Counting Colorful Necklaces and Bracelets in Three Colors

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Abstract. A necklace or bracelet is *colorful* if no pair of adjacent beads are the same color. In addition, two necklaces are *equivalent* if one results from the other by permuting its colors, and two bracelets are *equivalent* if one results from the other by either permuting its colors or reversing the order of the beads; a bracelet is thus a necklace that can be turned over. This note counts the number $K(n)$ of non-equivalent colorful necklaces and the number $K'(n)$ of colorful bracelets formed with n-beads in at most three colors. Expressions obtained for $K'(n)$ simplify expressions given by OEIS sequence A114438, while the expressions given for $K(n)$ appear to be new and are not included in OEIS.

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1. Introduction

A *necklace* with n beads and c colors is an n-tuple, each of whose components can assume one of c values, where not all c colors need appear. Two necklaces are *equivalent* if one results from the other by either rotating it cyclically or permuting its colors. The classical necklace problem asks to determine the number of non-equivalent necklaces formed with n beads of c colors. The answer to this problem is given by

$$N(n,c) = \frac{1}{n}\sum_{d|n}\varphi(d)c^{n/d}, \tag{1.1}$$

where φ is the Euler totient function.

A *bracelet* with n beads and c colors is a necklace of n beads and c colors that can be turned over, and thus the order of its beads is reversed.

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The number of non-equivalent bracelets with n beads of c colors is given by

$$N'(n,c) = \frac{N(n,c) + R(n,c)}{2}, \tag{1.2}$$

where

$$R(n,c) = \begin{cases} c^{(n+1)/2} & \text{if } n \text{ is odd}, \\ \frac{1+c}{2} c^{n/2} & \text{if } n \text{ is even}. \end{cases} \tag{1.3}$$

As expected, $N'(n,c) \le N(n,c)$. These results and further details are given in [9, 10] and the references therein.

In this work we consider a variation on this problem that arose from considering sequences of n coordinate-axis rotations defined by Euler angles, where $n = 7$ for aircraft [1]. For this problem, it is of interest to count the number of distinct coordinate-axis rotation sequences of length n that are closed in the sense of transforming the starting frame by a sequence of coordinate-axis rotations that lead back to the starting frame [2]. A pair of successive coordinate-axis rotations around the same axis can be combined into a single rotation, and the labeling of the axes of the starting frame is arbitrary. Counting the number of closed sequences consisting of n coordinate-axis rotations is thus equivalent to counting necklaces in 3 colors, where each color corresponds to an axis label. Furthermore, reversing a sequence of coordinate-axis rotations is equivalent to replacing each Euler angle in the sequence of coordinate-axis rotations by its negative. Hence, for the purpose of determining all feasible Euler angles for each closed sequence of coordinate-axis rotations, it suffices to count bracelets.

Motivated by the fact that successive coordinate-axis rotations around the same axis can be merged, the present paper considers colorful necklaces and bracelets formed with n-beads of three colors, where a necklace or bracelet is *colorful* if no pair of adjacent beads have the same color. Two colorful necklaces are *equivalent* if one results from the other by permuting its colors, and two colorful bracelets are *equivalent* if one results from the other by either permuting its colors or reversing the order of the beads; a colorful bracelet is thus a colorful necklace that can be turned over. In fact, the number of colorful bracelets with n beads in three colors appears in the On-line Encyclopedia of Integer Sequences (OEIS) as sequence A114438, which is the "Number of Barlow packings that repeat after n (or a divisor of n) layers." The provided references indicate that the problem of studying this sequence originates in crystallography [4, 5, 8].

The contribution of the present paper is twofold. First, we provide explicit formulas for the number of color bracelets that simplify those given by $P'(n)$ in [4, p. 272]. Furthermore, we provide expressions for the number of colorful necklaces with n beads in three colors; this sequence is currently unknown to OEIS.

2. n-Periodic Sequences

Instead of working with necklaces and bracelets of n beads we work with n-periodic sequences. To fix notation, let $\mathbb{N}_q$ denote the set of natural numbers from 1 to q. In particular, $\mathbb{N}_3$ represents the set of the three colors under consideration.

Definition 2.1. For a positive integer n, a *colorful* n-periodic sequence is a function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{N}_3$ defined on the set of integers modulo n, such that $f(i) \neq f(i+1)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. The set of colorful n-periodic sequences is denoted by $\mathcal{A}_n$.

The set $\mathcal{A}_n$ represents the colorful n-bead necklaces or bracelets of three colors.

Next, we consider the permutations r and s on $\mathbb{Z}/n\mathbb{Z}$ defined by $s(i) = i+1$ and $r(i) = -i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. The group generated by s is

$$\langle s \rangle = \{id, s, s^2, \dots, s^{n-1}\},$$

which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The group generated by r is $\langle r \rangle = \{id, r\} \simeq \mathbb{Z}/2\mathbb{Z}$. Furthermore, since $rsr = s^{-1}$, the group generated by r and s is

$$\langle s, r \rangle = \{id, s, s^2, \dots, s^{n-1}, r, rs, rs^2, \dots, rs^{n-1}\},$$

which is isomorphic to the dihedral group D_n of order $2n$.

We also consider the symmetric group $\mathfrak{S}_3$ of $\mathbb{N}_3$, where $\mathfrak{S}_3$ consists of the identity id, the three substitutions τ_{12}, τ_{13}, and τ_{23} (with τ_{ij} representing the 2-cycle (i, j)), and the two 3-cycles $c = (1, 2, 3)$ and $c^2 = c^{-1} = (1, 3, 2)$. We recall that two permutations σ and σ' are *conjugates* if there exists a permutation τ such that $\sigma' = \tau^{-1} \circ \sigma \circ \tau$; this is equivalent to the fact that σ and σ' have the same cyclic decomposition. Hence, for $\mathfrak{S}_3$ all transpositions are conjugates and all 3-cycles are conjugates, which can be checked directly.

The group acting on the elements of $\mathcal{A}_n$, considered as colorful n-bead necklaces, is $G = \mathfrak{S}_3 \times \langle s \rangle$, with group action defined by $\gamma f = \sigma \circ f \circ t^{-1}$ for $\gamma = (\sigma, t) \in G$ and $f \in \mathcal{A}_n$. Similarly, the group acting on the elements of $\mathcal{A}_n$, considered as colorful n-bead bracelets, is $G' = \mathfrak{S}_3 \times \langle s, r \rangle$, with group action defined again by $\gamma f = \sigma \circ f \circ t^{-1}$ for $\gamma = (\sigma, t) \in G'$ and $f \in \mathcal{A}_n$.

The action of G on $\mathcal{A}_n$ defines an equivalence relation $\sim_G$ on colorful n-bead necklaces given by

$$f \sim_G g \iff \text{there exists } \gamma \in G \text{ such that } \gamma f = g.$$

The set of equivalence classes of this relation is denoted by $\mathcal{A}_n/G$, and the desired number of non-equivalent colorful n-bead necklaces is exactly $K(n) = |\mathcal{A}_n/G|$. Similarly, $K'(n) = |\mathcal{A}_n/G'|$, where $\mathcal{A}_n/G'$ is the set of equivalence classes of colorful n-bead bracelets defined by the action of G' on $\mathcal{A}_n$.

The basic tool in this investigation is the classical Burnside's Lemma [7, Theorem 3.22]. (While this lemma bears the name of Burnside, it seems that it was well-known to Frobenius (1887) and before him to Cauchy (1845). An account on the history of this lemma can be found in [9], see also [11]).

