ALGORITHMIC COUNTING OF NONEQUIVALENT COMPACT HUFFMAN CODES

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ABSTRACT. It is known that the following five counting problems lead to the same integer sequence $f_t(n)$:
(1) the number of nonequivalent compact Huffman codes of length n over an alphabet of t letters,
(2) the number of "nonequivalent" canonical rooted t-ary trees (level-greedy trees) with n leaves,
(3) the number of "proper" words,
(4) the number of bounded degree sequences, and
(5) the number of ways of writing
$$1 = \frac{1}{t^{x_1}} + \cdots + \frac{1}{t^{x_n}}$$
with integers $0 \le x_1 \le x_2 \le \cdots \le x_n$.
In this work, we show that one can compute this sequence for **all** $n < N$ with essentially one power series division. In total we need at most $N^{1+\varepsilon}$ additions and multiplications of integers of cN bits, $c < 1$, or $N^{2+\varepsilon}$ bit operations, respectively. This improves an earlier bound by Even and Lempel who needed $O(N^3)$ operations in the integer ring or $O(N^4)$ bit operations, respectively.

1. INTRODUCTION

Codes, unit fractions and more. Elsholtz, Heuberger and Prodinger [8] studied the number $f_t(r)$ of solutions of the equation

$$\sum_{i=1}^{r} \frac{1}{t^{x_i}} = 1,$$

where the x_i are nonnegative integers and $0 \le x_1 \le \cdots \le x_r$. It is known that this counting problem is equivalent to several other counting problems, namely the number of "nonequivalent" complete rooted t-ary trees (also called "level-greedy trees"), the number of "proper words" (in the sense of Even and Lempel [9]), the number of bounded degree sequences, and the number of "nonequivalent" compact Huffman codes of length r over an alphabet of t letters. For a detailed survey on the existing results, applications and literature on these sequences; see [8]. As a small concrete example, we note that for $t = 2$, $r = 5$ we have $f_2(5) = 3$, as can be seen from working out the following:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}.$$

Since $f_t(r)$ is positive only when $r = 1 + n(t-1)$, it is more convenient to study $g_t(n) = f_t(1 + n(t-1))$ instead. For $t = 2$ the values of $g_t(n)$ start with

$$1, 1, 1, 2, 3, 5, 9, 16, 28, 50, 89, 159, \ldots,$$

for $t = 3$ with
$$1, 1, 1, 2, 4, 7, 13, 25, 48, 92, 176, 338, \ldots,$$
and the first terms of $g_4(n)$ are
$$1, 1, 1, 2, 4, 8, 15, 29, 57, 112, 220, 432, \ldots.$$
These are sequences A002572, A176485 and A176503 in the On-Line Encyclopedia of Integer Sequences [27].

Asymptotics. It has been proved (see Elsholtz, Heuberger, Prodinger [8]) that for fixed t, the asymptotic growth of these sequences can be described by two main terms and an error term as

$$g_t(n) = R\rho^n + R_2\rho_2^n + O\big(r_3^n\big), \tag{1.1}$$

where $1 < r_3 < \rho_2 < \rho < 2$. Here all constants depend on t. In particular, if $t = 2$, then

$$\rho = 1.794\ldots, \quad \rho_2 = 1.279\ldots, \quad r_3 = 1.123, \quad R = 0.14\ldots, \quad R_2 = 0.061\ldots.$$

Moreover, the authors of [8] also show that $\rho = 2 - 2^{-t-1} + O\big(t\,4^{-t}\big)$ as $t \to \infty$.

Beside the enumeration of all the objects, the asymptotic behavior like the expectation, the variance or the (local) limiting distribution for many different parameters have been studied; see [16, 17].

Algorithmic counting. It seems that the fastest known algorithm to produce the sequence $g_t(n)$ for $n < N$ is the one by Even and Lempel [9]. According to them, it takes $O(N^3)$ additions of integers bounded by $O(\rho^N)$ (with $\rho < 2$; so integers with roughly N bits in size), which are $O(N^4)$ bit operations[1]. They only studied the case $t = 2$ in detail, but mention that their result can be generalized to arbitrary t.

