Consecutive patterns in restricted permutations and involutions

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Abstract

It is well-known that the set $\mathbf{I}_n$ of involutions of the symmetric group $\mathbf{S}_n$ corresponds bijectively - by the Foata map F - to the set of n-permutations that avoid the two vincular patterns $1\underline{23}$ and $1\underline{32}$. We consider a bijection Γ from the set $\mathbf{S}_n$ to the set of histoires de Laguerre, namely, bicolored Motzkin paths with labelled steps, and study its properties when restricted to $\mathbf{S}_n(1\underline{23}, 1\underline{32})$. In particular, we show that the set $\mathbf{S}_n(\underline{123}, 132)$ of permutations that avoids the consecutive pattern $\underline{123}$ and the classical pattern 132 corresponds via Γ to the set of Motzkin paths, while its image under F is the set of restricted involutions $\mathbf{I}_n(3412)$. We exploit these results to determine the joint distribution of the statistics des and inv over $\mathbf{S}_n(\underline{123}, 132)$ and over $\mathbf{I}_n(3412)$. Moreover, we determine the distribution in these two sets of every consecutive pattern of length three. To this aim, we use a modified version of the well-known Goulden-Jacson cluster method.

Keywords: permutation pattern, involution, histoire de Laguerre, Motzkin path, cluster method.
MSC2010: 05A05, 05A15 (primary); 05A18 (secondary).

1 Introduction

In 1994 De Medicis and Viennot [5] introduced the definition of histoire de Laguerre, namely, a pair (d, l), where d is a Motzkin path of length n whose horizontal steps may have two different colors and $l = (l_1, \dots, l_n)$ is a sequence of non-negative integers with suitable constraints.

Many bijections are present in the literature between the set $\mathcal{H}_n$ of histoires de Laguerre and the symmetric group $\mathbf{S}_n$ (see [10] for a survey on this topic), as well as between a specific subset $\mathcal{L}_n$ of $\mathcal{H}_n$ and the set $\mathbf{I}_n$ of involutions in $\mathbf{S}_n$ (see, e.g., [2]). More recently, Claesson [3] proved that the set $\mathbf{I}_n$ corresponds bijectively - via the classical

Foata map - with the set $\mathbf{S}_n(1\underline{23}, 1\underline{32})$ of n-permutations that avoid the two vincular patterns $1\underline{23}$ and $1\underline{32}$.

In the first part of the present paper we consider the bijection Γ between the set of n-permutations and the set of historires de Laguerre described in [1], which is essentially the bijection defined in [4, p. 256], up to the reverse. This last bijection is in turn a slightly modified version of the well-known Françon-Viennot bijection [7]. We exploit Γ to connect the sets $\mathbf{S}_n(1\underline{23}, 1\underline{32})$, $\mathbf{I}_n$ and $\mathcal{L}_n$. More precisely we prove that the image under Γ of the set $\mathbf{S}_n(1\underline{23}, 1\underline{32})$ is precisely the set $\mathcal{L}_n$, namely, the set consisting of (uncolored) Motzkin paths whose down steps are labelled with an integer that does not exceed their height. Furthermore, we show that the bijection Ψ defined in [2] is nothing but the composition of the map F with Γ. Finally, the set of permutations avoiding the consecutive pattern $\underline{123}$ and the classical pattern 132 is mapped by Γ onto the set of unlabelled Motzkin paths and is mapped by F^{-1} onto the set of involutions avoiding the classical pattern 3412.

In the second part of the paper we exploit the properties of the maps Γ and Ψ to study in parallel some statistics over the two sets $\mathbf{S}_n(\underline{123}, 132)$ and $\mathbf{I}_n(3412)$. In particular, in both cases we determine the joint distribution of inversions and descents, as well as the distribution of the occurrences of every consecutive pattern of length three.

In many situations we take advantage of a particular instance of the Goulden-Jackson cluster method [8] for Motzkin paths. For the sake of completeness we describe this method in the Appendix.

2 The bijections

A *Motzkin path* of length n is a lattice path starting in $(0, 0)$, ending in $(n, 0)$, consisting of *up steps* U of the form $(1, 1)$, *down steps* D of the form $(1, -1)$ and *horizontal steps* H of the form $(1, 0)$ and lying weakly above the x-axis.

As usual, a Motzkin path can be identified with a *Motzkin word*, namely, a word $w = d_1 d_2 \dots d_n$ of length n in the alphabet $\{U, D, H\}$ with the constraint that the number of occurrences of the letter U is equal to the number of occurrences of the letter D and, for every i, the number of occurrences of U in the subword $d_1 d_2 \dots d_i$ is not smaller than the number of occurrences of D. In the following we will not distinguish between a Motzkin path and the corresponding word.

Now we consider the set of *bicolored* Motzkin paths, defined as Motzkin paths whose horizontal steps have two possible colors c_1 and c_2, such that horizontal steps lying on the x-axis cannot be colored with the color c_2. We will denote by H a horizontal step colored by c_1 and with $\widetilde{H}$ a horizontal step colored by c_2. It is well known that bicolored Motzkin paths are counted by Catalan numbers (see [12]).

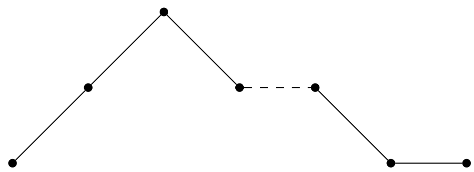
We will denote by $\mathcal{M}_n$ and $\mathcal{CM}_n$ the sets of Motzkin paths of length n and bicolored Motzkin paths of length n, respectively.

We associate to every $d = d_1 d_2 \dots d_n \in \mathcal{CM}_n$ the n-tuple $h(d) = (h_1, h_2, \dots, h_n)$, where for every $i = 1, \dots, n$, the integer h_i is defined as

$$\begin{cases} \text{the y-coordinate of the ending point of the step } d_i & \text{if } d_i \text{ is } D \\ \text{the y-coordinate of the starting point of the step } d_i & \text{otherwise} \end{cases}$$

We will call the integer h_i the *height* of the step d_i, and the n-tuple $h(d)$ the *height list* of d.

Example 2.1. Consider the bicolored Motzkin path $d = UUD\widetilde{H}DH$, namely,



where the horizontal step with color c_2 is represented by a dashed line. Then $h(d) = (0, 1, 1, 1, 0, 0)$.

We now describe a map from the set of permutations of length n to the set $\mathcal{CM}_n$. This map is a slight modification of the map described in [1] in terms of valued Dyck paths.

Let $\pi = \pi_1 \pi_2 \dots \pi_n$ be a permutation in $\mathbf{S}_n$ written in one-line notation. An *ascending run* in π is a maximal increasing subword of π. For example, the ascending runs of 346512 are $w_1 = 346$, $w_2 = 5$ and $w_3 = 12$. Write π as

$$\pi = w_1 w_2 \dots w_k,$$

where the $w_i's$ are the ascending runs in π. The first and the last element of an ascending run of length at least two are called a *head* and a *tail*, respectively. The only element of an ascending run of length one is called a *head-tail*. Every other element is called a *boarder*.

