere and Poisson processes with power-law intensity.** Let us restate Theorem 1.1 in terms of some random polytopes closely related to $Z^{(d)}$. Let $U_1, \dots, U_n$ be random points sampled uniformly and independently from the d-dimensional upper half-sphere

$$\mathbb{S}_+^d := \{x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1} : x_0 \geq 0, \|x\| = 1\}.$$

The polyhedral convex cone generated by these points (also known as their positive hull) is denoted by

$$C_n = \operatorname{pos}(U_1, \dots, U_n) = \left\{ \sum_{i=1}^n \lambda_i U_i : \lambda_1, \dots, \lambda_n \geq 0 \right\}.$$

The random spherical polytope $C_n \cap \mathbb{S}_+^d$ was first studied by Bárány, Hug, Reitzner and Schneider in [3]. Among other results, these authors computed the expected facet number $\mathbb{E} f_{d-1}(C_n \cap \mathbb{S}_+^d)$, the expected surface area and spherical mean width of $C_n \cap \mathbb{S}_+^d$, and showed that $\mathbb{E} f_0(C_n \cap \mathbb{S}_+^d)$ converges to a finite limit expressed as a multiple integral. These studies were continued in [11], where it was shown that C_n is closely related to convex hulls of certain Poisson processes. Namely, let $\Pi_{d,1}$ be the Poisson point process on $\mathbb{R}^d \backslash \{0\}$ with intensity $\|x\|^{-d-1}$; see the right panel of Figure 1.1 for a realization when $d = 2$. The convex hull of the atoms of this point process is denoted by $\operatorname{conv} \Pi_{d,1}$. Even though the number of atoms is a.s. infinite (because they cluster at 0), this convex hull is a (random) polytope containing the origin in its interior, with probability 1; see [11, Corollary 4.2]. It is known that

$$\mathbb{E} f_\ell(Z^{(d)}) = \mathbb{E} f_{d-\ell-1}(\operatorname{conv} \Pi_{d,1}) = \lim_{n \to \infty} \mathbb{E} f_{d-\ell-1}(C_n \cap \mathbb{S}_+^d) \tag{1.13}$$

for all $\ell \in \{0, 1, \dots, d-1\}$. The second equality was obtained in [11, Theorem 2.4]. In particular, the f-vector of $C_n \cap \mathbb{S}_+^d$ converges, as $n \to \infty$, to a finite limit without any normalization, which is in sharp contrast to what is known in the setting of random convex hulls in flat convex bodies, where the f-vectors diverge to ∞. The first equality in (1.13) follows from the observation made in [12, Theorem 1.23] that $\Pi_{d,1}$ is the dual polytope of $Z^{(d)}$. In fact, the polar hyperplanes of the points of $\Pi_{d,1}$, with respect to the unit sphere, form an isotropic and homogeneous Poisson hyperplane tessellation with intensity $\gamma = 1$; see the proof of Theorem 1.23 in [12].

Combining (1.13) with Theorem 1.1 and putting $k := d - \ell - 1$, we obtain the following theorem identifying the limit in (1.13).

Theorem 1.3. *For all* $d \in \mathbb{N}$ *and* ***odd*** $k \in \{1, \dots, d-1\}$,

$$\mathbb{E} f_k(\operatorname{conv} \Pi_{d,1}) = \lim_{n \to \infty} \mathbb{E} f_k(C_n \cap \mathbb{S}_+^d) = \frac{\pi^{k+1}}{(k+1)!} A[d, k+1].$$

Previously, only the following two special cases of Theorem 1.3 were known with explicit limits:

$$\lim_{n \to \infty} \mathbb{E} f_{d-1}(C_n \cap \mathbb{S}_+^d) = \frac{d!}{2^d} \kappa_d^2, \qquad \lim_{n \to \infty} \mathbb{E} f_1(C_n \cap \mathbb{S}_+^d) = \frac{1}{2} \binom{d+1}{3} \pi^2.$$

The first identity was established in [3, Theorem 3.1], while the second one can be found in [11, Remark 2.5]. Via the duality between the polytopes $Z^{(d)}$ and $\operatorname{conv} \Pi_{d,1}$, these identities are equivalent to the corresponding properties of the Poisson zero polytope $Z^{(d)}$ stated in (1.6) and (1.7).

In fact, we can even compute the complete expected f-vector of the spherical polytope $C_n \cap \mathbb{S}_+^d$ for every finite n.

Theorem 1.4. *For all $d \in \mathbb{N}$, $n \geq d+1$, and all **odd** $k \in \{1, \ldots, d-1\}$, we have*

$$\mathbb{E} f_k(C_n \cap \mathbb{S}_+^d) = \frac{n! \pi^{k+1-n}}{(k+1)!} \sum_{\substack{s=0,1,\ldots \\ d-2s \geq k+1}} B\{n, d-2s\}(d-2s-1)^2 A[d-2s-2, k-1],$$

where $A[n,k]$ and $B\{n,k\}$ were defined in (1.2) *and* (1.9)*, respectively.*

Similarly to the discussion of Section 1.2, the values of $\mathbb{E} f_k(C_n \cap \mathbb{S}_+^d)$ with even k can be determined uniquely by using the Dehn-Sommerville relations. Indeed, the f-vector of the spherical polytope $C_n \cap \mathbb{S}_+^d$ coincides with the f-vector of the flat, d-dimensional polytope $C_n \cap \{x_0 = 1\}$, which is simplicial with probability 1 and thus satisfies the Dehn-Sommerville relations. A more efficient method to treat the even values of k will be described in Section 1.6.

It is interesting to compare Theorems 1.3 and 1.4 with the results of Cover and Efron [5] (see also [9] for a recent work in this direction) who computed the expected f-vector of the random polyhedral cone $D_n := \operatorname{pos}(V_1, \ldots, V_n)$ generated by n i.i.d. random vectors $V_1, \ldots, V_n$ with uniform distribution on the *whole* sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. They gave an explicit formula for the conditional expectation $\mathbb{E}[f_k(D_n \cap \mathbb{S}^d) | \{D_n \neq \mathbb{R}^{d+1}\}]$ in terms of binomial coefficients and proved that

$$\lim_{n \to \infty} \mathbb{E}[f_k(D_n \cap \mathbb{S}^d) | \{D_n \neq \mathbb{R}^{d+1}\}] = 2^{k+1} \binom{d}{k+1}$$

for all $k \in \{0, \ldots, d-1\}$; see [5, Theorem 3']. The number on the right-hand side is the number of k-faces of the d-dimensional crosspolytope (as Cover and Efron [5, Section 4] observed in the setting of the dual cones). The event $\{D_n \neq \mathbb{R}^{d+1}\}$ occurs iff there is a (random) half-space containing the vectors $V_1, \ldots, V_n$, whereas in our Theorems 1.3 and 1.4 we condition on the event that these vectors are in some fixed (deterministic) half-space. These very similar looking types of conditioning lead to two completely different limits of the f-vector.

Let us now consider some special cases of Theorem 1.4. In the case $k = d-1$, Bárány et al. [3, Theorem 3.1] showed that

$$\mathbb{E} f_{d-1}(C_n \cap \mathbb{S}_+^d) = \binom{n}{d} \frac{2\omega_d}{\omega_{d+1}} \int_0^\pi (\sin x)^{d-1} \left(\frac{x}{\pi}\right)^{n-d} \mathrm{d}x \tag{1.14}$$

with $\omega_{d+1} = 2\pi^{(d+1)/2} / \Gamma(\frac{d+1}{2})$ being the surface measure of the d-dimensional unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. To see that this result is a special case of Theorem 1.4, let us first consider the case when d is even. Taking $k = d-1$ in Theorem 1.4 and observing that $A[d-2, d-2] = (d-3)!!^2$, we obtain

$$\mathbb{E} f_{d-1}(C_n \cap \mathbb{S}_+^d) = \frac{n! \pi^{d-n}}{d!} (d-1)!!^2 B\{n, d\} = \binom{n}{d} \frac{(d-1)!!^2}{(d-1)!} \int_0^\pi (\sin x)^{d-1} \left(\frac{x}{\pi}\right)^{n-d} \mathrm{d}x.$$

This is consistent with (1.14). Let now $d \geq 3$ be odd. Then, we may take $k = d-2$ in Theorem 1.4 and use the relation $f_{d-1}(C_n \cap \mathbb{S}_+^d) = \frac{2}{d} f_{d-2}(C_n \cap \mathbb{S}_+^d)$. Bearing in mind that

$A[d-2,d-3]=(d-3)!!^2$, we arrive at

$$\mathbb{E}f_{d-1}(C_n\cap\mathbb{S}^d_+)=\frac{2}{d}\,\mathbb{E}f_{d-2}(C_n\cap\mathbb{S}^d_+)=\frac{2}{d}\,\frac{n!\pi^{d-1-n}}{(d-1)!}(d-1)!!^2B\{n,d\}$$
$$=\binom{n}{d}\frac{2}{\pi}\frac{(d-1)!!^2}{(d-1)!}\int_0^\pi(\sin x)^{d-1}\left(\frac{x}{\pi}\right)^{n-d}\mathrm{d}x,$$

which is consistent with (1.14) and completes its verification.

Another two special cases in which the expression in Theorem 1.4 can be considerably simplified are given in the following

Proposition 1.5. *For all **odd** $k\in\{1,\dots,d-1\}$, we have*

$$\binom{d+2}{k+1}-\mathbb{E}f_k(C_{d+2}\cap\mathbb{S}^d_+)=\pi^{k-d-1}\frac{d+2}{(k+1)!}\cdot\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}\cdot(A[d+2,k+1]-A[d,k+1]),$$
$$\binom{d+3}{k+1}-\mathbb{E}f_k(C_{d+3}\cap\mathbb{S}^d_+)=\pi^{k-d-1}\frac{d+3}{(k+1)!}\cdot\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+4}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}\cdot(A[d+2,k+1]-A[d,k+1]).$$

Let us also mention that taking $k=1$ in Theorem 1.4 and observing that $A[d,0]=1$ for all $d\in\mathbb{N}$, we arrive at the following expression for the number of edges of $C_n\cap\mathbb{S}^d_+$:

$$\mathbb{E}f_1(C_n\cap\mathbb{S}^d_+)=\frac{1}{2}n!\pi^{2-n}\sum_{\substack{s=0,1,\dots\\ m:=d-2s\geq 2}}\frac{m-1}{(m-2)!(n-m)!}\int_0^\pi(\sin x)^{m-1}x^{n-m}\mathrm{d}x.$$

Let us finally observe that using the definition of $B\{n,k\}$ it is not difficult[2] to show that for every fixed $k\in\mathbb{N}$, we have

$$\lim_{n\to\infty}\frac{B\{n,k\}}{\pi^n/n!}=1.$$

Inserting this into the formula from Theorem 1.4, using the recursive property of $A[n,k]$ (see (1.8)) and evaluating the telescope sum, one can easily re-derive Theorem 1.3.