Theorem 2.2 (Burnside's Lemma). *Let G be a finite group acting on a finite set X. Then*

$$|X/G| = \frac{1}{|G|}\sum_{\gamma\in G}|\mathfrak{F}(\gamma)|,$$

where $\mathfrak{F}(\gamma)$ is the set of elements $x\in X$ fixed by γ (i.e. $\gamma x = x$.)

Noting that G is a subgroup of G', our task is to determine the numbers

$$\mathfrak{f}^{(n)}(\sigma,\varepsilon,j) = \left|\mathfrak{F}^{(n)}(\sigma,\varepsilon,j)\right| \tag{2.1}$$

with $\sigma\in\mathfrak{S}_3$, $\varepsilon\in\{0,1\}$, and $j\in\{0,\dots,n-1\}$, where

$$\mathfrak{F}^{(n)}(\sigma,\varepsilon,j) = \left\{f\in\mathcal{A}_n : f = \sigma\circ f\circ r^{\varepsilon}\circ s^{j}\right\}. \tag{2.2}$$

This paper is organized as follows: In Section 3, we gather some useful properties and lemmas. In Section 4 the case of necklaces is considered. Finally, in Section 5 we consider the case of bracelets.

3. Useful Properties and Lemmas

Lemma 3.1. *If n and m are positive integers, then $\mathcal{A}_n\cap\mathcal{A}_m = \mathcal{A}_{\gcd(n,m)}$.*

Proof. This result follows from the fact that $n\mathbb{Z}+m\mathbb{Z} = \gcd(n,m)\mathbb{Z}$. □

Our first step is to determine the number of n-periodic colorful sequences, that is $\alpha_n \overset{\text{def}}{=} |\mathcal{A}_n|$. This is the object of the next proposition.

Proposition 3.2. *For all $n\geq 1$,*

$$\alpha_n = |\mathcal{A}_n| = 2^n + 2(-1)^n. \tag{3.1}$$

Proof. Note that $\alpha_1 = 0$ and $\alpha_2 = 6$. Suppose that $n\geq 3$, and define

$$\mathcal{A}_n' = \{f\in\mathcal{A}_n : f(n-1)\neq f(1)\},$$
$$\mathcal{A}_n'' = \{f\in\mathcal{A}_n : f(n-1) = f(1)\}.$$

The mapping $\mathcal{A}_n'\to\mathcal{A}_{n-1} : f\mapsto\tilde{f}$ with $\tilde{f}$ defined by $\tilde{f}_{|\mathbb{N}_{n-1}} = f_{|\mathbb{N}_{n-1}}$ is bijective because $f(n)$ is uniquely defined by the knowledge of $f(n-1)$ and $f(1)$, indeed $\{f(1), f(n-1), f(n)\} = \mathbb{N}_3$. Hence $|\mathcal{A}_n'| = |\mathcal{A}_{n-1}| = \alpha_{n-1}$.

Also, the mapping $\mathcal{A}_n''\to\mathcal{A}_{n-2} : f\mapsto\hat{f}$ with $\hat{f}$ defined by $\hat{f}_{|\mathbb{N}_{n-2}} = f_{|\mathbb{N}_{n-2}}$ is surjective and the pre-image of each $g\in\mathcal{A}_{n-2}$ consists of exactly two elements, namely f_1 and f_2 defined by $f_{1|\mathbb{N}_{n-2}} = f_{2|\mathbb{N}_{n-2}} = g_{|\mathbb{N}_{n-2}}$, $f_1(n) = \min(\mathbb{N}_3\setminus\{f(1)\})$ and $f_2(n) = \max(\mathbb{N}_3\setminus\{f(1)\})$. Hence $|\mathcal{A}_n''| = 2\,|\mathcal{A}_{n-2}| = 2\alpha_{n-2}$. But $\{\mathcal{A}_n',\mathcal{A}_n''\}$ is a partition of $\mathcal{A}_n$, so

$$\alpha_n = |\mathcal{A}_n| = |\mathcal{A}_n'| + |\mathcal{A}_n''| = \alpha_{n-1} + 2\alpha_{n-2},$$

and the desired conclusion follows by induction. □

In particular, since the neutral element of G (or G') fixes the whole set $\mathcal{A}_n$, the next corollary is immediate.

Corollary 3.3.

$$\mathfrak{f}^{(n)}(id,0,0)=\alpha_n. \tag{3.2}$$

Corollary 3.4. *For distinct $i,j\in\mathbb{N}_3$, let $\mathcal{A}_n^{i\cdot\cdot j}$ denote the subset of $\mathcal{A}_n$ consisting of functions f satisfying $f(1)=i$ and $f(n)=j$. Then*

$$\left|\mathcal{A}_n^{i\cdot\cdot j}\right|=\frac{\alpha_n}{6}. \tag{3.3}$$

Proof. Given i and j, there is a unique permutation $\sigma\in\mathfrak{S}_3$ such that $\sigma(i)=1$ and $\sigma(j)=2$, and with this σ the mapping $f\mapsto\sigma\circ f$ defines a bijection between $\mathcal{A}_n^{i\cdot\cdot j}$ and $\mathcal{A}_n^{1\cdot\cdot 2}$. Thus

$$\left|\mathcal{A}_n^{i\cdot\cdot j}\right|=\left|\mathcal{A}_n^{1\cdot\cdot 2}\right|.$$

The conclusion follows since $\{\mathcal{A}_n^{1\cdot\cdot 2},\mathcal{A}_n^{1\cdot\cdot 3},\mathcal{A}_n^{2\cdot\cdot 1},\mathcal{A}_n^{2\cdot\cdot 3},\mathcal{A}_n^{3\cdot\cdot 1},\mathcal{A}_n^{3\cdot\cdot 2}\}$ constitutes a partition of $\mathcal{A}_n$. □

The next lemma helps to reduce the number of cases to be considered. The proof is immediate and left to the reader.

Lemma 3.5 (Reduction). *Suppose that a group G acts on a set X, and consider two elements g and g' from G. If there is $h\in G$ such that $g'=h^{-1}gh$, then the mapping $x\mapsto hx$ defines a bijection from $\mathfrak{F}(g')$ onto $\mathfrak{F}(g)$. In particular, if X and G are finite and if g and g' are conjugate elements from G then $|\mathfrak{F}(g)|=|\mathfrak{F}(g')|$.*

Remark 3.6. Another simple remark from group theory is that if $G=A\times B$ is the direct product of two groups A and B, and if a and a' are conjugate elements from A, then (a,e_B) and (a',e_B), (with e_B denoting the neutral element of B), are also conjugate elements in G.

With Lemma 3.5 and Remark 3.6 at hand, the next corollary is immediate:

Corollary 3.7.