Now we associate to π a bicolored Motzkin path d of length n defined as follows. For $i = 1, \dots, n$,

- if i is a head-tail, set $d_i = H$;
- if i is a head, set $d_i = U$;
- if i is a tail, set $d_i = D$;
- if i is a boarder, set $d_i = \widetilde{H}$.

Then $d = d_1 d_2 \dots d_n$.

Obviously the correspondence $\gamma : \pi \to d$ is far from being injective. For example, both the permutations 3124 and 1243 in $\mathbf{S}_4$ correspond to the bicolored Motzkin path $U\widetilde{H}HD$. In order to get a bijection, we associate to the permutation π a pair (d, l), where d is the bicolored Motzkin path defined above and $l = (l_1, l_2, \dots, l_n)$ is the sequence of non-negative integers

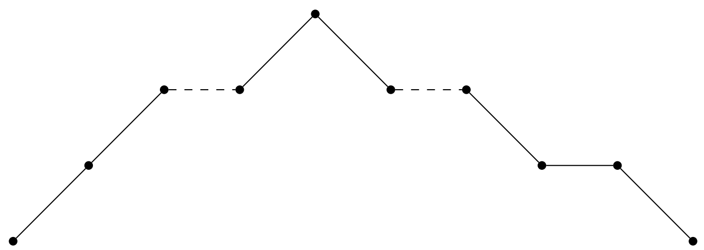
$$l_i = |\{j | s_j < i < t_j,\ t_j \text{ precedes } i \text{ in } \pi\}|$$

where s_j and t_j are the first and the last element of the j-th ascending run of π.

We denote by $\Gamma(\pi)$ the pair (d, l) associated with the permutation π.

Example 2.2. Consider the permutation $\pi = 826913547$. The ascending runs of π are $w_1 = 8$, $w_2 = 269$, $w_3 = 135$ and $w_4 = 47$.

We have $\Gamma(\pi) = (d, l)$, where $d = UU\widetilde{H}UD\widetilde{H}DHD =$



and $l = (0, 0, 1, 2, 1, 0, 1, 0, 0)$.

We observe that the above bijection is essentially the map defined in [4, p. 256], up to the reverse.

We recall that a pair (d, l), where d is a bicolored Motzkin path of length n and $l = (l_1, \dots, l_n)$ is a sequence of non-negative integers, is called a *histoire de Laguerre* provided that $l_i \le h_i$ for all $1 \le i \le n$, where h_i is the i-th element of the height list of d (see [5]). We denote by $\mathcal{H}_n$ the set of histoires de Laguerre of length n.

Theorem 2.3. *The map Γ is a bijection between $\mathbf{S}_n$ and $\mathcal{H}_n$.*

Proof. See [1, Theorem 2.6]. $\square$

We now describe the connection between the map Γ defined above and a bijection Ψ between the set $\mathbf{I}_n$ of involutions of length n and labeled Motzkin paths studied in [2]. In order to do this, we exploit a result proved by Claesson [3], namely, the fact that the classical Foata map induces a bijection between the set of involutions of length n and the set of permutations of the same length that avoid two vincular patterns.

Let $\pi \in \mathbf{S}_n$ and $\tau \in \mathbf{S}_m$. We say that $\pi = \pi_1 \dots \pi_n$ *contains the pattern* $\tau = \tau_1 \dots \tau_m$ *in the classical sense* if there exists an index subsequence $1 \le i_1 < i_2 < \dots < i_m \le n$

such that the words $\pi_{i_1}\pi_{i_2}\dots\pi_{i_m}$ and $\tau_1\tau_2\dots\tau_m$ are order isomorphic. Otherwise, π *avoids the pattern* τ.

A *vincular pattern* is a permutation τ in $\mathbf{S}_m$ some of whose consecutive letters may be underlined. If τ contains $\underline{\tau_i\tau_{i+1}\dots\tau_j}$ as a subword then the letters corresponding to $\tau_i, \tau_{i+1}, \dots, \tau_j$ in an occurrence of τ in a permutation σ must be adjacent, whereas there is no adjacency condition for non underlined consecutive letters (see [9, p. 10]).

For example, the permutation 431256 contains two occurrences of the vincular pattern $2\underline{13}$, namely, 425 and 325. Note that a vincular pattern without underlined letters is a pattern in the classical sense. On the other hand, the occurrences of a vincular pattern all of whose letters are underlined must be formed by adjacent letters. The set of permutations of length n that avoid the vincular pattern τ is denoted by $\mathbf{S}_n(\tau)$.

We now recall Claesson's result. Let π be an involution. Write π in *standard cycle notation,* i.e., so that each cycle is written with its least element first and the cycles are written in decreasing order of their least element. Define $F(\pi)$ to be the permutation obtained from π by erasing the parentheses separating the cycles. As an example consider $\pi = 47318625 \in \mathbf{I}_8$. The cycle notation for π is $(6)(5,8)(3)(2,7)(1,4)$ and $F(\pi) = 65832714$.

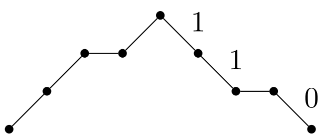
In [3] Claesson proved that the map F is a bijection between $\mathbf{I}_n$ and $\mathbf{S}_n(1\underline{32}, 1\underline{23})$. It is easily seen that this last set coincides with $\mathbf{S}_n(\underline{132}, \underline{123})$ (see [6]). On the other hand, in [2] the authors define a bijection Φ between the set $\mathbf{I}_n$ and the set of *labelled Motzkin paths* of length n, namely, Motzkin paths whose down steps are labelled with an integer that does not exceed their height, while the other steps are unlabelled. The set of labelled Motzkin paths of length n will be denoted by $\mathcal{L}_n$. Of course $\mathcal{L}_n \subset \mathcal{H}_n$.

In the present paper we need a bijection Ψ that is a slightly modified version of the bijection Φ. The map Ψ can be described as follows. Let $\pi \in \mathbf{I}_n$.

For every $i = 1, \dots, n$:

- if i is a fixed point for π , draw a horizontal step;
- if i is the first element in a 2-cycle, draw an up step;
- if i is the second element in a 2-cycle (j, i), draw a down step. Label this step with h, where h is the number of cycles (x, y) of π such that $j < x < i < y$.

For example, consider the involution $\pi = 65382174$ whose standard cycle notation is $(7)(48)(3)(25)(16)$. Then

$\Psi(\pi) =$ 

Our next aim is to prove the following result.

Theorem 2.4. *The image under Γ of the set $\mathbf{S}_n(\underline{132}, \underline{123})$ is $\mathcal{L}_n$, and the following diagram*

$$\begin{tikzcd}
\mathbf{I}_n \arrow[rr, "\Psi"] \arrow[dr, "F"'] & & \mathcal{L}_n \\
& \mathbf{S}_n(\underline{132}, \underline{123}) \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

commutes.

First of all, we characterize the image of the map Γ, when restricted to the set $\mathbf{S}_n(\underline{132}, \underline{123})$.

Proposition 2.5. *Let $\pi \in \mathbf{S}_n$ and let $\Gamma(\pi) = (d, l)$. Then $\pi \in \mathbf{S}_n(\underline{132}, \underline{123})$ if and only if*

- *the path $d = d_1 \dots d_n$ has no horizontal steps of color c_2, and*
- *for every index i, $l_i > 0$ implies that d_i is a down step.*

Proof. Firstly note that π avoids the pattern $\underline{123}$ if and only if the ascending runs of π have length at most two. In this case the set of boarders of π is empty and in d there are no horizontal steps of color c_2.

