Schur Algebras for the Alternating Group and Koszul Duality

Thangavelu Geetha, Amritanshu Prasad, and Shraddha Srivastava

Abstract. We introduce the alternating Schur algebra $\mathrm{AS}_F(n,d)$ as the commutant of the action of the alternating group $\mathbf{A}_d$ on the d-fold tensor power of an n-dimensional F-vector space. When F has characteristic different from 2, we give a basis of $\mathrm{AS}_F(n,d)$ in terms of bipartite graphs, and a graphical interpretation of the structure constants. We introduce the abstract Koszul duality functor on modules for the even part of any $\mathbf{Z}/2\mathbf{Z}$-graded algebra. The algebra $\mathrm{AS}_F(n,d)$ is $\mathbf{Z}/2\mathbf{Z}$-graded, having the classical Schur algebra $\mathrm{S}_F(n,d)$ as its even part. This leads to an approach to Koszul duality for $\mathrm{S}_F(n,d)$-modules that is amenable to combinatorial methods. We characterize the category of $\mathrm{AS}_F(n,d)$-modules in terms of $\mathrm{S}_F(n,d)$-modules and their Koszul duals. We use the graphical basis of $\mathrm{AS}_F(n,d)$ to study the dependence of the behavior of derived Koszul duality on n and d.

1. Introduction

1.1. Schur-Weyl duality and its variants. Frobenius determined the irreducible characters of the symmetric group $\mathbf{S}_d$ over $\mathbf{C}$, the field of complex numbers, in 1900 [**12**]. Building on this, Schur classified the irreducible polynomial representations of $\mathrm{GL}_n(\mathbf{C})$ and computed their characters in his PhD thesis [**27**]. The group $\mathrm{GL}_n(\mathbf{C})$ acts on the factors of $(\mathbf{C}^n)^{\otimes d}$, while $\mathbf{S}_d$ permutes the tensor factors. In 1927, Schur used these commuting actions to reprove the results of his dissertation [**28**]. Following Weyl's expositions of this method [**32, 33**], it is known as Schur-Weyl duality.

Over the years, several variants of Schur-Weyl duality have emerged. Shrinking $\mathrm{GL}_n(\mathbf{C})$ to the orthogonal group $\mathrm{O}_n(\mathbf{C})$, Brauer obtained the duality between Brauer algebras $\mathrm{Br}_d(n)$ and $\mathrm{O}_n(\mathbf{C})$ [**7**]. Motivated by the Potts model in statistical mechanics, Jones [**16**] and Martin [**20**] further shrunk $\mathrm{O}_n(\mathbf{C})$ down to $\mathbf{S}_n$, obtaining the partition algebras $\mathrm{P}_d(n)$. Bloss [**5**] reduced $\mathbf{S}_n$ to $\mathbf{A}_n$ to obtain an algebra $\mathrm{AP}_d(n)$ which coincides with the partition algebra when $n \geq 2d+2$. We take the smallest possible step in the opposite direction: we reveal what takes the place of the polynomial representations of $\mathrm{GL}_n(\mathbf{C})$ when the action of the symmetric group $\mathbf{S}_d$ is restricted to the alternating group $\mathbf{A}_d$. The situation is summarized in Table 1. The significance of this investigation lies in its connection with the

2010 *Mathematics Subject Classification.* 20G43,20G05,05E10.
Key words and phrases. Schur algebra, Koszul duality, Schur-Weyl duality, alternating group.

$F^{n\otimes d}$		
??	**This article**	$\mathbf{A}_d$
$\cup$		$\cap$
$\mathrm{GL}_n(F)$	Schur-Weyl	$\mathbf{S}_d$
$\cup$		$\cap$
$\mathrm{O}_n(F)$	Brauer	$\mathrm{Br}_d(n)$
$\cup$		$\cap$
$\mathbf{S}_n$	Martin and Jones	$\mathrm{P}_d(n)$
$\cup$		$\cap$
$\mathbf{A}_n$	Bloss	$\mathrm{AP}_d(n)$

TABLE 1. Dualities arising from tensor space

Koszul duality functor on the category of homogeneous polynomial representations of $\mathrm{GL}_n(\mathbf{C})$ of degree d.

1.2. Schur algebras for the alternating group. Motivated by Green [**14**, Theorem 2.6c], define the Schur algebra as

$$\mathrm{S}_F(n,d) = \mathrm{End}_{\mathbf{S}_d}((F^n)^{\otimes d})$$

for any field F, and positive integers n and d. When F is infinite, then $\mathrm{S}_F(n,d)$-modules are the same as homogeneous polynomial representations of $\mathrm{GL}_n(F)$ of degree d (see [**14**, Section 2.4] and [**23**, Section 6.2]). Define the alternating Schur algebra $\mathrm{AS}_F(n,d)$ by replacing $\mathbf{S}_d$ by $\mathbf{A}_d$ in the definition above:

$$\mathrm{AS}_F(n,d) = \mathrm{End}_{\mathbf{A}_d}((F^n)^{\otimes d}).$$

When F has characteristic different from 2, this algebra has a decomposition (Lemma 2.1)

$$\mathrm{AS}_F(n,d) = \mathrm{S}_F(n,d) \oplus \mathrm{S}_F^-(n,d) \tag{1}$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Here $\mathrm{S}_F^-(n,d) = \mathrm{Hom}_{\mathbf{S}_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \mathrm{sgn})$, where sgn denotes the sign character of $\mathbf{S}_d$. The subspace $\mathrm{S}_F^-(n,d)$ is an $(\mathrm{S}_F(n,d), \mathrm{S}_F(n,d))$-bimodule.

When $n^2 < d$, then $\mathrm{S}_F^-(n,d) = 0$, and $\mathrm{AS}_F(n,d) = \mathrm{S}_F(n,d)$, as observed by Regev [**25**, Theorem 1]. But when $n^2 \geq d$, $\mathrm{S}_F^-(n,d) \neq 0$, and in Lemma 2.2, we note that $\mathrm{S}_F^-(n,d)$ is a full tilting left $\mathrm{S}_F(n,d)$-module as studied by Donkin in [**9**, Section 3].

1.3. Bases and structure constants. Schur [**28**] gave a combinatorial description of a basis and the corresponding structure constants of the Schur algebra (see also [**14**, Section 2.3]). By indexing Schur's basis of $\mathrm{S}_F(n,d)$ by bipartite multigraphs with $n+n$ vertices and d edges, Méndez [**22**] (see also Geetha and Prasad [**13**]) gave a graphic interpretation of the structure constants. We describe a basis of $\mathrm{S}_F^-(n,d)$ in terms of bipartite simple graphs in Theorem 2.24. So from the decomposition (1), a basis of $\mathrm{AS}_F(n,d)$ is obtained. A graphic interpretation of the structure constants of $\mathrm{AS}_F(n,d)$ is given in Theorems 2.14 and 2.25. This will be used to derive properties of $\mathrm{AS}_F(n,d)$, its bimodule $\mathrm{S}_F^-(n,d)$, and Koszul duality.

1.4. Koszul duality and modules. The term *Koszul duality* is used for several constructions which interchange the roles of exterior and symmetric powers.

The earliest notion of Koszul duality was introduced by Priddy [**24**]. It applies to *pre-Koszul algebras*, which are also called *quadratic algebras*. A pre-Koszul algebra is a quotient of a tensor algebra $T(V) = \bigoplus_{n\geq 0} \otimes^n V$ by a two-sided ideal I that is generated in degree two. Its Koszul dual is the algebra $T(V^*)/(I \cap (V \otimes V))^{\perp}$; the quotient of the dual tensor algebra by the annihilator in degree two of I. In this setting the Koszul dual of the symmetric algebra of V is the exterior algebra of V^*.

Bernstein, Gelfand, and Gelfand [**3**, Theorem 3] introduced an equivalence between the bounded derived categories of graded modules over symmetric and exterior algebras, which was called the Koszul duality functor by Beilinson, Ginsburg, and Schectman [**4**].

Friedlander and Suslin [**11**] introduced the category of *strict polynomial functors* of degree d as the representations of the Schur category of degree d, for each non-negative integer d (see Section 4.1). The category of strict polynomial functors of degree d unifies the categories of homogeneous polynomial representations of $\mathrm{GL}_n(F)$ of degree d across all n. Standard examples of strict polynomial functors of degree d are the dth tensor power functor $\otimes^d$, the dth symmetric power functor Sym^d, and the dth exterior power functor $\wedge^d$. Evaluating a strict polynomial functor of degree d at F^n gives an $\mathrm{S}_F(n, d)$-module for each n. Friedlander and Suslin showed that this evaluation functor is an equivalence of categories when $n \geq d$. Chałupnik [**8**] and Touzé [**30**] used the term Koszul duality to refer to a functor on the category of strict polynomial functors of degree d which takes the Schur functor associated to the partition λ of d to the Weyl functor associated with the partition λ' *conjugate* to λ. Krause [**17**] discovered an internal tensor product on the category of strict polynomial functors of fixed degree d. Given such a tensor product it was then natural for him to define Koszul duality in this category as tensor product with $\wedge^d$. This definition is different from the Koszul duality functors defined earlier by Chałupnik and Touzé are not the same. Those coincide with a duality defined by Ringel [**26**] using tilting modules for quasi-hereditary algebras. This tilting module was described by Donkin [**9**] in the case of Schur algebras.

We introduce the term *abstract Koszul duality* to refer to a very simple functor which makes sense for any $\mathbf{Z}/2\mathbf{Z}$-graded algebra $\mathrm{AS} = \mathrm{S} \oplus \mathrm{S}^-$. The abstract Koszul dual of an S-module V is defined as

$$D(V) = \mathrm{S}^- \otimes_{\mathrm{S}} V.$$

The multiplication operation on AS gives rise to an (S, S)-bimodule homomorphism $\phi : \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{S}^- \to S$ and hence a natural transformation from $D \circ D$ to the identity functor on the category of S-modules. We prove (Theorem 3.4) that the category of AS-modules is same as the category of pairs (M, θ_M) where M is an S-module and $\theta_M : D(M) \to M$ is compatible with ϕ in the sense of (15).

In Section 4, we specialize to the case $\mathrm{AS}_F(n, d) = \mathrm{S}_F(n, d) \oplus \mathrm{S}_F^-(n, d)$ to obtain a Koszul duality functor D on the category of $\mathrm{S}_F(n, d)$-modules. In Theorem 4.5, we show that the evaluation at F^n of the Koszul duality functor of Krause is naturally isomorphic to our Koszul duality functor when $n \geq d$. In this sense, our abstract Koszul duality functor on Schur algebras coincides with Krause's Koszul duality.

Our description of the structure constants of $\mathrm{AS}_F(n,d)$ allow us to give a direct combinatorial proof of the well-known fact that, when $n \geq d$ and when the characteristic of F is 0 or greater than d, then abstract Koszul duality is an equivalence (Theorem 4.2).

Krause [**17**] showed that derived Koszul duality functor is an auto-equivalence of the unbounded derived category of strict polynomial functors. Since the evaluation functor is an equivalence, this implies that derived Koszul duality is an auto-equivalence at the level of unbounded derived category of $\mathrm{S}_F(n,d)$-modules when $n \geq d$. However, this does not address the case where $n < d$. Using our combinatorial methods, we show that derived Koszul duality is not an equivalence when $n < d$ (Theorem 4.9). This proof uses a criterion of Happel [**15**] for a tensor functor to be a derived equivalence. In the context of derived Koszul duality, this criterion requires that the canonical algebra homomorphism $\mathrm{S}_F(n,d) \to \mathrm{End}_{\mathrm{S}_F(n,d)}(\mathrm{S}_F^-(n,d))$ is an isomorphism. Donkin [**9**, Proposition 3.7] proved this when $n \geq d$. When the characteristic of F is not 2 we give a combinatorial proof of Donkin's result, and also show that it fails when $n < d$ (Theorem 4.7). Figure 1 describes the behavior of Koszul duality for all values of the parameters n and d.

We conclude this paper by discussing a possible application of our techniques to Bloss's alternating partition algebra, and a diagrammatic interpretation of the Schur category (Section 5).

2. The alternating Schur algebra

Let F be a field of characteristic different from 2, n and d be positive integers. The symmetric group $\mathbf{S}_d$ acts on the tensor space $(F^n)^{\otimes d}$ by permuting the tensor factors. The Schur algebra can be defined as

$$\mathrm{S}_F(n,d) := \mathrm{End}_{\mathbf{S}_d}((F^n)^{\otimes d}).$$

By restricting the action of $\mathbf{S}_d$ to the alternating group $\mathbf{A}_d$, define the *alternating Schur algebra* as

$$\mathrm{AS}_F(n,d) := \mathrm{End}_{\mathbf{A}_d}((F^n)^{\otimes d}).$$

Clearly, $\mathrm{S}_F(n,d)$ is a subalgebra of $\mathrm{AS}_F(n,d)$.