- **(a)** *For all $\varepsilon\in\{0,1\}$ and all $\ell\in\{0,1,\dots,n-1\}$ we have*
$$\mathfrak{f}^{(n)}(\tau_{12},\varepsilon,\ell)=\mathfrak{f}^{(n)}(\tau_{13},\varepsilon,\ell)=\mathfrak{f}^{(n)}(\tau_{23},\varepsilon,\ell). \tag{3.4}$$
- **(b)** *For all $\varepsilon\in\{0,1\}$ and all $\ell\in\{0,1,\dots,n-1\}$ we have*
$$\mathfrak{f}^{(n)}(c,\varepsilon,\ell)=\mathfrak{f}^{(n)}(c^2,\varepsilon,\ell) \tag{3.5}$$
- **(c)** *For all $\sigma\in\mathfrak{S}_3$ and all $\ell\in\{0,1,\dots,\lfloor(n-1)/2\rfloor\}$ we have*
$$\mathfrak{f}^{(n)}(\sigma,1,2\ell)=\mathfrak{f}^{(n)}(\sigma,1,0) \tag{3.6}$$
- **(d)** *For all $\sigma\in\mathfrak{S}_3$ and all $\ell\in\{0,1,\dots,\lfloor n/2-1\rfloor\}$ we have*
$$\mathfrak{f}^{(n)}(\sigma,1,2\ell+1)=\mathfrak{f}^{(n)}(\sigma,1,1) \tag{3.7}$$

Proof. Both **(a)** and **(b)** follow from the fact that all permutations of the same cycle structure are conjugate. On the other hand, since $rs^{2\ell}=s^{-\ell}rs^{\ell}$ and $rs^{2\ell+1}=s^{-\ell}(rs)s^{\ell}$ for all ℓ, both **(c)** and **(d)** follow from Corollary 3.7. □

The final result in this preliminary section is a simple formula concerning sums involving Euler's totient function φ (see [3, Chapter V, Section 5.5]), recall that $\varphi(n)$ is the number of integers in $\mathbb{N}_n$ coprime to n.

Lemma 3.8. *For every positive integer n we have*

$$\sum_{d|n}(-1)^d\varphi\left(\frac{n}{d}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is odd then all its divisors are odd and using [3, Theorem 63], we get

$$\sum_{d|n}(-1)^d\varphi\left(\frac{n}{d}\right) = -\sum_{d|n}\varphi\left(\frac{n}{d}\right) = -n.$$

Now, if $n = 2m$ for some positive integer m, then

$$\begin{aligned}\sum_{d|n}(-1)^d\varphi\left(\frac{n}{d}\right) &= \sum_{\substack{d|n \\ d \text{ is even}}}\varphi\left(\frac{n}{d}\right) - \sum_{\substack{d|n \\ d \text{ is odd}}}\varphi\left(\frac{n}{d}\right) \\ &= 2\sum_{\substack{d|n \\ d \text{ is even}}}\varphi\left(\frac{n}{d}\right) - \sum_{d|n}\varphi\left(\frac{n}{d}\right) \\ &= 2\sum_{d'|m}\varphi\left(\frac{m}{d'}\right) - \sum_{d|n}\varphi\left(\frac{n}{d}\right) \\ &= 2m - n = 0,\end{aligned}$$

where we used again [3, Theorem 63]. □

4. Counting Colorful Necklaces

In this section we consider $G = \mathfrak{S}_3 \times \langle s\rangle$. According to Corollary 3.7 we need to determine $\mathfrak{f}^{(n)}(\sigma, 0, \ell)$, for $\sigma \in \{id, \tau_{12}, c\}$ and $\ell \in \mathbb{Z}/n\mathbb{Z}$. The next proposition gives the answer.

Proposition 4.1.

(a) *If* $\gcd(\ell, n) = d$ *then* $\mathfrak{F}^{(n)}(id, 0, \ell) = \mathfrak{F}^{(n)}(id, 0, d) = \mathcal{A}_d$. *In particular,*

$$\mathfrak{f}^{(n)}(id, 0, \ell) = \alpha_{\gcd(\ell,n)}. \tag{4.1}$$

Thus, $\gcd(n, \ell) = 1$ *implies* $\mathfrak{F}^{(n)}(id, 0, \ell) = \emptyset$.

(b) *i. Suppose that* $3 \nmid n$, *then for all* $\ell \in \mathbb{Z}/n\mathbb{Z}$ *we have*

$$\mathfrak{f}^{(n)}(c, 0, \ell) = 0. \tag{4.2}$$

ii. Suppose that $n = 3m$ *for some positive integer* m, *then for all* $\ell \in \mathbb{Z}/n\mathbb{Z}$ *we have*

$$\mathfrak{f}^{(n)}(c, 0, \ell) = \begin{cases} 0 & \text{if } 3 \mid (\ell/d), \\ 2^d - (-1)^d & \text{if } 3 \nmid (\ell/d), \end{cases} \tag{4.3}$$

where $d = \gcd(m, \ell)$.

(c) *i. Suppose that $2 \nmid n$, then for all $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have*

$$\mathfrak{f}^{(n)}(\tau_{12}, 0, \ell) = 0. \tag{4.4}$$

ii. Suppose that $n = 2m$ for some positive integer m, then for all $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have

$$\mathfrak{f}^{(n)}(\tau_{12}, 0, \ell) = \begin{cases} 0 & \text{if } 2 \mid (\ell/d), \\ 2^d & \text{if } 2 \nmid (\ell/d), \end{cases} \tag{4.5}$$

where $d = \gcd(m, \ell)$.

Proof. **(a)** A sequence $f \in \mathfrak{F}^{(n)}((id, 0, \ell))$ satisfies $f \circ s^\ell = f$, so it belongs to $\mathcal{A}_\ell$. Thus, by Lemma 3.1, we have

$$\mathfrak{F}^{(n)}((id, 0, \ell)) \subset \mathcal{A}_n \cap \mathcal{A}_\ell = \mathcal{A}_d.$$

The converse inclusion: $\mathcal{A}_d \subset \mathfrak{F}^{(n)}((id, 0, \ell))$ is trivial, because both ℓ and n are multiples of d.

(b,c) *i.* Let σ be any permutation from $\mathfrak{S}_3$, and suppose that $\mathfrak{F}^{(n)}(\sigma, 0, \ell) \neq \emptyset$ so there is $f \in \mathfrak{F}^{(n)}(\sigma, 0, \ell)$. From

$$f \circ s^\ell = \sigma^{-1} \circ f, \tag{4.6}$$

we conclude by an easy induction that for all integers p we have

$$f \circ s^{p\ell} = \sigma^{-p} \circ f \tag{4.7}$$

- If $\sigma = c$ and $3 \nmid n$ we have $c^3 = id$, so (4.7) implies that $f \circ s^{3p\ell} = f$ for all integers p. But, because $3 \nmid n$ there is $r \in \{1, 2\}$ such that $n - r = 3p$ for some p. Consequently, $f \circ s^{(n-r)\ell} = f$, or equivalently $f = f \circ s^{r\ell} = c^{-r} \circ f$ because f is n-periodic. This is a contradiction because neither c nor c^2 has fixed points. Thus $\mathfrak{F}^{(n)}(c, 0, \ell) = \emptyset$. This proves **(b)** *i.*
- If $\sigma = \tau_{12}$ and n is odd, we have $\tau_{12}^2 = id$, so (4.7) implies that $f \circ s^{2p\ell} = f$ for all integers p. But, because $n = 2p + 1$ for some p we conclude that $f \circ s^{(n-1)\ell} = f$, or equivalently $f = f \circ s^\ell = \tau_{12} \circ f$. This is a contradiction because f takes two different values, and τ_{12} has only one fixed point. Thus $\mathfrak{F}^{(n)}(\tau_{12}, 0, \ell) = \emptyset$. This proves **(c)** *i.*