Main result. In this paper, we show that the cost of evaluating the generating function developed in [8] essentially comes from one division of power series of precision[2] N whose coefficients are integers bounded by $O(\rho^N)$ (with $\rho < 2$). It is not obvious that the cost for evaluation of numerator and denominator of the generating function are asymptotically (much) smaller than the total cost; see Theorem 1 for details. We in particular show that the cost for evaluating numerator and denominator are asymptotically almost (neglecting logarithmic factors) by a factor N smaller.

Using the multiplication algorithms of Schönhage and Strassen [29] or of Fürer [12, 13] (see Section 2 for an overview) our algorithm leads to $N(\log N)^2\,2^{O(\log^* N)}$ operations in the integer ring and $N^2(\log N)^4\,2^{O(\log^* N)}$ bit operations, where $\log^* N$ denotes the *iterated logarithm*.[3] In Remark 6.2 a discussion on the memory requirements can be found. An implementation of this algorithm, based on FLINT [11, 15] (which is, for example, included in the SageMath mathematics software [28]) is also available;[4] see also Appendix A for the relevant lines of code and remarks related to the implementation. In Appendix B, we discuss the running times of this implementation.

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[1]We did not study if a careful implementation of Even and Lempel's algorithm can lead to a lower number of bit operations.

[2]We say that a power series H has precision N if we can write it as $H(q) = \sum_{n=0}^{N-1} h_n q^n + O(q^N)$ with explicit coefficients h_n.

[3]The *iterated logarithm* (also called *log star*) gives the number of applications of the logarithm so that the result is at most 1. For example, we can define it recursively by $\log^* M = 1 + \log^*(\log M)$ if $M > 1$ and $\log^* M = 0$ otherwise.

[4]The code accompanying this article can be downloaded from **https://gitlab.com/dakrenn/count-nonequivalent-compact-huffman-codes**.

Task	Ring operations	Bit operations
addition	N	NM
multiplication	$N \log N \, 2^{O(\log^* N)}$	$N \log N \, 2^{O(\log^* N)} \cdot M \log M \, 2^{O(\log^* M)}$
division	$N \log N \, 2^{O(\log^* N)}$	$N^2 M (\log N)^2 2^{O(\log^* N)}$

TABLE 2.1. Cost of operations of power series with precision N and coefficients with bit size M. (We assume $M = O(N)$ and state a simpler expression for the bit operations for division.)

The literature describes a number of algorithms constructing the complete list of t-ary Huffman codes of length $r = 1 + n(t-1)$; see [18, 22, 25, 26]. There is no performance analysis given. But, as the number of such codes grows exponentially in r it is clear that listing all codes is not a fast method to determine the number of such codes only. The algorithm by Even and Lempel [9] computes the number $f_2(n)$ without listing all codes, and is to the best of our knowledge the fastest algorithm previously known. Our algorithm relies on calculations involving power series with large integer coefficients.

A quite general survey paper by Klazar [23] studies classes of problems where the output needs at most a polynomial number of steps, in terms of the input size. As we can compute $f_t(n)$ efficiently, this problem falls into the class considered by Klazar.

Notes. It should be pointed out that in this article, we derive and compare upper bounds. It might be that the actual cost are smaller. However, as we compute the first N coefficients all at the same time and the coefficients grow exponentially in N, a lower bound for the number of bit operations necessarily contains a factor N^2. Moreover, as multiplication of some sort is involved, lower order factors (growing with N) are expected as well.

We also mention that the following is open: How fast can a single coefficient $g_t(n)$ (in contrast to all coefficients with $n < N$) be computed?

2. Cost of the underlying operations

In this section, we give a brief overview on the time requirements for performing addition and multiplication of two integers and for performing multiplication and division of power series. The current state of the art is also summarized in Table 2.1.

Addition and multiplication. First, assume that we want to perform addition of two numbers bounded by 2^M, i.e., numbers with M bits. We have to look at each bit of the numbers exactly once and add those (maybe with a carry). Therefore, we need $O(M)$ bit operations.

Next, we look at multiplication of two numbers bounded by 2^M. It is clear that this can be achieved with $O(M^2)$ operations, but it can be done better. An overview is given in the survey article by Karatsuba [19]. The Karatsuba multiplication algorithm [20, 21] has a complexity of $O(M^{\log_2 3})$. A faster generalisation of it is the Toom–Cook-algorithm [4]. Combining Karatsuba multiplication with the Fast Fourier Transform algorithm (see Cooley and Tukey [5]) gives an algorithm with bit complexity $O(M(\log M)^{(2+\varepsilon)})$; see [1, 2, 3, 24].