LEMMA 2.1. *For any representations V and W of $\mathbf{S}_d$,*

$$\mathrm{Hom}_{\mathbf{A}_d}(V,W) = \mathrm{Hom}_{\mathbf{S}_d}(V,W) \oplus \mathrm{Hom}_{\mathbf{S}_d}(V, W \otimes \mathrm{sgn}). \tag{2}$$

Here $W \otimes \mathrm{sgn}$ denotes the twist of W by the sign character $\mathrm{sgn} : \mathbf{S}_d \to \{\pm 1\}$.

Define,

$$\mathrm{S}_F^-(n,d) := \mathrm{Hom}_{\mathbf{S}_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \mathrm{sgn}). \tag{3}$$

Lemma 2.1 gives a $\mathbf{Z}/2\mathbf{Z}$-grading of $\mathrm{AS}_F(n,d)$ in the sense of Bourbaki [**6**, Chapter III, Section 3.1]:

$$\mathrm{AS}_F(n,d) = \mathrm{S}_F(n,d) \oplus \mathrm{S}_F^-(n,d). \tag{4}$$

The summand $\mathrm{S}_F^-(n,d)$ is an $(\mathrm{S}_F(n,d), \mathrm{S}_F(n,d))$-bimodule. Recall that a *weak composition* of d with n parts is a vector $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative integers summing to d. Let $\Lambda(n,d)$ denote the set of weak compositions of d with n parts. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n,d)$, define

$$\wedge^\lambda F^n = \wedge^{\lambda_1} F^n \otimes \cdots \otimes \wedge^{\lambda_n} F^n.$$

where, for a non-negative integer s, $\wedge^s F^n$ is the sth exterior power of F^n. As a left $\mathrm{S}_F(n,d)$-module, $\mathrm{S}^-_F(n,d) = \bigoplus_{\lambda\in\Lambda(n,d)} \wedge^\lambda F^n$. By Donkin [**9**, Section 3], we have:

Lemma 2.2. *The left module $\mathrm{S}^-_F(n,d)$ is a full tilting module of $\mathrm{S}_F(n,d)$.*

2.1. Twisted permutation representations. Let X be a finite set on which a group G acts on the right (henceforth called a G-set). The space $F[X]$ of F-valued functions on X may be regarded as a representation of G:

$$\rho_X(g)f(x) = f(x\cdot g), \text{ for } x\in X,\ g\in G, \text{ and } f\in F[X]. \tag{5}$$

Let χ be a multiplicative character $G\to F^*$. One may twist the representation (5) by χ:

$$\rho_X^\chi(g)f(x) = \chi(g)f(x\cdot g). \tag{6}$$

Denote the representation space of this twisted action as $F[X]\otimes\chi$.

Suppose that X and Y are finite G-sets. Given a function $\kappa : X\times Y\to F$, the *integral operator* $\xi_\kappa : F[Y]\to F[X]$ associated to κ is defined as

$$\xi_\kappa f(x) = \sum_{y\in Y}\kappa(x,y)f(y), \text{ for } f\in F[Y]. \tag{7}$$

The function κ is known as the *integral kernel* of ξ_κ.

If Z is another finite G-set, $\kappa' : X\times Y\to F$ and $\kappa'' : Y\times Z\to F$ are functions. Then

$$\xi_{\kappa'}\circ\xi_{\kappa''} = \xi_{\kappa'*\kappa''},$$

where $\kappa'*\kappa'' : X\times Z\to F$ is the *convolution product*

$$\kappa'*\kappa''(x,z) = \sum_{y\in Y}\kappa'(x,y)\kappa''(y,z). \tag{8}$$

We have (see [**23**, Section 4.2]):

Theorem 2.3. *For any finite G-spaces X and Y, and any multiplicative character $\chi : G\to F^*$,*

$$\begin{aligned}\mathrm{Hom}_G(F[Y], F[X]\otimes\chi)\\ = \{\xi_\kappa \mid \kappa : X\times Y\to F \textit{ such that } \kappa(x\cdot g, y\cdot g) = \chi(g)\kappa(x,y)\}.\end{aligned} \tag{9}$$

The identity (9) implies that

$$\dim\mathrm{Hom}_G(F[Y], F[X]\otimes\chi) \le |(X\times Y)/G|,$$

with equality holding if χ is the trivial character. However, if $g\in G$, and $(x,y)\in X\times Y$ are such that $(x\cdot g, y\cdot g) = (x,y)$, then if $\xi_\kappa\in\mathrm{Hom}_G(F[Y], F[X]\otimes\chi)$,

$$\kappa(x,y) = \kappa(x\cdot g, y\cdot g) = \chi(g)\kappa(x,y),$$

so that either $\chi(g) = 1$ or κ vanishes on the G-orbit of (x,y).

Definition 2.4 (Transverse Pair). A pair $(x,y)\in X\times Y$ is said to be *transverse* with respect to χ if $G_x\cap G_y\subset\ker\chi$. If (x,y) is a transverse pair with respect to χ, we write $x \pitchfork y$.

If (x, y) is a transverse pair, then

$$\kappa(x \cdot g, y \cdot g) := \chi(g)\kappa(x, y)$$

is a well-defined non-zero function on the G-orbit of (x, y). Let

$$X \pitchfork Y = \{(x, y) \in X \times Y \mid x \pitchfork y\}.$$

Then $X \pitchfork Y$ is stable under the diagonal action of G on $X \times Y$. We have (see [**23**, Theorem 4.2.3]):

Theorem 2.5. *Let X and Y be finite G-sets, and $\chi : G \to F^*$ be a multiplicative character. For each orbit $O \in (X \pitchfork Y)/G$, choose a base point $(x_O, y_O) \in O$. Define*

$$\kappa_O(x, y) = \begin{cases} \chi(g) & \text{if } x = x_O \cdot g \text{ and } y = y_O \cdot g \text{ for some } g \in G, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, write ξ_O for ξ_{κ_O}. Then the set

$$\{\xi_O \mid O \in (X \pitchfork Y)/G\}$$

is a basis for $\mathrm{Hom}_G(F[Y], F[X] \otimes \chi)$*. Consequently,*

$$\dim \mathrm{Hom}_G(F[Y], F[X] \otimes \chi) = |(X \pitchfork Y)/G)|.$$

In the special case where χ is the trivial character, we get:

Corollary 2.6. *Let X and Y be finite G-sets. For each orbit O in $(X \times Y)/G$ define*

$$\kappa_O(x, y) = \begin{cases} 1 & \text{if } (x, y) \in O, \\ 0 & \text{otherwise.} \end{cases}$$

Write $\xi_O = \xi_{\kappa_O}$. Then the set

$$\{\xi_O \mid O \in (X \times Y)/G\}$$

is a basis for $\mathrm{Hom}_G(F[Y], F[X])$*. Consequently,*

$$\dim \mathrm{Hom}_G(F[Y], F[X]) = |(X \times Y)/G|.$$

Given a function $\kappa : X \times Y \to F$, define

$$\kappa^*(y, x) = \kappa(x, y) \text{ for } x \in X,\ y \in Y.$$

The following is easy to see:

Lemma 2.7. *For any G-set X, the map $\xi_\kappa \mapsto \xi_{\kappa^*}$ is an anti-involution on the algebra* $\mathrm{End}_G(F[X])$*.*

2.2. Structure constants of the Schur algebra. We recall the combinatorial interpretation of structure constants of the Schur algebra from [**13**]. Let $[n] = \{1, \dots, n\}$ and

$$I(n, d) = \{\underline{i} := (i_1, \dots, i_d) \mid i_s \in [n]\}.$$

An element $w \in \mathbf{S}_d$ acts on $I(n, d)$ by permuting the coordinates:

$$(i_1, \dots, i_d) \cdot w = (i_{w(1)}, \dots, i_{w(d)}).$$

For $\underline{i} = (i_1, \dots, i_d) \in I(n, d)$, define

$$e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_d},$$

where e_i is the ith coordinate vector in F^n. The vector space $(F^n)^{\otimes d}$ has a basis

$$\{e_{\underline{i}} \mid \underline{i} \in I(n,d)\}.$$

and $w \in \mathbf{S}_d$ acts on a basis vector $e_{\underline{i}}$ as follows:

$$w \cdot e_{\underline{i}} = w \cdot (e_{i_1} \otimes \cdots \otimes e_{i_d}) = e_{i_{w^{-1}(1)}} \otimes \cdots \otimes e_{i_{w^{-1}(d)}}.$$

Let $F[I(n,d)]$ denote the space of all F-valued functions on $I(n,d)$. Mapping $e_{\underline{i}}$ to the indicator function of $\underline{i} \in I(n,d)$ defines an isomorphism of $(F^n)^{\otimes d}$ onto $F[I(n,d)]$. Thus $(F^n)^{\otimes d}$ can be regarded as a permutation representation of $\mathbf{S}_d$.

Let $B(n,d)$ denote the set of all configurations of d distinguishable balls, numbered $1,\dots,d$ in n boxes, numbered $1,\dots,n$. The symmetric group $\mathbf{S}_d$ acts on such configurations by permuting the d balls. An element of $B(n,d)$ is a set partition

$$\{1,\dots,d\} = S_1 \coprod \cdots \coprod S_n,$$

where S_i is the set of balls in the ith box.

Lemma 2.8. *Given $\underline{i} \in I(n,d)$, let $b(\underline{i})$ denote the balls-in-boxes configuration in $B(n,d)$ where the ith box contains the balls $\{s \mid i_s = i\}$. Then $b : I(n,d) \to B(n,d)$ is an $\mathbf{S}_d$-equivariant bijection of $I(n,d)$ onto $B(n,d)$.*

By Corollary 2.6, a basis for $\mathrm{S}_F(n,d)$ is indexed by orbits for the diagonal action of $\mathbf{S}_d$ on $B(n,d) \times B(n,d)$.

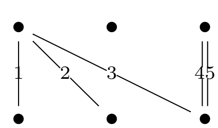
Definition 2.9 (Labelled bipartite multigraph). Let $[n] = \{1,\dots,n\}$ (as before) and $[n'] = \{1',\dots,n'\}$. A labelling of a bipartite multigraph with vertex set $[n'] \coprod [n]$ and d edges is a function $l : [d] \to [n'] \times [n]$ such that, for each $(i',j) \in [n'] \times [n]$, the cardinality of $l^{-1}(i',j)$ is the number of edges joining i' and j. In other words, labels are assigned to edges without distinguishing between edges joining the same pair of vertices.

Given a pair $S = (S_1,\dots,S_n)$ and $T = (T_1,\dots,T_n)$ in $B(n,d)$, define a labelled bipartite graph $\gamma_{S,T}$ with multiple edges on the vertex set $[n'] \coprod [n]$ as follows:

There are $|S_j \cap T_i|$ edges between i' and j, labelled by the numbers of the balls in $S_j \cap T_i$.

The bipartite multigraph is always drawn in two rows, with the vertices from $[n']$ in the upper row and vertices from $[n]$ in the lower row, numbered from left to right. Since the vertices are always labelled in this manner, the vertex labels can be omitted in the drawing.

Example 2.10. When $S = (\{1\},\{2\},\{3,4,5\})$ and $T = (\{1,2,3\},\emptyset,\{4,5\})$, the associated labelled multigraph is:



Clearly, $(S,T) \mapsto \gamma_{S,T}$ is a bijection from $B(n,d) \times B(n,d)$ onto the set of labelled bipartite multigraphs with vertex set $[n'] \coprod [n]$ and d edges. The symmetric group $\mathbf{S}_d$ acts on $B(n,d) \times B(n,d)$ by permuting labels. Therefore the $\mathbf{S}_d$ orbits in $B(n,d) \times B(n,d)$ are obtained by simply forgetting the labels, leaving only the underlying bipartite multigraph. We write $\Gamma_{S,T}$ for the bipartite multigraph underlying $\gamma_{S,T}$. Such a graph can also be represented by its *adjacency matrix* (whose (i,j)th entry is the number of edges joining i' and j), which is a matrix of non-negative integers that sum to d.

In view of Corollary 2.6, we recover a result of [**13, 22**]:

THEOREM 2.11. *Let $M(n,d)$ denote the set of all bipartite multigraphs with vertex set $[n'] \coprod [n]$ and d edges. For each $\Gamma \in M(n,d)$, define $\xi_\Gamma \in \mathrm{S}_F(n,d) = \mathrm{End}_{\mathbf{S}_d}(F[B(n,d)])$ by*

$$\xi_\Gamma f(S) = \sum_{\{T \mid \Gamma_{S,T} = \Gamma\}} f(T).$$

Then

$$\{\xi_\Gamma \mid \Gamma \in M(n,d)\}$$

is a basis for $\mathrm{S}_F(n,d)$.

