

“Choix de Bruxelles”: A New Operation on Positive Integers

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Abstract

The “Choix de Bruxelles” operation replaces a positive integer n by any of the numbers that can be obtained by halving or doubling a substring of the decimal representation of n . For example, 16 can become any of 16, 26, 13, 112, 8, or 32. We investigate the properties of this interesting operation and its iterates.

1 Introduction

Let n be a positive integer with decimal expansion $d_1d_2d_3\dots d_k$. The “Choix de Bruxelles” operation replaces n by any of the numbers that can be formed by taking a number s represented by a substring $d_pd_{p+1}\dots d_q$, with $1 \leq p \leq q \leq k$, where the initial digit d_p is not zero, and replacing the substring *in situ* by the decimal expansion of $2s$ or, if s is even, by the decimal expansion of $s/2$. One may also leave n unchanged (corresponding to choosing the empty substring).

Since the definition may be confusing at first glance, we give a detailed example. Suppose $n = 20218$. By choosing substrings of length 1, we can obtain any of

10218, 40218, 20118, 20418, 20228, 20214, and 202116(!).

(For the last of these, we replaced 8 by 16 *in situ*.) Using substrings of length 2 we can obtain

20218 & 40218 again, 20428, 2029(!), 20236,

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and using substrings of lengths 0, 3, 4, 5 we obtain

20218; 10118, 40418, 20109, 20436; 40428; 10109 and 40436.

It is the possibility of increasing or decreasing the number of digits by one that makes this operation interesting. Changing 16 to 112, for example, increases the number by a factor of 7. In fact, as we see in Theorem 2.3, n may change by a factor ranging from $\frac{1}{10}$ to 10. Note that the operation is symmetric: if n can be changed to m , then m can be changed to n .

The name “Choix de Bruxelles” arose from a combination of several ideas: this a kind of mathematical game, like “sprouts” [1], sprouts are “choux de Bruxelles”, the operation was proposed by the first author, who lives in Brussels, and involves making choices.

Our goal is to investigate the properties of this operation, and to study what happens when it is iterated.

Section 2 studies the range of numbers that can be reached by applying the operation once. Theorems 2.1 and 2.2 give the largest and smallest numbers that can be obtained from n , and Theorem 2.3 describes their range.

Section 3 shows that by repeatedly applying the operation, any number not ending with 0 or 5 can be transformed into any other such number, and any number ending in 0 or 5 can be transformed into any other number of that form, but the two classes always remain separate. The final section studies how many steps are need to reach n from 1, assuming that n does not end in 0 or 5.

One could also consider this operator in other bases. However, the binary version (see sequence [A323465](#)² in [2]) is not very interesting—the operation preserves the binary weight—so we have not pursued this.

Notation. For clarity we will sometimes use the symbol \parallel to indicate that the digits of two numbers are to be concatenated. For example, if $x = 7$ and $y = 8$, $(2x) \parallel y$ indicates the number 148.

Table 1: Numbers arising when “Choix de Bruxelles” is applied to the numbers 1 to 16.

n	goes to	n	goes to
1	1, 2	9	9, 18
2	1, 2, 4	10	5, 10, 20
3	3, 6	11	11, 12, 21, 22
4	2, 4, 8	12	6, 11, 12, 14, 22, 24
5	5, 10	13	13, 16, 23, 26
6	3, 6, 12	14	7, 12, 14, 18, 24, 28
7	7, 14	15	15, 25, 30, 110
8	4, 8, 16	16	8, 13, 16, 26, 32, 112

²Six-digit numbers prefixed by A refer to entries in the On-Line Encyclopedia of Integer Sequences.

2 Numbers that can be reached in one step.

Table 1 shows the numbers that can be reached from n in one step, for $1 \leq n \leq 16$. The rows of this table form sequence [A323460](#) in the OEIS [2].

The smallest ($M_0(n)$, [A323462](#)) and largest ($M_1(n)$, [A323288](#)) numbers in each row are as follows:

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$M_0(n)$:	1	1	3	2	5	3	7	4	9	5	11	6	13	7	15	8	...
$M_1(n)$:	2	4	6	8	10	12	14	16	18	20	22	24	26	28	110	112	...