1.6. **Removing restrictions on parity.** Most of the results stated above impose a restriction on the parity of the parameter k; see Theorems 1.1, 1.3, 1.4 and Proposition 1.5. This is due to the fact that $A[n,k]$ was defined to be 0 if k is odd; see (1.2). It is natural to ask whether the parity restrictions can be removed if we modify the definition of $A[n,k]$ appropriately. One way to do this is to turn Theorem 1.1 into a definition of $A[n,k]$. Let us agree, just for the purpose of the rest of Section 1, to **re-define** $A[n,k]$ as follows:

$$A[n,k]:=\frac{k!}{\pi^k}\mathbb{E}f_{n-k}(Z^{(n)}),\tag{1.15}$$

for all $n\in\mathbb{N}$ and all $k\in\{0,\dots,n\}$ irrespective of the parity. By Theorem 1.1, this is consistent with the original definition of $A[n,k]$ given in (1.2) if k is even, but gives a different (non-zero) value if k is odd. It turns out that, with this new definition, the parity restrictions can be removed in the remaining theorems.

Proposition 1.6. *With the above convention on $A[n,k]$, the formulae from Theorems 1.3, 1.4 and Proposition 1.5 continue to hold without restrictions on the parity of $k\in\{0,\dots,d-1\}$.*

[2] Make the change of variables $x=\pi(1-y/n)$ in the integral $\int_0^\pi(\sin x)^{k-1}x^{n-k}\mathrm{d}x$.

Remark 1.7. In particular, we claim that all these results apply to the case $k=0$ upon replacing $(d-2s-1)^2A[d-2s-2,-1]$ by $A[d-2s,1]-A[d-2s-2,1]$ in Theorem 1.4. Thus, the formula of Theorem 1.4 takes the form

$$\mathbb{E}f_0(C_n\cap\mathbb{S}^d_+) = n!\pi^{1-n}\sum_{\substack{s=0,1,\dots\\ d-2s\geq 1}} B\{n,d-2s\}(A[d-2s,1]-A[d-2s-2,1]),$$

where we put $A[0,1]:=A[-1,1]:=0$.

The case $k=0$ of Theorem 1.4 is especially interesting since it is related to the expected spherical volume of $C_n\cap\mathbb{S}^d_+$, or, which is the same up to a constant factor, the expected angle of the cone C_n. Let $\alpha(C_n)$ be the angle of the cone C_n (normalized such that the full solid angle is 1). Also, let σ_d denote the d-dimensional surface measure on the sphere $\mathbb{S}^d$. Recall that $\omega_{d+1}:=\sigma_d(\mathbb{S}^d)=2\pi^{(d+1)/2}/\Gamma(\frac{d+1}{2})$.

Theorem 1.8. *For all $d\in\mathbb{N}$ and $n\geq d+1$ we have*

$$\mathbb{E}\alpha(C_n)=\frac{\mathbb{E}\sigma_d(C_n\cap\mathbb{S}^d_+)}{\omega_{d+1}}=\frac{n!}{2\pi^n}\sum_{\substack{m\in\{d+2,\dots,n+1\}\\ m\equiv d \pmod 2}} B\{n+1,m\}(A[m,1]-A[m-2,1]).$$

The asymptotic rate of convergence of $\mathbb{E}\alpha(C_n)$ to $1/2$ (the solid angle of the half-space), as $n\to\infty$, was determined in [3, Theorem 7.1], where it was shown that

$$\frac{1}{2}\omega_{d+1}-\mathbb{E}\sigma_d(C_n\cap\mathbb{S}^d_+)=C_*(d)n^{-1}+O(n^{-2}),\qquad\text{and}\qquad \lim_{n\to\infty}\mathbb{E}f_0(C_n\cap\mathbb{S}^d_+)=\frac{2C_*(d)}{\omega_{d+1}}$$

for certain constant $C_*(d)$ expressed in [3, Equation (22)] as a multiple integral which is not clear how to evaluate. Comparing the second formula with Theorem 1.3 (where, bearing in mind Proposition 1.6, we take $k=0$), we conclude that $C_*(d)=\pi\omega_{d+1}A[d,1]/2$.

The proof of Theorem 1.8 is based on an Efron-type identity (see (3.14), below) linking the expected angle of C_n to $\mathbb{E}f_0(C_{n+1}\cap\mathbb{S}^d_+)$. More generally, Theorem 2.7 of [11] expresses the so-called expected Grassmann angles of the cones C_n, $n\in\mathbb{N}$, through their f-vectors. Combining this result with Theorem 1.4, it is possible to obtain explicit expressions for the expected Grassmann angles of C_n. Moreover, Theorem 2.8 of [11] gives asymptotic expressions for the expected Grassmann angles, expected conic intrinsic volumes and expected conic mean projection volumes of the random cone C_n, as $n\to\infty$, in terms of certain constants $B_{k,d}$. By combining our Theorem 1.3 with Theorem 2.4 of [11], we obtain the formula

$$B_{k,d}=\frac{k!}{2}\lim_{n\to\infty}\mathbb{E}f_{k-1}(C_n\cap\mathbb{S}^d_+)=\frac{\pi^k}{2}A[d,k],$$

for all $k\in\{1,\dots,d\}$. This turns all results of [11] that involve the constants $B_{k,d}$ into explicit formulae. We refrain from giving further details.

The next proposition provides a simple recursive algorithm for computing the numbers $A[n,k]$ and thus the expected f-vector of the Poisson zero polytope, without any restrictions on parity. The values of $A[n,k]$ for small n, computed by implementing this algorithm in Mathematica, are given in Table 3.

Proposition 1.9. *With convention* (1.15)*, the triangular array $A[n,k]$, where $n\in\mathbb{N}$, $k\in\{0,\dots,n\}$, is uniquely determined by the following properties:*

(i) *$A[n,0]=1$ for all $n\in\mathbb{N}$;*
(ii) *$A[n,n]=2^{-n}(n!)^2/\Gamma(\frac{n}{2}+1)^2$ and $A[n,n-1]=\frac{\pi}{2}A[n,n]$ for all $n\in\mathbb{N}$;*

(iii) $A[n,k] = A[n-2,k] + (n-1)^2 A[n-2,k-2]$ *for all* $n \geq 4$ *and* $k \in \{2,\dots,n-2\}$;
(iv) $A[n,1] = \frac{1}{\pi}((-1)^{n-1} + 1 + \sum_{k=2}^{n}(-1)^k(\pi^k/k!)A[n,k])$ *for all* $n \geq 3$.

Proof. Part (ii) is equivalent to the formula $\mathbb{E}f_0(Z^{(n)}) = 2^{-n}n!\pi^n/\Gamma(\frac{n}{2}+1)^2$ together with the relation $2f_1(Z^{(n)}) = nf_0(Z^{(n)})$. Both results were already mentioned in Section 1.3. Part (iii) is just a restatement of Proposition 1.2 in terms of the $A[n,k]$'s, whereas Part (iv) follows from the Euler relation $\sum_{k=0}^{n}(-1)^k\mathbb{E}f_k(Z^{(n)}) = 1$. The fact that (i)-(iv) determine the $A[n,k]$'s *uniquely* easily follows by induction over n. Indeed, (i) and (ii) determine $A[1,k]$ and $A[2,k]$ for all admissible k's, which is the base of induction. Assuming that the $A[m,k]$'s are determined uniquely for all $m \in \{1,\dots,n-1\}$, $k \in \{0,\dots,m\}$ with some $n \geq 3$, we can use (i), (ii) and (iii) to determine $A[n,k]$ for all $k \in \{0,\dots,n\}\backslash\{1\}$. Finally, (iv) determines $A[n,1]$, thus completing the induction. Note that the Euler-type relation (iv) cannot be removed without loosing uniqueness since the value $A[3,1]$ is not determined uniquely by the remaining conditions. $\square$

As we already know from (1.2), $A[n,k]$ is an integer number for even k. It turns out that in the case when k is odd, $A[n,k]$ is a polynomial of π^2, if n is even, and $1/\pi$ times a polynomial of π^2, if n is odd. This fact can easily be established by induction over n keeping in mind Proposition 1.9. It remains an open problem to find a closed-form expression for the coefficients of these polynomials.

1.7. **Sylvester problem on the half-sphere.** The classical Sylvester four point problem asks for the probability that four random points chosen uniformly and independently from some convex plane region have a convex hull which is a triangle. In the case when the region is a disk (or, more generally, any ellipse), the answer is $35/(12\pi^2)$. A d-dimensional version of this problem was solved by Kingman [13] who computed explicitly the probability that the convex hull of $d+2$ points chosen independently and uniformly from the d-dimensional ball is a simplex with $d+1$ vertices. Let us study a similar problem on the half-sphere. Let $U_1,\dots,U_{d+2}$ be random points sampled uniformly and independently from the upper half-sphere $\mathbb{S}_+^d$. We ask for the probability $P(d)$ that the spherical convex hull of these points, namely $C_{d+2}\cap\mathbb{S}_+^d$, is a spherical simplex with $d+1$ vertices.

Theorem 1.10. *For all* $d \in \mathbb{N}$ *we have*

$$P(d) = \pi^{-(d+1)}\,(d+2)\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}(A[d+2,1] - A[d,1]),$$

with the convention $A[d,1] := \frac{1}{\pi}\mathbb{E}f_{d-1}(Z^{(d)})$.

Proof. Since the number of vertices of the spherical polytope $C_{d+2}\cap\mathbb{S}_+^d$ is either $d+1$ (with probability $P(d)$) or $d+2$ (with probability $1-P(d)$), we have

$$\mathbb{E}f_0(C_{d+2}\cap\mathbb{S}_+^d) = (d+1)P(d) + (d+2)(1-P(d)) = (d+2) - P(d).$$

On the other hand, the first formula of Proposition 1.5 with $k=0$ (bearing in mind Proposition 1.6) yields

$$(d+2) - \mathbb{E}f_0(C_{d+2}\cap\mathbb{S}_+^d) = \pi^{-(d+1)}(d+2)\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}\cdot(A[d+2,1] - A[d,1]).$$

Resolving this w.r.t. $P(d)$ we arrive at the required formula. $\square$

The first few values of $P(d)$ are given in Table 5. For example, for four points on the two-dimensional half-sphere $\mathbb{S}^2_+$, the probability that the convex hull is a spherical triangle is $P(2) = \frac{24}{\pi^2} - 2 \approx 0.4317$.