REMARK 2.12. If $(\underline{i}, \underline{j})$ has image Γ under the composition $I(n,d)^2 \to B(n,d)^2 \to M(n,d)$, then the basis element $\xi_{\underline{i},\underline{j}}$ of [**14**, Section 2.6] coincides with the basis element ξ_Γ of Theorem 2.11.

The structure constants $c^\Gamma_{\Gamma_1\Gamma_2}$ are defined by

$$\xi_{\Gamma_1}\xi_{\Gamma_2} = \sum_{\Gamma \in M(n,d)} c^\Gamma_{\Gamma_1\Gamma_2}\xi_\Gamma.$$

DEFINITION 2.13. Let l, l_1 and l_2 be labellings of graphs Γ, Γ_1 and Γ_2 in $M(n,d)$, respectively. We say that (l_1, l_2) is compatible with l if, for all $s = 1, \dots, d$, if we write $l_1(s) = (i'_1, j_1)$ and $l_2(s) = (i'_2, j_2)$, then

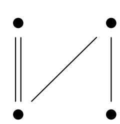
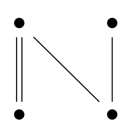
(2.13.a) $i'_2 = j_1$, and
(2.13.b) $l(s) = (i'_1, j_2)$.

THEOREM 2.14. *Let l be any labelling of Γ. The structure constant $c^\Gamma_{\Gamma'\Gamma''}$ is the number of pairs (l_1, l_2) of labellings of Γ_1 and Γ_2 that are compatible with l.*

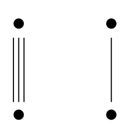
Before giving a proof, we illustrate the theorem with a few examples.

EXAMPLE 2.15. Let $w \in \mathbf{S}_d$ be a permutation, and assume that $n \geq d$. Let $\Gamma(w)$ denote the bipartite graph where (i', j) is an edge if and only if $1 \leq i \leq d$ and $w(i) = j$. Then, for all $w_1, w_2 \in \mathbf{S}_d$, $\xi_{\Gamma(w_1)}\xi_{\Gamma(w_2)} = \xi_{\Gamma(w_1 w_2)}$.

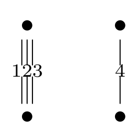
EXAMPLE 2.16. Consider

$\Gamma_1 =$  , $\Gamma_2 =$ 

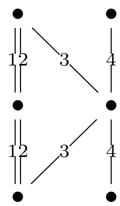
To find $c^\Gamma_{\Gamma_1\Gamma_2}$, with

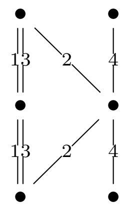
$\Gamma = \Gamma_3 =$  ,

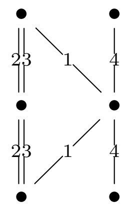
choose any labelling of Γ, such as



For this there are clearly three pairs of compatible labellings of Γ_1 and Γ_2, namely, we can choose which of the first three balls ends up in the second box of the middle row:

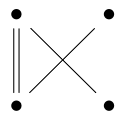
(10) 

, 

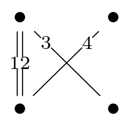
, and 

.

On the other hand, if

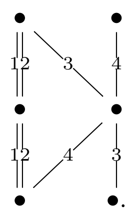
$\Gamma = \Gamma_4,$  ,

we may take the labelling:

(11) 

,

for which the only compatible labellings of Γ_1 and Γ_2 are:



.

It turns out that for no other $\Gamma \in \Gamma(n, d)$ is it possible to find even one compatible way of labelling Γ_1 and Γ_2, so we have:

$$\xi_{\Gamma_1}\xi_{\Gamma_2} = 3\xi_{\Gamma_3} + \xi_{\Gamma_4}.$$

EXAMPLE 2.17. Let $F_{n,n}$ denote the complete bipartite graph with vertex set $[n'] \coprod [n]$, where every vertex in $[n']$ is connected to every vertex in $[n]$. Then the coefficient of $\xi_{F_{n,n}}$ in $\xi_{F_{n,n}}\xi_{F_{n,n}}$ is the number of Latin squares of order n [**29**, Sequence A002860].

To see this, let l be any labelling of the edges of $F_{n,n}$. Given labellings l_1 and l_2 be of $F_{n,n}$ that are compatible with l, define the (i, j)th entry of the Latin square associated to (l_1, l_2) to be k if $l^{-1}(i', j) = l_2^{-1}(k', j) = l_1^{-1}(i', k)$. Remarkably, the number of Latin squares of order n is known only for $n = 1, \ldots, 11$.

PROOF OF THEOREM 2.14. Given a labelling l of Γ, define:

$$S_j = \cup_{i'=1}^{n} l^{-1}(i', j), \text{ and } U_i = \cup_{j=1}^{n} l^{-1}(i', j).$$

Then $S = (S_1, \ldots, S_n)$, and $U = (U_1, \ldots, U_n)$ are elements of $B(n, d)$, and by construction $\Gamma_{S,U} = \Gamma$. Now Equation (8) implies that

$$c^{\Gamma}_{\Gamma_1\Gamma_2} = \#\{T \in B(n, d) \mid \Gamma_{S,T} = \Gamma_1 \text{ and } \Gamma_{T,U} = \Gamma_2\}. \tag{12}$$

Given $T \in B(n,d)$ contributing to the above count, define labellings l_1 and l_2 of Γ_1 and Γ_2 by:

$$l_1^{-1}(i',j) = S_j \cap T_i, \text{ and } l_2^{-1}(i',j) = T_j \cap U_i.$$

Then (l_1, l_2) is compatible with l. Conversely, for every pair (l_1, l_2) compatible with l, take $T = (T_1, \dots, T_n)$ where

$$T_k = \cup_{i'=1}^n l_1^{-1}(i',k) = \cup_{j=1}^n l_2^{-1}(k',j).$$

Then T contributes to the count in (12). □

Example 2.18. In Example 2.16, the three compatible pairs of labels in (10) correspond to taking T as $(\{1,2\},\{3,4\})$, $(\{1,3\},\{2,4\})$, and $(\{2,3\},\{1,4\})$, respectively, and the compatible pair of labels in (11) corresponds to $T = (\{1,2\},\{3,4\})$.

2.3. A basis for $\mathrm{S}_F^-(n,d)$. By Theorem 2.5, a basis of $\mathrm{S}_F^-(n,d)$ is indexed by orbits in $B(n,d) \pitchfork B(n,d)/\mathbf{S}_d$.

Lemma 2.19. *A pair $(S,T) \in B(n,d)^2$ lies in $B(n,d) \pitchfork B(n,d)$ if and only if $\gamma_{S,T}$ is a simple bipartite graph.*

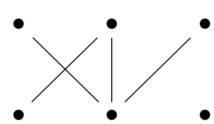
Proof. Let $S = (S_1, \dots, S_n)$, $T = (T_1, \dots, T_n)$. If $\gamma_{S,T}$ is not simple, then there exist indices i and j such that $S_j \cap T_i$ contains at least two elements, say k and l. The transposition $(kl) \in \mathbf{S}_d$ stabilized (S,T) but has $\mathrm{sgn}((kl)) = -1$, so $(S,T) \notin B(n,d) \pitchfork B(n,d)$.

However, if $\gamma_{S,T}$ is simple, then the simultaneous stabilizer of S and T in $\mathbf{S}_d$ is trivial, so $(S,T) \in B(n,d) \pitchfork B(n,d)$. □

In order to specify a basis for $\mathrm{S}_F^-(n,d)$ using Theorem 2.5, we need to choose a base point for each $\mathbf{S}_d$-orbit in $B(n,d) \pitchfork B(n,d)$. We do this using the following definition:

Definition 2.20 (Standard labelling of a bipartite simple graph). Given a bipartite simple graph Γ with vertex set $[n'] \coprod [n]$, label each edge by its index when the edges (i',j) are arranged in increasing lexicographic order, with priority given to the upper index, i.e., $(i',j) < (r',s)$ if either $i' < r'$ or $i' = r'$ and $j < s$.

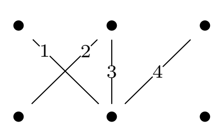
Example 2.21. Take

$\Gamma =$ 

The edges, written in lexicographic order, are:

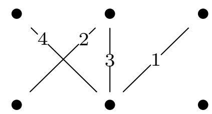
$$(1',2), (2',1), (2',2), (3',2).$$

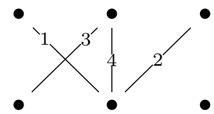
Therefore the standard labelling is:



Definition 2.22 (Sign of a labelling of a bipartite simple graph). Let l_0 denote the standard labelling of a simple bipartite graph Γ on $[n'] \coprod [n]$. Let $l : [d] \to [n'] \times [n]$ be a labelling of Γ (see Definition 2.9). The sign $\epsilon(\Gamma, l)$ of l is the sign of the permutation on $[d]$ which takes $l_0(i)$ to $l(i)$ for each i.

Example 2.23. For the graph from Example 2.21, the labellings:

$l_1 =$ 

and $l_2 =$ 

give rise to permutations 4231 and 1342 respectively, so that $\epsilon(\Gamma, l_1) = -1$, and $\epsilon(\Gamma, l_2) = +1$.

Recall, from Section 2.2, that $\gamma_{S,T}$ is a labelled bipartite graph associated to $(S, T) \in B(n, d) \times B(n, d)$, whose underlying unlabelled graph is denoted by $\Gamma_{S,T}$.

Theorem 2.24. *Let $N(n, d)$ denote the set of all bipartite simple graphs with vertex set $[n'] \coprod [n]$ and d edges. For each $\Gamma \in N(n, d)$, define $\zeta_\Gamma \in \mathrm{S}_F^-(n, d) = \mathrm{Hom}_{\mathbf{S}_d}(F[B(n, d)], F[B(n, d)] \otimes \mathrm{sgn})$ by*

$$\zeta_\Gamma f(S) = \sum_{\{T \mid \Gamma_{S,T} = \Gamma\}} \epsilon(\gamma_{S,T}) f(T).$$

The set

$$\{\zeta_\Gamma \mid \Gamma \in N(n, d)\}$$

forms a basis of $\mathrm{S}_F^-(n, d)$.

Proof. Recall that we choose the pair (S_0, T_0) corresponding to the standard labelling l_0 of Γ as the base point of the orbit associated to Γ. A pair (S, T) is in the orbit of (S_0, T_0) if and only if $\Gamma_{S,T} = \Gamma$. And the sign of the permutation $w \in \mathbf{S}_d$ such that $S = S_0.w$ and $T = T_0.w$ is the sign of the labelled bipartite graph $\gamma_{S,T}$. So the integral kernel κ_Γ of the operator ζ_Γ is:

$$\kappa_\Gamma(S, T) = \begin{cases} \epsilon(\gamma_{S,T}) & \text{if } \Gamma_{S,T} = \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

So theorem follows from Theorem 2.5. □

Theorem 2.14 tells us how to multiply two elements of the subalgebra $\mathrm{S}_F(n, d)$ of $\mathrm{AS}_F(n, d)$. The remaining structure constants are given by the following theorem.

Theorem 2.25. *The remaining structure constants are given as follows:*

(2.25.a) *Given $\Gamma_1 \in M(n, d)$, and $\Gamma_2 \in N(n, d)$,*

$$\xi_{\Gamma_1} \zeta_{\Gamma_2} = \sum_{\Gamma \in N(n,d)} c^\Gamma_{\Gamma_1 \Gamma_2} \zeta_\Gamma,$$

where

$$c^\Gamma_{\Gamma_1 \Gamma_2} = \sum_{l_1, l_2} \epsilon(\Gamma_2, l_2),$$

and the sum runs over all labellings l_1 and l_2 of Γ_1 and Γ_2 respectively, that are compatible with the standard labelling l of Γ.

(2.25.b) *Given $\Gamma_1 \in N(n, d)$, and $\Gamma_2 \in M(n, d)$,*

$$\zeta_{\Gamma_1} \xi_{\Gamma_2} = \sum_{\Gamma \in N(n,d)} c^\Gamma_{\Gamma_1 \Gamma_2} \zeta_\Gamma,$$

where

$$c^\Gamma_{\Gamma_1 \Gamma_2} = \sum_{l_1, l_2} \epsilon(\Gamma_1, l_1),$$

and the sum runs over all labellings l_1 *and* l_2 *of* Γ_1 *and* Γ_2 *respectively, that are compatible with the standard labelling* l *of* Γ.