(2.1)

Theorem 2.1. *The largest number $M_1(n)$ that can be obtained from*

$$n = d_1 d_2 d_3 \dots d_k \tag{2.2}$$

by the Choix de Bruxelles operation is either $2n$, if all $d_i < 5$, and otherwise is obtained by doubling the substring $d_p \dots d_k$ where d_p is the right-most digit ≥ 5 .

Proof. If all $d_i < 5$, the number of digits will not change during the operation, and so the best we can do is to double every digit, getting $M_1(n) = 2n$.

If there are digits $d_i \geq 5$, then doubling any substring beginning with such a d_i will increase the number of digits by one. The doubled substring will begin with 1 instead of d_i , and so should be as far to the right as possible. (E.g., if we double a substring starting at $d_3 = 7$, say, we get a number $d_1 d_2 1 \dots$, whereas if we double a substring starting at $d_5 = 6$ we get $d_1 d_2 d_3 d_4 1 \dots$, which is a larger number since $d_3 \geq 5$.)

Finally, if d_p is the right-most digit ≥ 5 , let t_q (where $p \leq q \leq k$) denote the result of doubling just the substring from d_p through d_q . Then

$$t_q = d_1 \dots d_{p-1} \parallel 2(d_p \dots d_q) \parallel d_{q+1} \dots d_k,$$

and this number is maximized by taking $q = k$. □

Theorem 2.2. *The smallest number $M_0(n)$ that can be obtained from (2.2) by the Choix de Bruxelles operation is as follows: If there is a substring $d_r \dots d_s$ of (2.2) which starts with $d_r = 1$ and ends with an even digit d_s , take the string of that form which starts with the left-most 1 and ends with the right-most even digit, and halve it. Otherwise, if there is any even digit, take the substring from d_1 to the right-most even digit and halve it. In the remaining case, all d_i are odd, and $M_0(n) = n$.*

The proof uses similar arguments to the proof of Theorem 2.1, and is omitted.

We can use Theorems 2.1 and 2.2 to get precise bounds on the range of numbers produced by the operation.

Theorem 2.3. *The numbers m obtained by applying the Choix de Bruxelles operation to n lie in the range*

$$\frac{n}{10} < m < 10n, \tag{2.3}$$

and there are values of n for which m is arbitrarily close to either bound.

Proof. For the upper bound, by Theorem 2.1 we can assume there is a digit of n that is ≥ 5 . Let d be the right-most such digit, and write $n = A \parallel d \parallel B = A10^i + d10^{i-1} + B$. Let $2d = 10 + x$, $x \in \{0, 2, 4, 6, 8\}$. Then

$$M_1(n) = A10^{i+1} + 10^i + x10^{i-1} + 2B$$

and $M_1(n) < 10n$ follows immediately. Numbers of the form $n = 99 \dots 9$ with $d = 9$, $i = 1$, $B = 0$ have $M_1(n) = 99 \dots 918 = 10n - 72$, so $M_1(n)/n = 10 - 72/n$, which comes arbitrarily close to the bound.

For the lower bound, from Theorem 2.1 the only nontrivial case is when there is a substring $B = 1 \dots e$, e even, and then $n = A10^{i+j} + B10^j + C$, $M_0(n) = A10^{i+j-1} + \frac{B}{2}10^j + C$, which implies $n < 10M_0(n)$. Numbers of the form $n = 10^t + 10$, $t \geq 2$, with $M_0(n) = 10^{t-1} + 5$, $M_0(n)/n = \frac{1}{10}(1 + 4/10^{t-1} + \dots)$ come arbitrarily close to this bound. \square

3 Which numbers can be reached from 1?

If we start with 1 and repeatedly apply the Choix de Bruxelles operation, which numbers can we reach? From³

$$1 - 2 - 4 - 8 - 16 - 112 - 56 - 28 - 14 - 12 - 6 - 3 \tag{3.1}$$

we can reach 3 in 11 steps⁴, as well as 2, 4, 8 and 6, and $\dots - 28 - 14 - 7$ and $\dots - 28 - 14 - 18 - 9$ give us all the numbers less than 10 except 5. Further experimenting suggests that 5 and 10 may be impossible to reach. This is true, as we now show.