2. Proof of Theorem 1.1

2.1. **Beta' polytopes.** The proofs strongly rely on the results of the paper [12] whose notation we follow. A random point X in $\mathbb{R}^d$ is said to have the *beta distribution* with parameter $\beta > -1$ if its density is given by

$$f_{d,\beta}(x) = c_{d,\beta}\left(1 - \|x\|^2\right)^{\beta} \mathbb{1}_{\{\|x\|<1\}}, \qquad x \in \mathbb{R}^d, \qquad c_{d,\beta} = \frac{\Gamma\left(\frac{d}{2} + \beta + 1\right)}{\pi^{\frac{d}{2}}\Gamma\left(\beta + 1\right)}. \tag{2.1}$$

Similarly, X has *beta' distribution* with parameter $\beta > d/2$ if its density has the form

$$\tilde{f}_{d,\beta}(x) = \tilde{c}_{d,\beta}\left(1 + \|x\|^2\right)^{-\beta}, \qquad x \in \mathbb{R}^d, \qquad \tilde{c}_{d,\beta} = \frac{\Gamma\left(\beta\right)}{\pi^{\frac{d}{2}}\Gamma\left(\beta - \frac{d}{2}\right)}. \tag{2.2}$$

In order to conform with the notation of [12], we usually supply objects and quantities related to the beta' case with the tilde, even though we shall deal almost exclusively with the beta' case here.

Let $X_1, X_2, \ldots$ be i.i.d. random points in $\mathbb{R}^d$ with density $\tilde{f}_{d,\beta}$. The convex hull of $X_1, \ldots, X_n$ is called the *beta' polytope* and denoted by $\tilde{P}^{\beta}_{n,d} := [X_1, \ldots, X_n]$. These random polytopes were introduced in the works of Miles [22] and Ruben and Miles [23], and further studied in [10, 6, 4, 11, 12]. In [12], expected values of various functionals of these polytopes (including the f-vector as well as the internal and external angles) were expressed through quantities of two sorts. The quantities of the first sort, denoted by $\tilde{I}_{n,k}(\alpha)$, are given by the explicit formula

$$\tilde{I}_{n,k}(\alpha) = \int_{-\pi/2}^{+\pi/2} \tilde{c}_{1,\frac{\alpha k+1}{2}} (\cos x)^{\alpha k - 1} \left(\int_{-\pi/2}^{x} \tilde{c}_{1,\frac{\alpha+1}{2}} (\cos y)^{\alpha-1} \mathrm{d}y\right)^{n-k} \mathrm{d}x, \tag{2.3}$$

see [12, Remark 1.17], and are closely related to the external angles of the beta' polytopes; see [12, Theorem 1.16]. The quantities of the second sort, denoted by $\tilde{J}_{n,k}(\beta)$, are defined as follows. Let $Z_1, \ldots, Z_n$ be n independent random points in $\mathbb{R}^{n-1}$ distributed according to the density $\tilde{f}_{n-1,\beta}$. Then, $\tilde{J}_{n,k}(\beta)$ is the expected internal angle of the simplex $[Z_1, \ldots, Z_n]$ at its face $[Z_1, \ldots, Z_k]$, for $k \in \{1, \ldots, n\}$. By definition, $\tilde{J}_{n,n}(\beta) = 1$.

For the purposes of the present paper, it will be more convenient to work with the quantities

$$\tilde{\mathbb{I}}_{n,k}(\beta) := \binom{n}{k} \tilde{I}_{n,k}(\beta) \quad \text{and} \quad \tilde{\mathbb{J}}_{n,k}(\beta) := \binom{n}{k} \tilde{J}_{n,k}(\beta), \qquad k \in \{1, \ldots, n\}.$$

Note that, by definition, $\tilde{\mathbb{J}}_{n,k}(\beta)$ is the expected sum of internal angles at all k-vertex faces $[Z_{i_1}, \ldots, Z_{i_k}]$ of the beta' simplex $[Z_1, \ldots, Z_n] \subset \mathbb{R}^{n-1}$ with n vertices. On the other hand, $\tilde{\mathbb{I}}_{n,k}(2\beta - n + 1)$ is the expected sum of external angles at all k-vertex faces of the same random simplex; see [12, Theorem 1.16].

2.2. **Expected internal angle sums.** As explained in Section 1.5, to prove Theorem 1.1 it suffices to show that for all $d \in \mathbb{N}$ and all **odd** $k \in \{1, \ldots, d-1\}$, we have

$$\mathbb{E} f_k(\operatorname{conv} \Pi_{d,1}) = \frac{\pi^{k+1}}{(k+1)!} A[d, k+1], \tag{2.4}$$

where $\operatorname{conv}\Pi_{d,1}$ is the convex hull of the Poisson process $\Pi_{d,1}$ defined in Section 1.5. The numbers $A[n,k]$ are defined by their generating function (1.2). The starting point of our proof is the following explicit formula derived in [12, Theorem 1.21 and Section 1.5]:

$$\mathbb{E} f_k(\operatorname{conv}\Pi_{d,1}) = \sum_{\substack{s=0,1,\dots\\ n:=d-2s\geq k+1}} \frac{2}{n}\pi^n \tilde{c}_{1,\frac{n+1}{2}} \tilde{\mathbb{J}}_{n,k+1}\left(\frac{n}{2}\right), \tag{2.5}$$

for all $k\in\{0,\dots,d-1\}$. The main contribution of the present paper is the evaluation of the expected internal angle sums $\tilde{\mathbb{J}}_{n,k+1}(n/2)$.

Proposition 2.1. *For all **even** $k\in\{1,\dots,n\}$, the expected sum of internal angles at faces with k vertices of the beta' simplex $\tilde{P}_{n,n-1}^{n/2}\subset\mathbb{R}^{n-1}$ with n vertices and $\beta=n/2$ is given by*

$$\tilde{\mathbb{J}}_{n,k}\left(\frac{n}{2}\right) = \frac{\pi^{k-n}}{k!}\cdot\frac{n}{2\tilde{c}_{1,\frac{n+1}{2}}}\cdot(n-1)^2 A[n-2,k-2]. \tag{2.6}$$

Proof of Theorem 1.1 given Proposition 2.1. Using (1.2), it is easy to check that $A[n,k+1]-A[n-2,k+1]=(n-1)^2A[n-2,k-1]$; see, e.g., Lemma 2.5. Replacing k by $k+1$ and using this relation, we can write (2.6) as

$$\tilde{\mathbb{J}}_{n,k+1}\left(\frac{n}{2}\right) = \frac{\pi^{k+1-n}}{(k+1)!}\cdot\frac{n}{2\tilde{c}_{1,\frac{n+1}{2}}}\cdot(A[n,k+1]-A[n-2,k+1]),$$

for all odd $k\in\{1,\dots,n-1\}$.

We need to prove (2.4). Fix some odd $k\in\{1,\dots,d-1\}$. Plugging this formula for $\tilde{\mathbb{J}}_{n,k+1}(n/2)$ into (2.5), we obtain

$$\mathbb{E} f_k(\operatorname{conv}\Pi_{d,1}) = \sum_{\substack{s=0,1,\dots\\ n:=d-2s\geq k+1}} \frac{\pi^{k+1}}{(k+1)!}\cdot(A[n,k+1]-A[n-2,k+1]) = \frac{\pi^{k+1}}{(k+1)!}A[d,k+1]$$

because the last term in the telescope sum, which is either $-A[k-1,k+1]$ or $-A[k,k+1]$, vanishes. This establishes (2.4). ☐

2.3. **System of equations for expected internal angles.** In the rest of the paper we prove Proposition 2.1. As a first step, we shall provide a system of relations between the quantities $\tilde{\mathbb{I}}_{n,k}(\beta)$ and $\tilde{\mathbb{J}}_{n,k}(\beta)$ which leads to a recursive algorithm for computing $\tilde{\mathbb{J}}_{n,k}(\beta)$.

Proposition 2.2. *For every $n\in\{2,3,\dots\}$, $k\in\{1,\dots,n-1\}$ and for every $\beta>(n-1)/2$ we have*

$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} \tilde{\mathbb{I}}_{n,n-2s}(2\beta-n+1)\tilde{\mathbb{J}}_{n-2s,k}(\beta-s) = \frac{1}{2}\binom{n}{k}, \tag{2.7}$$

$$\sum_{\substack{s=0,1,\dots\\ n-2s-1\geq k}} \tilde{\mathbb{I}}_{n,n-2s-1}(2\beta-n+1)\tilde{\mathbb{J}}_{n-2s-1,k}\left(\beta-s-\frac{1}{2}\right) = \frac{1}{2}\binom{n}{k}. \tag{2.8}$$

Proof. Given a d-dimensional polyhedral cone C, we denote by $\upsilon_0(C),\dots,\upsilon_d(C)$ its conic intrinsic volumes. For their definition and a review of their properties we refer to [2, 1] (whose notation we follow) and to [25, Section 6.5] (where slightly different notation is used). Here we shall need only the *Gauss-Bonnet relation* [2, Equation (5.3)] which states that

$$\sum_{\substack{s\geq 0\\ j:=d-2s\geq 0}} \upsilon_j(C) = \sum_{\substack{s\geq 0\\ j:=d-2s-1\geq 0}} \upsilon_j(C) = \frac{1}{2}$$

for every d-dimensional polyhedral cone C that is not a linear subspace.

Consider the $(n-1)$-dimensional beta' simplex $\tilde{P}^{\beta}_{n,n-1}$ defined as the convex hull $[X_1,\ldots,X_n]$ of n independent random points $X_1,\ldots,X_n$ having the probability density $\tilde{f}_{n-1,\beta}$ on $\mathbb{R}^{n-1}$. The tangent cone at its k-vertex face $G=[X_1,\ldots,X_k]$ is defined as

$$\tilde{T}^{\beta}_{n,k} := \{v\in\mathbb{R}^{n-1}:\ \text{there exists } \varepsilon>0 \text{ such that } g_0+\varepsilon v\in \tilde{P}^{\beta}_{n,n-1}\},$$

where g_0 is any point in the relative interior of G, for example $g_0=(X_1+\ldots+X_k)/k$.

The expected conic intrinsic volumes of the tangent cone were computed in [12, Theorem 1.18]: For all $k\in\{1,\ldots,n-1\}$ and $j\in\{k-1,\ldots,n-1\}$ we have

$$\mathbb{E}\upsilon_j(\tilde{T}^{\beta}_{n,k}) = \frac{1}{\binom{n}{k}}\tilde{\mathbb{I}}_{n,j+1}(2\beta-n+1)\tilde{\mathbb{J}}_{j+1,k}\left(\beta-\frac{n-1-j}{2}\right). \tag{2.9}$$

For $j\notin\{k-1,\ldots,n-1\}$ we have $\mathbb{E}\upsilon_j(\tilde{T}^{\beta}_{n,k})=0$. In particular, all intrinsic volumes with $j<k-1$ vanish, which is due to the fact that the lineality space of the tangent cone, defined as the intersection of $\tilde{T}^{\beta}_{n,k}$ with $-\tilde{T}^{\beta}_{n,k}$, coincides with the affine hull of G shifted to the origin and has dimension $k-1$.