(2.25.c) *Given* $\Gamma_1 \in N(n,d)$, *and* $\Gamma_2 \in N(n,d)$,

$$\zeta_{\Gamma_1}\zeta_{\Gamma_2} = \sum_{\Gamma \in M(n,d)} c^{\Gamma}_{\Gamma_1\Gamma_2}\xi_{\Gamma},$$

where

$$c^{\Gamma}_{\Gamma_1\Gamma_2} = \sum_{l_1,l_2} \epsilon(\Gamma_1,l_1)\epsilon(\Gamma_2,l_2),$$

and the sum runs over all labellings l_1 *and* l_2 *of* Γ_1 *and* Γ_2 *respectively, that are compatible with a fixed labelling* l *of* Γ.

Proof. Given a labelling l of Γ, construct S and U in $B(n,d)$ as in the proof of Theorem 2.14. Define $\kappa_{\Gamma_2} : B(n,d) \times B(n,d) \to F$ by

$$\kappa_{\Gamma_2}(T,U) = \epsilon(\gamma_{T,U}).$$

Then ζ_{Γ_2} is the integral operator $\xi_{\kappa_{\Gamma_2}}$, as in (7). Then, by Equation (8), the structure constant in Part (2.25.a) of the theorem is given by:

$$c^{\Gamma}_{\Gamma_1\Gamma_2} = \sum_{T} \zeta_{\Gamma_2}(T,U),$$

where the sum runs over all $T \in B(n,d)$ such that $\Gamma_{S,T} = \Gamma_1$, and $\Gamma_{T,U} = \Gamma_2$. Defining labelling l_1 and l_2 of Γ_1 and Γ_2 as in the proof of Theorem 2.14, we find that $\zeta_{\Gamma_2}(T,U) = \epsilon(\Gamma_2,l_2)$, proving (2.25.a). The proofs of the remaining assertions are similar. □

Definition 2.26. Given $\Gamma \in M(n,d) \sqcup N(n,d)$, we define Γ^* to be the horizontal reflection of Γ, i.e., i' is connected to j in Γ^* if and only if j' is connected to i in Γ. The operation $*$ on the set $M(n,d) \sqcup N(n,d)$ is an involution.

Lemma 2.27. *For every* $\Gamma \in N(n,d)$, *let* l_0 *denote its standard labelling. Let* l_0^* *denote the labelling of* Γ^* *given by* $l_0^*(i',j) = l_0(j',i)$. *Then the linear map* $\mathrm{AS}_F(n,d) \to \mathrm{AS}_F(n,d)$ *defined by:*

$$\xi_\Gamma \mapsto \xi_{\Gamma^*} \text{ for } \Gamma \in M(n,d),$$
$$\zeta_\Gamma \mapsto \epsilon(\Gamma^*, l_0^*)\zeta_{\Gamma^*} \text{ for } \Gamma \in N(n,d)$$

is an anti-involution of $\mathrm{AS}_F(n,d)$.

Remark 2.28. The above involution, when restricted to the Schur algebra, is the same as the one described by Green [**14**, Section 2.7].

Proof. We show that the linear map in Lemma 2.27 is the same as the anti-involution in Lemma 2.7 with $X = B(n,d)$ and $G = \mathbf{A}_d$.

For $\Gamma \in M(n,d)$, ξ_Γ is the integral operator with kernel:

$$\kappa_\Gamma(S,T) = \begin{cases} 1 & \text{if } \Gamma_{S,T} = \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Gamma_{T,S} = \Gamma^*_{S,T}$, $\kappa^*_\Gamma = \kappa_{\Gamma^*}$.

For $\Gamma \in N(n,d)$, ζ_Γ is the integral operator with kernel:

$$\kappa_\Gamma(S,T) = \begin{cases} \epsilon(\gamma_{S,T}) & \text{if } \Gamma_{S,T} = \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $\gamma_{S,T} = (\Gamma, l_0)$, then $\gamma_{T,S} = (\Gamma^*, l_0^*)$. Therefore,

$$\begin{aligned}\kappa_{\Gamma^*}(T,S) &= \epsilon(\gamma_{T,S}) \\ &= \epsilon(\Gamma^*, l_0^*)\kappa_\Gamma(S,T).\end{aligned}$$

So the kernels κ_{Γ^*} and $\epsilon(\Gamma^*, l_0^*)\kappa_\Gamma^*$ coincide at (T,S), and hence on its entire $\mathbf{S}_d$-orbit in $B(n,d)$. □

We illustrate the above results with an example that will be used in the proof of Lemma 4.1.

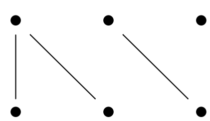
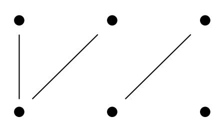
Example 2.29. Recall that $\Lambda(n,d)$ denotes the set of all weak compositions of d with at most n parts. For $\lambda \in \Lambda(n,d)$ with $n \geq d$, let $\Gamma_\lambda \in N(n,d)$ denote the bipartite graph where i' is connected to j if

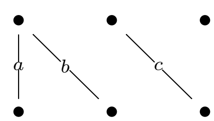
$$\lambda_1 + \cdots + \lambda_{i'-1} < j \leq \lambda_1 + \cdots + \lambda_{i'}.$$

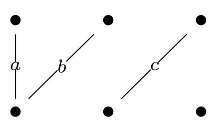
Then we have:

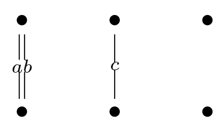
$$\zeta_{\Gamma_\lambda}\zeta_{\Gamma^*_\lambda} = \lambda_1! \cdots \lambda_n! \xi_{\Gamma^0_\lambda},$$

where $\Gamma^0_\lambda \in M(n,d)$ is the bipartite multigraph where i' is connected to i by λ_i edges. For example,

$\Gamma^*_{(2,1)} =$  , and $\Gamma_{(2,1)} =$ 

For a labelling $l_1 =$ 

of $\Gamma^*_{(2,1)}$ only the labelling $l_2 =$ 

of $\Gamma_{(2,1)}$, and the labelling $l =$ 

of $\Gamma^0_{(2,1)}$ are such that (l_1, l_2) are compatible with l. Moreover, $\epsilon(\Gamma^*_{(2,1)}, l_1) = \epsilon(\Gamma_{(2,1)}, l_2)$. Interchanging the labels a and b in l_1 and l_2, respectively, gives another pair of labels compatible with l, so that $\zeta_{\Gamma_\lambda}\zeta_{\Gamma^*_\lambda} = 2\xi_{\Gamma^0_\lambda}$.

The remaining results in this section help us understand the structure of $\mathrm{S}^-_F(n,d)$ as an $\mathrm{S}_F(n,d)$-module. Some of them will play an important role in understanding Koszul duality (Section 4).

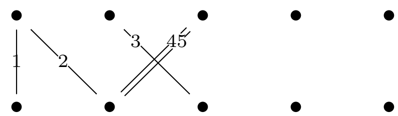
The notion of standard labelling (Definition 2.20) of graphs in $N(n,d)$ can be extended to graphs in $M(n,d)$ as follows: when an edge (i', j) occurs with multiplicity m, it is simply listed m times when the edges are arranged in lexicographic order with priority given to the upper index. Example 2.10 is the standard labelling of its underlying graph.

Definition 2.30. For $n \geq d$, define the following simple bipartite graphs associated to $\Gamma \in M(n,d)$:

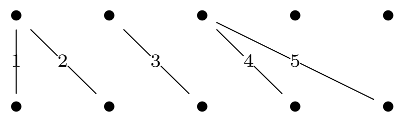
(2.30.a) Let $D(\Gamma) \in N(n,d)$ be the graph with edges (i', s) for every edge (i', j) with label s under the standard labelling of Γ.

(2.30.b) Let $U(\Gamma) \in N(n,d)$ be the graph with edges (s', j) for every edge (i', j) with label s under the standard labelling of Γ.

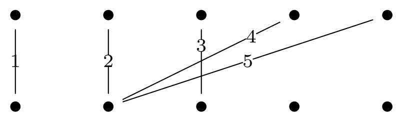
Example 2.31. Let $n = 5$, $d = 5$, and Γ (with its standard labelling) is given by:



Then

$D(\Gamma) =$ 

and

$U(\Gamma) =$ 

The significance of the elements $U(\Gamma)$ and $D(\Gamma)$, for $\Gamma \in M(n,d)$, is elaborated in the following lemmas.

Lemma 2.32. *Let $n \geq d$ and $\Gamma \in N(n,d)$. Then $\zeta_\Gamma = \xi_{U(\Gamma)}\zeta_{\Gamma_{\lambda_0}}\xi_{D(\Gamma)}$, where $\lambda_0 = (1^d, 0^{n-d}) \in \Lambda(n,d)$. Consequently, $\mathrm{S}_F^-(n,d)$ is a cyclic $(\mathrm{S}_F(n,d), \mathrm{S}_F(n,d))$-bimodule.*

Proof. This can be done in two steps. Firstly, $\zeta_{\Gamma_{\lambda_0}}\xi_{D(\Gamma)} = \zeta_{D(\Gamma)}$, and secondly $\xi_{U(\Gamma)}\zeta_{D(\Gamma)} = \zeta_\Gamma$. We indicate the proof of the second identity (the first is similar): Let l_0, l_1, and l_2 be the standard labellings of Γ, $U(\Gamma)$ and $D(\Gamma)$ respectively. The labellings (l_1, l_2) of $D(\Gamma)$ and $U(\Gamma)$ are the only ones that are compatible with l_0. This is because, for the edge (i', j) of Γ labelled s, s is the unique vertex such i' is connected to s in $D(\Gamma)$ and s' is connected to j in $U(\Gamma)$. The identity now follows from (2.25.a). □

Similarly, we have:

Lemma 2.33. *For $n \geq d$ and $\Gamma \in N(n,d)$, we have $\zeta_\Gamma = \zeta_{U(\Gamma)}\xi_{D(\Gamma)}$ and $\zeta_\Gamma = \xi_{U(\Gamma)}\zeta_{D(\Gamma)}$.*

Corollary 2.34. *As a left $\mathrm{S}_F(n,d)$-module, $\mathrm{S}_F^-(n,d)$ is generated by $\{\zeta_{\Gamma_\lambda^*} \mid \lambda \in \Lambda(n,d)\}$, and as a right $\mathrm{S}_F(n,d)$-module, it is generated by $\{\zeta_{\Gamma_\lambda} \mid \lambda \in \Lambda(n,d)\}$. Here Γ_λ is the graph associated to λ in Example 2.29.*

Proof. For any $\Gamma \in M(n,d)$, $D(\Gamma)$ is of the form Γ_λ^* for some $\lambda \in \Lambda(n,d)$, so the statement for left modules follows from the second identity in Lemma 2.33. The statement for right modules follows by applying Lemma 2.27 to the first identity in Lemma 2.33. □

Lemma 2.35. *Let $n \geq d$ and $\Gamma \in M(n,d)$. Then, for $\Gamma' \in M(n,d)$, the structure constant of $\zeta_{U(\Gamma)}$ in the product $\xi_{\Gamma'}\zeta_{D(\Gamma)^*}$ is $\delta_{\Gamma',\Gamma}$.*

Proof. The edge (i', j) with label s in the standard labelling of Γ gives rise to an edge (s', j) with standard label s in $U(\Gamma)$. The graph $D(\Gamma)^*$ has only one edge originating at s', namely (s', i). Therefore, for any compatible pair (l_1, l_2) of labellings of $D(\Gamma)^*$ and Γ', this edge must have label s. Thus Γ' must have an edge (i', j) labelled s. In other words, $\Gamma' = \Gamma$ and l_2 is its standard labelling. □

3. Abstract Koszul duality

3.1. The algebra. Recall [**6**, Chapter III, Section 3.1] that a $\mathbf{Z}/2\mathbf{Z}$ grading on a ring AS is a decomposition $\mathrm{AS} = \mathrm{S} \oplus \mathrm{S}^-$ into additive subgroups such that S is a subring, S^- is closed under left and right multiplication by elements of S, and for any $\alpha, \beta \in \mathrm{S}^-$, $\alpha\beta \in \mathrm{S}$. This $\mathbf{Z}/2\mathbf{Z}$-grading gives rise to:

(a) an (S, S)-bimodule structure on S^-,
(b) and an (S, S)-bimodule homomorphism $\phi : \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \to \mathrm{S}$ (induced by the S-balanced bilinear map $(\alpha, \beta) \mapsto \alpha\beta$ for $\alpha, \beta \in \mathrm{S}^-$).