(b) *ii.* Assume that $\mathfrak{F}^{(n)}(c, 0, \ell) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(c, 0, \ell)$. From $c \circ f \circ s^\ell = f$ we conclude that $f \circ s^{3\ell} = f$. Thus $f \in \mathcal{A}_{3j} \cap \mathcal{A}_{3m} = \mathcal{A}_{3d}$, with $d = \gcd(m, \ell)$. Further, if $\ell/d = 3q + r$ with $r \in \{0, 1, 2\}$ then $\ell = 3dq + dr$ and consequently

$$f \circ s^\ell = f \circ s^{rd} = c^2 \circ f. \tag{4.8}$$

- If $r = 0$ then (4.8) implies $f = c^2 \circ f$ which is impossible since f is not constant. Thus $\mathfrak{F}^{(n)}(c, 0, \ell) = \emptyset$ in this case.
- If $r = 1$ then (4.8) shows that $f \in \mathfrak{F}^{(3d)}(c, 0, d)$. Conversely, it is easy to check that any $f \in \mathfrak{F}^{(3d)}(c, 0, d)$ belongs to $\mathfrak{F}^{(n)}(c, 0, \ell)$. Thus, we have shown that

$$\mathfrak{F}^{(n)}(c, 0, \ell) = \mathfrak{F}^{(3d)}(c, 0, d). \tag{4.9}$$

Now, when f belongs to $\mathfrak{F}^{(3d)}(c,0,d)$ it is completely determined by its restriction to $\mathbb{N}_d$, and the mapping (see Figure 1):

$$\Phi : \mathcal{A}_{d+1}^{1\cdot\cdot3} \cup \mathcal{A}_{d+1}^{2\cdot\cdot1} \cup \mathcal{A}_{d+1}^{3\cdot\cdot2} \to \mathfrak{F}^{(3d)}(c,0,d), f \mapsto \tilde{f},$$

where $\tilde{f}$ is the unique sequence from $\mathfrak{F}^{(3d)}(c,0,d)$ which coincides with f on $\mathbb{N}_d$, is a bijection.

$$\underbrace{(x_1,\dots,x_d)}_{f_{|\mathbb{N}_d}} \mapsto \underbrace{(x_1,\dots,x_d,c^2(x_1),\dots,c^2(x_d),c(x_1),\dots,c(x_d))}_{\tilde{f}_{|\mathbb{N}_{3d}}}$$

FIGURE 1. The bijection $\Phi : \mathcal{A}_{d+1}^{1\cdot\cdot3} \cup \mathcal{A}_{d+1}^{2\cdot\cdot1} \cup \mathcal{A}_{d+1}^{3\cdot\cdot2} \to \mathfrak{F}^{(3d)}(c,0,d)$

We conclude, according to Corollary 3.4 that

$$\mathfrak{f}^{(3d)}(c,0,d) = \left|\mathcal{A}_{d+1}^{1\cdot\cdot3}\right| + \left|\mathcal{A}_{d+1}^{2\cdot\cdot1}\right| + \left|\mathcal{A}_{d+1}^{3\cdot\cdot2}\right| = \frac{\alpha_{d+1}}{2}.$$

- If $r = 2$, then a similar argument to the previous one (with c replaced by c^2), yields the desired conclusion. This completes the proof of **(b)** *ii*.

(c) *ii*. For simplicity we write τ for τ_{12}. Suppose that $\mathfrak{F}^{(n)}(\tau,0,\ell) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(\tau,0,\ell)$. We have

$$f \circ s^\ell = \tau \circ f. \tag{4.10}$$

Hence $f \circ s^{2\ell} = f$ and consequently $f \in \mathcal{A}_{2\ell}$, this implies that $f \in \mathcal{A}_{2m} \cap \mathcal{A}_{2\ell} = \mathcal{A}_{2d}$ where $d = \gcd(m,\ell)$. Now write $\ell/d = 2q + r$ with $r \in \{0,1\}$, then $\ell = 2dq + dr$ and consequently

$$f \circ s^\ell = f \circ s^{rd} = \tau \circ f. \tag{4.11}$$

- If $r = 0$, then (4.11) implies $f = \tau \circ f$ which is impossible since f is not constant. Thus $\mathfrak{F}^{(n)}(\tau,0,\ell) = \emptyset$ in this case.
- If $r = 1$, then (4.11) implies $f \circ s^d = \tau \circ f$, that is $f \in \mathfrak{F}^{(2d)}(\tau,0,d)$. Conversely, it is easy to check that any $f \in \mathfrak{F}^{(2d)}(\tau,0,d)$ belongs to $\mathfrak{F}^{(n)}(\tau,0,\ell)$. Thus, we have shown that in this case

$$\mathfrak{F}^{(n)}(\tau,0,\ell) = \mathfrak{F}^{(2d)}(\tau,0,d). \tag{4.12}$$

Clearly, if $d = 1$ then $\mathfrak{F}^{(2d)}(\tau,0,d)$ consists exactly of two elements: namely f_1, defined by $f_1(1) = 1, f_1(2) = 2$, and $f_2 = \tau \circ f_1$. So,

$$\mathfrak{f}^{(2)}(\tau,0,1)) = 2.$$

Now suppose that $d > 1$. Any $f \in \mathfrak{F}^{(2d)}((\tau,d))$ is completely determined its restriction to $\mathbb{N}_d$ (note that $f(d)$ should be different from $\tau(f(1))$ and $f(d-1)$,) so considering the different possibilities for $f(1)$ we see that the mapping Ψ, (see Figure 2):

$$\Psi : \mathcal{A}_{d+1}^{1\cdot\cdot2} \cup \mathcal{A}_{d+1}^{2\cdot\cdot1} \cup \mathcal{A}_{d}^{3\cdot\cdot1} \cup \mathcal{A}_{d}^{3\cdot\cdot2} \to \mathfrak{F}^{(2d)}(\tau,0,d), f \mapsto \hat{f},$$

where $\hat{f}$ is the unique sequence from $\mathfrak{F}^{(2d)}(\tau, 0, d)$ that coincides with f on $\mathbb{N}_d$, is a bijection.

$$\boxed{\underbrace{(x_1, \ldots, x_d)}_{f_{|\mathbb{N}_d}} \mapsto \underbrace{(x_1, \ldots, x_d, \tau(x_1), \ldots, \tau(x_d))}_{\hat{f}_{|\mathbb{N}_{2d}}}}$$

FIGURE 2. The bijection $\Psi : \mathcal{A}_{d+1}^{1\cdot\cdot 2} \cup \mathcal{A}_{d+1}^{2\cdot\cdot 1} \cup \mathcal{A}_{d}^{3\cdot\cdot 1} \cup \mathcal{A}_{d}^{3\cdot\cdot 2} \to \mathfrak{F}^{(2d)}(\tau, 0, d)$

Thus, according to Corollary 3.4, we have

$$\mathfrak{f}^{(2d)}(\tau, 0, d) = \frac{\alpha_{d+1} + \alpha_d}{3} = 2^d.$$

This concludes the proof of **(c)** *ii*, in view of (4.12). □

The final step is to put all the pieces together to get the expression of K_n in terms of n using Burnside's Lemma.

