ON THE DIFFERENCES BETWEEN ZUMKELLER NUMBERS

F. JOKAR

Abstract. In this paper, we prove that for every $\ell \in \mathbb{N}$ there are infinitely many (a, b) that both a and b are Zumkeller numbers and $b - a = \ell$

0. Introduction

A positive integer n is said to be a Zumkeller number if the positive divisors of n can be partitioned into two disjoint subsets of equal sum [1]. In this paper, we prove that for every even integer ℓ there are infinitely many (a, b), in which both a and b are even zumkeller numbers and $a - b = \ell$. We also prove that for every odd integer ℓ, there are infinitely many (a, b) that b is an odd Zumkeller number, a is an even Zumkeller number, and $b - a = \ell$

1. differences between Zumkeller numbers

Definition 1.1 (Definition 1 in [1])**.** A positive integer n is said to be a Zumkeller number if the positive divisors of n can be partitioned into two disjoint subsets of equal sum. A Zumkeller partition for a Zumkeller number n is a partition $\{A, B\}$ of the set of positive divisors of n so that each of A and B sums to the same value.

Proposition 1.2 (Corollary 5 in [1])**.** *If the integer n is Zumkeller and w is relatively prime to n, then nw is Zumkeller*

Example 1.3. It is easy to verify that 6 is a Zumkeller number. On the other hand, for every $k \in \mathbb{N}$, $gcd(6, 3k + 2) = gcd(6, 3k + 1) = 1$. Hence, $18k + 6 = 6 \times (3k + 1)$ and $(18k + 12) = 6 \times (3k + 2)$ are two Zumkeller numbers.

Definition 1.4 (Definition 2 in [1])**.** A positive integer n is said to be a practical number if every positive integer less than n can be represented as a sum of distinct positive divisors of n.

Proposition 1.5 (Proposition 7 in [1])**.** *A positive integer n with the prime factorization $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ and $p_1 < p_2 < \cdots < p_m$ is a practical number if and only if $p_1 = 2$ and $p_{i+1} \leq \sigma(p_1^{k_1} \dots p_i^{k_i}) + 1$ for $1 \leq i \leq m - 1$*

Proposition 1.6 (Proposition 10 in [1])**.** *A practical number n is Zumkeller if and only if $\sigma(n)$ is even.*

Theorem 1.7. *Let ℓ be an even integer. Then there is a Zumkeller number a that $a + \ell$ is also a Zumkeller number*

Proof. Suppose that ℓ is a an even integer. Then there are $k \in \mathbb{N}$ and an even integer $0 \leq r < 18$ that $\ell = 18k + r$. By Example 1.3, we have:

2010 *Mathematics Subject Classification.* 11R04.
Key words and phrases. Zumkeller numbers, practical numbers, Zumkeller partition.

(1) If $r = 0$, then $a = 6$ and $a+\ell = 18k'+6$ are Zumkellers (k' is a nonnegative integer which varies according to a and r),
(2) If $r = 2$, then $a = 28$ and $a+\ell = 18k'+12$ are Zumkellers,
(3) If $r = 4$, then $a = 20$ and $a+\ell = 18k'+6$ are Zumkellers,
(4) If $r = 6$, then $a = 6$ and $a+\ell = 18k'+12$ are Zumkellers,
(5) If $r = 8$, then $a = 40$ and $a+\ell = 18k'+12$ are Zumkellers,
(6) If $r = 10$, then $a = 20$ and $a+\ell = 18k'+12$ are Zumkellers,
(7) If $r = 12$, then $a = 54$ and $a+\ell = 18k'+12$ are Zumkellers,
(8) If $r = 14$, then $a = 28$ and $a+\ell = 18k'+6$ are Zumkellers,
(9) If $r = 16$, then $a = 80$ and $a+\ell = 18k'+6$ are Zumkellers.