Example 3.1. We may take $\mathrm{AS} = \mathrm{AS}_F(n, d) = \mathrm{End}_{\mathbf{A}_d}((F^n)^{\otimes d})$, and $\mathrm{S} = \mathrm{S}_F(n, d) = \mathrm{End}_{\mathbf{S}_d}((F^n)^{\otimes d})$, for any field F with characteristic different from 2.

3.2. Modules. Let M be an AS-module. The AS-module structure can be viewed as a linear map:

$$\mathrm{AS} \otimes_\mathbf{Z} M = (\mathrm{S} \oplus \mathrm{S}^-) \otimes_\mathbf{Z} M = (\mathrm{S} \otimes_\mathbf{Z} M) \oplus (\mathrm{S}^- \otimes_\mathbf{Z} M) \to M.$$

So M is an S-module, and restriction of the module action $\mathrm{AS} \otimes_\mathbf{Z} M \to M$ to $\mathrm{S}^- \otimes_\mathbf{Z} M$ induces an S-module homomorphism:

$$\theta_M : \mathrm{S}^- \otimes_\mathrm{S} M \to M. \tag{13}$$

Furthermore, this homomorphism θ_M has the property that the diagram

$$\begin{array}{ccc} \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \otimes_\mathrm{S} M & \xrightarrow{\phi \otimes \mathrm{id}_M} & S \otimes_\mathrm{S} M \\ {\scriptstyle \mathrm{id}_{\mathrm{S}^-} \otimes \theta_M} \downarrow & & \| \\ \mathrm{S}^- \otimes_\mathrm{S} M & \xrightarrow[\theta_M]{} & M \end{array} \tag{14}$$

commutes.

Definition 3.2. Given an (S, S)-bimodule S^- and an (S, S)-bimodule homomorphism $\phi : \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \to \mathrm{S}$, for an S-module N, an S-module homomorphism $\theta : \mathrm{S}^- \otimes_\mathrm{S} N \to N$ is said to be compatible with ϕ if the diagram

$$\begin{array}{ccc} \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \otimes_\mathrm{S} N & \xrightarrow{\phi \otimes \mathrm{id}_N} & S \otimes_\mathrm{S} N \\ {\scriptstyle \mathrm{id}_{\mathrm{S}^-} \otimes \theta} \downarrow & & \| \\ \mathrm{S}^- \otimes_\mathrm{S} N & \xrightarrow[\theta]{} & N \end{array} \tag{15}$$

commutes.

3.3. Duality. Let $\mathrm{AS} = \mathrm{S} \oplus \mathrm{S}^-$ be as before. Define a functor $D : \text{S-Mod} \to \text{S-Mod}$ by:

$$D(M) = \mathrm{S}^- \otimes_\mathrm{S} M,$$

for every S-module M. Given S-modules M and N, and an S-module homomorphism $f : M \to N$, let $D(f) = \mathrm{id}_{\mathrm{S}^-} \otimes f : D(M) \to D(N)$. We call the resulting functor $D : \text{S-Mod} \to \text{S-Mod}$ an *abstract Koszul duality functor*. In Section 4 it will be shown that, in the setting of Example 3.1 (the alternating Schur algebra), abstract Koszul duality is essentially the Koszul duality functor of Krause[**17**].

The commutative diagram (15), defining the compatibility of θ with ϕ, can be rewritten in terms of abstract Koszul duality as:

(16)
$$\begin{array}{ccc} D^2(N) & \xrightarrow{\phi\otimes\mathrm{id}_N} & N \\ {\scriptstyle D(\theta)}\big\downarrow & \nearrow{\scriptstyle\theta} & \\ D(N) & & \end{array}$$

Definition 3.3. Given an (S,S)-bimodule S^- and an (S,S)-bimodule homomorphism $\phi : \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \to \mathrm{S}$, let (S,ϕ)-Mod denote the category whose objects are pairs (N,θ), where N is an S-module, and $\theta : D(N) \to N$ is compatible with ϕ. A morphism $(N,\theta) \to (N',\theta')$ is an S-module homomorphism $f : N \to N'$ such that the diagram

$$\begin{array}{ccc} D(N) & \xrightarrow{\theta} & N \\ {\scriptstyle D(f)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D(N') & \xrightarrow{\theta'} & N' \end{array}$$

commutes.

Theorem 3.4. *Given an* AS*-module M, let θ_M be as in* (13)*. Then $M \mapsto (M,\theta_M)$ is an isomorphism of categories* AS-Mod $\to (\mathrm{S},\phi)$-Mod*.*

Proof. Given an object (N,θ) in (S,ϕ)-Mod, the compatibility of θ with ϕ allows the S-module structure on N to be extended to an AS-module structure. This constructs the inverse of the functor in the theorem. □

Given an S-module N, the morphism $\phi : \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \to \mathrm{S}$ gives rise to a natural transformation $\eta_N : D^2 \to \mathrm{id}_{\text{S-Mod}}$, defined as the composition:

(17)
$$\begin{array}{ccc} D^2(N) & \xrightarrow{\phi\otimes\mathrm{id}_N} & \mathrm{S}\otimes_\mathrm{S} N \\ & \searrow{\scriptstyle\eta_N} & \big\| \\ & & N \end{array}$$

Theorem 3.5. *Let* AS*,* S*,* S^- *and ϕ be as in Section 3.1. The following are equivalent:*

(3.5.a) *The map $\phi : \mathrm{S}^- \otimes_\mathrm{S} \mathrm{S}^- \to \mathrm{S}$ is an isomorphism.*
(3.5.b) *The natural transformation $\eta : D^2 \to \mathrm{id}_{\text{S-Mod}}$ is a natural isomorphism.*
(3.5.c) *For every object (N,θ) in (S,ϕ)-Mod, $\theta : \mathrm{S}^- \otimes_\mathrm{S} N \to N$ is an isomorphism of* S*-modules.*

Proof. To see that (3.5.a) implies (3.5.b), observe from the diagram (17) that if ϕ is an isomorphism, then η_N is an isomorphism for every N. It follows that η is a natural isomorphism. For the converse, taking $N = \mathrm{S}$, the commutativity of (17) shows that ϕ is an isomorphism.

To see that (3.5.a) implies (3.5.c), note that the commutativity of (15) implies that, if ϕ is an isomorphism, then θ is an epimorphism, and $\mathrm{id}_{\mathrm{S}^-} \otimes \theta$ is a monomorphism. Since tensoring is a right-exact functor, it follows that $\mathrm{id}_{\mathrm{S}^-} \otimes \theta$ is also an epimorphism, hence an isomorphism. Since $\phi \otimes \mathrm{id}_N$ is also an isomorphism the inverse of θ can be constructed by reversing the arrows in (15). For the converse, just take $N = \mathrm{S}$ in (3.5.c). □

3.4. Abstract Ringel duality. Let S^- be an (S,S)-bimodule. Denote the left S-module S^- by ${}_{\mathrm{S}}\mathrm{S}^-$. For a left S-module M, the homomorphism space $\mathrm{Hom}_{\mathrm{S}}({}_{\mathrm{S}}\mathrm{S}^-, M)$ inherits the structure of a left S-module from the right S-module structure on S^-. Motivated by [**26**, Section 6], we call the functor

$$\mathrm{Hom}_{\mathrm{S}}({}_{\mathrm{S}}\mathrm{S}^-, -) : \mathrm{S}\text{-Mod} \to \mathrm{S}\text{-Mod}$$

the abstract Ringel duality functor on S-Mod. It is clear that the abstract Koszul duality functor is the left adjoint of abstract Ringel duality functor.

3.5. Abstract simple modules. In general, it is not clear how simple AS-modules can be classified using simple S-modules and Koszul duality. In this section, we give some results in this direction. These are enough to give a complete solution in the semisimple case.

Let M be a simple S-module. We consider the following cases:

3.5.1. *DM is isomorphic to M.* If $\eta_M : D^2M \to M$ is zero, then $(M, 0)$ (where 0 is the zero map from $DM \to M$) is the unique ϕ-compatible morphism. Otherwise, any non-zero morphism $\theta : DM \to M$ is an isomorphism. Schur's lemma implies that $\theta \circ D\theta = a\eta_M$ for some $a \in (\mathrm{End}_{\mathrm{S}} M)^*$ (the multiplicative group of non-zero elements in the division algebra $\mathrm{End}_{\mathrm{S}} M$). If a has a square root in $(\mathrm{End}_{\mathrm{S}} M)^*$, then θ can be normalized to make it a ϕ-compatible morphism. Moreover, after normalization, $\pm\theta$ are two ϕ-compatible morphisms, leading to two non-isomorphic AS-modules. Also, in this case, $(M, \pm\theta)$ are simple, because their restrictions to S are simple. On the other hand, if a does not have a square root in $\mathrm{End}_{\mathrm{S}}(M)$, then there is no simple AS-module whose restriction to S is isomorphic to M.

3.5.2. *$DM = 0$.* In this case $(M, 0)$ is the unique AS-module whose restriction to S is isomorphic to M.

3.5.3. *DM is simple, but not isomorphic to M, $\eta_M \neq 0$.* Let $\tilde{M} = DM \oplus M$. We have $D\tilde{M} = D^2M \oplus DM$. Any morphism $\theta : D\tilde{M} \to \tilde{M}$ can be written in matrix form as

$$\theta = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where $X : D^2M \to DM$, $Y : DM \to DM$, $Z : D^2M \to M$, and $W : DM \to M$. By Schur's lemma, $W = 0$. The compatibility of θ with ϕ becomes:

$$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix} \begin{pmatrix} DX & DY \\ DZ & 0 \end{pmatrix} = \begin{pmatrix} \eta_{DM} & 0 \\ 0 & \eta_M \end{pmatrix}.$$

Multiplying out the left hand side gives:

$$\begin{pmatrix} XDX + YDZ & XDY \\ ZDX & ZDY \end{pmatrix} = \begin{pmatrix} \eta_{DM} & 0 \\ 0 & \eta_M \end{pmatrix}.$$

Since $\eta_M \neq 0$, $DY \neq 0$, and so $Y \neq 0$. Since DM is simple, by Schur's lemma, Y is invertible. Since D is a functor, DY is also invertible. Hence, equality of top right entries implies that $X = 0$. Moreover, $Z = \eta_M DY^{-1}$. In other words, θ is of the form:

$$\theta_Y = \begin{pmatrix} 0 & Y \\ \eta_M DY^{-1} & 0 \end{pmatrix}.$$

LEMMA 3.6. *For all* $Y, Y' \in (\mathrm{End}_{\mathrm{S}} M)^*$, $\mathrm{Hom}_{\mathrm{AS}}((\tilde{M}, \theta_Y), (\tilde{M}, \theta_{Y'}))$ *is non-zero, and* $\mathrm{End}_{\mathrm{AS}}(\tilde{M}, \theta_Y)$ *is a division ring.*

PROOF. Any AS-module morphism $(\tilde{M}, \theta_y) \to (\tilde{M}, \theta_{y'})$ can be written in matrix form as

$$\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}, \text{ where } X \in \mathrm{End}_{\mathrm{S}} DM \text{ and } W \in \mathrm{End}_{\mathrm{S}} M,$$

and must satisfy:

$$\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} 0 & Y \\ \eta_M DY^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y' \\ \eta_M DY'^{-1} & 0 \end{pmatrix} \begin{pmatrix} DX & 0 \\ 0 & DW \end{pmatrix}$$

We get $XY = Y'DW$, and $W\eta_M DY^{-1} = \eta_M DY'^{-1}DX$. Taking $W = \mathrm{id}_M$, and $X = Y'Y^{-1}$ gives a non-zero element of $\mathrm{Hom}_{\mathrm{AS}}((\tilde{M}, \theta_Y), (\tilde{M}, \theta_{Y'}))$. When $Y = Y'$, then we have $X = YDWY^{-1}$, so that X is non-zero (and hence invertible) if and only if W is. It follows that every non-zero element of $\mathrm{End}_{\mathrm{AS}}(\tilde{M}, \theta_Y)$ is invertible. □

LEMMA 3.7. *Let M be an* S*-module. Then the* AS*-module* $\mathrm{AS} \otimes_{\mathrm{S}} M$ *is isomorphic to* $(DM \oplus M, \theta)$*, where θ is given by the matrix:*

$$\theta = \begin{pmatrix} 0 & \mathrm{id}_{DM} \\ \eta_M & 0 \end{pmatrix}.$$

PROOF. Note that

$$\mathrm{AS} \otimes_{\mathrm{S}} M = (\mathrm{S}^- \otimes_{\mathrm{S}} M) \oplus (\mathrm{S} \otimes_{\mathrm{S}} M) = DM \oplus M.$$

The map θ comes from the action of S^- on this AS-module, which gives $\eta_M : D^2M \to M$ on the first summand, and $\mathrm{id}_{DM} : DM \to DM$ on the second summand. □