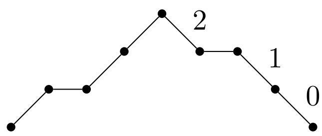
Now let $\pi \in \mathbf{S}_n(\underline{132}, \underline{123})$. Suppose that there exists an integer i such that $l_i > 0$ and d_i is either an up step or a horizontal step. By definition of the sequence l, this implies that the permutation π contains three elements π_s, π_{s+1}, π_r, with $\pi_r = i$, such that

- $r > s + 1$,
- π_s and π_{s+1} are the head and the tail of an ascending run,
- $\pi_s < \pi_r < \pi_{s+1}$,
- π_r is either a head or a head-tail.

If π_r is a head, then π_{r+1} is the corresponding tail and π_s, π_r, π_{r+1} form an occurrence of $1\underline{23}$. If π_r is a head-tail, $\pi_{r-1} > \pi_r$ and π_s, π_{r-1}, π_r form an occurrence of $1\underline{32}$. But this is impossible since, as noted above, $\mathbf{S}_n(\underline{132}, \underline{123}) = \mathbf{S}_n(1\underline{32}, 1\underline{23})$. On the other hand, if the permutation π contains an occurrence of the pattern $\underline{132}$ corresponding to the elements $\pi_j, \pi_{j+1}, \pi_{j+2}$, then $\pi_j < \pi_{j+2} < \pi_{j+1}$, π_{j+2} is a head or a head-tail, and π_j, π_{j+1} are the head and the tail of an ascending run. Hence $l_{\pi_{j+2}} > 0$. $\square$

Theorem 2.4 now follows immediately from the description of the maps F, Γ and Ψ and from the previous Proposition.

As an example consider the involution $\pi = (6)(48)(37)(2)(15) \in \mathbf{I}_8$. Then the corresponding permutation in $\mathbf{S}_8(\underline{132}, \underline{123})$ is $F(\pi) = 64837215$, and

$\Psi(\pi) =$  $= \Gamma(F(\pi))$.

The subset of $\mathcal{L}_n$ of labelled Motzkin paths of length n all of whose labels are zero is obviously isomorphic to the set $\mathbf{M}_n$ of Motzkin paths. It is possible to characterize the preimage of this set under the maps Γ and Ψ in terms of pattern avoiding permutations. In fact, in [1] the following result is proved.

Proposition 2.6. *Let $\pi \in \mathbf{S}_n$ and let $\Gamma(\pi) = (d, l)$. Then*

$$\pi \in \mathbf{S}_n(132) \textit{ if and only if } l = (0, \ldots, 0).$$

As a consequence Γ induces a bijection between $S_n(132, \underline{123})$ and $\mathbf{M}_n$.

Moreover, in [2, Theorem 9] it has been shown that the map Ψ induces a bijection between the set of involutions avoiding the pattern 3412 and the set of labelled Motzkin paths of length n all of whose labels are zero. These results imply that the following diagram

$$\begin{tikzcd}
\mathbf{I}_n(3412) \arrow[rr, "\Psi"] \arrow[dr, "F"'] & & \mathbf{M}_n \\
& \mathbf{S}_n(132, \underline{123}) \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

commutes.

In the following sections we show how some statistics over the sets $\mathbf{S}_n(132, \underline{123})$ and $\mathbf{I}_n(3412)$ can be translated into statistics over Motzkin paths.

3 Inversions and descents over $\mathbf{I}_n(3412)$

Let d be a Motzkin path. A *tunnel* in d is a horizontal segment between two lattice points of d lying weakly below d and containing exactly two lattice points of d. Note that each horizontal step of d is a tunnel. We will call the horizontal steps *trivial tunnels.*

We recall that each non-empty Motzkin path m can be decomposed either as Hm', where m' is an arbitrary Motzkin path, or as $Um'Dm''$, where m' and m'' are arbitrary Motzkin paths. This decomposition is called *first return decomposition.* The definition of the map Ψ implies that each 2-cycle of an involution $\pi \in \mathbf{I}_n(3412)$ corresponds to a non-trivial tunnel of $\Psi(\pi)$ and vice-versa.

Let $\pi = \pi_1 \dots \pi_n \in \mathbf{S}_n$. An *inversion* of π is a pair (i, j) with $i < j$ such that $\pi_i > \pi_j$. In this case we will say that the symbol π_i *is in inversion* with the symbol π_j. The number of inversions of the permutation π will be denoted by $\mathrm{inv}(\pi)$.

The permutation π has a *descent* at position i if $\pi_i > \pi_{i+1}$. Otherwise, π has an *ascent* at position i.The number of descents of π will be denoted by $\mathrm{des}(\pi)$.

Now we want to study the joint distribution of the statistics inv, des, fix over the set

$$\mathbf{I}(3412) := \bigcup_{n\geq 0} \mathbf{I}_n(3412),$$

where $\mathrm{fix}(\pi)$ denotes the number of fixed points of π, namely, determine an expression for the generating function

$$F(x, y, z, w) = \sum_{n\geq 0} \sum_{\pi\in\mathbf{I}_n(3412)} x^n y^{\mathrm{inv}(\pi)} z^{\mathrm{des}(\pi)} w^{\mathrm{fix}(\pi)}. \tag{1}$$

First of all we prove a preliminary result.

Lemma 3.1. *Let $\pi \in \mathbf{I}_n(3412)$. Then*

$$inv(\pi) = 2A - t,$$

where A is the area between the path $\Psi(\pi)$ and the x-axis and t is the number of non-trivial tunnels of $\Psi(\pi)$.

Proof. Write π as $\pi_1\pi_2 \dots \pi_n$. Let i be the least index such that $\pi_i > i$. Then (i, π_i) is a cycle of π. Hence the symbol π_i is in inversion with all the symbols $\pi_{i+1}, \pi_{i+2} \dots \pi_{i+k} = i$, where $k = \pi_i - i$. In fact, suppose by contradiction that there exist an index r, with $1 \leq r \leq k - 1$, such that $\pi_{i+r} > \pi_i$, then $\pi_i, \pi_{i+r}, i, r.$ would be an occurrence of the pattern 3412.

For the same reason the symbol i is in inversion with the $k-1$ elements $\pi_{i+1}, \dots, \pi_{i+k-1}$. Here we excluded the inversion (i, π_i).

On the other hand, let T be the tunnel in the Motzkin path $\Psi(\pi)$ corresponding to the cycle (i, π_i). The area of the trapezoid with height one and T as a basis is precisely $k = \pi_i - i$.