□

Theorem 1.8. *Let ℓ be an odd integer. Then there is a Zumkeller number a that $a+\ell$ is also Zumkeller.*

Proof. Suppose that ℓ is an odd integer. There is a prime number p which $p \nmid 945$ and $p \nmid \ell$. Let that t is an integer which $p < 2 \times 2^t - 1 = \sigma(2^t)$. Hence by Proposition 1.5 and Proposition 1.6, $2^t \times p$ is a Zumkeller number. Therefore, there are an odd integer r_1 that $1 \le r_1 < (2^t p)^2$ and $k_1 \in \mathbb{N}$ which $\ell = (2^t p)^2 k_1 + r_1$. Let $r_2 = 2^t p - r_1$. There are $1 \le r_3 < 945$ and $k_2 \in \mathbb{N}$ that $945 = (2^t p)^2 k_2 + r_3$ and since $gcd(r_3, 2^t p) = 1$, so there is $r_4 \in \mathbb{N}$ which $r_4 r_3 \equiv r_2 \pmod{(2^t p)^2}$ and $gcd(r_3, 2^t p) = 1$. Let $k_4 \in \mathbb{N}$, $q = (2^t p)^2 k_4 + r_4$ be a prime number, and $gcd(q, 945) = 1$ (by Dirichlet's Theorem we can find such a prime number). Since 945 is Zumkeller and $gcd(q, 945) = 1$, $a = q \times 945$ is a Zumkeller number. There is also $m \in \mathbb{N}$ that $gcd(m, 2^t q) = 1$ and $a + \ell = 2^t q m$. Hence, $a + \ell$ is a Zumkeller number. □

Proposition 1.9. *Let $(a_1, a_2, \ldots, a_k)$ be a k-tuple of Zumkeller numbers which $a_1 < a_2 < \cdots < a_k$ and for every $1 \le i \ne j \le k$, $\ell_{ij} = a_i - a_j$. there are infinitely many k-tuples of Zumkeller numbers like $(a'_1, a'_2, \ldots, a'_k)$ that $a'_1 < a'_2 < \cdots < a'_k$ and for every $1 \le i \ne j \le k$, $\ell_{ij} = a'_i - a'_j$*

Proof. Suppose that $a_1 < a_2 < \cdots < a_k$ are Zumkeller numbers and for every $1 \le i \ne j \le k$, $\ell_{ij} = a_i - a_j$. Then $a'_1 = a_1^n a_2^n \ldots a_k^n + a_1$, $a'_2 = a_1^n a_2^n \ldots a_k^n + a_2$, $\ldots$, $a'_k = a_1^n a_2^n \ldots a_k^n + a_k$ are Zumkeller numbers and for every $1 \le i \ne j \le k$, $\ell_{ij} = a'_i - a'_j$. □

Corollary 1.10. *For every $\ell \in \mathbb{N}$, there are infinitely many Zumkeller numbers like a which $a + \ell$ is also a Zumkellern number.*

Theorem 1.11 (See [2]). *Let a be a Zumkeller and b be the smallest Zumkeller number which is greater than a. Then, $b - a \le 12$*

Proof. Suppose that a is a Zumkeller number. There are $a, k \in \mathbb{N}$ that $a = 18k + r$ and $0 \le r < 18$. If $0 \le r \le 12$, then it is clear that there is $r' \in \mathbb{N}$ that $a + r' = 18k + 12$. Hence, by Example 1.3, it is a Zumkeller number. If $13 \le r \le 18$, then it is clear that there is a $r' \in \mathbb{N}$, $0 \le r' \le 12$ that $a + r' = 18(k+1) + 6$. Therefore, by Example 1.3 it is a Zumkeller number. □

Corollary 1.12 (See [2]). *The difference between consecutive Zumkeller numbers is at most 12.*

Remark 1.13. There are Zumkeller numbers a and b that b is the smallest Zumkeller-number which is greater than a and $b - a = 12$. For instance, $a = 222$ is a Zumkeller number. $b = 224$ is the smallest Zumkeller number which is greater that a and the difference between them is 12.

References

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Faculty of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: jokar.farid@gmail.com