Applying the Gauss-Bonnet relation to the tangent cone $\tilde{T}^{\beta}_{n,k}$ and taking the expectation, we arrive at the required relation (2.7). □

In view of the interpretation of $\tilde{\mathbb{I}}_{n,k}(\beta)$ and $\tilde{\mathbb{J}}_{n,k}(\beta)$ as expected sums of internal/external angles, Relations (2.7) and (2.8) can be seen as a stochastic version of McMullen's non-linear angle-sum relations [17] in the setting of beta' polytopes.

The above proposition leads to a recursive algorithm which can be used to compute the quantities $\tilde{\mathbb{J}}_{n,k}(\beta)$, both numerically and exactly. First, recall that $\tilde{\mathbb{J}}_{n,n}(\beta)=1$ by definition. Separating in (2.7) the term with $s=0$ and noting that $\tilde{\mathbb{I}}_{n,n}(2\beta-n+1)=1$ by (2.3), we can write

$$\tilde{\mathbb{J}}_{n,k}(\beta) = \frac{1}{2}\binom{n}{k} - \sum_{\substack{s=1,2,\ldots\\ n-2s\geq k}} \tilde{\mathbb{I}}_{n,n-2s}(2\beta-n+1)\tilde{\mathbb{J}}_{n-2s,k}(\beta-s)$$

for $k\in\{1,\ldots,n-1\}$. This gives an expression for $\tilde{\mathbb{J}}_{n,k}(\beta)$ in terms of the quantities of the form $\tilde{\mathbb{J}}_{\ell,k}(\alpha)$ with $\ell<n$ and the quantities of the form $\tilde{\mathbb{I}}_{n,k}(\alpha)$ for which we have explicit expression (2.3). Proceeding recursively, we can compute $\tilde{\mathbb{J}}_{n,k}(\beta)$.

Using this algorithm together with (2.5) we computed exactly the expected vectors of the Poisson zero polytopes in low dimensions; see Table 1. Then, we guessed the formula stated in Theorem 1.1 by the method of trials and errors.

2.4. **Uniqueness of the solution and reduction to a combinatorial identity.** To prove Proposition 2.1, we shall proceed as follows. First, we shall observe that the system of linear equations (2.7), which is triangular with 1's on the diagonal, determines the unknown quantities $\tilde{\mathbb{J}}_{n,k}(\beta)$ uniquely. This will be stated more precisely in the next proposition. Thus, in order to prove Proposition 2.1, it suffices to check that equation (2.7) continues to hold if we replace $\tilde{\mathbb{J}}_{n,k}(n/2)$ by their conjectured values given by the right-hand side of (2.6). This reduces Proposition 2.1 to certain combinatorial identity which will be verified in Section 2.7.

First of all, we are interested in the particular case $\beta = n/2$ of the above setting, in which case equation (2.7) simplifies to

$$\sum_{\substack{s=0,1,\ldots\\ n-2s\geq k}} \tilde{\mathbb{I}}_{n,n-2s}(1)\tilde{\mathbb{J}}_{n-2s,k}\left(\frac{n}{2}-s\right) = \frac{1}{2}\binom{n}{k}, \tag{2.10}$$

for all $n \in \{2,3,\ldots\}$, $k \in \{1,\ldots,n-1\}$.

Proposition 2.3. *Consider the following system of linear equations for the unknown quantities $v_{n,k}$, where $n \in \{2,3,\ldots\}$ and k is an even number in $\{1,\ldots,n\}$:*

$$\sum_{\substack{s=0,1,\ldots\\ n-2s\geq k}} \tilde{\mathbb{I}}_{n,n-2s}(1)v_{n-2s,k} = \frac{1}{2}\binom{n}{k}, \text{ for all } n \in \{2,3,\ldots\},\ k \in 2\mathbb{N},\ k < n, \tag{2.11}$$

$$v_{n,n} = 1, \text{ for all even } n \in \{2,4,6,\ldots\}. \tag{2.12}$$

Then, the unique solution of this system is given by $v_{n,k} = \tilde{\mathbb{J}}_{n,k}(n/2)$, for all $n \in \{2,3,\ldots\}$ and all even $k \in \{1,\ldots,n\}$.

Proof. Since $v_{n,k} = \tilde{\mathbb{J}}_{n,k}(n/2)$ is indeed a solution according to (2.10), it remains to show that the solution is unique. This will be done by induction. For the base case $n = 2$, note that the quantity $v_{2,2} = 1$ is determined uniquely by (2.12). Assume now that $n \in \{3,4,\ldots\}$ and that we have shown that the quantities $v_{m,\ell}$ are uniquely determined by (2.11), (2.12) for all $m \in \{2,3,\ldots,n-1\}$ and all even $\ell \in \{1,\ldots,m\}$. We are going to show that $v_{n,k}$ are determined uniquely for all even $k \in \{1,\ldots,n\}$. If n is even and $n = k$, then $v_{n,n} = 1$ by (2.12). So, let $k \in 2\mathbb{N}$ be even with $k < n$. Then, separating the term with $s = 0$ in (2.11) and observing that $\tilde{\mathbb{I}}_{n,n}(1) = 1$, we can express $v_{n,k}$ through $v_{m,\ell}$'s with $m = n-2s$ strictly smaller than n, thus completing the induction. □

In view of the above, in order to prove Proposition 2.1, it suffices to check that for all $n \in \{2,3,\ldots\}$, and all even $k \in \{1,\ldots,n-1\}$

$$\sum_{\substack{s=0,1,\ldots\\ n-2s\geq k}} \left(\tilde{\mathbb{I}}_{n,n-2s}(1)\cdot\frac{\pi^{k-n+2s}}{k!}\cdot\frac{n-2s}{\tilde{c}_{1,\frac{n-2s+1}{2}}}\cdot(n-2s-1)^2\cdot A[n-2s-2,k-2]\right) = \binom{n}{k}. \tag{2.13}$$

The rest of the paper is devoted to the proof of (2.13).

2.5. **Definition of $B\{n,k\}$.** For $n \in \mathbb{N}$ and $k \in \{1,\ldots,n\}$ introduce the numbers

$$B\{n,k\} := \frac{1}{(k-1)!(n-k)!}\int_0^\pi (\sin x)^{k-1}x^{n-k}\mathrm{d}x. \tag{2.14}$$

Note that $B\{n,1\} = \pi^n/n!$. The values of $B\{n,k\}$ for small n and k are given in Table 4. Let us extend this definition by putting

$$B\{n,k\} := \begin{cases}\pi^n/n!, & \text{for all } n \in \mathbb{N},\ k = 0,\\ 0, & \text{for all } n \in \mathbb{N},\ k \in \{n+1,n+2,\ldots\}.\end{cases} \tag{2.15}$$

The next lemma expresses $\tilde{\mathbb{I}}_{n,k}(1)$ through $B\{n,k\}$.

Lemma 2.4. *For all $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$ we have*
$$\tilde{\mathbb{I}}_{n,k}(1) = \binom{n}{k} \tilde{I}_{n,k}(1) = \frac{n!}{k} \tilde{c}_{1,\frac{k+1}{2}} \pi^{k-n} B\{n,k\}.$$

Proof. Using (2.3) with $\alpha = 1$, the fact that $\tilde{c}_{1,1} = 1/\pi$ (see (2.2)) and finally the variable change $x = \varphi - \frac{\pi}{2}$, we obtain
$$\tilde{I}_{n,k}(1) = \int_{-\pi/2}^{+\pi/2} \tilde{c}_{1,\frac{k+1}{2}} (\cos x)^{k-1} \left(\frac{x}{\pi} + \frac{1}{2}\right)^{n-k} \mathrm{d}x = \tilde{c}_{1,\frac{k+1}{2}} \pi^{k-n} \int_0^\pi (\sin\varphi)^{k-1} \varphi^{n-k} \mathrm{d}\varphi,$$
which proves the claim after recalling that $\tilde{\mathbb{I}}_{n,k}(1) = \binom{n}{k} \tilde{I}_{n,k}(1)$. $\square$

In view of Lemma 2.4, we can rewrite (2.16) in the following form:
$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n, n-2s\}(n-2s-1)^2 A[n-2s-2, k-2] = \frac{\pi^{n-k}}{(n-k)!}, \tag{2.16}$$
for all $n \in \{2, 3, \dots\}$ and all even $k \in \{1, \dots, n-1\}$.

2.6. **Recurrence relations for $A[n,k]$ and $B\{n,k\}$.** First we establish recurrence relations for $A[n,k]$ and $B\{n,k\}$. These are similar to the relations satisfied by the Stirling numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$; see (1.11).

Lemma 2.5. *For all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$ we have*
$$A[n+2, k] - A[n,k] = (n+1)^2 A[n, k-2]. \tag{2.17}$$

Proof. By definition of $A[n,k]$, see (1.2), we have
$$A[n+2,k] = [x^k]\Big((1+(n+1)^2x^2)(1+(n-1)^2x^2)(1+(n-3)^2x^2)\dots\Big),$$
$$A[n,k] = [x^k]\Big((1+(n-1)^2x^2)(1+(n-3)^2x^2)(1+(n-5)^2x^2)\dots\Big).$$
Subtracting these identities, we obtain
$$\begin{aligned}A[n+2,k] - A[n,k] &= [x^k]\Big((n+1)^2x^2(1+(n-1)^2x^2)(1+(n-3)^2x^2)\dots\Big)\\ &= (n+1)^2 A[n,k-2],\end{aligned}$$
thus proving the claim. $\square$

Recall that the numbers $B\{n,k\}$ were defined in (2.14) and (2.15).

Lemma 2.6. *For all $n \in \mathbb{N}$ and $k \in \{2, 3, \dots\}$, we have*
$$B\{n, k-2\} - B\{n,k\} = (k-1)^2 B\{n+2, k\}. \tag{2.18}$$

Proof. *Case 1.* Let first $k \in \{3, \dots, n\}$. Integrating by parts, we obtain
$$\begin{aligned}B\{n,k\} &= \frac{1}{(k-1)!(n-k)!} \int_0^\pi (\sin x)^{k-1} \mathrm{d}\left(\frac{x^{n-k+1}}{n-k+1}\right)\\ &= -\frac{1}{(k-1)!(n-k+1)!} \int_0^\pi x^{n-k+1}(k-1)(\sin x)^{k-2}(\cos x)\mathrm{d}x.\end{aligned}$$

Applying partial integration for the second time, we arrive at

$$\begin{aligned} B\{n,k\} &= -\frac{1}{(k-2)!(n-k+1)!}\int_0^\pi (\sin x)^{k-2}(\cos x)\mathrm{d}\left(\frac{x^{n-k+2}}{n-k+2}\right) \\ &= \frac{1}{(k-2)!(n-k+2)!}\int_0^\pi x^{n-k+2}\frac{\mathrm{d}}{\mathrm{d}x}\Big((\sin x)^{k-2}(\cos x)\Big)\mathrm{d}x \\ &= \frac{1}{(k-2)!(n-k+2)!}\int_0^\pi x^{n-k+2}\Big((k-2)(\sin x)^{k-3}(\cos x)^2 - (\sin x)^{k-1}\Big)\mathrm{d}x. \end{aligned}$$

Observe that we required $k \geq 3$ because, for $k = 2$, the term $x^{n-k+2}(\sin x)^{k-2}(\cos x)$ appearing in the partial integration formula does not vanish at $x = \pi$.