The multiplication algorithm given by Schönhage and Strassen (see [29]) takes $O(M \log M \log \log M)$ time. It also uses fast Fourier transform. The asymptotically fastest known multiplication algorithm is given by Fürer [12, 13]. It has computational complexity $M \log M \, 2^{O(\log^* M)}$, where we again denote the *iterated logarithm* by $\log^* M$. Fürer's algorithm uses complex arithmetic. A related algorithm of

the same complexity but using modular arithmetic is due to De, Kurur, Saha and Saptharishi [7].

Power series operations. Let us also summarize the complexity of power series computations; for references see the books of Cormen, Leiserson, Rivest and Stein [6] or Knuth [24]. The multiplication can, again, be speeded up by using fast Fourier transform. We can use the algorithms for integer multiplication presented above; see von zur Gathen and Gerhard [30]. Also, the computational complexity can be adapted: Given power series with precision N (i.e., the first N terms) over a ring, we can perform multiplication with $N \log N \, 2^{O(\log^* N)}$ ring operations using Fürer's algorithm.

In order to perform division (inversion) of power series with precision N, we can use the Newton–Raphson-method. We need at most $4 \, m(N) + N$ ring operations, where $m(N)$ denotes the number of operations needed to multiply two power series with precision N; see von zur Gathen and Gerhard [30, Theorem 9.4] for details; the additional summand $m(N)$ in comparison to that theorem comes from the multiplication with the numerator. Therefore, by using Fürer's algorithm, we can invert/divide with $N \log N \, 2^{O(\log^* N)}$ ring operations.

The bit size occuring in the ring operations for a division of power series with precision N and coefficients of bit size M is NM by the remarks after [30, Theorem 9.6]. Therefore and by assuming $M = O(N)$ for simpler expressions with respect to the logarithms, we end up with

$$N \log N \, 2^{O(\log^* N)} \cdot NM \log\big(NM\big) 2^{O(\log^*(NM))} = N^2 M (\log N)^2 2^{O(\log^* N)}$$

bit operations.

3. Cost for extracting coefficients

We start by noting that for $n < N$ each coefficient $g_t(n)$ can be written with $M := \lfloor \log_2 g_t(N) \rfloor + 1$ bits and that by using the asymptotics (1.1) we can bound this by $N \log_2 \rho + O(1)$ when N tends to ∞. Here the constant $\rho < 2$ depends on t; see [8] for details on ρ.

We denote the cost (number of operations in the integer ring)

- of an *addition* (or *subtraction*) of two power series of precision N by **A**,
- of a *multiplication* of two power series of precision N by **M** and
- of a *division* of two power series of precision N by **D**.

All these operations are performed on power series of precision N with coefficients written by M bits (numbers bounded by 2^M). However, we will see during our main proof (Section 5) that the coefficients appearing in power series additions and multiplications are actually smaller; we will take this into account for counting bit operations.

Moreover, we denote the cost

- of *other power series operations* of precision N (for example, memory allocation or writing initial values) by **S**.

Finally, we denote the cost

- of *other operations*, more precisely operations of numbers with less than $\log_2 N$ bits (for example additions of indices) by **O**.

Thus, an operation with cost **O** is performed on numbers bounded by N only (in contrast to the bounded-by-2^M-operations).

With these notions we can write down the precise formulation of our main theorem.

Theorem 1. *Calculating the first N terms of $g_t(n)$ can be done with*

$$\mathbf{D} + \big(\log_t N + O(1)\big)\mathbf{M} + 2\big(\log_t N + O(1)\big)\mathbf{A} + O(\log N)\mathbf{S} + O(\log N)\mathbf{O} \quad (3.1)$$

power series operations,

$$N(\log N)^2\, 2^{O(\log^* N)}$$

operations in the ring of integers, and with

$$N^2(\log N)^4\, 2^{O(\log^* N)} \tag{3.2}$$

bit operations.

In order to prove Theorem 1, we look at the cost of calculating the first N terms, which is done by extracting coefficients of the power series

$$H(q) = \frac{\sum_{j=0}^{\infty} q^{[j]}(-1)^j \prod_{i=1}^{j} \frac{q^{[i]}}{1-q^{[i]}}}{\sum_{j=0}^{\infty} (-1)^j \prod_{i=1}^{j} \frac{q^{[i]}}{1-q^{[i]}}} \tag{3.3}$$

with

$$[j] := 1 + t + \cdots + t^{j-1}. \tag{3.4}$$

This generating function (3.3) can be found in Flajolet and Prodinger [10, Theorem 2] for $t = 2$ and in Elsholtz, Heuberger and Prodinger [8, Theorem 6] for general t. It is derived from the equivalent formulation as counting problem on trees, which was mentioned in the introduction.