Repeating the preceding argument on the involution obtained from π by deleting the symbols i and π_i we get the assertion. □

Lemma 3.2. *Let $\pi \in \mathbf{I}_n(3412)$. The descents of π correspond bijectively to the occurrences in $\Psi(\pi)$ of the following subwords: UU, DD, UH, HD and UD.*

Proof. Suppose that one of these subwords occurs in the Motzkin path $\Psi(\pi)$. Let v be this subword and i be the position of the first step of v. If $v = UU$, then both π_i and π_{i+1} are the greater elements in their respective 2-cycles and hence $\pi_i > \pi_{i+1}$. If $v = UH$, then π_i is the greater element in its 2-cycle while π_{i+1} is a fixed point and hence $\pi_i > \pi_{i+1}$. The other cases can be treated in a similar way. □

We recall that a *weak valley* of a Motzkin path is an occurrence of one of the following consecutive patterns:

$$HH, \quad HU, \quad DH, \quad DU.$$

The preceding result yields immediately:

Corollary 3.3. *The distribution of ascents over n-involutions avoiding the pattern 3412 is the same as the distribution of weak valleys over Motzkin paths of length n.*

This implies that the generating function $G(x,z)$ of Motzkin paths according to the length (x) and the number of weak valleys (z) can be deduced from the function $F(x,y,z,w)$ appearing in Formula (1) as follows:

$$G(x,z) = 1 + \frac{F(xz,1,1/z,1)-1}{z}.$$

Similarly, $1+\frac{F(xz,1,1/z,0)-1}{z}$ gives the generating function of Dyck paths according to the length and the number of valleys, which is essentially given by Narayana polynomials.

The above lemmas imply that the generating function F satisfies the following equation obtained by the first return decomposition for Motzkin paths.

$$\begin{aligned} F(x,y,z,w) = 1 &+ xwF(x,y,z,w) + x^2yzF(x,y,z,w) \\ &+ x^2yz^2(F(xy^2,y,z,w)-1)\cdot F(x,y,z,w). \end{aligned} \tag{2}$$

In fact, the terms in the right hand side of the previous equation correspond to Motzkin paths either empty or of the form Hm, UDm, $Um'Dm$, with m' non-empty, respectively.

From equation (2) we get easily the following continued fraction expression for F.

Theorem 3.4.

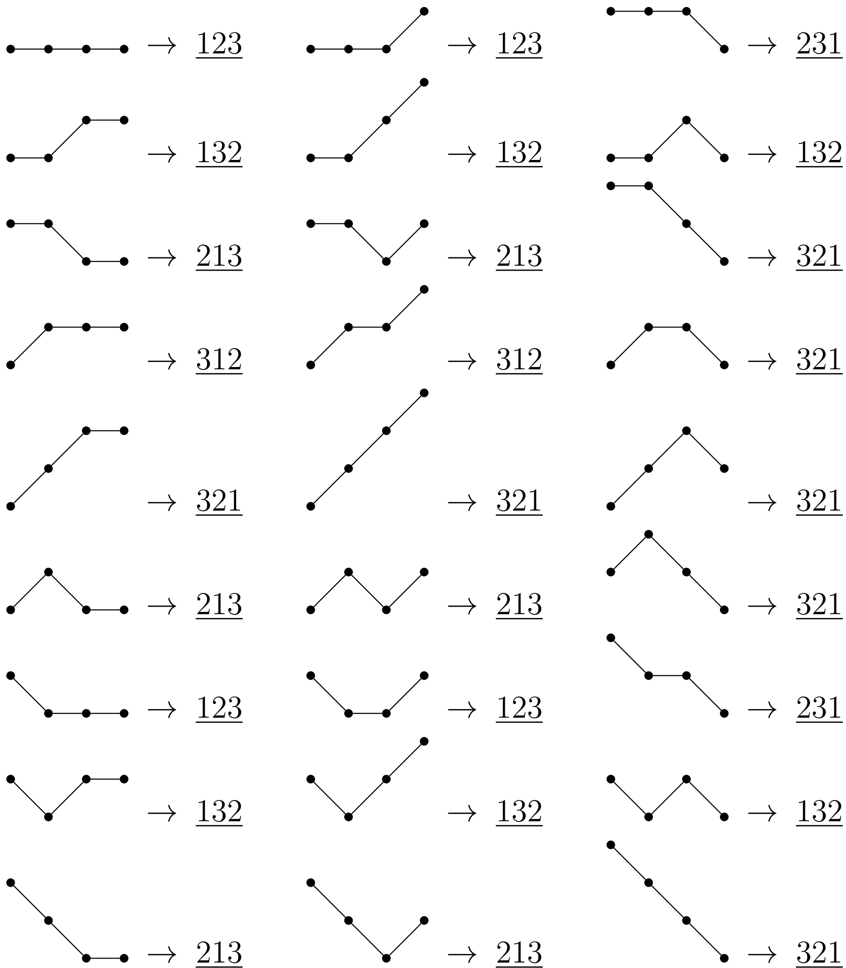
$$\begin{aligned} F(x,y,z,w) =& \frac{1}{1-xw-x^2yz+x^2yz^2-x^2yz^2F(xy^2,y,z,w)} = \\ =& \cfrac{1}{1+b_0-\cfrac{c_0}{1+b_1-\cfrac{c_1}{1+b_2-\cfrac{c_2}{1+\dots}}}} \end{aligned}$$

where $b_i = -xy^{2i}w - x^2y^{2i+1}z + x^2y^{2i+1}z^2$ and $c_i = x^2y^{2i+1}z^2$, $i \geq 0$.

4 The distribution of consecutive patterns in $\mathbf{I}_n(3412)$

Lemma 3.2 allows us to translate every three-letter subword of a Motzkin path into an occurrence of a consecutive pattern of the corresponding involution in $\mathbf{I}_n(3412)$.

Theorem 4.1. *Let $\pi \in \mathbf{I}_n(3412)$ and let $\Psi(\pi)$ be the corresponding Motzkin path. Then a subword of $\Psi(\pi)$ of length three corresponds to an occurrence of a consecutive pattern in π according to the following table.*



Now we enumerate the involutions in $\mathbf{I}_n(3412)$ according to the occurrences of a given consecutive pattern of length three and the number of fixed points.

First of all, observe that an involution $\pi \in \mathbf{I}_n(3412)$ has k occurrences of $\underline{213}$ ($\underline{312}$) and f fixed points if and only if $RC(\pi)$ has k occurrences of $\underline{132}$ ($\underline{231}$, respectively) and f fixed points, where $RC(\pi)$ is the *reverse-complement* of the permutation π, namely, $RC(\pi = \pi_1 \dots \pi_n) = n+1-\pi_n \dots n+1-\pi_1$.

Hence we can restrict our attention to the consecutive patterns $\underline{123}$, $\underline{321}$, $\underline{132}$ and $\underline{231}$.

4.1 The pattern $\underline{123}$

Let $a_{n,k,f}$ be the number of involutions $\pi \in \mathbf{I}_n(3412)$ with k occurrences of $\underline{123}$ and with f fixed points.

Our goal is to find a formula for the generating function

$$F = \sum a_{n,k,f} x^n t^k z^f.$$

To this aim we use a variation of the Goulden-Jackson cluster method (see [8, p. 128]). In the Appendix we give a detailed description of the notations and the results that will be used.

By Theorem 4.1, the pattern $\underline{123}$ in π corresponds to occurrences in $\Psi(\pi)$ of subwords in the set $S = \{H^3, H^2U, DH^2, DHU\}$. These subwords give rise to the clusters of type H^j with $j \geq 3$, H^jU with $j \geq 2$, DH^j with $j \geq 2$, and DH^jU with $j \geq 1$.