Using the identity $\cos^2 x = 1 - \sin^2 x$ and simplifying, we arrive at

$$\begin{aligned} B\{n,k\} = \frac{(k-2)}{(k-2)!(n-k+2)!}\int_0^\pi x^{n-k+2}(\sin x)^{k-3}\mathrm{d}x \\ - \frac{(k-1)}{(k-2)!(n-k+2)!}\int_0^\pi x^{n-k+2}(\sin x)^{k-1}\mathrm{d}x. \end{aligned}$$

We now easily recognize that the first term on the right-hand side is $B\{n, k-2\}$, whereas the second term is $(k-1)^2 B\{n+2, k\}$. Thus, we proved that $B\{n,k\} = B\{n,k-2\} - (k-1)^2 B\{n+2,k\}$ for $k \in \{3,\dots,n\}$.

Case 2. If $k \geq n+3$, then all terms in (2.18) vanish by definition.

Case 3. If $k = n+2$ or $k = n+1$, then $B\{n,k\} = 0$ by definition and we need to verify that

$$B\{n,n\} = (n+1)^2 B\{n+2,n+2\} \qquad \text{and} \qquad B\{n,n-1\} = n^2 B\{n+2,n+1\}$$

for all $n \in \mathbb{N}$. The second identity holds for $n = 1$ since $B\{1,0\} = \pi = B\{3,2\}$, and we ignore this case in the following. Recalling the definition of $B\{n,k\}$ stated in (2.14), we can write these identities as

$$\int_0^\pi (\sin x)^{n+1}\mathrm{d}x = \frac{n}{n+1}\int_0^\pi (\sin x)^{n-1}\mathrm{d}x, \qquad n \in \mathbb{N}, \tag{2.19}$$

$$\int_0^\pi x(\sin x)^{n}\mathrm{d}x = \frac{n-1}{n}\int_0^\pi x(\sin x)^{n-2}\mathrm{d}x, \qquad n \geq 2. \tag{2.20}$$

To verify (2.19), we use partial integration as follows:

$$\int_0^\pi (\cos^2 x)(\sin x)^{n-1}\mathrm{d}x = \frac{1}{n}\int_0^\pi (\cos x)\mathrm{d}(\sin x)^n = \frac{1}{n}\int_0^\pi (\sin x)(\sin x)^n\mathrm{d}x.$$

Replacing $\cos^2 x$ by $1 - \sin^2 x$ on the left-hand side, we arrive at (2.19). To verify (2.20), write

$$\begin{aligned} \int_0^\pi x(\cos^2 x)(\sin x)^{n-2}\mathrm{d}x &= \frac{1}{n-1}\int_0^\pi (x\cos x)\mathrm{d}(\sin x)^{n-1} \\ &= \frac{1}{n-1}\int_0^\pi (x\sin x - \cos x)(\sin x)^{n-1}\mathrm{d}x = \frac{1}{n-1}\int_0^\pi x(\sin x)^n\mathrm{d}x \end{aligned}$$

because $\int_0^\pi (\cos x)(\sin x)^{n-1}\mathrm{d}x = 0$. Replacing $\cos^2 x$ by $1 - \sin^2 x$ on the left-hand side, we arrive at (2.20).

Case 4. Let $k = 2$. If $n = 1$, identity (2.18) takes the form $B\{1,0\} = B\{3,2\}$, which is true because both terms are equal to π. In the case when $n \geq 2$, we need to verify the identity

$\pi^n/n! = B\{n,2\} + B\{n+2,2\}$, or, after recalling (2.14) and multiplying by $n!$,

$$\pi^n = \int_0^\pi (\sin x)(n(n-1)x^{n-2} + x^n)\mathrm{d}x, \qquad n \geq 2.$$

This is an easy exercise in partial integration:

$$\begin{aligned}\int_0^\pi (\sin x) n(n-1)x^{n-2}\mathrm{d}x = \int_0^\pi n(\sin x)\mathrm{d}x^{n-1} = -\int_0^\pi n(\cos x)x^{n-1}\mathrm{d}x = -\int_0^\pi (\cos x)\mathrm{d}x^n \\ = -(\cos x)x^n\Big|_0^\pi - \int_0^\pi (\sin x)x^n\mathrm{d}x = \pi^n - \int_0^\pi (\sin x)x^n\mathrm{d}x.\end{aligned}$$

□

2.7. **The basic combinatorial identity.** In the next lemma we prove (2.16), thereby completing the proof of Proposition 2.1 and Theorem 1.1.

Lemma 2.7. *For all $n \in \mathbb{N}$ and all **even** $k \in \{1, 2, \dots, n-1\}$,*

$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n, n-2s\}(n-2s-1)^2 A[n-2s-2, k-2] = \frac{\pi^{n-k}}{(n-k)!}. \tag{2.21}$$

Remark 2.8. Another formula of the same type will be established in Lemma 3.1.

Proof of Lemma 2.7. We argue by induction, assuming the identity for some n and proving it for $n+2$.

Base cases. We start by verifying the cases $n = 3$ and $n = 4$ (because for $n = 1$ and $n = 2$ the set of admissible k's is empty).

Case $n = 3$. Then, $k = 2$ and (2.21) turns into $B\{3,3\}2^2 A[1,0] = \pi$, which is true because $A[1,0] = 1$ and $B\{3,3\} = \pi/4$.

Case $n = 4$. Then, $k = 2$ and (2.21) turns into

$$B\{4,4\}3^2 A[2,0] + B\{4,2\}1^2 A[0,0] = \frac{\pi^2}{2},$$

which is true because $A[2,0] = A[0,0] = 1$, while $B\{4,4\} = 2/9$ and $B\{4,2\} = \pi^2/2 - 2$.

Induction assumption. Assume that identity (2.21) holds for some $n \in \{3, 4, \dots\}$ and all even $k \in \{1, \dots, n-1\}$. By Lemma 2.5, we have

$$(n-2s-1)^2 A[n-2s-2, k-2] = A[n-2s, k] - A[n-2s-2, k],$$

so that we can write the induction assumption in the form

$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n, n-2s\}\Big(A[n-2s, k] - A[n-2s-2, k]\Big) = \frac{\pi^{n-k}}{(n-k)!}$$

or, more conveniently,

$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n, n-2s\}A[n-2s, k] = \frac{\pi^{n-k}}{(n-k)!} + \sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n, n-2s\}A[n-2s-2, k], \tag{2.22}$$

for all even $k \in \{1, \dots, n-1\}$.

Induction step. We need to prove that identity (2.21) holds with n replaced by $n+2$, that is

$$S := \sum_{\substack{s=0,1,\dots\\ n+2-2s\geq k}} B\{n+2, n+2-2s\}(n-2s+1)^2 A[n-2s, k-2] = \frac{\pi^{n+2-k}}{(n+2-k)!}, \tag{2.23}$$

for all even $k \in \{1, \dots, n+1\}$. By Lemma 2.6, for all s such that $n+2-2s \geq k \geq 2$ we have

$$(n-2s+1)^2 B\{n+2, n+2-2s\} = B\{n, n-2s\} - B\{n, n-2s+2\}.$$

Inserting this into the above definition S, we obtain

$$\begin{aligned} S &= \sum_{\substack{s=0,1,\dots\\ n+2-2s\geq k}} \left(B\{n, n-2s\} - B\{n, n-2s+2\}\right) A[n-2s, k-2] \\ &= \sum_{\substack{s=0,1,\dots\\ n-2s\geq k-2}} B\{n, n-2s\}A[n-2s, k-2] - \sum_{\substack{s=0,1,\dots\\ n+2-2s\geq k}} B\{n, n-2s+2\}A[n-2s, k-2]. \end{aligned}$$

Let first $k \neq 2$. To the first sum we apply the induction assumption (2.22) with k replaced by $k-2$ (which is an even number in the range $\{1, \dots, n-1\}$):

$$\begin{aligned} S = \frac{\pi^{n-(k-2)}}{(n-(k-2))!} &+ \sum_{\substack{s=0,1,\dots\\ n-2s\geq k-2}} B\{n, n-2s\}A[n-2s-2, k-2] \\ &- \sum_{\substack{s=0,1,\dots\\ n-2s\geq k-2}} B\{n, n-2s+2\}A[n-2s, k-2]. \end{aligned}$$

Introducing the new summation index $s' := s-1$ in the second sum and leaving the first sum unchanged, we obtain

$$\begin{aligned} S = \frac{\pi^{n-(k-2)}}{(n-(k-2))!} &+ \sum_{\substack{s=0,1,\dots\\ n-2s\geq k-2}} B\{n, n-2s\}A[n-2s-2, k-2] \\ &- \sum_{\substack{s'=-1,0,\dots\\ n-2s'\geq k}} B\{n, n-2s'\}A[n-2s'-2, k-2]. \end{aligned}$$

The sums on the right-hand side differ by just two terms corresponding to $s' = -1$ (in the second sum) and s such that $n-2s \in \{k-1, k-2\}$ (in the first sum). The term with $s' = -1$ is $B\{n, n+2\}A[n, k-2]$, which vanishes by definition. The term in the first sum for which $n-2s \in \{k-1, k-2\}$ also vanishes because then $n-2s-2 \in \{k-3, k-4\}$ and consequently $A[n-2s-2, k-2] = 0$. So, the sums cancel each other and we are left with

$$S = \frac{\pi^{n+2-k}}{(n+2-k)!},$$

which verifies (2.23). To complete the induction, it remains to check the case $k = 2$. Since $A[n-2s, 0] = 1$, we have

$$S = \sum_{\substack{s=0,1,\dots\\ n-2s\geq 0}} B\{n, n-2s\} - \sum_{\substack{s=0,1,\dots\\ n-2s\geq 0}} B\{n, n-2s+2\}.$$

Again, the sums differ by just two terms. One of them is $-B\{n, n+2\} = 0$. The other term is $B\{n, 0\} = \pi^n/n!$ (if n is even) or $B\{n, 1\} = \pi^n/n!$ (if n is odd). In both cases, we have $S = \pi^n/n!$, which completes the induction. $\square$

The proof of Theorem 1.1 is thus complete.