4. AUXILIARY RESULTS

When extracting the first N coefficients, we do not need the "full" generating function, i.e., the infinite sums in the numerator and denominator of (3.3) can be truncated to finite sums. The following lemma tells us how many coefficients we need. We use this asymptotic result in our analysis of the algorithm; for the actual computer programme, we can check indices and exponents by a direct computation.

Lemma 4.1. *To calculate numerator and denominator of the generating function (3.3) with precision N, we need only summands with*

$$j \le J = \log_t N + O(1).$$

Proof. Because of an additional factor $q^{[j]}$ in each summand of the numerator, it is sufficient that the largest index of the denominator is less than N. Therefore, we will only look at the indices of the denominator.

Consider the summand of the denominator with index j. The lowest index of a non-zero coefficient of the denominator is

$$\sigma_j = \sum_{i=1}^{j} [i] = \frac{t^{j+1} - t}{(t-1)^2} - \frac{j}{t-1} = \frac{t^{j+1}}{(t-1)^2}\Big(1 - \frac{j(t-1)}{t^{j+1}} - \frac{1}{t^j}\Big)$$

where the notation $[i]$ is defined in Equation (3.4). We only need summands with $\sigma_j < N$. Taking the logarithm yields

$$j - 1 + \log_t\Big(1 - \frac{j(t-1)}{t^{j+1}} - \frac{1}{t^j}\Big) - 2\log_t\Big(1 - \frac{1}{t}\Big) < \log_t N$$

As the first logarithm tends to 0 as $j \to \infty$ and the second is bounded, the error term $O(1)$ is large enough and the result follows. $\square$

While the bit size of the coefficients $g_t(n)$ is linear in N, the size of the coefficients of numerator and denominator of (3.3) is much smaller. We make this precise by using the following lemma.

Lemma 4.2. *For $n \leq N$, the nth coefficient of*

$$\frac{1}{1-q^{[1]}} \frac{1}{1-q^{[2]}} \frac{1}{1-q^{[3]}} \cdots \frac{1}{1-q^{[j]}} \tag{4.1}$$

with $j \leq J$ and J of Lemma 4.1 as well as the nth coefficients of numerator and denominator can be written with

$$\frac{(\log N)^2}{2(\log 2)(\log t)} + O(\log N) \tag{4.2}$$

bits.

Proof. It suffices to show the result for $j = J$ due to nonnegativity and therefore monotonicity in j of our bounds of all coefficients.

Each factor of (4.1) is a geometric series whose coefficients are either 0 or 1. Thus, the nth coefficient of the product equals the cardinality of the set

$$\left\{ (a_1, a_2, \ldots, a_J) \in \mathbb{N}_0^J : \sum_{i=1}^{J} a_i[i] = n \right\}.$$

By using the crude estimate $a_i \leq n/[i]$, we see that we have at most $2N/[i]$ choices for a_i because $n < N$ and $[i] < N$ by construction. Thus we can bound the cardinality of the set above by

$$\frac{2^J N^J}{[1][2]\cdots[J]} \leq \frac{2^J N^J}{1 \cdot t \cdot t^2 \cdots t^{J-1}} = \frac{2^J N^J}{t^{(J-1)J/2}}.$$

We use $J = \log_t N + O(1)$ of Lemma 4.1 to obtain

$$\frac{2^J N^J}{t^{(J-1)J/2}} = \exp\Big(J(\log N) - J^2 \frac{\log t}{2} + J \frac{\log t}{2} + J(\log 2) \Big)$$

$$\leq \exp\Big(\frac{(\log N)^2}{\log t} - \frac{(\log N)^2}{2 \log t} + O(\log N) \Big)$$

from which follows that the nth coefficient of (4.1) is bounded by

$$\exp\Big(\frac{(\log N)^2}{2 \log t} + O(\log N) \Big). \tag{4.3}$$

The result in terms of bit size follows by logarithmizing.