Note that

- the cluster H^j reduces to a horizontal step and has depth 0,
- the cluster H^jU reduces to an up step and has depth 0,
- the cluster DH^j reduces to a down step and has depth 0,
- the cluster DH^jU reduces to a horizontal step and has depth -1.

To find F we will use Theorem 7.1 of the Appendix.

First of all, we determine the generating functions $A_H(x,t,z)$, $A'_H(x,t,z)$, $A_D(x,t,z)$, $A'_D(x,t,z)$, $A_U(x,t,z)$ and $A'_U(x,t,z)$.

Consider the cluster H^j, $j \geq 5$. This can be obtained by either juxtaposing a horizontal step to the right of H^{j-1} and adding an occurrence of the subword H^3 that covers the last two letters of H^{j-1}

$$H^j = \underbrace{HH\ldots \overbrace{HH}}_{H^{j-1}}\!\!\overbrace{H}^{H^3}$$

or juxtaposing two horizontal steps to the right of H^{j-2} and adding an occurrence of the subword H^3 that covers the last letter of H^{j-2}

$$H^j = \underbrace{HH\ldots H}_{H^{j-2}}\!\!\overbrace{HH}^{H^3}.$$

Note that in these two cases the number of occurrences of H^3 increases by one, the number of horizontal steps and the length increase by one in the first case and by two in the second case. As a consequence, the generating function for clusters of this kind is

$$\frac{x^3z^3t}{1 - xzt - x^2z^2t},$$

where the variables x, t, and z represent length, number of occurrences of the subwords in S and number of H, respectively.

Similarly, the cluster DH^j, $j \geq 4$, can be obtained in the two ways depicted below

$$DH^j = \underbrace{DH\ldots \mathrlap{\overbrace{}^{H^3}}HH}_{DH^{j-1}}H$$

or

$$DH^j = \underbrace{DH\ldots \mathrlap{\overbrace{}^{H^3}}H}_{DH^{j-2}}HH$$

hence, the corresponding generating function is

$$\frac{x^3z^2t}{1 - xzt - x^2z^2t}. \tag{3}$$

By similar arguments the generating function for the cluster H^jU is

$$\frac{x^3z^2t}{1 - xzt - x^2z^2t}.$$

Lastly, the cluster DH^jU, $j \geq 3$ can be obtained by either juxtaposing the letter U to the right of DH^j and adding an occurrence of H^2U that covers the last two letters of DH^{j-1}, or juxtaposing the letters HU to the right of DH^{j-1} and adding an occurrence of H^2U that covers the last letter of DH^{j-1}. By formula (3) we get the following expression for the generating function of the cluster of the form DH^jU, $j \geq 2$

$$\frac{x^3z^2t}{1 - xzt - x^2z^2t} \cdot (xt + x^2tz)$$

The cluster DHU must be considered separately. Its contribution is x^3tz.

As a consequence we have

$$A_H(x, t, k) = \frac{x^3z^2t(z + xt + x^2tz)}{1 - xzt - x^2z^2t} + x^3tz,$$

$$A'_H(x, t, z) = \frac{x^3z^3t}{1 - xzt - x^2z^2t}$$

and

$$A_D(x, t, z) = A'_D(x, t, z) = A_U(x, t, z) = A'_U(x, t, z) = \frac{x^3z^2t}{1 - xzt - x^2z^2t}.$$

Now we are in position to apply Theorem 7.1, hence finding the generating function $F_{\underline{123}}$ evaluated in $x, t{+}1, z$. After the substitution $t \leftarrow t{-}1$ we get the following expression for $F_{\underline{123}}(x, t, z)$.

Theorem 4.2.

$$F_{\underline{123}}(x,t,z) = \frac{2A^2}{2(1-B)A^2 - A^2 + A^2C + A^2\sqrt{(1-C)^2 - 4A^2}}$$

where

$$A = x + \frac{x^3z^2(t-1)}{1 - xz(t-1) - x^2z^2(t-1)},$$

$$B = xz + \frac{x^3z^3(t-1)}{1 - xz(t-1) - x^2z^2(t-1)}$$

and

$$C = xz + \frac{x^3z^2(t-1)(z + x(t-1) + x^2(t-1)z)}{1 - xz(t-1) - x^2z^2(t-1)} + x^3(t-1)z.$$

4.2 The pattern $\underline{132}$

Now we consider the pattern $\underline{132}$. By Theorem 4.1, an occurrence of this pattern in $\pi \in \mathbf{I}_n(3412)$ corresponds to six possible subwords in $\Psi(\pi)$, namely, DUY and HUY', where Y and Y' can be any letters in $\{U, D, H\}$. The occurrences of such words correspond to the occurrences of DU and HU.

Also in this case we use the cluster method. Here we have $S = \{HU, DU\}$. Note that the only possible clusters formed by these two words are HU and DU themselves. The first of these two clusters reduces to an up step and has depth 0, the second one reduces to a horizontal step and has depth -1. Hence we have

$$A_H(x,t,k) = x^2t,$$

$$A_U(x,t,z) = A'_U(x,t,z) = x^2tz$$

and

$$A_D(x,t,z) = A'_D(x,t,z) = A'_H(x,t,z) = 0.$$

Theorem 7.1 allows us to determine $F_{\underline{132}}(x, t+1, z)$. After the substitution $t \leftarrow t-1$, we get

Theorem 4.3.

$$F_{\underline{132}}(x,t,z) = \frac{2}{1 - xz + x^2t - x^2 + \sqrt{(1 - xz - x^2(t-1))^2 - 4x(x + x^2z(t-1))}}.$$

Notice that this generating function in the case $z = 1$ encodes the distribution of *weakly descending subpaths* over the set of Motzkin paths (see sequence A114690 in [11]), where a weakly descending subpath is a maximal subword consisting of H and D steps. In fact, every occurrence either of HU or DU breaks a weakly descending subpath. Hence, in every Motzkin path the number of weakly descending subpaths equals the number of occurrences of these two patterns increased by one.

4.3 The pattern $\underline{321}$

Theorem 4.1 shows that the occurrences of the pattern $\underline{321}$ in π correspond to the occurrences of UUX, $X'DD$ and UHD in $\Psi(\pi)$, where X and X' can be arbitrary letters in $\{U, H, D\}$. Hence the occurrences of this pattern corresponds to the occurrences of UU, DD and UHD in $\Psi(\pi)$.

Let $F_{\underline{321}}(x, t, z)$ be the corresponding generating function.

Denote by X_w and Y_w be the first and last step of a Motzkin path w. We define

- $A(x, t, z)$ to be the generating function of the set of Motzkin paths such that $(X_w, Y_w) = (U, D)$,
- $B(x, t, z)$ to be the generating function of the set of Motzkin paths such that (X_w, Y_w) is either (U, H) or (H, D),
- $C(x, t, z)$ to be the generating function of the set of Motzkin paths such that $(X_w, Y_w) = (H, H)$.