Theorem 4.2. *The number of non-equivalent colorful n-bead necklaces with three colors is given by*

$$K(n) = \frac{1}{6n} \sum_{d|n} (1 + \mathbb{I}_{2\mathbb{Z}\setminus 3\mathbb{Z}}(d)) \gcd(d, 6)\, \varphi(d) 2^{n/d} - \frac{1}{3^{1+\nu_3(n)}} \mathbb{I}_{\mathbb{Z}\setminus 2\mathbb{Z}}(n) \tag{4.13}$$

$$= \left\lfloor \frac{1}{6n} \sum_{d|n} (1 + \mathbb{I}_{2\mathbb{Z}\setminus 3\mathbb{Z}}(d)) \gcd(d, 6)\, \varphi(d) 2^{n/d} \right\rfloor \tag{4.14}$$

where $\mathbb{I}_X$ is the indicator function of the set X, (i.e. $\mathbb{I}_X(k) = 1$ if $k \in X$ and $\mathbb{I}_X(k) = 0$ if $k \notin X$,) and $3^{\nu_3(n)}$ is the largest power of 3 dividing n.

Proof. According to Corollary 3.7 and Burnside's Lemma 2.2 we have

$$K(n) = \frac{A_n + 3B_n + 2C_n}{6n} \tag{4.15}$$

with

$$A_n = \sum_{\ell=0}^{n-1} \mathfrak{f}^{(n)}(id, 0, \ell), \tag{4.16}$$

$$B_n = \sum_{\ell=0}^{n-1} \mathfrak{f}^{(n)}(\tau_{12}, 0, \ell), \tag{4.17}$$

$$C_n = \sum_{\ell=0}^{n-1} \mathfrak{f}^{(n)}(c, 0, \ell). \tag{4.18}$$

Using part **(a)** of Proposition 4.1 we have

$$\begin{aligned} A_n &= \sum_{\ell=0}^{n-1} \alpha_{\gcd(\ell,n)} = \sum_{d|n} \alpha_d \left|\{\ell : 0 \le \ell < n, \gcd(\ell,n) = d\}\right| \\ &= \sum_{d|n} \alpha_d \left|\left\{\ell' : 0 \le \ell' < \frac{n}{d}, \gcd\left(\ell', \frac{n}{d}\right) = 1\right\}\right| = \sum_{d|n} \varphi\left(\frac{n}{d}\right) \alpha_d. \end{aligned}$$

Thus, using the expression of α_n from Proposition 3.2, we get

$$A_n = \sum_{d|n} \left(2^d + 2(-1)^d\right) \varphi\left(\frac{n}{d}\right). \tag{4.19}$$

Similarly, according part **(c)** of Proposition 4.1 we know that $B_n = 0$ if n is odd, while we have the following when $n = 2m$:

$$\begin{aligned} B_n &= \sum_{\ell=0}^{n-1} 2^{\gcd(\ell,m)} \mathbb{I}_{\mathbb{Z}\setminus 2\mathbb{Z}}\left(\frac{\ell}{\gcd(\ell,m)}\right) \\ &= \sum_{d|m} 2^d \left|\{0 \le \ell < 2m : \gcd(\ell,m) = d, \text{and } \ell/d \text{ is odd}\}\right| \\ &= \sum_{d|m} 2^d \left|\left\{0 \le \ell' < 2\frac{m}{d} : \gcd(\ell', \frac{m}{d}) = 1, \text{and } \ell' \text{ is odd}\right\}\right| \\ &= \sum_{d|m} 2^d \left|\left\{0 \le \ell' < 2\frac{m}{d} : \gcd(\ell', 2\frac{m}{d}) = 1\right\}\right| = \sum_{d|m} 2^d \varphi\left(\frac{n}{d}\right). \end{aligned}$$

Finally we get

$$B_n = \mathbb{I}_{2\mathbb{Z}}(n) \sum_{d|(n/2)} 2^d \varphi\left(\frac{n}{d}\right) \tag{4.20}$$

Now, we come to C_n. According to part **(b)** of Proposition 4.1 we know that $C_n = 0$ if n is not a multiple of 3 while if $n = 3m$ we have

$$\begin{aligned} C_n &= \sum_{\ell=0}^{n-1} \left(2^{\gcd(\ell,m)} - (-1)^{\gcd(\ell,m)}\right) \mathbb{I}_{\mathbb{Z}\setminus 3\mathbb{Z}}\left(\frac{\ell}{\gcd(\ell,m)}\right) \\ &= \sum_{d|m} \left(2^d - (-1)^d\right) \left|\{0 \le \ell < 3m : \gcd(\ell,m) = d, \text{and } 3 \nmid \ell/d\}\right| \\ &= \sum_{d|m} \left(2^d - (-1)^d\right) \left|\left\{0 \le \ell' < 3\frac{m}{d} : \gcd(\ell', \frac{m}{d}) = 1, \text{and } 3 \nmid \ell'\right\}\right| \\ &= \sum_{d|m} \left(2^d - (-1)^d\right) \left|\left\{0 \le \ell' < 3\frac{m}{d} : \gcd(\ell', 3\frac{m}{d}) = 1\right\}\right| \\ &= \sum_{d|m} \left(2^d - (-1)^d\right) \varphi\left(\frac{n}{d}\right). \end{aligned}$$

Thus,

$$C_n = \mathbb{I}_{3\mathbb{Z}}(n) \sum_{d|(n/3)} \left(2^d - (-1)^d\right) \varphi\left(\frac{n}{d}\right) \tag{4.21}$$

Replacing (4.19), (4.20) and (4.21) in (4.15) we get

$$K(n) = b_n + \varepsilon_n, \tag{4.22}$$

with

$$b_n = \frac{1}{6n}\left(\sum_{d|n} 2^d \varphi\left(\frac{n}{d}\right) + 3\mathbb{I}_{2\mathbb{Z}}(n) \sum_{d|(n/2)} 2^d \varphi\left(\frac{n}{d}\right) + 2\mathbb{I}_{3\mathbb{Z}}(n) \sum_{d|(n/3)} 2^d \varphi\left(\frac{n}{d}\right)\right) \tag{4.23}$$

and

$$\varepsilon_n = \frac{1}{3n}\left(\sum_{d|n} (-1)^d \varphi\left(\frac{n}{d}\right) - \mathbb{I}_{3\mathbb{Z}}(n) \sum_{d|(n/3)} (-1)^d \varphi\left(\frac{n}{d}\right)\right) \tag{4.24}$$

In order to reduce a little bit the expression of $K(n)n$ we use Lemma 3.8. Indeed, Suppose that $n = 3^\nu m$ where $\nu = \nu_3(n)$ is the exponent of 3 in the prime factorization of n, thus $3 \nmid m$. Clearly if $\nu = 0$ then using Lemma 3.8 we get

$$\varepsilon_n = \frac{1}{3n} \sum_{d|n} (-1)^d \varphi\left(\frac{n}{d}\right) = -\frac{1}{3}\mathbb{I}_{\mathbb{Z}\setminus 2\mathbb{Z}}(n) \tag{4.25}$$

Now if $\nu > 0$ then

$$\begin{aligned}
\varepsilon_n &= \frac{1}{3n}\left(\sum_{d|(3^\nu m)} (-1)^d \varphi\left(\frac{n}{d}\right) - \sum_{d|(3^{\nu-1} m)} (-1)^d \varphi\left(\frac{n}{d}\right)\right) \\
&= \frac{1}{3n} \sum_{d|(3^\nu m),\, d\nmid(3^{\nu-1} m)} (-1)^d \varphi\left(\frac{n}{d}\right) \\
&= \frac{1}{3n} \sum_{d=3^\nu q,\, q|m} (-1)^d \varphi\left(\frac{n}{d}\right) \\
&= \frac{1}{3n} \sum_{q|m} (-1)^{3^\nu q} \varphi\left(\frac{m}{q}\right) = \frac{1}{3n} \sum_{q|m} (-1)^q \varphi\left(\frac{m}{q}\right) \\
&= -\frac{m}{3n}\mathbb{I}_{\mathbb{Z}\setminus 2\mathbb{Z}}(m) = -\frac{1}{3^{1+\nu}}\mathbb{I}_{\mathbb{Z}\setminus 2\mathbb{Z}}(m).
\end{aligned}$$