Numerator and denominator are sums where J summands are added up (or subtracted). This corresponds to an additional factor J in the bound (4.3) or an additional summand $\log_2 J$ in the formula (4.2), respectively. As $J = O(\log_t N)$ by Lemma 4.1, this is absorbed by the error term, so the same formula holds. $\quad\square$

5. PROOF OF THEOREM 1

We start with an overview of our strategy. For computing the first N coefficients of the generating function $H(q)$ (see (3.3)), we only need the summands of the numerator and the denominator with $j < J$ according to Lemma 4.1.

First, consider the denominator of $H(q)$. We compute the products

$$\prod_{i=1}^{j} \frac{q^{[i]}}{1-q^{[i]}},$$

iteratively by expanding the J different terms $q^{[i]}/(1 - q^{[i]})$ as geometric series and perform power series multiplications. After each multiplication, we accumulate the result by using one power series addition.

We deal with the numerator in the same fashion. However, by performing the computation of numerator and denominator simultaneously, the above products are only needed to be evaluated once.

Finally, to obtain the first N coefficients of $H(q)$, we need one power series division of numerator and denominator.

Pseudocode for our algorithm is given in Algorithm 1; an efficient implementation using the FLINT library is presented in Appendix A. The actual analysis of this algorithm is done by counting the operations needed, in particular the power series operations, and providing bounds for the bit sizes of the variables.

Algorithm 1 Pseudocode

```python
    j = 0; hp_j = 0; sigma_j = 0; sign = 1

    numerator = 1; denominator = 1; coefficient_product = 1

    while True:
        j += 1; hp_j = 1 + t*hp_j; sigma_j += hp_j; sign = -sign
        # hp_j=[j]; sigma_j = sum_{i=1}^j [j]; sign = (-1)^j

        if sigma_j > N:
            break

        new_factor = 0
        for i in range(hp_j, hp_j, N):
            new_factor[i] = 1
        # new_factor = q^[j]/(1-q^[j])

        coefficient_product *= new_factor
        # coefficient_product=prod_{i=1}^j q^[i]/(1-q^[i])

        denominator += sign * coefficient_product
        numerator += sign * coefficient_product.shifted(hp_j)
        # numerator up to summand j; denominator up to summand j

    result = numerator / denominator
```

The above said, let us come to the actual proof.

Proof of Theorem 1. We analyse the code of Algorithm 1; see Appendix A for the details. It starts by initialising variables (memory allocation and initial values) for the power series operations, which contributes $O(1)\mathbf{S}$. Further initialisation is done by $O(1)\mathbf{O}$ operations.

For computing the first N coefficients of $H(q)$ (see (3.3)), we only need the summands of numerator and denominator with $j < J = \log_t N + O(1)$ according to Lemma 4.1. Speaking in terms of our computer programme, our outer loop needs J passes. We now describe what happens in each of these passes; the final cost need then to be multiplied by J.

Suppose we are in step j. After some update of auxiliary variables (cost $O(1)\mathbf{O}$), we compute the product

$$\prod_{i=1}^{j} \frac{q^{[i]}}{1 - q^{[i]}},$$

out of the product with factors up to index $i = j - 1$. Expanding $q^{[j]}/(1 - q^{[j]})$ as geometric series contributes at most $\mathbf{S}$ and performing a power series multiplication contributes $\mathbf{M}$ and additionally one swap $O(1)\mathbf{S}$. For obtaining the number of bit

operations, we need estimates of the coefficients appearing in the multiplication. Lemma 4.2 bounds their value by

$$\frac{(\log N)^2}{2(\log 2)(\log t)} + O(\log N) \tag{5.1}$$

bits. Therefore each of our power series multiplications $\mathbf{M}$ needs

$$N \log N \, 2^{O(\log^* N)} \times (\log N)^2 \log \log N \, 2^{O(\log^* N)} = N(\log N)^3 \log \log N \, 2^{O(\log^* N)}$$

bit operations by Fürer's result [12, 13]; see also Section 2.

After each multiplication, we accumulate the results for numerator and denominator by using one power series addition $\mathbf{A}$ for each of the two. For the numerator, we additionally need $O(1)\mathbf{S}$ for the multiplication by $q^{[j]}$ performed by shifting. Concerning bit operations, we use the bound of the coefficients for numerator and denominator provided by Lemma 4.2. In terms of bit size, this leads to the number of bits given in (5.1). Therefore a power series addition needs

$$O(N) \, O\big((\log N)^2\big) = O\big(N(\log N)^2\big)$$

bit operations.