THEOREM 3.8. *The* AS*-module* $(DM \oplus M, \theta_Y)$ *defined above is isomorphic to* $\mathrm{AS} \otimes_{\mathrm{S}} M$ *for every* $Y \in (\mathrm{End}_{\mathrm{S}} DM)^*$*. Consequently, whenever M and DM are simple, non-isomorphic* S*-modules, and* $\eta_M \neq 0$*, then* $\mathrm{AS} \otimes_{\mathrm{S}} M$ *is, up to isomorphism, the unique simple* AS*-module whose restriction to* S *contains M.*

PROOF. To see that $\mathrm{AS} \otimes_{\mathrm{S}} M = DM \oplus M$ is simple, note that its only proper non-trivial S-submodules are M and DM. But M is not AS-invariant because S^- maps M onto DM. Also, DM is not AS-invariant, because S^- maps DM onto D^2M. Since $\eta_M \neq 0$, D^2M cannot be contained in DM. The theorem now follows from Lemma 3.6. □

3.5.4. *The case where ϕ is an isomorphism.* When $\phi : \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{S}^- \to \mathrm{S}$ is an isomorphism the preceding results, using Theorem 3.5, can be summarized in the following form:

THEOREM 3.9. *Suppose that* AS *is endowed with a* $\mathbf{Z}/2\mathbf{Z}$*-grading* $\mathrm{AS} = \mathrm{S} \oplus \mathrm{S}^-$*, and* $\phi : \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{S}^- \to \mathrm{S}$ *(as defined in Section 3.1) is an isomorphism. Let M be a simple* S*-module. Then*

(3.9.a) *Suppose there exists an isomorphism* $\theta : DM \to M$*. Then θ can be scaled to become compatible with ϕ. There exist at most two isomorphism classes of simple* AS*-modules* $(M, \pm\theta)$ *whose restrictions to* S *are isomorphic to M. If* $(\mathrm{End}_{\mathrm{S}} M)^*$ *is a 2-divisible group, then these two classes always exist.*

(3.9.b) *Otherwise, up to isomorphism,* $\mathrm{AS} \otimes_{\mathrm{S}} M$ *is the unique simple* AS-*module whose restriction to* S *contains* M *as a submodule. Also,* $\mathrm{AS}\otimes_{\mathrm{S}} M$ *and* $\mathrm{AS} \otimes_{\mathrm{S}} DM$ *are isomorphic as* AS-*modules.*

Corollary 3.10. *Suppose* F *is an algebraically closed field of characteristic not equal to* 2*. Let* $\mathrm{AS} = \mathrm{S} \oplus \mathrm{S}^-$ *be a* $\mathbf{Z}/2\mathbf{Z}$*-graded* F*-algebra. A complete set of isomorphism classes of simple* AS-*modules is given by:*

(3.10.c) $(M, \pm\theta)$ *(defined in Section 3.5.1), as* M *runs over isomorphism classes of simple* S-*modules such that* DM *is isomorphic to* M*,*

(3.10.d) $\mathrm{AS} \otimes_{\mathrm{S}} M$*, as* M *runs over isomorphism classes of all unordered pairs* $\{M, M'\}$ *of non-isomorphic mutually dual simple* S-*modules.*

4. Koszul duality for modules over Schur algebra

In this section, let S denote the Schur algebra $\mathrm{S}_F(n, d)$, and let S^- denote the (S, S)-bimodule $\mathrm{S}^-_F(n, d)$. We now use our combinatorial methods from Section 2 to determine when abstract Koszul duality is an equivalence.

Lemma 4.1. *When the characteristic of* F *is* 0 *or greater than* d*, and* $n \geq d$*, the map* $\phi : \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{S}^- \to \mathrm{S}$ *is an isomorphism.*

Proof. For each $\lambda \in \Lambda(n, d)$, let $\Gamma^0_\lambda \in M(n, d)$ be the bipartite multigraph with λ_i edges from i' to i (and no other edges), as in Example 2.29. Then

$$\mathrm{id}_{\mathrm{S}} = \sum_{\lambda\in\Lambda(n,d)} \xi_{\Gamma^0_\lambda}. \tag{18}$$

Therefore by Example 2.29,

$$\mathrm{id}_S = \sum_{\lambda\in\Lambda(n,d)} \frac{1}{\lambda_1! \cdots \lambda_n!} \zeta_{\Gamma_\lambda}\zeta^*_{\Gamma_\lambda}. \tag{19}$$

Therefore the image of ϕ, which is a two-sided ideal of S, contains the identity element, and therefore is all of S.

The injectivity of ϕ can be proved using a dimension count. Let N^d (resp., N_d) denote the graphs in $N(n, d)$ with upper (resp., lower) degree sequence $(1^d, 0^{n-d})$. Let $\Gamma(w) \in N(n, d)$ be as in Example 2.15. For $\Gamma \in N^d$, define $\Gamma \cdot w \in N^d$ by $\xi_\Gamma\xi_{\Gamma(w)} = \xi_{\Gamma\cdot w}$. Similarly, for $\Gamma \in N_d$, define $w \cdot \Gamma \in N_d$ by $\xi_{\Gamma(w)}\xi_\Gamma = \xi_{w\cdot\Gamma}$. Consider the equivalence relation on $N^d \times N_d$ where $(\Gamma, \Gamma') \sim (\Gamma \cdot w^{-1}, w \cdot \Gamma')$ for $w \in \mathbf{S}_d$. Let $N^d \times_{\mathbf{S}_d} N_d$ denote the set of equivalence classes.

Now, given $(\Gamma', \Gamma'') \in N^d \times N_d$, define $\Gamma = \Phi(\Gamma', \Gamma'') \in M(n, d)$ to be the graph for which the number of edges joining (i', j) is the number of indices $1 \leq k \leq n$ such that (i', k) is an edge of Γ'' and (k', j) is an edge of Γ'. This map induces an injective function $\bar{\Phi} : N^d \times_{\mathbf{S}_d} N_d \to M(n, d)$. Moreover, $\Gamma = \Phi(U(\Gamma), D(\Gamma))$, so $\bar{\Phi} : N^d \times_{\mathbf{S}_d} N_d \to M(n, d)$ is a bijection.

The elements $\zeta_{\Gamma'} \otimes \zeta_{\Gamma''}$ as Γ' and Γ'' run over $N(n, d)$, span $\mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{S}^-$. We have:

$$\begin{aligned}\zeta_\Gamma \otimes \zeta_{\Gamma'} &= \zeta_{U(\Gamma)}\xi_{D(\Gamma)} \otimes \xi_{U(\Gamma')}\zeta_{D(\Gamma')} && \text{(from Lemma 2.33)}\\ &= \zeta_{U(\Gamma)} \otimes \xi_{D(\Gamma)}\xi_{U(\Gamma')}\zeta_{D(\Gamma')}\end{aligned}$$

Now $U(\Gamma) \in N^d$ and $\xi_{D(\Gamma)}\xi_{U(\Gamma')}\zeta_{D(\Gamma')}$ lies in the span of $\zeta_{\Gamma'''}$, for $\Gamma''' \in N_d$. Therefore $\zeta_{\Gamma'}\otimes\zeta_{\Gamma''}$ span $\mathrm{S}^-\otimes_{\mathrm{S}}\mathrm{S}^-$ as $(\Gamma', \Gamma'') \in N^d\times N_d$. Moreover, $\zeta_{\Gamma'\cdot w}\otimes\zeta_{w^{-1}\cdot\Gamma''} = \zeta_{\Gamma'} \otimes \zeta_{\Gamma''}$, so $\dim \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{S}^- \leq |N^d \times_{\mathbf{S}_d} N_d| = |M(n, d)|$. □

Now, using Theorem 3.5 we have established a direct combinatorial proof of the following theorem:

THEOREM 4.2. *For a field F of characteristic* 0 *or greater than d, and $n \geq d$, the Koszul duality functor D :* S-Mod $\to$ S-Mod *is an equivalence of categories.*

4.1. Strict polynomial functor. Friedlander and Suslin [**11**] introduced strict polynomial functors in order to establish the finite generation of the full cohomology ring of a finite group scheme. They also showed that the strict polynomial functors of degree d unify modules over the Schur algebras $\mathrm{S}_F(n,d)$ across all n. In this section, we briefly recall the definition of strict polynomial functors and some useful functors on the category of strict polynomial functors.

Following [**17, 31**], define the *Schur category* (also known as the *divided power category*) $\mathbf{\Gamma}_F^d$ as the category whose objects are finite dimensional vector spaces over F. For objects V and W, the morphism space is:

$$\mathrm{Hom}_{\mathbf{\Gamma}_F^d}(V,W) := \mathrm{Hom}_{\mathbf{S}_d}(V^{\otimes d}, W^{\otimes d}).$$

The category, $\mathrm{Rep}\,\mathbf{\Gamma}_F^d$ of strict polynomial functors is the functor category $\mathrm{Func}(\mathbf{\Gamma}_F^d, F\text{-Mod})$. Thus it is an abelian, complete, and co-complete category.

EXAMPLE 4.3. Let V and W be objects of $\mathbf{\Gamma}_F^d$. Some examples of strict polynomial functors are:

(4.3.a) The dth tensor power functor $\otimes^d : \mathbf{\Gamma}_F^d \to F\text{-Mod}$. On objects, $\otimes^d(V) = V^{\otimes d}$. On the morphism space, the map

$$\mathrm{Hom}_{\mathbf{S}_d}(V^{\otimes d}, W^{\otimes d}) \to \mathrm{Hom}_{\mathbf{S}_d}(V^{\otimes d}, W^{\otimes d})$$

is the identity map.

(4.3.b) The dth divided power functor $\Gamma^d : \mathbf{\Gamma}_F^d \to F\text{-Mod}$. On objects $\Gamma^d(V) = (V^{\otimes d})^{\mathbf{S}_d}$ and on the morphism space, the map

$$\mathrm{Hom}_{\mathbf{\Gamma}_F^d}(V,W) \to \mathrm{Hom}_{\mathbf{S}_d}((V^{\otimes d})^{\mathbf{S}_d}, (W^{\otimes d})^{\mathbf{S}_d})$$

is given by the restriction.

(4.3.c) Similarly, the dth exterior power functor $\wedge^d : \mathbf{\Gamma}_F^d \to F\text{-Mod}$, and the dth symmetric power functor $\mathrm{Sym}^d : \mathbf{\Gamma}_F^d \to F\text{-Mod}$ are strict polynomial functors of degree d.

(4.3.d) Let U be an object in $\mathbf{\Gamma}_F^d$. Then define $\mathbf{h}_U : \mathbf{\Gamma}_F^d \to F\text{-Mod}$ as follows:

$$\mathbf{h}_U(W) = \mathrm{Hom}_{\mathbf{\Gamma}_F^d}(U,W) = \mathrm{Hom}_{\mathbf{S}_d}(U^{\otimes d}, W^{\otimes d}).$$

The functor $\mathbf{h}_U \in \mathrm{Rep}\,\mathbf{\Gamma}_F^d$ is called a *representable functor*. The functor $\mathbf{h} : U \mapsto \mathbf{h}_U$ is the contravariant Yoneda embedding.

(4.3.e) For any object U of $\mathbf{\Gamma}_F^d$ and any $X \in \mathrm{Rep}\,\mathbf{\Gamma}_F^d$, define a functor $X^U : \mathbf{\Gamma}_F^d \to F\text{-Mod}$ by

$$X^U(W) = X(\mathrm{Hom}_F(U,W)). \tag{20}$$

When $X = \Gamma^d$, $X^U = h_U$.

Given a strict polynomial functor X, $X(F^n)$ inherits the structure of an $\mathrm{S}_F(n,d)$-module. For every non-negative integer n, we have the evaluation functor $\mathrm{ev}_n : \mathrm{Rep}\,\mathbf{\Gamma}_F^d \to \mathrm{S}_F(n,d)\text{-Mod}$ as:

$$\mathrm{ev}_n(X) = X(F^n) \text{ for } X \in \mathrm{Rep}\,\mathbf{\Gamma}_F^d.$$

THEOREM 4.4. [**11**, Theorem 3.2] *The functor* $\mathrm{ev}_n : \mathrm{Rep}\,\mathbf{\Gamma}_F^d \to \mathrm{S}_F(n,d)\text{-Mod}$ *is an equivalence of categories whenever* $n \geq d$.