Finally, noting that $n = m \mod 2$, we obtain the following formula for ε_n which is also valid when $\nu = 0$ according to (4.25):

$$\varepsilon_n = -\frac{1}{3^{1+\nu_3(n)}}\mathbb{I}_{\mathbb{Z}\setminus 2\mathbb{Z}}(n). \tag{4.26}$$

Now note that b_n can be written ans follows

$$b_n = \frac{1}{6n} \sum_{d|n} 2^d \lambda(n, d) \varphi\left(\frac{n}{d}\right) \tag{4.27}$$

with

$$\lambda(n, d) = 1 + 3J(n, d) + 2K(n, d)$$

where

$$J(n,d)=\begin{cases}1 & \text{if } 2|n \text{ and } d|(n/2),\\ 0 & \text{otherwise.}\end{cases}$$

and

$$K(n,d)=\begin{cases}1 & \text{if } 3|n \text{ and } d|(n/3),\\ 0 & \text{otherwise.}\end{cases}$$

equivalently

$$J(n,d)=\mathbb{I}_{2\mathbb{Z}}\left(\frac{n}{d}\right),\quad \text{and}\quad K(n,d)=\mathbb{I}_{3\mathbb{Z}}\left(\frac{n}{d}\right).$$

Thus

$$\lambda(n,d)=1+3\mathbb{I}_{2\mathbb{Z}}\left(\frac{n}{d}\right)+2\mathbb{I}_{3\mathbb{Z}}\left(\frac{n}{d}\right) \tag{4.28}$$

So, we may write b_n in the following form

$$b_n=\frac{1}{6n}\sum_{d|n}2^d\chi\left(\frac{n}{d}\right)\varphi\left(\frac{n}{d}\right) \tag{4.29}$$

with $\chi:\mathbb{Z}\to\mathbb{N}_6$ defined by

$$\chi(k)=(1+3\mathbb{I}_{2\mathbb{Z}}(k)+2\mathbb{I}_{3\mathbb{Z}}(k))=\begin{cases}1 & \text{if } \gcd(k,6)=1,\\ 4 & \text{if } \gcd(k,6)=2,\\ 3 & \text{if } \gcd(k,6)=3,\\ 6 & \text{if } \gcd(k,6)=6.\end{cases} \tag{4.30}$$

This can also be written in the form $\chi(k)=(1+\mathbb{I}_{2\mathbb{Z}\setminus 3\mathbb{Z}}(k))\gcd(k,6)$, and the announced expression (4.13) for $K(n)$ is obtained. Finally, the formula $K(n)=\lfloor b_n\rfloor$, follows from the fact that. $-\frac{1}{3}\le\varepsilon_n\le 0$. □

We conclude our discussion of the case of necklaces by noting that there are some simple cases where the formula for $K(n)$ is particularly appealing, for example, if $n=p>3$ is prime, then

$$K(p)=\frac{2^p-2}{6p},$$

and if $\gcd(n,6)=1$, then

$$K(n)=\left\lfloor\frac{1}{6n}\sum_{d|n}\varphi(d)2^{n/d}\right\rfloor=\frac{1}{6n}\sum_{d|n}\varphi(d)2^{n/d}-\frac{1}{3}.$$

Remark 4.3. If 6 and n are coprime, then $K(n)$ is related to the number $N(n,2)$ of n-bead necklaces of two colors (1.1) by the formula

$$K(n)=\lfloor N(n,2)/6\rfloor=(N(n,2)-2)/6.$$

Remark 4.4. An equivalent formula for $K(n)$ that does not use the indicator function of the set $2\mathbb{Z} \setminus 3\mathbb{Z}$ is the following

$$K(n) = \left\lfloor \frac{1}{6n} \sum_{d|n} \left(1 + \frac{4}{3}\cos^2\left(\frac{d\pi}{2}\right)\sin^2\left(\frac{d\pi}{3}\right)\right) \gcd(d,6)\,\varphi(d) 2^{n/d} \right\rfloor .$$

Table 1 lists the first 40 terms of the sequence $(K(n))_{n\geq 1}$.

n	$K(n)$	n	$K(n)$	n	$K(n)$	n	$K(n)$
1	0	11	31	21	16651	31	11545611
2	1	12	64	22	31838	32	22371000
3	1	13	105	23	60787	33	43383571
4	2	14	202	24	116640	34	84217616
5	1	15	367	25	223697	35	163617805
6	4	16	696	26	430396	36	318150720
7	3	17	1285	27	828525	37	619094385
8	8	18	2452	28	1598228	38	1205614054
9	11	19	4599	29	3085465	39	2349384031
10	20	20	8776	30	5966000	40	4581315968

TABLE 1. List of $K(1), \dots, K(40)$, which counts colorful necklaces.

5. Counting Colorful Bracelets

As we explained before, bracelets are turnover necklaces. It is the action of the group $G' = \mathfrak{S}_3 \times \langle r, s\rangle$ on the set of n-periodic colorful sequences $\mathcal{A}_n$ that is considered.

We are interested in the number of orbits $\mathcal{A}_n/G'$ denoted by $K'(n)$. Again Burnside's Lemma comes to our rescue. We need to determine the numbers $\mathfrak{f}^{(n)}(\sigma, \varepsilon, \ell)$ with $\sigma \in \mathfrak{S}_3$, $\varepsilon \in \{0, 1\}$ and $\ell \in \mathbb{Z}/n\mathbb{Z}$, but we have already done this in the case $\varepsilon = 0$ in the previous section.

Further, based Corollary 3.7, we only need to determine $\mathfrak{f}^{(n)}(\sigma, 1, 0)$ and $\mathfrak{f}^{(n)}(\sigma, 1, 1)$ for σ in $\{id, \tau_{12}, c\}$. This is the object of the next proposition.

Proposition 5.1.

(a) *i. If n is odd then $\mathfrak{f}^{(n)}(id, 1, 0) = 0$, otherwise $\mathfrak{f}^{(n)}(id, 1, 0) = 3 \times 2^{n/2}$.*
ii. $\mathfrak{f}^{(n)}(id, 1, 1) = 0$.

(b) *i. $\mathfrak{f}^{(n)}(\tau_{12}, 1, 0) = \alpha_{\lfloor (n+1)/2 \rfloor}/3$.*
ii. $\mathfrak{f}^{(n)}(\tau_{12}, 1, 1) = \alpha_{\lfloor n/2+1 \rfloor}/3$.

