# Ulam-Warburton Automaton - Counting Cells with Quadratics

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**Abstract**

This paper is about a sequence of quadratic functions that enumerate the total number of ON cells up to and including generation $n$ of the Ulam-Warburton cellular automaton, where $n$ has the form $n_m = m \cdot 2^k$.

Keywords: cellular automata (CA), enumeration, Ulam-Warburton, UWCA.

## 1 Introduction

The origins of cellular automata go back to Stanislaw Ulam in 1929 [1], he later explored these ideas with J. C. Holladay and Robert Schrandt [2]. The first function counting the total number of ON cells is the quadratic for the sharp upper bound occurring at generations $n = 2^k$ [3], this sparked interest and the fractal like object was named the Ulam-Warburton Cellular Automaton (UWCA) in 2003 [4]. Since then mathematicians have connected the UWCA with various objects including the Toothpick sequence [5], Nim Fractals [6] and the Sierpinski triangle [7].

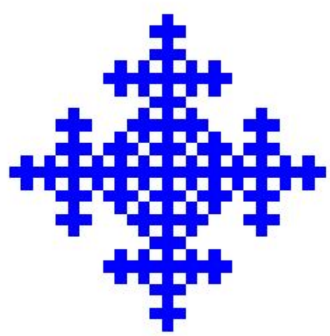

Figure 1: The Ulam-Warburton cellular automaton ($n = 14$)

The Ulam-Warburton cellular automaton is a 2-dimensional fractal pattern that grows on a grid of cells consisting of squares. Starting with one square

initially ON and all others OFF, successive iterations are generated by turning ON all squares that share precisely one edge with an ON square. This is the von Neumann neighbourhood.
Starting the generation index when the first cell is ON, the sharp upper bound function agrees with the total number of ON cells when $n = 2^k$ and is described by.

$$U_{sub}(n) = \frac{4}{3}n^2 - \frac{1}{3}.$$

This paper extends this result to all generations of the UWCA.

## 2 The Development of the Quadratics

Let $u(n)$ denote the number of ON cells at the $n^{th}$ stage.
$u(0) = 0, u(1) = 1$ and for $n \geq 2$.

$$u(n) = \frac{4}{3}3^{\mathrm{wt}(n-1)}.$$

Where $\mathrm{wt}(n)$ is the Hamming weight function which counts the number of 1's in the binary expansion of $n$ [5].

Let $U(n)$ denote the total number of ON cells after $n$ stages.

$$U(n) = \sum_{i=0}^{n} u(i),$$

$$U(n) = \frac{4}{3}\sum_{i=0}^{n-1} 3^{\mathrm{wt}(i)} - \frac{1}{3}.$$

We now consider integer sequences $n_m$ based on the form $n_m = m \cdot 2^k$ where $m \geq 1$ and $k \geq 0$. The total number of ON cells for these sequences becomes.

$$U_m(n_m) = \frac{4}{3}\sum_{i=0}^{m2^k-1} 3^{\mathrm{wt}(i)} - \frac{1}{3}. \tag{1}$$

Using this notation the expression for the sharp upper bound $U_{sub}(n)$ becomes $U_1(n_1)$ and therefore.

$$\frac{4}{3}\sum_{i=0}^{2^k-1} 3^{\mathrm{wt}(i)} - \frac{1}{3} = \frac{4}{3}n_1^2 - \frac{1}{3}. \tag{2}$$

Which is the first of a sequence of quadratics, and in terms of $k$ we have.

$$U_1(k) = \frac{4}{3}2^{2k} - \frac{1}{3}.$$

Returning to equation (1) for $U_m(n_m)$ and introducing the relationship.

$$\frac{4}{3}\sum_{i=0}^{m2^k-1} 3^{\mathrm{wt}(i)} - \frac{1}{3} = \sum_{i=0}^{m-1} 3^{\mathrm{wt}(i)} \frac{4}{3}\sum_{i=0}^{2^k-1} 3^{\mathrm{wt}(i)} - \frac{1}{3}. \tag{3}$$

Let

$$a_m = \sum_{i=0}^{m-1} 3^{\mathrm{wt}(i)}$$

This is OEIS [8] sequence A130665.

Substituting $a_m$ in to equation (3) and using equation (1) we have.

$$U_m(n_m) = a_m \frac{4}{3}\sum_{i=0}^{2^k-1} 3^{\mathrm{wt}(i)} - \frac{1}{3}.$$

As $n_m = m \cdot 2^k = m \cdot n_1$ therefore $n_1^2 = \frac{n_m^2}{m^2}$ using this with equation (2) we have.

$$U_m(n_m) = \frac{a_m}{m^2}\frac{4}{3}n_m^2 - \frac{1}{3}. \tag{4}$$

This is the result we were aiming for and in terms of k we have.

$$U_m(k) = a_m \frac{4}{3} 2^{2k} - \frac{1}{3}.$$

| $k$ | $n_1$ | $U_1$ | $n_3$ | $U_3$ | $n_5$ | $U_5$ | $n_7$ | $U_7$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 9 | 5 | 25 | 7 | 49 |
| 1 | 2 | 5 | 6 | 37 | 10 | 101 | 14 | 197 |
| 2 | 4 | 21 | 12 | 149 | 20 | 405 | 28 | 789 |
| 3 | 8 | 85 | 24 | 597 | 40 | 1,621 | 56 | 3,157 |
| 4 | 16 | 341 | 48 | 2,389 | 80 | 6,485 | 112 | 12,629 |
| 5 | 32 | 1,365 | 96 | 9,557 | 160 | 25,941 | 224 | 50,517 |
| 6 | 64 | 5,461 | 192 | 38,229 | 320 | 103,765 | 448 | 202,069 |
| 7 | 128 | 21,845 | 384 | 152,917 | 640 | 415,061 | 896 | 808,277 |
| 8 | 256 | 87,381 | 768 | 611,669 | 1,280 | 1,660,245 | 1792 | 3,233,109 |

Table 1: The odd numbered sequences $U_1$ to $U_7$ that give the total number of ON cells in the sequences $n_1$ to $n_7$.

# 3 The Limit Inferior and Limit Superior

We have [9].

$$0.9026116569... = \liminf_{n\to\infty} \frac{U(n)}{n^2} < \limsup_{n\to\infty} \frac{U(n)}{n^2} = \frac{4}{3}.$$

We can see from equation (4) that the coefficient of the quadratic term is the dominant expression in $\frac{U(n)}{n^2}$ and is a rational number.

## References


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