Let us define an undirected graph G with vertices labeled by the positive integers, where vertices n and m are joined by an edge if Choix de Bruxelles takes n to m .

Theorem 3.1. *The graph G has two connected components, one containing all numbers whose decimal expansion does not end in 0 or 5, the other containing all numbers which do end in 0 or 5.*

Proof. (i) All numbers not ending in 0 or 5 are connected to 1. If not, let n be the smallest number, not ending in 0 or 5, that cannot be reached from 1. Any number m that is reachable from n must be larger than n , or else by symmetry m would be a smaller counter-example. This means that all digits of n are odd, and cannot be 3, 7, or 9 (since by (3.1) etc. they could be reduced to 1). If n has an internal digit 5, we can use $51 - 102 - 52 - 26$ to get a smaller number. So any 5s must appear in a string at the end of n , which contradicts our assumptions. The only remaining possibility is that $n = 11 \dots 11$. But even these numbers can be reduced by using $11 - 12 - 6$.

(ii) Any number ending in 0 or 5 is connected to 5. This follows from (i), using $n0 - 10 - 5$ and $n5 - n \parallel 10 - 10 - 5$.

(iii) The Choix de Bruxelles operations never change a final 0 or 5 into any digit other than 0 or 5. \square

³We write $-$ rather than \rightarrow in describing these transformations, since the operation is symmetrical.

⁴Found by Lorenzo Angelini.

4 How many steps to reach n ?

Let $\tau(n)$ be the number of steps to reach n from 1 using the Choix de Bruxelles operation, or -1 if n cannot be reached from 1. We found the values of $\tau(n)$ for $n \leq 10000$ by computer ([A323454](#)). The initial values are shown in Table 2.

Table 2: $\tau(n)$ = number of steps to reach n from 1, or -1 if n cannot be reached.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\tau(n)$	0	1	11	2	-1	10	9	3	9	-1	10	9
n	13	14	15	16	17	18	19	20	21	22	23	24
$\tau(n)$	5	8	-1	4	7	8	8	-1	10	9	6	8
n	25	26	27	28	29	30	31	32	33	34	35	36
$\tau(n)$	-1	5	8	7	9	-1	6	5	10	6	-1	9

We will derive upper and lower bounds on $\tau(n)$ for large n . The record high values of $\tau(n)$ that are presently known, and the values of n where they occur, are as follows (see [A323463](#)):

$$\begin{array}{r} n: 1 \quad 2 \quad 3 \quad 99 \quad 369 \quad 999 \quad 1999 \quad 9879 \\ \tau(n): 0 \quad 1 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \end{array} \quad (4.1)$$

In particular, the entry $\tau(n) = 12$ for $n = 99$ means that every number less than 100 and not ending in 0 or 5 can be reached from 1 in at most 12 steps.

We also need analogous data for the other class of numbers. The numbers of steps to reach $5m$ from 5 ([A323484](#)) are:

$$\begin{array}{r} 5m: 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 \quad 45 \quad 50 \quad 55 \quad 60 \quad \dots \\ \text{steps: } 0 \quad 1 \quad 11 \quad 2 \quad 11 \quad 12 \quad 8 \quad 3 \quad 10 \quad 12 \quad 9 \quad 11 \quad \dots \end{array} \quad (4.2)$$

and the record high values that are presently known, and the values of $5m$ where they occur, are as follows (see [A323464](#)):

$$\begin{array}{r} 5m: 5 \quad 10 \quad 15 \quad 30 \quad 100 \quad 200 \quad 400 \quad 9875 \quad 19995 \\ \text{steps: } 0 \quad 1 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \end{array} \quad (4.3)$$

In particular, the entry 13 for $n = 100$ means that every multiple of 5 less than 100 can be reached from 5 in at most 12 steps, and so *all* numbers less than 100 can be reached from either 1 or 5 in at most 12 steps.