(c) *$\mathfrak{f}^{(n)}(c, 1, 0) = \mathfrak{f}^{(n)}(c, 1, 1) = 0$.*

Proof. **(a)** Suppose that $\mathfrak{F}^{(n)}(id, 1, 0) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(id, 1, 0)$. Write $n = 2m + t$ with $t \in \{0, 1\}$. Because $f(k) = f(-k) = f(n - k)$ for every k, we conclude by considering $k = m$ that $f(m + t) = f(m)$. But

$f(m) \neq f(m+1)$ so we must have $t = 0$ and $n = 2m$. Now, from the fact that $f(2m-k) = f(k)$ for every k we conclude that

$$\underbrace{\big(f(0), \ldots, f(m), f(m+1), \ldots, f(2m-1)\big)}_{\text{a period of } n=2m} = \big(f(0), \ldots, f(m), f(m-1), \ldots, f(1)\big).$$

So, the mapping

$$f \mapsto (f(0), f(1), \ldots, f(m))$$

defines a bijection between $\mathfrak{F}^{(n)}(id, 1, 0)$ and the set

$$\{(x_0, \ldots, x_m) \in \mathbb{N}_3 : x_{i+1} \neq x_i, i = 0, \ldots, m-1\}$$

Now, x_0 may take any one of three possible values and each other x_i has two possible values. So, the cardinality of this set is 3×2^m. Thus **(a)** *i.* is proved.

Now suppose that $\mathfrak{F}^{(n)}(id, 1, 1) \neq \emptyset$ and consider f from $\mathfrak{F}^{(n)}(id, 1, 1)$. We have $f(-k-1) = f(k)$ for every k, in particular, for $k = 0$ we get $f(-1) = f(0)$ which is absurd, and **(a)** *ii.* follows.

(b) *i.* we write τ for τ_{12}. Suppose that $\mathfrak{F}^{(n)}(\tau, 1, 0) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(\tau, 1, 0)$. We have

$$\forall\, k \in \mathbb{Z}, \quad f(-k) = \tau(f(k)).$$

Taking $k = 0$ we get $f(0) = \tau(f(0))$, and this implies that $f(0) = 3$.

- If $n = 2m$ then $f(m) = f(m-n) = f(-m) = \tau(f(m))$ and consequently $f(m) = 3$. The restriction of f to the period $\{-m+1, \ldots, m-1, m\}$ has the form

  $$(\tau(f(m-1)), \ldots, \tau(f(1)), 3, f(1), \ldots, f(m-1), 3).$$

  So, f is completely determined by the knowledge of $(f(1), \ldots, f(m-1))$ and consequently there is a bijection between $\mathfrak{F}^{(2m)}(\tau, 1, 0)$ and $\mathcal{A}_m^{3\cdot\cdot 1} \cup \mathcal{A}_m^{3\cdot\cdot 2}$. Thus, by Corollary 3.4, we have

  $$\mathfrak{f}^{(n)}(\tau, 1, 0) = \frac{\alpha_m}{3} = \frac{1}{3}\alpha_{\lfloor (n+1)/2 \rfloor}.$$

- If $n = 2m+1$, then $f(m+1) = f(m+1-n) = f(-m) = \tau(f(m))$ and consequently $f(m) \neq 3$. The restriction of f to the period $\{-m, \ldots, m\}$ takes the form

  $$(\tau(f(m)), \tau(f(m-1)), \ldots, \tau(f(1)), 3, f(1), \ldots, f(m))$$

  So, f is completely determined by the knowledge of $(f(1), \ldots, f(m))$ and consequently there is a bijection between $\mathfrak{F}^{(2m+1)}(\tau, 1, 0)$ and $\mathcal{A}_{m+1}^{3\cdot\cdot 1} \cup \mathcal{A}_{m+1}^{3\cdot\cdot 2}$. Thus

  $$\mathfrak{f}^{(n)}(\tau, 1, 0) = \frac{\alpha_{m+1}}{3} = \frac{1}{3}\alpha_{\lfloor (n+1)/2 \rfloor}.$$

(b) *ii.* Now suppose that $\mathfrak{F}^{(n)}(\tau, 1, 1) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(\tau, 1, 1)$. We have

$$\forall\, k \in \mathbb{Z}, \quad f(-k-1) = \tau(f(k))$$

Taking $k = 0$ we get $f(-1) = \tau(f(0))$, but $f(-1) \neq f(0)$ thus $f(0) \in \{1, 2\}$.

- If $n = 2m$, then $f(m-1) = f(m-1-n) = f(-m-1) = \tau(f(m))$ but $f(m-1) \neq f(m)$ thus $f(m-1) \in \{1,2\}$. The restriction of f to the period $\{-m, \dots, m-1\}$ takes the form

$$(\tau(f(m-1)), \dots, \tau(f(0)), f(0), f(1), \dots, f(m-1)).$$

So, f is completely determined by the knowledge of $(f(0), \dots, f(m-1))$. We can partition the set $\mathfrak{F}^{(2m)}(\tau,1,1)$ according to the values taken by $(f(0), f(m-1))$, and we have obvious bijective mappings:

$$\begin{aligned}
&\mathfrak{F}^{(2m)}(\tau,1,1) \cap \{f : f(0)=1, f(m-1)=1\} \to \mathcal{A}_{m-1}^{1\cdot\cdot 2} \cup \mathcal{A}_{m-1}^{1\cdot\cdot 3}\\
&\mathfrak{F}^{(2m)}(\tau,1,1) \cap \{f : f(0)=2, f(m-1)=2\} \to \mathcal{A}_{m-1}^{2\cdot\cdot 1} \cup \mathcal{A}_{m-1}^{2\cdot\cdot 3}\\
&\mathfrak{F}^{(2m)}(\tau,1,1) \cap \{f : f(0)=1, f(m-1)=2\} \to \mathcal{A}_{m}^{1\cdot\cdot 2}\\
&\mathfrak{F}^{(2m)}(\tau,1,1) \cap \{f : f(0)=2, f(m-1)=1\} \to \mathcal{A}_{m}^{2\cdot\cdot 1}
\end{aligned}$$

Thus

$$\mathfrak{f}^{(n)}(\tau,1,1) = \frac{1}{6}(4\alpha_{m-1} + 2\alpha_m) = \frac{2}{3}\left(2^m - (-1)^m\right) = \frac{1}{3}\alpha_{\lfloor n/2+1 \rfloor}.$$

- If $n = 2m+1$ then $f(m) = f(m-n) = f(-m-1) = \tau(f(m))$ and consequently $f(m) = 3$. The restriction of f to the set $\{-m, \dots, m\}$ takes the form

$$(\tau(f(m-1)), \dots, \tau(f(0)), f(0), f(1), \dots, f(m-1), 3).$$

So, f is completely determined by the knowledge of $(f(0), \dots, f(m-1))$ and there is an obvious bijective mapping between $\mathfrak{F}^{(2m+1)}(\tau,1,1)$ and $\mathcal{A}_{m+1}^{1\cdot\cdot 3} \cup \mathcal{A}_{m+1}^{2\cdot\cdot 3}$. Thus

$$\mathfrak{f}^{(n)}(\tau,1,1) = \frac{\alpha_{m+1}}{3} = \frac{1}{3}\alpha_{\lfloor n/2+1 \rfloor}.$$

This concludes the proof of part **(b)**.

(c) First, suppose that $\mathfrak{F}^{(n)}(c,1,0) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(c,1,0)$. We have $f(k) = c(f(-k))$ for all $k \in \mathbb{Z}$. In particular, $f(0) = c(f(0))$ which is absurd because c has no fixed points.

Next suppose that $\mathfrak{F}^{(n)}(c,1,1) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(c,1,1)$. We have $f(k) = c(f(-k-1))$ for all $k \in \mathbb{Z}$.

- If $n = 2m+1$ then

$$f(m) = f(m-n) = f(-m-1) = c^{-1}(f(m)),$$

which is absurd because c has no fixed points.