Observe that the first return decomposition implies that

$$F_{\underline{321}} = 1 + xzF_{\underline{321}} + x^2(1 + xzt + Bt + At^2 + C)F_{\underline{321}}. \tag{4}$$

A simple inclusion-exclusion argument yields

$$A = F_{\underline{321}} - 2xzF_{\underline{321}} + (xz)^2F_{\underline{321}} + xz - 1.$$

Moreover it is easily seen that

$$B = 2A\frac{xz}{1 - xz} = 2(F_{\underline{321}} - 2xzF_{\underline{321}} + (xz)^2F_{\underline{321}} + xz - 1)\frac{xz}{1 - xz} = 2xz(F_{\underline{321}} - xzF_{\underline{321}} - 1)$$

and

$$C = (xz)^2F_{\underline{321}}.$$

Substituting in (4), we get

Theorem 4.4. $F_{\underline{321}}$ *satisfies the following functional equation*

$$aF_{\underline{321}}^2 + bF_{\underline{321}} + c = 0,$$

where

$$a = 2x^3zt - 2x^4z^2t + x^2t^2 - 2x^3zt^2 + x^4z^2t^2 + x^4z^2,$$
$$b = -1 + xz - x^3zt - x^2t^2 + x^2 + x^3zt^2, \qquad c = 1.$$

4.4 The pattern $\underline{312}$

This pattern correspond to occurrences of UHH and UHU in $\Psi(\pi)$. Let $F(x,t,z)$ be the corresponding generating function.

Set

$$G(x,t_1,t_2,z) = \sum_n \sum_{d\in\mathbf{M}_n} x^n t_1^{o(UH)} t_2^{o(UHD)} z^{o(H)},$$

where $o(UH), o(UHD)$ and $o(H)$ denote the number of occurrences of the subwords UH, UHD and H in d.

By the first return decomposition we get the following recurrence for G.

$$G = 1 + xzG + x^2G + x^3t_1t_2zG + x^3zt_1G(G-1) + x^2G(G - xzG - 1).$$

In fact, a Motzkin path can either

- be empty, or
- start by UD, UHD, or
- be of the form $UHmDd$ or $Um'Dd$, where m is a non empty Motzkin path, m' is a non empty path starting with U, and d an arbitrary path.

Hence we get a functional equation satisfied by $F_{\underline{312}}(x,t,z)$ substituting in the previous equation $t_1 \leftarrow t$ and $t_2 \leftarrow \frac{1}{t}$:

$$F_{\underline{312}} = 1 + xzF_{\underline{312}} + x^2F_{\underline{312}} + x^3zF_{\underline{312}} + x^3ztF_{\underline{312}}(F_{\underline{312}} - 1) + x^2F_{\underline{312}}(F_{\underline{312}} - xzF_{\underline{312}} - 1),$$

namely,

Theorem 4.5. *The generating function $F_{\underline{312}}$ satisfies the following functional equation*

$$aF_{\underline{312}}^2 + bF_{\underline{312}} + c = 0,$$

where

$$a = x^3zt + x^2 - x^3z,$$
$$b = xz + x^2 + x^3z - x^3zt - x^2 - 1, \qquad c = 1.$$

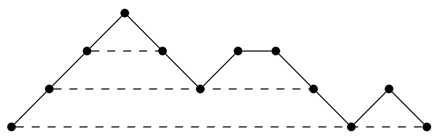
5 Inversions and descents over $\mathbf{S}_n(132, \underline{123})$

We now turn to the case of permutations in $\mathbf{S}_n(132, \underline{123})$.

First of all we recall that, given a permutation $\pi \in \mathbf{S}_n(132, \underline{123})$, if $\pi = w_1 \dots w_k$ is the decomposition of π into ascending runs, then the $w_i's$ have length at most 2 and the sequence of the heads of π is a decreasing sequence. Moreover, the inverse of the map Γ has an easy description in terms of tunnels of the Motzkin path, as in the case of the map Ψ.

Proposition 5.1. *Let d be a Motzkin path and π the corresponding permutation in $\mathbf{S}_n(132, \underline{123})$. Let $t_1 t_2 \dots t_k$ be the sequence of tunnels of d, listed in decreasing order of the x-coordinate of their leftmost point. The decomposition of π into ascending runs is $\pi = w_1 w_2 \dots w_k$ with $w_i = x_i x_i'$, where x_i is the x-coordinate of the first point of t_i, increased by one, and x_i' is the x-coordinate of the last point of t_i.*

As an example consider the following Motzkin path

$d =$ 

The sequence of tunnels of d is given by 9-11, 6-7, 5-8, 2-4, 1-5, 0-9, where each tunnel is represented by the x-coordinates of its first and last point. Hence the corresponding permutation is $\pi = 10\,11\,7\,6\,8\,3\,4\,2\,5\,1\,9$.

Recall that a *coinversion* in a permutation π is a pair (i, j) such that $i < j$ and $\pi_i < \pi_j$. The number of coinversions of a permutation π will be denoted by $\mathrm{coinv}(\pi)$. Of course a permutation π has k coinversions if and only if it has $\binom{n}{2} - k$ inversions.

Now we are interested in the generating function for permutations in $\mathbf{S}(132, \underline{123}) := \cup_{n \geq 0} S_n(132, \underline{123})$ enumerated by number of coinversions and number of descents:

$$F(x, y, z) = \sum_{\pi \in \mathbf{S}(132, \underline{123})} x^n y^{\mathrm{coinv}(\pi)} z^{\mathrm{des}(\pi)}.$$

We have the following.

Proposition 5.2. *Let $\pi \in \mathbf{S}_n(132, \underline{123})$ and let $\Gamma(\pi)$ be the corresponding Motzkin path. Then*

- *$coinv(\pi)$ is the area of $\Gamma(\pi)$ and*
- *$des(\pi)$ is equal to the number of tunnels of $\Gamma(\pi)$ minus one.*

Proof. Let (i, j) be a coinversion of the permutation π. Since the sequence of heads of π is decreasing, π_j is a tail, hence it corresponds to a down step in $\Gamma(\pi)$. Furthermore, given a down step $\bar{D}$ in $\Gamma(\pi)$ at position k, consider the up step $\bar{U}$ that forms a tunnel with $\bar{D}$, and denote by h the position of $\bar{U}$. Then, by the construction of the map Γ, the coinversions of π having k as second element are precisely (x, k) where $h \leq x \leq k - 1$. The number of such elements equals the area of the trapezoid determined by the tunnel between $\bar{U}$ and $\bar{D}$.

The second statement follows immediately form the fact that every descent in π corresponds to a non initial head or head-tail. $\square$

The above Proposition and the first return decomposition for Motzkin paths yield the following recurrence equation for the generating function F.

$$\begin{aligned} F(x,y,z) = 1 + x + zx(F(x,y,z)-1) + yz^2x^2(F(x,y,xz)-1)(F(x,y,z)-1) \\ + yzx^2(F(x,y,z)-1) + yzx^2(F(x,y,xz)-1) + yx^2. \end{aligned} \tag{5}$$

Notice that $F(x,1,z)$ is the generating function of sequence A107131 in [11], while $F(x,y,1)$ is the generating function of sequence A129181 in [11].

6 The distribution of consecutive patterns in $\mathbf{S}_n(132, \underline{123})$

Now we enumerate permutations $\pi \in \mathbf{S}_n(132, \underline{123})$ by the number of occurrences of a consecutive pattern of length three. Needless to say, we consider only the patterns $\underline{213}$, $\underline{231}$, $\underline{312}$ and $\underline{321}$.