In total, we end up with

$$J\big(\mathbf{M} + 2\mathbf{A} + O(1)\mathbf{S} + O(1)\mathbf{O}\big)$$

operations to evaluate the outer loop; these operations translate to

$$N(\log N)^2 \, 2^{O(\log^* N)}$$

operations in the ring of integers and to

$$N(\log N)^4 \log \log N \, 2^{O(\log^* N)}$$

bit operations.

We are now ready to collect all costs for proving the first part of Theorem 1. Additionally to the above, we divide the numerator by the denominator and need one power series division $\mathbf{D}$. The clean-up accounts to $O(1)\mathbf{S}$. This yields (3.1).

Using the Newton–Raphson-method and Fürer's algorithm (see Section 2 and Table 2.1) a power series division $\mathbf{D}$ results in

$$N \log N \, 2^{O(\log^* N)}$$

operations in the ring. Its operands[5] have bit size

$$\frac{(\log N)^2}{2(\log 2)(\log t)} + O(\log N)$$

which results in

$$N^2 (\log N)^4 2^{O(\log^* N)}$$

bit operations for our computations.

We note that the number of bit operations of a power series operation $\mathbf{S}$ is linear in N as the coefficients are bounded and that $\mathbf{O}$ is an operation on numbers with $O(\log N)$ bits. The error term includes all these. Collecting all bit operation results gives the upper bound (3.2). $\qquad\square$

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[5]The actual bit size during the division is $\frac{N(\log N)^2}{2(\log 2)(\log t)} + O(N \log N)$; see the end of Section 2 for details.

6. Remarks

In this last section, we provide a some remarks related to the above proof and coefficient extraction algorithm.

Remark 6.1. In the proof above, we have seen that the cost (bit operations) of the power series division is asymptotically roughly (not taking logarithms and smaller factors into accout) a factor N larger than the cost for computing numerator and denominator, and all the overhead cost.

Moreover, only focusing on the computation of numerator and denominator, the costs (again bit operations) for computing these two are asymptotically dominated by power series multiplication, albeit only by roughly (again not taking into account logarithmically smaller factors) a factor $\log N$ compared to addition and other power series operations.

Note that when only considering operations in the integer ring, then the multiplications performed in the evaluation of numerator and denominator take the asymptotically leading role by $N(\log N)^2 \, 2^{O(\log^* N)}$ operations compared to $N(\log N) \, 2^{O(\log^* N)}$ ring operations of the power series division.

At the end of this article we make a short remark on the memory requirements for the presented coefficient extraction algorithm.

Remark 6.2. Our algorithm needs $O(N)$ units of memory—a unit stands for the memory requirements of storing a number of bounded by ρ^N—plus the memory needed for the power series multiplication and division.[6] The above means that we can bound the memory requirements by $O(N^2)$ bits.

References

[1] Allan Borodin and Ian Munro, *The computational complexity of algebraic and numeric problems*, American Elsevier Publishing Co., Inc., New York-London-Amsterdam, 1975, Elsevier Computer Science Library; Theory of Computation Series, No. 1.

[2] Jonathan M. Borwein, Peter B. Borwein, and David H. Bailey, *Ramanujan, modular equations, and approximations to pi, or how to compute one billion digits of pi*, Amer. Math. Monthly **96** (1989), 201–219.

[3] E. Oran Brigham, *The Fast Fourier transform*, Prentice-Hall, Englewood Cliffs, NJ, 1974.

[4] Stephen A. Cook, *On the minimum computation time of functions*, Ph.D. thesis, Harvard University, 1966.

[5] James William Cooley and Tukey John Wilder, *An algorithm for the machine calculation of complex Fourier series*, Math. Comput. **19** (1965), 297–301.

[6] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, *Introduction to algorithms*, second ed., The MIT Press, 2001.

[7] Anindya De, Piyush P. Kurur, Chandan Saha, and Ramprasad Saptharishi, *Fast integer multiplication using modular arithmetic*, Symposium on Theory of Computation (STOC), 2008.

[8] Christian Elsholtz, Clemens Heuberger, and Helmut Prodinger, *The number of Huffman codes, compact trees, and sums of unit fractions*, IEEE Trans. Inf. Theory **59** (2013), 1065–1075.