3. Further proofs

3.1. **Proof of Proposition 1.2.** We are going to show that for all $n \in \mathbb{N}$ and all $m \in \{1, \dots, n+1\}$, we have

$$m(m-1)\left(\mathbb{E}f_{n+2-m}(Z^{(n+2)}) - \mathbb{E}f_{n-m}(Z^{(n)})\right) = \pi^2(n+1)^2\mathbb{E}f_{n+2-m}(Z^{(n)}). \tag{3.1}$$

In the case when $m = n+1$ and with the convention $\mathbb{E}f_{-1}(Z^{(n)}) = 0$, the relation reduces to

$$n\mathbb{E}f_1(Z^{(n+2)}) = \pi^2(n+1)\mathbb{E}f_1(Z^{(n)}). \tag{3.2}$$

This relation easily follows from the formula

$$\mathbb{E}f_1(Z^{(n)}) = \frac{n}{2}\mathbb{E}f_0(Z^{(n)}) = \frac{n \cdot n!\pi^n}{2^{n+1}\Gamma(\frac{n}{2}+1)^2}. \tag{3.3}$$

To settle the general case, we argue by induction. Assume that we established (3.1) for all $m \in \{k+1, \dots, n+1\}$ with some $k \in \{1, \dots, n\}$. We need to show that (3.1) holds with $m = k$, that is

$$k(k-1)\left(\mathbb{E}f_{n+2-k}(Z^{(n+2)}) - \mathbb{E}f_{n-k}(Z^{(n)})\right) = \pi^2(n+1)^2\mathbb{E}f_{n+2-k}(Z^{(n)}). \tag{3.4}$$

It suffices to assume that k is **odd** because the even case has been settled in Section 1.4. Note that (3.4) becomes trivial for $k = 1$ because both sides vanish. Let in the following $k \geq 3$.

Case 1: n is even. Let us write the Dehn-Sommerville relation (1.4) in the form

$$\mathbb{E}f_{n-j}(Z^{(n)}) = \frac{1}{2}\sum_{m=j+1}^{n}(-1)^m\binom{m}{j}\mathbb{E}f_{n-m}(Z^{(n)}),$$

for all odd $j \in \{1, \dots, n-1\}$. Using this formula 3 times with $j \in \{k, k-2\}$, we can write (3.4) in the following form:

$$\begin{aligned} k(k-1)&\left(\sum_{m=k+1}^{n+2}(-1)^m\binom{m}{k}\mathbb{E}f_{n+2-m}(Z^{(n+2)}) - \sum_{m=k+1}^{n}(-1)^m\binom{m}{k}\mathbb{E}f_{n-m}(Z^{(n)})\right) \\ &= \pi^2(n+1)^2\sum_{m=k-1}^{n}(-1)^m\binom{m}{k-2}\mathbb{E}f_{n-m}(Z^{(n)}). \end{aligned} \tag{3.5}$$

Applying to the left-hand side induction assumption (3.1) and introducing on the right-hand side the new summation index $m' := m+2$, we write the above equation in the form

$$\begin{aligned} k(k-1)&\left(\sum_{m=k+1}^{n}\frac{\pi^2(n+1)^2}{m(m-1)}(-1)^m\binom{m}{k}\mathbb{E}f_{n+2-m}(Z^{(n)}) - \binom{n+1}{k}\mathbb{E}f_1(Z^{(n+2)})\right. \\ &\left.+\binom{n+2}{k}\mathbb{E}f_0(Z^{(n+2)})\right) = \pi^2(n+1)^2\sum_{m'=k+1}^{n+2}(-1)^{m'}\binom{m'-2}{k-2}\mathbb{E}f_{n+2-m'}(Z^{(n)}). \end{aligned}$$

Observing that $\frac{k(k-1)}{m(m-1)}\binom{m}{k} = \binom{m-2}{k-2}$ and cancelling the sums over $m, m' \in \{k, \dots, n\}$, we arrive at

$$k(k-1)\left(-\binom{n+1}{k}\mathbb{E}f_1(Z^{(n+2)}) + \binom{n+2}{k}\mathbb{E}f_0(Z^{(n+2)})\right) = \pi^2(n+1)^2\left(-\binom{n-1}{k-2}\mathbb{E}f_1(Z^{(n)}) + \binom{n}{k-2}\mathbb{E}f_0(Z^{(n)})\right).$$

The equality of terms involving $\mathbb{E}f_1(Z^{(n+2)})$ and $\mathbb{E}f_1(Z^{(n)})$ follows from (3.2), whereas the equality of terms involving $\mathbb{E}f_0(Z^{(n+2)})$ and $\mathbb{E}f_0(Z^{(n)})$ follows from the identity $(n+2)\mathbb{E}f_0(Z^{(n+2)}) = \pi^2(n+1)\mathbb{E}f_0(Z^{(n)})$ which, in turn, is a consequence of (3.3). This completes the proof of (3.4).

Case 2: n is odd. Let us write the Dehn-Sommerville relations (1.5) in the form

$$\mathbb{E}f_{n-j}(Z^{(n)}) = \frac{2}{j}\mathbb{E}f_{n-j+1}(Z^{(n)}) + \frac{1}{j}\sum_{m=j+1}^{n}(-1)^m\binom{m}{j-1}\mathbb{E}f_{n-m}(Z^{(n)})$$

for all odd $j \in \{1, \dots, n\}$. Using this formula 3 times with $j \in \{k, k-2\}$, we can write (3.4) in the following form:

$$\begin{aligned}
&2(k-1)\left(\mathbb{E}f_{n+3-k}(Z^{(n+2)}) - \mathbb{E}f_{n+1-k}(Z^{(n)})\right)\\
&\quad+(k-1)\left(\sum_{m=k+1}^{n+2}(-1)^m\binom{m}{k-1}\mathbb{E}f_{n+2-m}(Z^{(n+2)}) - \sum_{m=k+1}^{n}(-1)^m\binom{m}{k-1}\mathbb{E}f_{n-m}(Z^{(n)})\right)\\
&\qquad= \frac{\pi^2(n+1)^2}{k-2}\left(2\mathbb{E}f_{n+3-k}(Z^{(n)}) + \sum_{m=k-1}^{n}(-1)^m\binom{m}{k-3}\mathbb{E}f_{n-m}(Z^{(n)})\right).
\end{aligned}$$

Since $k-1$ is even, we can use the result established in Section 1.4:

$$2(k-1)(k-2)\left(\mathbb{E}f_{n+3-k}(Z^{(n+2)}) - \mathbb{E}f_{n+1-k}(Z^{(n)})\right) = 2\pi^2(n+1)^2\mathbb{E}f_{n+3-k}(Z^{(n)}).$$

In view of this, it remains to show that

$$\begin{aligned}
&(k-1)(k-2)\left(\sum_{m=k+1}^{n+2}(-1)^m\binom{m}{k-1}\mathbb{E}f_{n+2-m}(Z^{(n+2)}) - \sum_{m=k+1}^{n}(-1)^m\binom{m}{k-1}\mathbb{E}f_{n-m}(Z^{(n)})\right)\\
&\qquad= \pi^2(n+1)^2\sum_{m=k-1}^{n}(-1)^m\binom{m}{k-3}\mathbb{E}f_{n-m}(Z^{(n)}).
\end{aligned}$$

The proof of this is analogous to the proof of (3.5). □

3.2. **Proof of Theorem 1.4.** As it was observed in [11], the f-vector of $C_n \cap \mathbb{S}^d_+$ has the same distribution as the f-vector of the beta' polytope $\tilde{P}^{(d+1)/2}_{n,d}$. Indeed, the intersection of the random cone C_n with the hyperplane $\{x_0 = 1\}$ (which is the tangent hyperplane to the half-sphere $\mathbb{S}^d_+$ at its north pole) has the same distribution as the random polytope $\tilde{P}^{(d+1)/2}_{n,d}$; see [11, Proposition 2.2].

For the beta' polytope $\tilde{P}^{(d+1)/2}_{n,d}$ it was shown in [12, Theorem 1.14] that

$$\mathbb{E}f_k(\tilde{P}^{(d+1)/2}_{n,d}) = 2\sum_{\substack{s=0,1,\dots\\ d-2s\geq k+1}}\tilde{\mathbb{I}}_{n,d-2s}(1)\tilde{\mathbb{J}}_{d-2s,k+1}\left(\frac{d}{2}-s\right) \tag{3.6}$$

for all $k \in \{0, \dots, d-1\}$. By Lemma 2.4 and Proposition 2.1,

$$\tilde{\mathbb{I}}_{n,d-2s}(1) = \frac{n!}{d-2s} \tilde{c}_{1,\frac{d-2s+1}{2}} \pi^{d-2s-n} B\{n, d-2s\},$$

$$\tilde{\mathbb{J}}_{d-2s,k+1}\left(\frac{d}{2} - s\right) = \frac{\pi^{k+1-d+2s}}{(k+1)!} \cdot \frac{d-2s}{2\tilde{c}_{1,\frac{d-2s+1}{2}}} \cdot (d-2s-1)^2 A[d-2s-2, k-1],$$

where for the second identity we bear in mind that k is assumed to be odd. Plugging these values into (3.6), and performing numerous cancellations, we arrive at

$$\mathbb{E} f_k(\tilde{P}_{n,d}^{(d+1)/2}) = \frac{n! \pi^{k+1-n}}{(k+1)!} \sum_{\substack{s=0,1,\dots \\ d-2s \geq k+1}} B\{n, d-2s\}(d-2s-1)^2 A[d-2s-2, k-1], \tag{3.7}$$

for all odd $k \in \{1, \dots, d-1\}$. This completes the proof of Theorem 1.4. □

3.3. **A complementary combinatorial identity.** As a consequence of Theorem 1.4, we shall derive the following combinatorial identity complementing Lemma 2.7.