- If $n = 2m$ then

$$\begin{aligned}
f(m) &= f(m-n) = f(-m) = c^{-1}(f(m-1))\\
&= c^{-1}(f(m-1-n)) = c^{-1}(f(-m-1))\\
&= c^{-2}(f(m))
\end{aligned}$$

which is also absurd because c^2 has no fixed points.

This achieves the proof of the proposition. □

Finally we arrive to the main theorem of this section.

Theorem 5.2. *The number of non-equivalent colorful n-bead Bracelets with three colors is given by*

$$K'(n) = \frac{K(n) + R(n)}{2} \tag{5.1}$$

with

$$R(n) = \begin{cases} 2^{n/2-1} & \text{if } n \text{ is even,} \\ \frac{1}{3}(2^{(n-1)/2} - (-1)^{(n-1)/2}) & \text{if } n \text{ is odd.} \end{cases} \tag{5.2}$$

where $K(n)$ is given by Theorem 4.2.

Proof. We only need to put things together. We know that

$$K'(n) = \frac{1}{12n}\left(\sum_{(\sigma,j)\in\mathfrak{S}_3\times\mathbb{Z}/n\mathbb{Z}} \mathfrak{f}^{(n)}(\sigma, 0, j) + \sum_{(\sigma,j)\in\mathfrak{S}_3\times\mathbb{Z}/n\mathbb{Z}} \mathfrak{f}^{(n)}(\sigma, 1, j)\right).$$

Thus $K'(n) = (K(n) + R(n))/2$ with

$$\begin{aligned} R(n) &= \frac{1}{6n}\sum_{(\sigma,j)\in\mathfrak{S}_3\times\mathbb{Z}/n\mathbb{Z}} \mathfrak{f}^{(n)}(\sigma, 1, j) \\ &= \frac{1}{6n}\left(\sum_{\substack{\sigma\in\mathfrak{S}_3 \\ 0\le 2j\le n-1}} \mathfrak{f}^{(n)}(\sigma, 1, 2j) + \sum_{\substack{\sigma\in\mathfrak{S}_3 \\ 0\le 2j+1\le n-1}} \mathfrak{f}^{(n)}(\sigma, 1, 2j+1)\right) \\ &= \frac{1}{6n}\left(\sum_{\substack{\sigma\in\mathfrak{S}_3 \\ 0\le 2j\le n-1}} \mathfrak{f}^{(n)}(\sigma, 1, 0) + \sum_{\substack{\sigma\in\mathfrak{S}_3 \\ 0\le 2j+1\le n-1}} \mathfrak{f}^{(n)}(\sigma, 1, 1)\right) \\ &= \frac{1}{6n}\left(\left\lfloor\frac{n+1}{2}\right\rfloor \sum_{\sigma\in\mathfrak{S}_3} \mathfrak{f}^{(n)}(\sigma, 1, 0) + \left\lfloor\frac{n}{2}\right\rfloor \sum_{\sigma\in\mathfrak{S}_3} \mathfrak{f}^{(n)}(\sigma, 1, 1)\right) \end{aligned}$$

where we used Corollary 3.7. Now using Proposition 5.1 we get

$$\begin{aligned} \sum_{\sigma\in\mathfrak{S}_3} \mathfrak{f}^{(n)}(\sigma, 1, 0) &= \mathfrak{f}^{(n)}(id, 1, 0) + 3\mathfrak{f}^{(n)}(\tau_{12}, 1, 0) \\ &= \begin{cases} 3\times 2^m + \alpha_m & \text{if } n = 2m, \\ \alpha_{m+1} & \text{if } n = 2m+1, \end{cases} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \sum_{\sigma\in\mathfrak{S}_3} \mathfrak{f}^{(n)}(\sigma, 1, 1) &= 3\mathfrak{f}^{(n)}(\tau_{1,2}, 1, 1) \\ &= \begin{cases} \alpha_{m+1} & \text{if } n = 2m, \\ \alpha_{m+1} & \text{if } n = 2m+1. \end{cases} \end{aligned} \tag{5.4}$$

Replacing in the expression of $R(n)$ we obtain

$$R(n) = \begin{cases} 2^{m-1} & \text{if } n = 2m, \\ \alpha_{m+1}/6 & \text{if } n = 2m+1. \end{cases}$$

and the announced result follows. □

Table 2 lists the first 40 terms of the sequence $(K'(n))_n$.

n	$K'(n)$	n	$K'(n)$	n	$K'(n)$	n	$K'(n)$
1	0	11	21	21	8496	31	5778267
2	1	12	48	22	16431	32	11201884
3	1	13	63	23	30735	33	21702708
4	2	14	133	24	59344	34	42141576
5	1	15	205	25	112531	35	81830748
6	4	16	412	26	217246	36	159140896
7	3	17	685	27	415628	37	309590883
8	8	18	1354	28	803210	38	602938099
9	8	19	2385	29	1545463	39	1174779397
10	18	20	4644	30	2991192	40	2290920128

TABLE 2. List of $K'(1), \ldots, K'(40)$, which counts colorful bracelets.

Remark 5.3. Although $K(n) \leq K'(n)$ for all $n \geq 1$, a surprising fact about $(K(n))_{n\geq 1}$ and $(K'(n))_{n\geq 1}$ is that they coincide for the first 8 values!.

Remark 5.4. The equality $2K'(n) = K(n) + R(n)$ and the easy-to-prove fact that $R(n) = n \mod 2$ for $n \geq 3$, allow us to find the parity pattern of the $K(n)$'s. The fact that $K(n) = n \mod 2$ for $n \geq 3$ seems difficult to prove directly.

6. Related Combinatorial Sequences

Colorful necklaces or bracelets with n beads and two colors are easy to determine. There are none when n is odd and just one equivalence class when n is even. Thus the sequences $(K^*(n))_{n\geq 1}$ and $(K'^*(n))_{n\geq 1}$ defined by

$$K^*(n) = K(n) - \frac{1+(-1)^n}{2}, \quad \text{and} \quad K'^*(n) = K'(n) - \frac{1+(-1)^n}{2} \tag{6.1}$$

represent the number of non-equivalent colorful necklaces in n beads with exactly 3 colors and the number of non-equivalent colorful bracelets in n beads with exactly 3 colors, respectively. Both sequences $(K^*(n))_{n\geq 1}$, and $(K'^*(n))_{n\geq 1}$ are currently not recognized by the OEIS.

Further, if we are interested in periodic colorful sequences of *exact* period n in at most 3 colors then the number $\widetilde{K}(n)$ of non-equivalent such sequences assuming that reversing is not allowed is given by

$$\widetilde{K}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) K(d)$$

where μ is the well known Moebius function. Indeed, this follows from the classical result [6, Theorem 1.5], because clearly $K(n) = \sum_{d|n} \widetilde{K}(d)$.

OEIS recognizes $(\widetilde{K}(n))_{n\geq 1}$ as the "Number of Zn S polytypes that repeat after n layers" A011957.

Similarly, if we are interested in periodic colorful sequences of *exact* period n in at most 3 colors then the number $\widetilde{K}'(n)$ of non-equivalent such sequences assuming that reversing is allowed is given by

$$\widetilde{K}'(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) K'(d).$$

OEIS recognizes $(\widetilde{K}'(n))_{n\geq 1}$ as the "Number of Barlow packings that repeat after exactly n layers" A011768.

7. Future Research

This paper has counted non-equivalent colorful necklaces and non-equivalent colorful bracelets in n beads with 3 colors. An open problem is to count non-equivalent colorful necklaces and colorful bracelets in n beads with $c \geq 4$ colors.

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