6.1 The pattern $\underline{213}$

Let $F_{\underline{213}}(x,t)$ be the generating function of permutations $\pi \in \mathbf{S}_n(132, \underline{123})$ enumerated by length and number of occurrences of $\underline{213}$.

Note that an occurrence of this pattern in a permutation π corresponds to an occurrence in $\Gamma(\pi)$ of a sequence of the form $U\alpha D$, where α is any non-empty Motzkin path. We call such a sequence a *long tunnel*.

In fact, an occurrence of $\underline{213}$ in π is a sequence of consecutive letters bac, with $a < b < c$. Here, ac is an ascending run w_{i+1}, while b is either the tail or the head-tail of the preceding ascending run w_i.

By Proposition 5.1, w_i and w_{i+1} correspond to two tunnels t_i, t_{i+1} such that t_i lies above t_{i+1}. Hence, the occurrence bac of the pattern $\underline{213}$ corresponds to the long tunnel t_{i+1}.

Let $\widehat{F}(x,t,y)$ be the generating function for Motzkin paths enumerated by length (x), occurrences of long tunnels (t) and peaks (y), i.e., occurrences of the sequence UD.

Notice that each non-empty Motzkin path can be either a horizontal step followed by any Motzkin path, or a peak followed by any Motzkin path, or a long tunnel followed by any Motzkin path. Hence, the generating function $\widehat{F}(x,t,y)$ satisfies

$$\widehat{F}(x,t,y) = 1 + x\widehat{F}(x,t,y) + x^2y\widehat{F}(x,t,y) + x^2t(\widehat{F}(x,t,y)-1)\widehat{F}(x,t,y).$$

With the substitution $y \leftarrow 1$ we get

Theorem 6.1. *The generating function $F_{\underline{213}}(x,t)$ satisfies the following equation:*

$$aF_{\underline{213}}^2 + bF_{\underline{213}} + 1 = 0,$$

where

$$a = x^2t, \qquad \text{and} \qquad b = -1 + x + x^2 - x^2t.$$

6.2 The pattern $\underline{231}$

By Proposition 5.1 an occurrence in π of the pattern $\underline{231}$ corresponds to an occurrence in $\Gamma(\pi)$ of an up step in a non-initial position. Let $\widehat{F}(x,t,y)$ be the generating function of Motzkin paths enumerated by length (x), number of non-initial up steps (t), number of initial up steps (y).

We have the following recurrence for $\widehat{F}(x,t,y)$:

$$\widehat{F}(x,t,y) = 1 + x\widehat{F}(x,t,t) + x^2y\widehat{F}(x,t,t). \tag{6}$$

In fact, every non-empty Motzkin d path can be decomposed either as Hm, or $Um'Dm''$, where m, m', and m'' are arbitrary Motzkin paths. Note that each up step in m, m', or m'' cannot be at the initial position of d.

Substituting $y \leftarrow t$ in (6) we find an expression for $\widehat{F}(x,t,t)$:

$$\widehat{F}(x,t,t) = \frac{1 - x - \sqrt{1 + x^2 - 2x - 4x^2t}}{2x^2t}.$$

Substituting this expression in (6) and then replacing $y \leftarrow 1$ we get an expression for the generating function for permutations $\pi \in \mathbf{S}_n(132, \underline{123})$ enumerated by length (x) and number of occurrences of $\underline{231}$ (t):

$$F_{\underline{231}}(x,t) = \widehat{F}(x,t,1) = 1 + \widehat{F}(x,t,t)(x + x^2\widehat{F}(x,t,t)).$$

6.3 The pattern $\underline{312}$

An occurrence of the pattern $\underline{312}$ corresponds to an occurrence in $\Gamma(\pi)$ of a peak $p = UD$ such that $\Gamma(\pi) = \alpha p \beta$ where $\beta \neq D^k$, $k \geq 0$. We call such peak a *non-final peak*.

Let $\widehat{F}(x,t,y)$ be the generating function for Motzkin paths enumerated by length (x), number of non-final peaks (t), number of final peaks (y).

The first return decomposition implies that

$$\begin{aligned}
\widehat{F}(x,t,y) &= \\
&= 1 + x\widehat{F}(x,t,y) + x^2y + x^2(\widehat{F}(x,t,y) - 1)(\widehat{F}(x,t,t) - 1) \\
&+ x^2(\widehat{F}(x,t,y) - 1) + x^2t(\widehat{F}(x,y,t) - 1).
\end{aligned}$$

Using the same arguments of the previous Subsection we get the following expression for the generating function $\widehat{F}(x,t,1)$:

$$F_{\underline{312}}(x,t) = \widehat{F}(x,t,1) = \frac{1 - x^2\widehat{F}(x,t,t) + x^2 - x^2t}{1 - x - x^2\widehat{F}(x,t,t) - x^2t}$$

where

$$\widehat{F}(x,t,t) = \frac{-b - \sqrt{b^2 - 4a}}{2a}$$

with

$$a = x^2, \qquad \text{and} \qquad b = -1 + x - x^2 + x^2 t.$$

6.4 The pattern $\underline{321}$

An occurrence of the pattern $\underline{321}$ corresponds to an occurrence in $\Gamma(\pi)$ of a horizontal step that is neither in the first nor in last position nor followed only by down steps. We call such a horizontal step a *distinguished* horizontal step.

Let $\widehat{F}(x,t,y,z)$ be the generating function for Motzkin paths enumerated by length (x), number of distinguished horizontal steps (t), number of horizontal steps in the first position (y), number of horizontal steps followed only by a (possibly empty) sequence of down steps (z).

We have the following recurrence for $\widehat{F}(x,t,y,z)$:

$$\begin{aligned}\widehat{F}(x,t,y,z) =& \\ &= 1 + xy(\widehat{F}(x,t,t,z) - 1) + xz + x^2\widehat{F}(x,t,t,z) \\ &+ x^2\widehat{F}(x,t,t,t)(\widehat{F}(x,t,t,z) - 1)\end{aligned}$$

As a consequence

$$F_{\underline{321}}(x,t) = \widehat{F}(x,t,1,1) = \frac{1 - xt - x^2G + x - x^2t + x^2 - x^3t + x^3}{-x^2 + 1 - xt - x^2G}$$

where

$$G = \frac{-b - \sqrt{b^2 - 4a}}{2a}$$

with

$$a = x^2, \qquad \text{and} \qquad b = -1 + xt.$$

7 Appendix

In this appendix we describe the method that we used in Section 4 to count Motzkin paths by occurrences of a set of given subpatterns. This method is a slight modification of the *Goulden-Jackson cluster method* used to enumerate words over an arbitrary finite alphabet by occurrences of given subwords ([8, p. 128]). In this context the Goulden-Jackson cluster method does not apply directly, since the words we are considering correspond to Motzkin paths, hence, they have particular constraints. Our method is inspired by those presented in [13], where the author uses similar ideas to count Dyck words by occurrences of given subwords.

Let $\mathcal{A}$ be the set of words in the alphabet $\{U, D, H\}$, and let $S \subseteq \mathcal{A}$.

Given $w \in \mathcal{A}$, let $|w|$ be the length of w, $|w|_L$ its number of steps of type $L \in \{U, D, H\}$ and $|w|_S$ the total number of occurrences in w of subwords from S.