Lemma 3.1. *For all $n \in \mathbb{N}$ and all **even** $k \in \{1, 2, \dots, n-1\}$,*

$$\sum_{\substack{s=0,1,\dots \\ n-2s \geq k+1}} B\{n, n-2s-1\}(n-2s-2)^2 A[n-2s-3, k-2] = \frac{\pi^{n-k}}{(n-k)!}. \tag{3.8}$$

Proof. The beta' polytope $\tilde{P}_{d+1,d}^{(d+1)/2}$ is a d-dimensional simplex with probability 1, so that

$$f_j\left(\tilde{P}_{d+1,d}^{(d+1)/2}\right) = \binom{d+1}{j+1}, \quad j = 0, \dots, d-1.$$

Comparing this with (3.7) (where we take $n = d+1$ and replace k by j), we arrive at the identity

$$\binom{n}{j+1} = \frac{n! \pi^{j+1-n}}{(j+1)!} \sum_{\substack{s=0,1,\dots \\ n-1-2s \geq j+1}} B\{n, n-2s-1\}(n-2s-2)^2 A[n-2s-3, j-1],$$

for all odd $j \in \{1, \dots, n-2\}$. After some cancellations, this yields

$$\sum_{\substack{s=0,1,\dots \\ n-1-2s \geq j+1}} B\{n, n-2s-1\}(n-2s-2)^2 A[n-2s-3, j-1] = \frac{\pi^{n-j-1}}{(n-j-1)!},$$

for all odd $j \in \{1, \dots, n-2\}$. Taking $k = j+1$ (which is even), we arrive at the required identity (3.8). □

3.4. **Proof of Proposition 1.5.** Let us prove the first identity. By Theorem 1.4 with $n = d+2$, for all odd $k \in \{1, \dots, d-1\}$ we have

$$\mathbb{E} f_k(C_{d+2} \cap \mathbb{S}_+^d) = \frac{(d+2)! \pi^{k-d-1}}{(k+1)!} \sum_{\substack{s=0,1,\dots \\ d-2s \geq k+1}} B\{d+2, d-2s\}(d-2s-1)^2 A[d-2s-2, k-1].$$

Lemma 2.7 with $n = d+2$ and k replaced by the even number $k+1$ states that

$$\sum_{\substack{s=0,1,\dots \\ d+2-2s \geq k+1}} B\{d+2, d+2-2s\}(d+2-2s-1)^2 A[d-2s, k-1] = \frac{\pi^{d+1-k}}{(d+1-k)!}.$$

Since the sums in the above two equations differ by just one term, we can write

$$\begin{aligned}\mathbb{E}f_k(C_{d+2}\cap\mathbb{S}^d_+) &= \frac{(d+2)!\pi^{k-d-1}}{(k+1)!}\left(\frac{\pi^{d+1-k}}{(d+1-k)!} - B\{d+2,d+2\}(d+1)^2A[d,k-1]\right)\\ &= \binom{d+2}{k+1} - \frac{(d+2)!\pi^{k-d-1}}{(k+1)!}B\{d+2,d+2\}(d+1)^2A[d,k-1]\\ &= \binom{d+2}{k+1} - \frac{(d+2)\pi^{k-d-1}}{(k+1)!}\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}(d+1)^2A[d,k-1]\end{aligned}$$

upon using the formula

$$B\{d+2,d+2\} = \frac{1}{(d+1)!}\int_0^\pi(\sin x)^{d+1}\mathrm{d}x = \frac{1}{(d+1)!}\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}.$$

To complete the proof of the first identity, recall that $(d+1)^2A[d,k-1] = A[d+2,k+1] - A[d,k+1]$ by Lemma 2.5.

The proof of the second identity is similar. By Theorem 1.4 with $n=d+3$, for all odd $k\in\{1,\dots,d-1\}$ we have

$$\mathbb{E}f_k(C_{d+3}\cap\mathbb{S}^d_+) = \frac{(d+3)!\pi^{k-d-2}}{(k+1)!}\sum_{\substack{s=0,1,\dots\\ d-2s\geq k+1}} B\{d+3,d-2s\}(d-2s-1)^2A[d-2s-2,k-1].$$

Lemma 3.1 with $n=d+3$ and k replaced by the even number $k+1$ states that

$$\sum_{\substack{s=0,1,\dots\\ d+3-2s\geq k+2}} B\{d+3,d+2-2s\}(d+1-2s)^2A[d-2s,k-1] = \frac{\pi^{d+2-k}}{(d+2-k)!}.$$

Again, the sums in the above two equations differ by just one term, so that we can write

$$\begin{aligned}\mathbb{E}f_k(C_{d+3}\cap\mathbb{S}^d_+) &= \frac{(d+3)!\pi^{k-d-2}}{(k+1)!}\left(\frac{\pi^{d+2-k}}{(d+2-k)!} - B\{d+3,d+2\}(d+1)^2A[d,k-1]\right)\\ &= \binom{d+3}{k+1} - \frac{(d+3)!\pi^{k-d-2}}{(k+1)!}B\{d+3,d+2\}(d+1)^2A[d,k-1]\\ &= \binom{d+3}{k+1} - \frac{(d+3)\pi^{k-d-1}}{(k+1)!}\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+4}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)}(d+1)^2A[d,k-1]\end{aligned}$$

upon using the formula

$$B\{d+3,d+2\} = \frac{1}{(d+1)!}\int_0^\pi(\sin x)^{d+1}x\mathrm{d}x = \frac{1}{(d+1)!}\frac{\pi^{3/2}\,\Gamma\left(\frac{d+2}{2}\right)}{2\Gamma\left(\frac{d+3}{2}\right)}.$$

The proof of the second identity is completed by recalling that $(d+1)^2A[d,k-1] = A[d+2,k+1] - A[d,k+1]$. □

3.5. **Proof of Proposition 1.6.** Define $A[n,k]$ by (1.15). Let us first show that the formula for the quantities $\tilde{\mathbb{J}}_{n,k}(\frac{n}{2})$ stated in Proposition 2.1, namely

$$\tilde{\mathbb{J}}_{n,k}\left(\frac{n}{2}\right) = \frac{\pi^{k-n}}{k!}\cdot\frac{n}{2\tilde{c}_{1,\frac{n+1}{2}}}\cdot(A[n,k]-A[n-2,k]), \tag{3.9}$$

holds for all $k\in\{1,\dots,n\}$ irrespective of the parity. Observe that we intentionally did not write the expression in the brackets as $(n-1)^2A[n-2,k-2]$ in order to include the case $k=1$. Also, we agree to define $A[n,k] := 0$ if $k > n$. Since the identity is known for even k, let k be odd. The proof is essentially a repetition of the proof of Theorem 1.1 (given after Proposition 2.1) in reversed order. Using the new definition of $A[n,k]$ given in (1.15), recalling (1.13) and finally applying (2.5), we obtain

$$\frac{\pi^k}{k!}A[n,k] = \mathbb{E}f_{n-k}(Z^{(n)}) = \mathbb{E}f_{k-1}(\operatorname{conv}\Pi_{n,1}) = \sum_{\substack{s=0,1,\dots\\ m:=n-2s\geq k}} \frac{2}{m}\pi^m \tilde{c}_{1,\frac{m+1}{2}} \tilde{\mathbb{J}}_{m,k}\left(\frac{m}{2}\right), \tag{3.10}$$

for all $k \in \{1,\dots,n\}$. Replacing n by $n-2$, we can write

$$\frac{\pi^k}{k!}A[n-2,k] = \sum_{\substack{s=0,1,\dots\\ m:=n-2-2s\geq k}} \frac{2}{m}\pi^m \tilde{c}_{1,\frac{m+1}{2}} \tilde{\mathbb{J}}_{m,k}\left(\frac{m}{2}\right), \tag{3.11}$$

for all $k \in \{1,\dots,n-2\}$. Subtracting (3.10) and (3.11), we arrive at the required identity (3.9) for all $k \in \{1,\dots,n-2\}$. The remaining cases $k = n$ and $k = n-1$ can be verified directly since $\tilde{\mathbb{J}}_{n,n}(n/2) = 1$ and $\tilde{\mathbb{J}}_{n,n-1}(n/2) = n/2$. The formulae for $A[n,n]$ and $A[n,n-1]$ can be found in Proposition 1.9 (ii).

Now, an easy inspection shows that the only ingredient in the proofs of Theorems 1.3 and 1.4 which depends on the parity of k is the formula for the quantities $\tilde{\mathbb{J}}_{n,k}(n/2)$ which we have just shown to hold without parity restrictions. (In fact, Theorem 1.3 does not require even this formula). The same applies to Lemmas 2.7 and 3.1 (the former needs only the recurrence relation for the $A[n,k]$'s in which the parity restriction was removed in Proposition 1.2), and, consequently, to Proposition 1.5, if we write these lemmas in the form

$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n,n-2s\}(A[n-2s,k] - A[n-2s-2,k]) = \frac{\pi^{n-k}}{(n-k)!}, \tag{3.12}$$

$$\sum_{\substack{s=0,1,\dots\\ n-2s\geq k}} B\{n,n-2s-1\}(A[n-2s-1,k] - A[n-2s-3,k]) = \frac{\pi^{n-k}}{(n-k)!} \tag{3.13}$$

for all $n \in \mathbb{N}$ and all $k \in \{1,2,\dots,n-1\}$ regardless of the parity. We need only to verify the induction base in the proof of Lemma 2.7, since it is different in the case of odd k.

Case $n = 2$. Then, $k = 1$, and the identity takes the form

$$B\{2,2\}(A[2,1] - A[0,1]) = \pi,$$

which is true because $B\{2,2\} = 2$ and $A[2,1] = \pi/2$.

Case $n = 3$. Then, the only admissible odd value of k is $k = 1$ and the identity takes the form

$$B\{3,3\}(A[3,1] - A[1,1]) + B\{3,1\}(A[1,1] - A[-1,1]) = \frac{\pi^2}{2},$$

which is true because $B\{3,3\} = \pi/4$, $B\{3,1\} = \pi^3/6$, $A[3,1] - A[1,1] = 2\pi/3$, $A[1,1] = 2/\pi$.

It follows that all statements listed above, namely Theorems 1.3, 1.4, Lemmas 2.7, 3.1 and Proposition 1.5, continue to hold without any parity restrictions on k. ∎

3.6. **Proof of Theorem 1.8.** Recall that C_{n+1} is defined as the positive hull of the points $U_1, \dots, U_{n+1}$ that are independent and uniformly distributed on the half-sphere $\mathbb{S}^d_+$. For every $k \in \{1, \dots, n+1\}$, the point U_k is not a vertex of $C_{n+1} \cap \mathbb{S}^d_+$ if and only if it is contained in the cone generated by the remaining points U_i, $i \in \{1, \dots, n+1\} \backslash \{k\}$. Hence,

$$(n+1) - \mathbb{E} f_0(C_{n+1} \cap \mathbb{S}^d_+) = (n+1)\mathbb{P}[U_{n+1} \in \operatorname{pos}(U_1, \dots, U_n)] = 2(n+1)\mathbb{E}\alpha(C_n). \tag{3.14}$$

This spherical Efron-type identity was obtained in [3, Equation (26)] and is a special case of the more general identity proved in [11, Theorem 2.7]. It follows from (3.14) that

$$\begin{aligned} \mathbb{E}\alpha(C_n) &= \frac{1}{2}\left(1 - \frac{1}{n+1}\mathbb{E} f_0(C_{n+1} \cap \mathbb{S}^d_+)\right) \\ &= \frac{1}{2}\Big(1 - \frac{n!}{\pi^n} \sum_{\substack{s=0,1,\dots \\ d-2s \geq 1}} B\{n+1, d-2s\}(A[d-2s, 1] - A[d-2s-2, 1])\Big), \end{aligned} \tag{3.15}$$

where in the second equality we applied Theorem 1.4 with $k = 0$ bearing in mind Remark 1.7. Lemmas 2.7 and 3.1 in the general form given in (3.12), (3.13), with n replaced by $n+1$ and $k = 1$ state that

$$\begin{aligned} &\frac{n!}{\pi^n} \sum_{\substack{m \in \{1, \dots, n+1\} \\ m \not\equiv n \pmod 2}} B\{n+1, m\}(A[m, 1] - A[m-2, 1]) \\ &\qquad = \frac{n!}{\pi^n} \sum_{\substack{m \in \{1, \dots, n+1\} \\ m \equiv n \pmod 2}} B\{n+1, m\}(A[m, 1] - A[m-2, 1] = 1. \end{aligned}$$

Replacing the term 1 in (3.15) by one of the above sums depending on the parity of d, we can write (3.15) in the form

$$\mathbb{E}\alpha(C_n) = \frac{n!}{2\pi^n} \sum_{\substack{m \in \{d+2, \dots, n+1\} \\ m \equiv d \pmod 2}} B\{n+1, m\}(A[m, 1] - A[m-2, 1]),$$

which completes the proof. $\square$

Acknowledgements

Supported by the German Research Foundation under Germany's Excellence Strategy EXC 2044 – 390685587, Mathematics Münster: Dynamics - Geometry - Structure. The author is grateful to Christoph Thäle for useful comments.