To get an upper bound on $\tau(n)$ for larger n , consider a k -digit number n given by (2.2). We can repeatedly replace the two leading digits by a single digit (by 1 or 5, in fact) at the cost of at most 12 steps. So we can reach 1 or 5 in at most $12(k-1)$ steps. Since $k = \lfloor 1 + \log_{10} n \rfloor$, this takes at most $12 \log_{10} n$ steps. In particular, $\tau(n) \leq 12 \log_{10} n$.

To get a lower bound on $\tau(n)$, we note that Theorem 2.3 implies that at least $\log_{10} n$ steps will be needed to reach n from 1. We can make this more precise by finding $R(n)$, the largest number that can be reached from 1 in n steps. The values of $R(n)$ can *almost* be found by a greedy algorithm. Set $r(0) = 1$, and, for $n > 0$, let $r(n)$ be the largest number that can be

obtained by applying Choix de Bruxelles to $r(n-1)$ (of course $R(n) \geq r(n)$). Theorem 2.1 tells us how to calculate $r(n)$, and the first 20 values are shown in Table 3. We find that $r(n+4) = 8112 \parallel r(n)$ for $n \geq 10$.

Table 3: Values of $r(n)$.

n	$r(n)$	n	$r(n)$
0	1	10	88224
1	2	11	816448
2	4	12	8164416
3	8	13	81644112
4	16	14	811288224
5	112	15	8112816448
6	224	16	81128164416
7	448	17	811281644112
8	4416	18	8112811288224
9	44112	19	81128112816448

Theorem 4.1. $R(n)$, the largest number that can be reached from 1 in n steps, is equal to $r(n)$ for $n \neq 7$, and $R(7) = 512$.

Proof. By computer calculation, the result is true for $n \leq 16$. Suppose $n \geq 17$. The candidates for $R(n)$ are all the numbers that can be obtained by applying Choix de Bruxelles to the $(n-1)$ st generation numbers—numbers that can be reached from 1 in $n-1$ steps. In view of Theorem 2.3, we can discard any $(n-1)$ st generation numbers that are less than $\frac{1}{10}r(n)$. The sets of remaining candidates form a repeating pattern of period four, which, starting in generation 14, are as follows (here $i = 1, 2, 3, \dots$ and P denote i copies of the string 8112):

Generation	Remaining candidates
$10 + 4i$	$P81646, P81652, P81662, P81664, P84112, P88112, P88212, P88222, P88224$
$11 + 4i$	$P816442, P816444, P816448$
$12 + 4i$	$P8164416$
$13 + 4i$	$P81128826, P81128832, P81644112$

At each new generation, the largest number is obtained by expanding the final number in each row (using Theorem 2.3), and the resulting numbers are the $r(n)$, as claimed. \square

In summary, we have:

Theorem 4.2. For $n \geq 14$,

$$8.112 \cdot 10^{n-6} < R(n) \leq 8.113 \cdot 10^{n-6}. \quad (4.4)$$

The upper bound in Theorem 4.2 states that, starting at 1, we cannot reach a number greater than $c10^{n-6}$ in n steps, where $c = 8.113$. By solving $c10^{n-6} = m$, we see that to reach m from 1 we require at least $n = 6 + \log_{10}(m/c) = \log_{10} m + 5.09 \dots$ steps.

By combining this with our earlier result, we have our final theorem.

Theorem 4.3. *The number of steps needed to reach n from 1 by the Choix de Bruxelles operation, assuming n does not end in 0 or 5, is bounded by*

$$\log_{10} n + 5 < \tau(n) \leq 12 \log_{10} n. \quad (4.5)$$

References

- [1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, 2nd ed., 4 vols., A. K. Peters, Boston, 2004.
- [2] The OEIS Foundation Inc. (2019), *The On-Line Encyclopedia of Integer Sequences*, <http://oeis.org>.

Concerned with sequences [A323288](#), [A323453](#), [A323454](#), [A323460](#), [A323462](#), [A323463](#), [A323464](#), [A323465](#), [A323484](#).

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