A *marked subword* of a word $w = a_1 \dots a_n \in \mathcal{A}$ with respect to the set S is a pair (i, v) where i is a positive integer, $v = a_i a_{i+1} \dots a_{i+|v|-1}$ and $v \in S$. A *marked word* is a word $w \in \mathcal{A}$ with a (possibly empty) set of marked subwords of w. A *cluster* with respect to S is a marked word that is not the concatenation of two nonempty marked words.

As an example, consider $S = \{UUU, DHU\}$. The marked subwords of the word $w = UUUHUDDUUUDDDDHUHDD$ are $(1, UUU)$, $(8, UUU)$ and $(14, DHU)$. Hence

$$(w, \{(1, UUU), (14, DHU)\})$$

is an example of a marked word.

Two clusters for S are the marked words

$$(DHUUUUU, \{(1, DHU), (3, UUU), (4, UUU), (5, UUU)\})$$

$$\mathrm{D\ H\ U\ U\ U\ U\ U}$$

and

$$(DHUUUUU, \{(1, DHU), (3, UUU), (5, UUU)\})$$

$$\mathrm{D\ H\ U\ U\ U\ U\ U}$$

whereas

$$(DHUUUUU, \{(1, DHU), (4, UUU), (5, UUU)\})$$

$$\mathrm{D\ H\ U\ U\ U\ U\ U}$$

is not a cluster, because it can be seen as the juxtaposition of the marked words $(DHU, \{(1, DHU)\})$ and $(UUUU, \{(1, UUU), (2, UUU)\})$.

Note that, if $w' \in S$, $(w', \{(1, w')\})$ is trivially a cluster.

We say that a word $w \in \mathcal{A}$ *reduces to an up step* (*to a down step*, *to a horizontal step*)if $|w|_U - |w|_D = 1$ ($-1, 0$, respectively). If one of these three cases occur we say that w *reduces to a single step.* A cluster *reduces to a single step* if the underlying word does.

Now we define the *depth* of a word $w \in \mathcal{A}$ that reduces to a single step. Draw the word w in the lattice plane starting at the origin and assigning to the letters U, D, H the usual steps. Let k be the minimal y-coordinate reached by the resulting path. We say that w *has depth* k if it reduces to an up or horizontal step and $k+1$ if it reduces to a down step. For example, the word DDU has depth -1, the word DU has depth -1, and the word UDU has depth 0.

Recall that the height of a step d_i of a Motzkin path is

$$\begin{cases} \text{the } y\text{-coordinate of the starting point of the step } d_i & \text{if } d_i \text{ is either } U \text{ or } H, \\ \text{the } y\text{-coordinate of the ending point of the step } d_i & \text{otherwise.} \end{cases}$$

Observe that if a Motzkin path can be decomposed as $Um'Dm''$ then all the steps in m' have height at least 1.

Theorem 7.1. *Let $S = \{w_1, \ldots, w_k\}$ be a subset of $\mathcal{A}$ such that no words w_i are proper subwords of other words in S. Suppose that each cluster formed by these words reduces to a single step and has depth greater than or equal to -1. Let $A_H(x,t,z)$ be the generating function of clusters that reduce to a horizontal step enumerated by length (x), occurrences of $w_1, \ldots, w_k$ as subwords (t), and horizontal steps (z). Denote by $A'_H(x,t,z)$ the generating function of clusters that reduce to a horizontal step with depth* 0*. The generating functions $A_D(x,t,z)$, $A'_D(x,t,z)$, and $A_U(x,t,z)$, $A'_U(x,t,z)$ are defined in the same way for clusters that reduce to a down step and an up step, respectively. Then the generating function $F(x,t,z)$ for Motzkin paths enumerated by length, occurrences of the words $w_1, \ldots, w_k$ and number of horizontal steps satisfies*

$$F(x,t+1,z) = \frac{2ys}{2(1-l')ys - y's' + y's'l + y's'\sqrt{(1-l)^2 - 4ys}}, \tag{7}$$

where

- $l = xz + A_H(x,t,z)$,
- $l' = xz + A'_H(x,t,z)$,
- $y = x + A_U(x,t,z)$,
- $y' = x + A'_U(x,t,z)$,
- $s = x + A_D(x,t,z)$,
- $s' = x + A'_D(x,t,z)$.

Proof.

$$F(x,t+1,z) = \sum_{w\in\mathcal{M}} x^{|w|} z^{|w|_H} (t+1)^{|w|_S} =$$

$$\sum_{w\in\mathcal{M}} x^{|w|} z^{|w|_H} \sum_{k\geq 0} \binom{|w|_S}{k} t^k =$$

$$\sum_{w\in\mathcal{M}} x^{|w|} z^{|w|_H} \sum_{T\subseteq S_w} t^{|T|},$$

where S_w is the set of words in S contained in w as subwords.

Hence $F(x,t+1,z)$ counts Motzkin words weighted by the number of marked subwords contained therein, by length and number of horizontal steps.

We want to show that the right-hand side of (7) counts the same objects.

Let $G(y,s,l,y',s',l')$ be the generating function for Motzkin paths enumerated by occurrences of U, D and H at non-zero height and by occurrences of U, D and H at zero height. Hence, the formal power series $G_1(y,s,l) := G(y,s,l,y,s,l)$ is the generating function of Motzkin paths enumerated by occurrences of U, D and H.

By the first return decomposition we get immediately

$$G_1(y,s,l) = 1 + lG_1(y,s,l) + ysG_1^2(y,s,l)$$

and

$$G(y,s,l,y',s',l') = 1 + l'G(y,s,z,y',s',l') + y's'G(y,s,l,y',s',l')\cdot G_1(y,s,l).$$

As a consequence

$$G_1(y,s,l) = \frac{1-l-\sqrt{(1-l)^2-4ys}}{2ys}$$

and

$$\begin{aligned}G(y,s,l,y',s',l') &= \frac{1}{1-l'-y's'G_1(y,s,l)}\\ &= \frac{2ys}{2(1-l')ys - y's' + y's'l + y's'\sqrt{(1-l)^2-4ys}}.\end{aligned}$$

Let $\widehat{G}(x,t,z)$ be the generating function obtained from $G(y,s,l,y',s',l')$ by replacing

- the variable l by $xz + A_H(x,t,z)$
- the variable l' by $xz + A'_H(x,t,z)$
- the variable y by $x + A_U(x,t,z)$
- the variable y' by $x + A'_U(x,t,z)$
- the variable s by $x + A_D(x,t,z)$
- the variable s' by $x + A'_D(x,t,z)$.

Note that $\widehat{G}(x,t,z)$ is precisely the right-hand side of Equation (7).

Let w be a Motzkin word. Choose in w some clusters $c_1, \ldots, c_k$. By hypothesis these clusters have depth -1 or 0.

If in w we replace each cluster c_i with the step that c_i reduces to, we get another Motzkin word.

Conversely, given a Motzkin word v we can choose in v some up, down and horizontal steps, and replace them by a cluster that reduces to an up, down and horizontal step, respectively, with the constraint that a step of height 0 can be only replaced by a cluster of depth 0.

As a consequence the generating function $\widehat{G}(x,t,z)$ counts marked Motzkin words weighted by the number of marked subwords contained therein.

□

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