4.2. Koszul duality of strict polynomial functors: In [**17**], Krause defined an internal tensor product ($\underline{\otimes}$) on the category of strict polynomial functors of a fixed degree d. Kulkarni, Srivastava, and Subrahmanyam [**18**], and independently, Acquilino and Reischuk [**1**] showed that this internal tensor product, via the Schur functor, is related to the Kronecker tensor product of representations of the symmetric group $\mathbf{S}_d$. Krause used this internal tensor product to introduce Koszul duality as the functor $(\wedge^d \underline{\otimes} -) : \mathrm{Rep}\,\mathbf{\Gamma}_F^d \to \mathrm{Rep}\,\mathbf{\Gamma}_F^d$. We can think about this functor as follows, for the representable functor $\mathbf{h}_V \in \mathrm{Rep}\,\mathbf{\Gamma}_F^d$, we have, using the notation of (20),

$$\wedge^d \underline{\otimes}\, \mathbf{h}_V = \wedge^{d,V}. \tag{21}$$

For arbitrary $X \in \mathrm{Rep}\,\mathbf{\Gamma}_F^d$, following [**17**], we exploit a theorem of Mac Lane [**19**, III.7,Theorem 1], namely:

$$X = \operatorname*{colim}_{\mathbf{h}_V \to X} \mathbf{h}_V. \tag{22}$$

Using this we have:

$$\wedge^d \underline{\otimes}\, X = \operatorname*{colim}_{\mathbf{h}_V \to X} \wedge^d \underline{\otimes}\, \mathbf{h}_V = \operatorname*{colim}_{\mathbf{h}_V \to X} \wedge^{d,V}. \tag{23}$$

In the following theorem, we relate the abstract Koszul duality of Schur algebra with the Koszul duality of strict polynomial functors.

THEOREM 4.5. *Consider the functors,*

$$(\mathrm{S}^- \otimes_S \mathrm{ev}_n(-)) : \mathrm{Rep}\,\mathbf{\Gamma}_F^d \to \mathrm{S}\text{-Mod},$$
$$\mathrm{ev}_n(\wedge^d \underline{\otimes} -) : \mathrm{Rep}\,\mathbf{\Gamma}_F^d \to \mathrm{S}\text{-Mod}.$$

Then there exists a natural transformation

$$\eta : (\mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(-)) \longrightarrow \mathrm{ev}_n(\wedge^d \underline{\otimes} -),$$

which is an isomorphism when $n \geq d$.

PROOF. Let $X = \mathbf{h}_V$. Then,

$$\begin{aligned} \mathrm{ev}_n(\wedge^d \underline{\otimes}\, \mathbf{h}_V) &= \mathrm{ev}_n(\wedge^{d,V}) \text{ by (21)} \\ &= \wedge^{d,V}(F^n) \\ &= \wedge^d \mathrm{Hom}_F(V, F^n) \text{ by Equation (20)} \\ &\simeq \mathrm{Hom}_{\mathbf{S}_d}(V^{\otimes d}, (F^n)^{\otimes d} \otimes \mathrm{sgn}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(\mathbf{h}_V) &= \mathrm{S}^- \otimes_{\mathrm{S}} \mathbf{h}_V(F^n) \\ &= \mathrm{Hom}_{\mathbf{S}_d}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \mathrm{sgn}) \otimes_{\mathrm{S}} \mathrm{Hom}_{\mathbf{S}_d}(V^{\otimes d}, (F^n)^{\otimes d}). \end{aligned}$$

Using these identifications, $\eta_{\mathbf{h}_V}(g_1 \otimes g_2) = g_1 \circ g_2$, for $g_1 \in \mathrm{S}^-$ and $g_2 \in \mathrm{ev}_n(\mathbf{h}_V)$, defines an S-linear map

$$\eta_{\mathbf{h}_V} : \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(\mathbf{h}_V) \longrightarrow \mathrm{ev}_n(\wedge^d \underline{\otimes}\, \mathbf{h}_V). \tag{24}$$

For arbitrary $X \in \mathrm{Rep}\,\mathbf{\Gamma}_F^d$, we construct η_X using Equation (22):

$$\eta_X = \operatorname*{colim}_{\mathbf{h}_V \to X} \eta_{\mathbf{h}_V}.$$

From the Yoneda lemma [**19**, Page 59], every morphism $\mathbf{h}_V \to \mathbf{h}_W$ between the representable functors is of the form $\mathbf{h}_f$ for a unique morphism $f \in \mathrm{Hom}_{\mathbf{F}^d_F}(W, V)$. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(\mathbf{h}_V) & \xrightarrow{\eta_{\mathbf{h}_V}} & \mathrm{ev}_n(\wedge^d \underline{\otimes}\, \mathbf{h}_V) \\ {\scriptstyle \mathrm{id}_{\mathrm{S}^-} \otimes \mathrm{ev}_n(\mathbf{h}_f)} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_n(\mathrm{id}_{\wedge^d} \underline{\otimes}\, \mathbf{h}_f)} \\ \mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(\mathbf{h}_W) & \xrightarrow{\eta_W} & \mathrm{ev}_n(\wedge^d \underline{\otimes}\, \mathbf{h}_V) \end{array}$$

Taking colimits then gives the naturality of η.

If $n \geq d$, $\mathbf{h}_{F^n}$ is a small projective generator of $\mathrm{Rep}\,\mathbf{\Gamma}^d_F$, i.e., every object has a presentation by $\mathbf{h}_{F^n}$, (see [**17**]). Note that the map $\eta_{\mathbf{h}_{F^n}}$ (24) is surjective because $\eta_{\mathbf{h}_{F^n}}(f \otimes \mathrm{id}_{\mathrm{S}}) = f$ for $f \in \mathrm{S}^-$ and hence an isomorphism because $\mathrm{ev}_n(\wedge^d \underline{\otimes}\, \mathbf{h}_{F^n})$ is isomorphic to S^-. By the construction of η_X, this implies that each η_X is an isomorphism for $X \in \mathrm{Rep}\,\mathbf{\Gamma}^d_F$. □

4.3. Derived abstract Koszul duality. For a finite dimensional associative algebra A, let $\mathcal{D}(A\text{-Mod})$ be the unbounded derived category of A-Mod. For a (A, A)-bimodule M, the functor $(M \otimes_A -)$ is a right exact functor so the total left derived functor $(M \otimes^{\mathbf{L}}_A -) : \mathcal{D}(A\text{-Mod}) \to \mathcal{D}(A\text{-Mod})$ exists. From Happel [**15**], we recall necessary and sufficient conditions for the functor $(M \otimes^{\mathbf{L}}_A -)$ to be an equivalence of categories.

For each $x \in A$, let $\psi_x \in \mathrm{End}_A(M_A)$ be defined by

$$\psi_x(y) = xy.$$

Taking x to ψ_x gives rise to a homomorphism of algebras:

$$\psi : A \to \mathrm{End}_A(M_A). \tag{25}$$

Recall

Theorem 4.6 (Happel [**15**]). *For a finite dimensional algebra A and a (A, A)-bimodule M, the functor $(M \otimes^{\mathbf{L}}_A -) : \mathcal{D}(A\text{-Mod}) \to \mathcal{D}(A\text{-Mod})$ is an equivalence of categories if and only if*

(4.6.a) *The module M_A admits a finite resolution by finitely generated projective right modules over A.*

(4.6.b) *The canonical map $\psi : A \to \mathrm{End}_A(M_A)$ is an isomorphism, and for $i \geq 1$, $\mathrm{Ext}^i_A(M, M) = 0$.*

(4.6.c) *There exists an exact sequence consisting of right A-modules:*

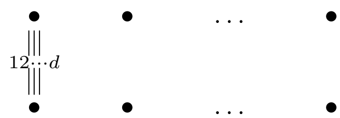
$$0 \to A \to M_1 \to \cdots \to M_l \to 0,$$

where for $1 \leq i \leq l$, M_i is a direct summand of finite direct sum of copies of M.

Theorem 4.7. *Let $A = \mathrm{S}$ and M be the (S, S)-bimodule S^-. Then the map ψ in Equation (25) is an isomorphism if and only if $n \geq d$.*

Remark 4.8. When $n \geq d$, it is known that ψ is an isomorphism, even for q-Schur algebras (see Donkin [**10**, p. 82]).

Proof. Suppose $n < d$. Consider the following labelled bipartite multigraph,

$\Gamma_1 =$ 

Let $\Gamma_2 \in N(n,d)$. Since Γ_2 is a simple bipartite graph therefore any labelling of Γ_2 which satisfies the condition 1 in Definition 2.13 requires d balls to place into d distinct boxes out of n. This is not possible as $n < d$. Thus we get that $\xi_{\Gamma_1}\zeta_{\Gamma_2} = 0$. Since ζ_Γ for $\Gamma \in N(n,d)$ forms a basis of S^- we get ξ_{Γ_1} is in the kernel of ψ, and so ψ is not injective.

For the converse, suppose $n \geq d$. Then the map ψ is an isomorphism is known from [**9**, Proposition 3.7]. But we give a combinatorial proof here. For $\theta \in \mathrm{End}_{\mathrm{S}}(\mathrm{S}^-_{\mathrm{S}})$, we denote the coefficient of ζ_{Γ_1} in $\theta(\zeta_{\Gamma_2})$ by $\langle \theta(\zeta_{\Gamma_2}), \zeta_{\Gamma_1}\rangle$.

To see that ψ is injective, note that any element of S is of the form $s = \sum_{\Gamma \in M(n,d)} \alpha_\Gamma \xi_\Gamma$. Now $\psi(s) = 0$ if and only if

$$\sum_{\Gamma \in M(n,d)} \alpha_\Gamma \xi_\Gamma \zeta_{\Gamma'} = 0,$$

for every $\Gamma' \in N(n,d)$. Fix $\Gamma_1 \in M(n,d)$ and let $\Gamma' = D(\Gamma_1)^*$. Then by Lemma 2.35,

$$\alpha_{\Gamma_1} = \Big\langle \sum_{\Gamma \in M(n,d)} \alpha_\Gamma \xi_\Gamma \zeta_{D(\Gamma_1)^*}, \zeta_{U(\Gamma_1)} \Big\rangle.$$

Thus $\alpha_{\Gamma_1} = 0$.

To see that ψ is surjective, we will show that $\dim_F \mathrm{End}_{\mathrm{S}}(\mathrm{S}^-_{\mathrm{S}}) \leq |M(n,d)|$. Firstly, by Lemma 2.33, $\mathrm{S}^-_{\mathrm{S}}$ is generated by $G = \{\zeta_\Gamma \mid \Gamma \in N^d\}$. Recall that N^d denotes the set of graphs in $N(n,d)$ with upper degree sequence $(1^d, 0^{n-d})$. Therefore any $\theta \in \mathrm{End}_{\mathrm{S}}(\mathrm{S}^-_{\mathrm{S}})$ is determined by its values on this set. Since θ is an S-module homomorphism $\theta(\zeta_\Gamma)$ again lies in the span of G. Therefore θ is completely determined by the values:

$$\{\langle \theta(\zeta_\Gamma), \zeta_{\Gamma'}\rangle \mid \Gamma, \Gamma' \in N^d\}.$$

Moreover, for any $w \in \mathbf{S}_d$,

$$\langle \theta(\zeta_\Gamma), \zeta_{\Gamma'}\rangle = \langle \theta(\zeta_{\Gamma\cdot w}), \zeta_{\Gamma'\cdot w}\rangle.$$

Therefore $\dim_F \mathrm{End}_{\mathrm{S}}(\mathrm{S}^-_{\mathrm{S}}) \leq |(N^d \times N^d)/\mathbf{S}_d| = |M(n,d)|$. □

Theorem 4.9. *Let F be any field of characteristic different from 2. The functor*

$$(\mathrm{S}^- \otimes^{\mathbf{L}}_{\mathrm{S}} -) : \mathcal{D}(\text{S-Mod}) \to \mathcal{D}(\text{S-Mod}) \tag{26}$$

is an equivalence of categories if and only if $n \geq d$.

Proof. If $n \geq d$ then from Theorem 4.5, $\mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(-)$ is isomorphic to $\mathrm{ev}_n(\wedge^d \underline{\otimes} -)$. Since the functor ev_n is exact so the total left derived functors of $\mathrm{S}^- \otimes_{\mathrm{S}} \mathrm{ev}_n(-)$ and $\mathrm{ev}_n(\wedge^d \underline{\otimes} -)$ are isomorphic to $\mathrm{S}^- \otimes^{\mathbf{L}}_{\mathrm{S}} \mathrm{ev}_n(-)$ and $\mathrm{ev}_n(\wedge^d \underline{\otimes}^{\mathbf{L}} -)$ respectively. From [**17**, Theorem 4.9], $(\wedge^d \underline{\otimes}^{\mathbf{L}} -)$ is an equivalence. From Theorem 4.4 $\mathrm{ev}_n(-)$ is an equivalence. Therefore $\mathrm{S}^- \otimes^{\mathbf{L}}_{\mathrm{S}} \mathrm{ev}_n(-)$ is an equivalence. So $\mathrm{ev}_n(-)$ being an equivalence forces $(\mathrm{S}^- \otimes^{\mathbf{L}}_{\mathrm{S}} -)$ to be an equivalence.