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4. Tables

d	$\mathbb{E}f_0(Z^{(d)}),\ \mathbb{E}f_1(Z^{(d)}),\ \ldots,\ \mathbb{E}f_{d-1}(Z^{(d)})$
1	2
2	$\frac{\pi^2}{2}, \frac{\pi^2}{2}$
3	$\frac{4\pi^2}{3}, 2\pi^2, 2\left(1+\frac{\pi^2}{3}\right)$
4	$\frac{3\pi^4}{8}, \frac{3\pi^4}{4}, 5\pi^2, 5\pi^2-\frac{3\pi^4}{8}$
5	$\frac{16\pi^4}{15}, \frac{8\pi^4}{3}, \frac{4}{9}\pi^2\left(15+4\pi^2\right), 10\pi^2, 2+\frac{10\pi^2}{3}-\frac{8\pi^4}{45}$
6	$\frac{5\pi^6}{16}, \frac{15\pi^6}{16}, \frac{259\pi^4}{24}, \frac{1}{48}\left(1036\pi^4-75\pi^6\right), \frac{35\pi^2}{2}, \frac{1}{48}\pi^2\left(840-518\pi^2+45\pi^4\right)$
7	$\frac{32\pi^6}{35}, \frac{16\pi^6}{5}, \frac{4}{15}\pi^4\left(49+12\pi^2\right), \frac{98\pi^4}{3}, \frac{4}{45}\pi^2\left(210+245\pi^2-12\pi^4\right), 28\pi^2, 2+\frac{28\pi^2}{3}-\frac{98\pi^4}{45}+\frac{16\pi^6}{105}$
8	$\frac{35\pi^8}{128}, \frac{35\pi^8}{32}, \frac{3229\pi^6}{180}, \frac{3229\pi^6}{60}-\frac{245\pi^8}{64}, \frac{329\pi^4}{4}, \frac{329\pi^4}{2}-\frac{3229\pi^6}{36}+\frac{245\pi^8}{32}, 42\pi^2,$ $42\pi^2-\frac{329\pi^4}{4}+\frac{3229\pi^6}{60}-\frac{595\pi^8}{128}$
9	$\frac{256\pi^8}{315}, \frac{128\pi^8}{35}, \frac{32}{315}\pi^6\left(205+48\pi^2\right), \frac{656\pi^6}{9}, \frac{364\pi^4}{5}+\frac{656\pi^6}{9}-\frac{256\pi^8}{75}, 182\pi^4,$ $40\pi^2+\frac{364\pi^4}{3}-\frac{656\pi^6}{27}+\frac{512\pi^8}{315}, 60\pi^2, 2+20\pi^2-\frac{182\pi^4}{15}+\frac{656\pi^6}{189}-\frac{128\pi^8}{525}$
10	$\frac{63\pi^{10}}{256}, \frac{315\pi^{10}}{256}, \frac{117469\pi^8}{4480}, \frac{\pi^8\left(469876-33075\pi^2\right)}{4480}, \frac{17281\pi^6}{72}, \frac{\pi^6\left(1382480-704814\pi^2+59535\pi^4\right)}{1920}, \frac{1463\pi^4}{4},$ $\frac{1463\pi^4}{2}-\frac{86405\pi^6}{72}+\frac{117469\pi^8}{160}-\frac{16065\pi^{10}}{256}, \frac{165\pi^2}{2}, \frac{165\pi^2}{2}-\frac{1463\pi^4}{4}+\frac{17281\pi^6}{24}-\frac{1996973\pi^8}{4480}+\frac{9765\pi^{10}}{256}$

Table 1. Expected f-vector of the Poisson zero polytope in dimensions $d \in \{1, \ldots, 10\}$

$A[n,k]$	$k=0$	$k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$	$k=14$
$n=1$	1	0	0	0	0	0	0	0
$n=2$	1	1	0	0	0	0	0	0
$n=3$	1	4	0	0	0	0	0	0
$n=4$	1	10	9	0	0	0	0	0
$n=5$	1	20	64	0	0	0	0	0
$n=6$	1	35	259	225	0	0	0	0
$n=7$	1	56	784	2304	0	0	0	0
$n=8$	1	84	1974	12916	11025	0	0	0
$n=9$	1	120	4368	52480	147456	0	0	0
$n=10$	1	165	8778	172810	1057221	893025	0	0
$n=11$	1	220	16368	489280	5395456	14745600	0	0
$n=12$	1	286	28743	1234948	21967231	128816766	108056025	0
$n=13$	1	364	48048	2846272	75851776	791691264	2123366400	0
$n=14$	1	455	77077	6092515	230673443	3841278805	21878089479	18261468225

TABLE 2. The values of $A[n,k]$ for $n \in \{1,\dots,14\}$ and even $k \in \{0,2,4,\dots,14\}$.

$A[n,k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=1$	1	$\frac{2}{\pi}$	0	0	0	0
$n=2$	1	$\frac{\pi}{2}$	1	0	0	0
$n=3$	1	$\frac{2}{\pi}+\frac{2\pi}{3}$	4	$\frac{8}{\pi}$	0	0
$n=4$	1	$5\pi-\frac{3\pi^3}{8}$	10	$\frac{9\pi}{2}$	9	0
$n=5$	1	$\frac{2}{\pi}+\frac{10\pi}{3}-\frac{8\pi^3}{45}$	20	$\frac{40}{\pi}+\frac{32\pi}{3}$	64	$\frac{128}{\pi}$
$n=6$	1	$\frac{1}{48}\pi\left(840-518\pi^2+45\pi^4\right)$	35	$\frac{259\pi}{2}-\frac{75\pi^3}{8}$	259	$\frac{225\pi}{2}$
$n=7$	1	$\frac{2}{\pi}+\frac{28\pi}{3}-\frac{98\pi^3}{45}+\frac{16\pi^5}{105}$	56	$\frac{112}{\pi}+\frac{392\pi}{3}-\frac{32\pi^3}{5}$	784	$\frac{1568}{\pi}+384\pi$
$n=8$	1	$42\pi-\frac{329\pi^3}{4}+\frac{3229\pi^5}{60}-\frac{595\pi^7}{128}$	84	$987\pi-\frac{3229\pi^3}{6}+\frac{735\pi^5}{16}$	1974	$6458\pi-\frac{3675\pi^3}{8}$

TABLE 3. The values of $A[n,k]$ for $n\in\{1,\dots,8\}$ and $k\in\{0,\dots,5\}$. For odd k, the values are defined by the convention from Section 1.6.

$B\{n,k\}$	$k=1$	$k=2$	$k=3$	$k=4$
$n=1$	π	0	0	0
$n=2$	$\frac{\pi^2}{2}$	2	0	0
$n=3$	$\frac{\pi^3}{6}$	π	$\frac{\pi}{4}$	0
$n=4$	$\frac{\pi^4}{24}$	$\frac{1}{2}\left(-4+\pi^2\right)$	$\frac{\pi^2}{8}$	$\frac{2}{9}$
$n=5$	$\frac{\pi^5}{120}$	$\frac{1}{6}\pi\left(-6+\pi^2\right)$	$\frac{1}{48}\pi\left(-3+2\pi^2\right)$	$\frac{\pi}{9}$
$n=6$	$\frac{\pi^6}{720}$	$\frac{1}{24}\left(48-12\pi^2+\pi^4\right)$	$\frac{1}{96}\pi^2\left(-3+\pi^2\right)$	$\frac{1}{162}\left(-40+9\pi^2\right)$
$n=7$	$\frac{\pi^7}{5040}$	$\pi-\frac{\pi^3}{6}+\frac{\pi^5}{120}$	$\frac{1}{960}\pi\left(15-10\pi^2+2\pi^4\right)$	$\frac{1}{162}\pi\left(-20+3\pi^2\right)$
$n=8$	$\frac{\pi^8}{40320}$	$-2+\frac{\pi^2}{2}-\frac{\pi^4}{24}+\frac{\pi^6}{720}$	$\frac{\pi^2\left(45-15\pi^2+2\pi^4\right)}{5760}$	$\frac{1456-360\pi^2+27\pi^4}{5832}$
$n=9$	$\frac{\pi^9}{362880}$	$\frac{\pi\left(-5040+840\pi^2-42\pi^4+\pi^6\right)}{5040}$	$\frac{\pi\left(-315+210\pi^2-42\pi^4+4\pi^6\right)}{80640}$	$\frac{\pi\left(3640-600\pi^2+27\pi^4\right)}{29160}$
$n=10$	$\frac{\pi^{10}}{3628800}$	$2-\frac{\pi^2}{2}+\frac{\pi^4}{24}-\frac{\pi^6}{720}+\frac{\pi^8}{40320}$	$\frac{\pi^2\left(-315+105\pi^2-14\pi^4+\pi^6\right)}{161280}$	$\frac{-131200+32760\pi^2-2700\pi^4+81\pi^6}{524880}$

TABLE 4. The values of $B\{n,k\}$ for $n \in \{1,\dots,10\}$ and $k \in \{1,\dots,4\}$.

d	$P(d)$
1	1
2	$-2+\frac{24}{\pi^2}$
3	$\frac{5}{\pi^2}-\frac{1}{3}$
4	$\frac{80}{\pi^4}+6-\frac{200}{3\pi^2}$
5	$\frac{1}{3}+\frac{105}{8\pi^4}-\frac{35}{8\pi^2}$
6	$-\frac{1568}{3\pi^4}+\frac{896}{5\pi^6}-34+\frac{29008}{75\pi^2}$
7	$-\frac{3}{5}-\frac{49}{2\pi^4}+\frac{105}{4\pi^6}+\frac{49}{6\pi^2}$
8	$\frac{27072}{5\pi^4}-\frac{2304}{\pi^6}+\frac{2304}{7\pi^8}+310-\frac{878288}{245\pi^2}$
9	$\frac{5\left(3465-6930\pi^2+6006\pi^4-1804\pi^6+128\pi^8\right)}{384\pi^8}$

Table 5. The first few values of the probability $P(d)$ defined in Section 1.7.

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