[9] Shimon Even and Abraham Lempel, *Generation and enumeration of all solutions of the characteristic sum condition*, Information and Control **21** (1972), 476–482.

[10] Philippe Flajolet and Helmut Prodinger, *Level number sequences for trees*, Discrete Math. **65** (1987), no. 2, 149–156.

[11] William B. Hart, Fredrik Johansson, and Sebastian Pancratz, *FLINT: Fast Library for Number Theory*, 2015, Version 2.5.2, `http://flintlib.org`.

[12] Martin Fürer, *Faster integer multiplication*, STOC'07—Proceedings of the 39th Annual ACM Symposium on Theory of Computing, ACM, New York, 2007, pp. 57–66.

[13] ———, *Faster integer multiplication*, SIAM J. Comput. **39** (2009), no. 3, 979–1005.

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[6]We have been unable to find a reference for the memory requirements of, for example, the Schönhage–Strassen-algorithm. It seems that the *GNU Multiple Precision Arithmetic Library (GMP)* can do this with $12N$ units of memory; see [14] for a comment of one of its authors.

[14] Torbjörn Granlund, *Schönhage-Strassen*, gmp-discuss mailing list, August 2010, `http://gmplib.org/list-archives/gmp-discuss/2010-August/004277.html`.

[15] William B. Hart, *Fast library for number theory: an introduction*, Mathematical software—ICMS 2010, Lecture Notes in Comput. Sci., vol. 6327, Springer, Berlin, 2010, pp. 88–91.

[16] Clemens Heuberger, Daniel Krenn, and Stephan Wagner, *Analysis of parameters of trees corresponding to Huffman codes and sums of unit fractions*, Proceedings of the Meeting on Analytic Algorithmics & Combinatorics (ANALCO), New Orleans, Louisiana, USA, January 6, 2013, SIAM, Philadelphia PA, 2013, pp. 33–42.

[17] _____, *Canonical trees, compact prefix-free codes and sums of unit fractions: A probabilistic analysis*, SIAM J. Discrete Math. **29** (2015), no. 3, 1600–1653.

[18] Dean Hoffman, Peter Johnson, and Nadine Wilson, *Generating Huffman sequences*, J. Algorithms **54** (2005), no. 1, 115–121.

[19] Anatolii Alexeevitch Karatsuba, *The complexity of computations*, Proceedings of the Steklov Institute of Mathematics, vol. 211, 1995, pp. 169–183.

[20] Anatolii Alexeevitch Karatsuba and Ofman Yuri Petrovich, *Multiplication of many-digital numbers by automatic computers*, Proceedings of the USSR Academy of Sciences, vol. 145, 1962, pp. 293–294.

[21] _____, *Multiplication of many-digital numbers by automatic computers*, Doklady Physics **7** (1963), 595–596.

[22] Mohammadali Khosravifard, Morteza Esmaeili, Hossein Saidi, and T. Aaron Gulliver, *A tree based algorithm for generating all possible binary compact codes with n codewords*, IEICE Trans. Fundamentals (2003), no. 10, 2510–2516.

[23] Martin Klazar, *What is an answer? - remarks, results and problems on PIO formulas in combinatorial enumeration, part I*, arXiv:1808.08449 [math.CO].

[24] Donald E. Knuth, *Seminumerical algorithms*, third ed., The Art of Computer Programming, vol. 2, Addison-Wesley, 1998.

[25] Hajar Narimani and Mohammadali Khosravifard, *The supertree of the compact codes*, Proc. Int. Symposium on Telecommunications, 2008, pp. 649–655.

[26] Othmane Niyaoui and Oussama Mohamed Reda, *A new algorithm for generating all binary Huffman codes based on path-length extensions*, International Journal of Applied Engineering Research **11** (2016), no. 21, 10618–10623.

[27] *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`, 2013.

[28] The SageMath Developers, *SageMath Mathematics Software (Version 8.3)*, 2018, `http://www.sagemath.org`.

[29] Arnold Schönhage and Volker Strassen, *Schnelle Multiplikation großer Zahlen*, Computing (Arch. Elektron. Rechnen) **7** (1971), 281–292.

[30] Joachim von zur Gathen and Jürgen Gerhard, *Modern computer algebra*, second ed., Cambridge University Press, 2003.