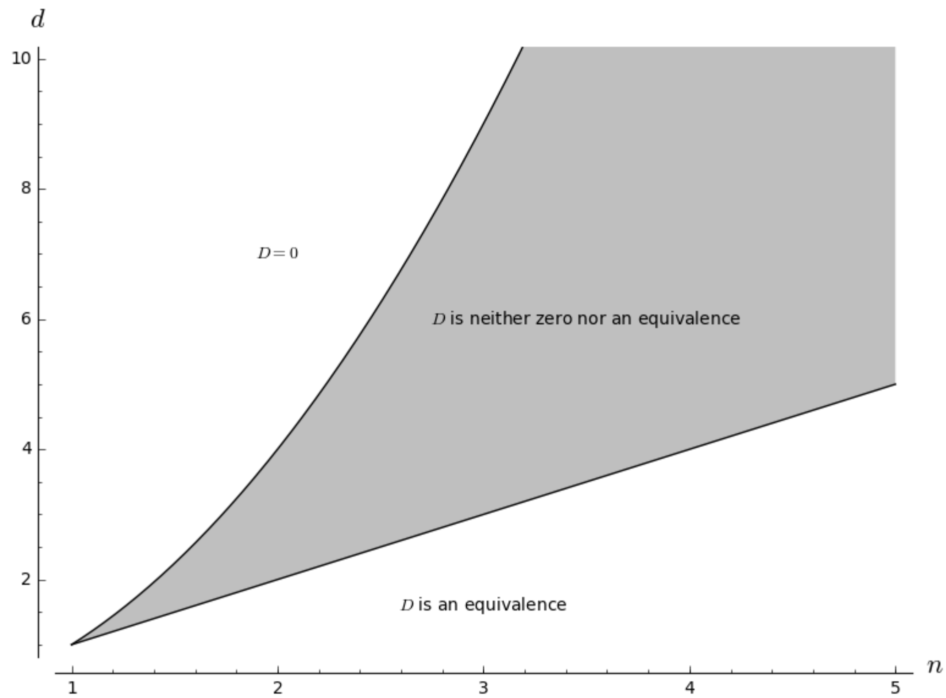


FIGURE 1. Dependence of derived Koszul duality on n and d

For the converse, suppose that $(\mathrm{S}^- \otimes^{\mathbf{L}}_{\mathrm{S}} -)$ is an equivalence. Then from Theorem 4.6 the map $\psi : \mathrm{S} \to \mathrm{End}_{\mathrm{S}}(\mathrm{S}^-)$ is an isomorphism. Theorem 4.7 now implies that $n \geq d$. □

The behavior of derived Koszul duality for different values of n and d is summarized in Figure 1.

5. Concluding remarks

5.1. Towards alternating partition algebras. The centralizer algebra $\mathrm{End}_{\mathbf{S}_n}((F^n)^{\otimes d})$ is a quotient of the partition algebra $\mathrm{P}_d(n)$ of Jones [**16**] and Martin [**20**]. Further restricting the action of $\mathbf{S}_n$ to the alternating group $\mathbf{A}_n$ we get the *alternating partition algebra* $\mathrm{AP}_d(n) = \mathrm{End}_{\mathbf{A}_n}((F^n)^{\otimes d})$, which from the isomorphism (2) decomposes as follows:

$$\mathrm{End}_{\mathbf{A}_n}((F^n)^{\otimes d}) = \mathrm{End}_{\mathbf{S}_n}((F^n)^{\otimes d}) \oplus \mathrm{Hom}_{\mathbf{A}_n}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \mathrm{sgn}) \tag{27}$$

Let $\mathrm{P}^-_d(n) = \mathrm{Hom}_{\mathbf{A}_n}((F^n)^{\otimes d}, (F^n)^{\otimes d} \otimes \mathrm{sgn})$. Then $\mathrm{P}^-_d(n)$ becomes a $(\mathrm{P}_d(n), \mathrm{P}_d(n))$-bimdoule by inflation. Bloss [**5**] showed $\mathrm{P}^-_d(n)$ is non-zero if and only if $n < 2d+2$. So we get an abstract Koszul duality $(\mathrm{P}^-_d(n) \otimes_{\mathrm{P}_d(n)} -)$ on the category of modules over the partition algebra $\mathrm{P}_d(n)$ when $n < 2d+2$. Many of the ideas and techniques in this article can be used to study bases, structure constants, and abstract Koszul duality for partition algebras. For $F = \mathbf{C}$, the dimensions of simple modules of $\mathrm{AP}_d(n)$ are given combinatorially by Benkart, Halverson, and Harman, see [**2**].

5.2. A diagrammatic interpretation of the Schur category. The following is one possible way to define the notion of a diagram category in the spirit of Martin's discussion in [**21**].

DEFINITION 5.1 (Diagram Category). A category $\mathcal{C}$ is called a *diagram category* if there exists a sequence $\{V_n\}_{n\geq 0}$ of objects which constitute a skeleton of $\mathcal{C}$, and for each pair (m,n) of non-negative integers, a class of "diagrams" $M(m,n)$, a basis

$$\mathcal{B}_{m,n} = \{\xi_\Gamma \mid \Gamma \in M(m,n)\}$$

of $\mathrm{Hom}_{\mathcal{C}}(V_n, V_m)$, and a combinatorial rule for computing the structure constants $c^{\Gamma}_{\Gamma'\Gamma''}$ that are defined by:

$$\xi_{\Gamma'} \circ \xi_{\Gamma''} = \sum_{\Gamma} c^{\Gamma}_{\Gamma'\Gamma''} \xi_\Gamma$$

for $\Gamma' \in M(l,m)$, $\Gamma'' \in M(m,n)$ and $\Gamma \in M(l,n)$.

REMARK 5.2. In the examples discussed by Martin [**21**], given diagrams Γ' and Γ'', there can exist more than one diagram Γ such that $c^{\Gamma}_{\Gamma'\Gamma''} > 0$. This is not a requirement in the above definition.

Consider the Schur category $\mathbf{\Gamma}^d_F$ defined in Section 4.1. Take $V_n = F^n$. Define $M_d(m,n)$ to be the set of all bipartite multigraphs with vertex set $[n'] \coprod [m]$ with d edges. Mimicking the discussion in Section 2.2, one may endow the Schur category with the structure of a diagram category in the sense of Definition 5.1.

Acknowledgements

GT was supported by the Humboldt Foundation, Institute of Algebra and Number Theory, University of Stuttgart, and by a SERB MATRICS grant (MTR/2017/000424) of the Department of Science & Technology, India. AP was supported by a swarnajayanti fellowship (DST/SJF/MSA-02/2014-15) of the Department of Science & Technology, India. SS was supported by a national postdoctoral fellowship (PDF/2017/000861) of the Department of Science & Technology, India. The authors thank Steffen König and Upendra Kulkarni for many helpful suggestions.

References

[1] C. Aquilino and R. Reischuk. The monoidal structure on strict polynomial functors. *J. Algebra*, 485:213–229, 2017. DOI: 10.1016/j.jalgebra.2017.05.009.

[2] G. Benkart, T. Halverson, and N. Harman. Dimensions of irreducible modules for partition algebras and tensor power multiplicities for symmetric and alternating groups. *J. Algebraic Combin.*, 46(1):77–108, 2017. DOI: 10.1007/s10801-017-0748-4.

[3] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Algebraic vector bundles on $\mathbf{P}^n$ and problems of linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):66–67, 1978. Available from `http://www.math.tau.ac.il/~bernstei/Publication_list/publication_texts/BGG-3.pdf`.

[4] A. A. Beilinson, V. A. Ginsburg, and V. V. Schechtman. Koszul duality. *J. Geom. Phys.*, 5(3):317–350, 1988. DOI: 10.1016/0393-0440(88)90028-9.

[5] M. Bloss. The partition algebra as a centralizer algebra of the alternating group. *Comm. Algebra*, 33(7):2219–2229, 2005. DOI: 10.1081/AGB-200063579.

[6] N. Bourbaki. *Algebra I. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[7] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)*, 38(4):857–872, 1937. DOI: 10.2307/1968843

[8] M. Chałupnik. Koszul duality and extensions of exponential functors. *Adv. Math.*, 218(3):969–982, 2008. DOI: 10.1016/j.aim.2008.02.008

[9] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993. DOI: 10.1007/BF02571640.

[10] S. Donkin. *The q-Schur algebra.* No. 253 in London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998.

[11] E. M. Friedlander and A. Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997. DOI: 10.1007/s002220050119.

[12] F. G. Frobenius. Über die Charaktere der symmetrischen Gruppe. *S'ber Akad. Wiss. Berlin*, pages 516–534, 1900. DOI: 10.3931/e-rara-18862.

[13] T. Geetha and A. Prasad. Graphic interpretation of the structure constants of the Schur algebra. In *Electronic booklet of Proceedings of the International Congress of Women Mathematicians*, 2014. Abstract no. 20140020; available at `http://arxiv.org/abs/1409.1414`.

[14] J. A. Green. *Polynomial representations of* GL_n, volume 830 of *Lecture Notes in Mathematics.* Springer, Berlin, augmented edition, 2007.

[15] D. Happel. On the derived category of a finite-dimensional algebra. *Comment. Math. Helv.*, 62(3):339–389, 1987. DOI: 10.1007/BF02564452.

[16] V. F. R. Jones. The Potts model and the symmetric group. In *Subfactors (Kyuzeso, 1993)*, pages 259–267. World Sci. Publ., River Edge, NJ, 1994.

[17] H. Krause. Koszul, Ringel and Serre duality for strict polynomial functors. *Compos. Math.*, 149(6):996–1018, 2013. DOI: 10.1112/S0010437X12000814.

[18] U. Kulkarni, S. Srivastava, and K. V. Subrahmanyam. Relating tensor structures on representations of general linear and symmetric groups. *Transformation Groups*, 23(2):437–461, June 2018. DOI: 10.1007/s00031-018-9481-x.

[19] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[20] P. Martin. *Potts models and related problems in statistical mechanics*, volume 5 of *Series on Advances in Statistical Mechanics.* World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[21] P. Martin. On diagram categories, representation theory and. In *Noncommutative Rings, Group Rings, Diagram Algebras, and Their Applications, December 18-22, 2006, University of Madras, Chennai, India*, volume 456, page 99. American Mathematical Soc., 2008.

[22] M. A. Méndez. Directed graphs and the combinatorics of the polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. *Ann. Comb.*, 5(3-4):459–478, 2001. Dedicated to the memory of Gian-Carlo Rota (Tianjin, 1999). DOI: 10.1007/s00026-001-8022-8.

[23] A. Prasad. *Representation Theory: A Combinatorial Viewpoint.* Number 147 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, Delhi, 2015.

[24] S. B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970. DOI: 10.1090/S0002-9947-1970-0265437-8.

[25] A. Regev. Double centralizing theorems for the alternating groups. *J. Algebra*, 250(1):335–352, 2002. DOI: 10.1006/jabr.2001.9091.

[26] C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991. DOI: 10.1007/BF02571521

[27] I. Schur. *Klasse von Matrizen die sich einer gegeben Matrix zuordnen lassen.* PhD thesis, Friedrich-Wilhelms-Universität zu Berlin, 1901. Available from `https://archive.org/details/bereineklassevo00schugoog`.

[28] I. Schur. Über die rationalen Darstellungen der allgemeinen linearen Gruppe. *S'ber Akad. Wiss. Berlin*, pages 100–124, 1927.

[29] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences. `http://oeis.org`.

[30] A. Touzé. Bar complexes and extensions of classical exponential functors. *Ann. Inst. Fourier (Grenoble)*, 64(6):2563–2637, 2014. DOI: 10.5802/aif.2921.

[31] W. van der Kallen. *Lectures on Bifunctors and Finite Generation of Rational Cohomology Algebras*, pages 41–65. Springer International Publishing, Cham, 2015.

[32] H. Weyl. *The Classical Groups: Their Invariants and Representations.* Princeton University Press, 1939.

[33] H. Weyl. *The Theory of Groups and Quantum Mechanics.* Dover Publications, 1950.

Indian Institute of Science Education and Research, Thiruvananthapuram
E-mail address: geetha_curie@yahoo.co.in

The Institute of Mathematical Sciences, HBNI, Chennai
E-mail address: amri@imsc.res.in

The Institute of Mathematical Sciences, HBNI, Chennai
E-mail address: maths.shraddha@gmail.com