Appendix A. Code

Below are the relevant lines of a programme written in C for computing the coefficients of $g_t(n)$ with $n < N$ are shown. The code can be found at `https://gitlab.com/dakrenn/count-nonequivalent-compact-huffman-codes`. The programme uses FLINT [11, 15]. Note that we do not use aliasing of input and output arguments in multiplication because providing our own auxiliary polynomial brings tiny performance improvements.

Appendix B. Timing

The table below contains timings (in seconds) for computing the first N coefficients with $t = 2$.

t	N	$t_{\text{n\&d}}$	t_{division}	t_{total}
2	256	0.000	0.001	0.001
2	512	0.001	0.005	0.006
2	1024	0.002	0.014	0.016
2	2048	0.005	0.045	0.050
2	4096	0.013	0.126	0.139
2	8192	0.029	0.562	0.591
2	16384	0.067	2.225	2.292
2	32768	0.251	10.709	10.960
2	65536	0.617	45.259	45.877
2	131072	1.189	198.995	200.184

Here, $t_{\text{n\&d}}$ is the time for generating numerator and denominator, t_{division} for the one power series division and $t_{\text{total}} = t_{\text{n\&d}} + t_{\text{division}}$.

The benchmark was executed on an Intel(R) Xeon(R) CPU E5-2630 v3 at 2.40GHz. The limiting factor for our computations is the memory requirement; it is the reason computing at most $N = 2^{17} = 131072$ coefficients.

The timings in the table and the theoretical result of this article fit together; we can see the $O(N^{2+\varepsilon})$ running time of the algorithm in our implementation.

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Algorithm 2 C-code using FLINT

```
fmpz_poly_init(result);
fmpz_poly_init2(coefficient_product, N);
fmpz_poly_init2(coefficient_product_shifted, 2*N);
fmpz_poly_init2(new_factor, N);
fmpz_poly_init2(numerator, N);
fmpz_poly_init2(denominator, N);
fmpz_poly_init2(aux, N);

fmpz_poly_set_ui(numerator, 1);
fmpz_poly_set_ui(denominator, 1);
fmpz_poly_set_ui(coefficient_product, 1);

j=0;
hp_j=0;
sigma_j=0;
sign=1;

while(-1) {
    j++;
    hp_j=1+t*hp_j;
    sigma_j+=hp_j;
    sign=-sign;
    /* hp_j=[j]; sigma_j=sum_{i=1}^j [j]; sign=(-1)^j */

    if(sigma_j>=N) {
        break;
    }

    for(i=hp_j; i<N; i+=hp_j) {
        fmpz_poly_set_coeff_ui(new_factor, i, 1);
    }
    /* new_factor = q^[j]/(1-q^[j]) */

    fmpz_poly_mullow(aux, coefficient_product, new_factor, N);
    fmpz_poly_swap(coefficient_product, aux);
    /* coefficient_product=prod_{i=1}^j q^[i]/(1-q^[i]) */

    for(i=hp_j; i<N; i+=hp_j) {
        fmpz_poly_set_coeff_ui(new_factor, i, 0);
    }
    /* new_factor = 0 */

    if(sign==1) {
        fmpz_poly_add(denominator, denominator, coefficient_product);

        if(hp_j+sigma_j<N) {
            fmpz_poly_set(coefficient_product_shifted, coefficient_product);
            fmpz_poly_truncate(coefficient_product_shifted, N-hp_j);
            fmpz_poly_shift_left(coefficient_product_shifted, coefficient_product_shifted, hp_j);
            fmpz_poly_add(numerator, numerator, coefficient_product_shifted);
        }
    } else {
        fmpz_poly_sub(denominator, denominator, coefficient_product);

        if(hp_j+sigma_j<N) {
            fmpz_poly_set(coefficient_product_shifted, coefficient_product);
            fmpz_poly_truncate(coefficient_product_shifted, N-hp_j);
            fmpz_poly_shift_left(coefficient_product_shifted, coefficient_product_shifted, hp_j);
            fmpz_poly_sub(numerator, numerator, coefficient_product_shifted);
        }
    }
    /* numerator= up to summand j; denominator up to summand j */
}

fmpz_poly_clear(coefficient_product);
fmpz_poly_clear(coefficient_product_shifted);
fmpz_poly_clear(new_factor);
fmpz_poly_clear(aux);

fmpz_poly_div_series(result, numerator, denominator, N);

fmpz_poly_clear(numerator);
fmpz_poly_clear(denominator);
```