NONCOMMUTATIVE CYCLIC ISOLATED SINGULARITIES

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ABSTRACT. The question of whether a noncommutative graded quotient singularity A^G is isolated depends on a subtle invariant of the G-action on A, called the pertinency. We prove a partial dichotomy theorem for isolatedness, which applies to a family of noncommutative quotient singularities arising from a graded cyclic action on the (-1)-skew polynomial ring. Our results generalize and extend some results of Bao, He and the third-named author and results of Gaddis, Kirkman, Moore and Won.

0. INTRODUCTION

Auslander [Au] proved that if G is a small finite subgroup of $\mathrm{GL}_n(\mathbb{C})$, acting linearly on the symmetric algebra over $\mathbb{C}$ (namely, the commutative polynomial ring) $R := \mathbb{C}[\mathbb{C}^{\oplus n}]$, with fixed subring R^G, then the natural map

$$R\#G \to \mathrm{End}_{R^G}(R)$$

is an isomorphism of graded algebras. Here $R\#G$ denotes the skew group algebra associated to the G-action on R and the hypothesis of G being *small* means that G does not contain any pseudo-reflections (e.g. G is a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$). This theorem plays an important role in the McKay correspondence, relating representations of G and those of R^G; and in the special case of dimension two, further relating configuration of the exceptional fibers in the minimal resolution of $\mathrm{Spec}\, R^G$. The noncommutative version of this theorem of Auslander is an important ingredient in establishing a noncommutative McKay correspondence, see [CKWZ1, CKWZ2] for some recent developments. In [BHZ1, BHZ2], a numerical invariant was introduced for a semisimple Hopf algebra action on a (not necessarily commutative) algebra R with finite Gelfand-Kirillov dimension (or GKdimension for short). The *pertinency* of a Hopf algebra H-action on R [BHZ1, Definition 0.1] is defined to be

$$\mathsf{p}(R,H) := \mathrm{GKdim}(R) - \mathrm{GKdim}(R\#H/(e_0))$$

where (e_0) is the two-sided ideal of the smash product $R\#H$ generated by the element $e_0 := 1\#\int$, where $\int$ denotes an integral of H. One of the main results in [BHZ1, BHZ2] is the following.

Theorem 0.1. [BHZ1, Theorem 0.3] *Let R be a noetherian, connected graded, Artin-Schelter regular, Cohen-Macaulay algebra of GKdimension at least 2. Let H be a semisimple Hopf algebra acting on R inner-faithfully and homogeneously. Then the following are equivalent:*

(1) $\mathsf{p}(R,H) \geq 2$.
(2) *The natural map $R\#H \to \mathrm{End}_{R^H}(R)$ is an isomorphism of graded algebras.*

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The above theorem is useful for studying quotient singularities R^H and for connecting the representation theory of H and that of R^H. Several groups of researchers have computed the pertinency $\mathsf{p}(R, H)$ in different situations. A lower bound of the pertinency for the cyclic permutation action on the (-1)-skew polynomial rings and for the group actions on the universal enveloping algebra of some Lie algebras was given in [BHZ1, BHZ2]; in [GKMW], the authors computed the pertinency for many new examples; the authors in [HZ] introduced a new method of computing pertinency by using pertinent sequences; the paper [CKZ] provided a lower bound of the pertinency for group coactions on noetherian graded down-up algebras.

Although many of these ideas can be applied to the Hopf algebra setting, in this paper we only consider group actions, namely, H is a group algebra over a finite group G. When G is acting on an algebra R, we usually assume that this action is inner-faithful.

In algebraic geometry, singularities have been studied extensively. We recall the following basic result. When a small finite subgroup $G \subseteq \mathrm{GL}_n(\mathbb{C})$ acts naturally on the vector space $V := \mathbb{C}^{\oplus n}$, the quotient $V/G := \mathrm{Spec}(\mathbb{C}[V]^G)$ has isolated singularities if and only if G acts freely on $V \setminus \{0\}$, see [MSt, Lemma 2.1], [Fu, Corollary to Lemma 2] and [MU1, p.7359].

In noncommutative algebraic geometry, Ueyama gave the following definition of a graded isolated singularity [Ue, Definition 2.2]. Let B be a noetherian connected graded algebra. Then B is a *graded isolated singularity* if the associated noncommutative projective scheme tails B (in the sense of [AZ]) has finite global dimension. Let R be a noetherian Artin-Schelter regular algebra and G a finite subgroup of the graded algebra automorphism group $\mathrm{Aut}_{gr}(R)$. Mori-Ueyama [MU1, Theorem 3.10] proved that if $\mathsf{p}(R, G) \geq 2$, then R^G is a graded isolated singularity if and only if $\mathsf{p}(R, G) = \mathrm{GKdim}\, R$ (which is the largest possible). This result was extended to the Hopf algebra setting, namely, replacing G by a semisimple Hopf algebra, in [BHZ1]. The first few examples of graded isolated singularities in the noncommutative setting were given in [Ue, Theorem 1.4, Examples 3.1, 4.7 and 5.5] by mimicking the commutative criterion of free action of G on $V \setminus \{0\}$. More examples of graded isolated singularities were given in [CKWZ1, CKWZ2, BHZ2, GKMW]. One example of graded isolated singularities in dimension three was given in [CKZ, Lemma 2.11(1)]. A more interesting example is [Ue, Examples 5.4] or [KKZ1, Example 3.1], where the G-action on the degree one piece of the regular algebra R is not free. We call such a graded isolated singularity *non-conventional* [Definition 10.1].

Since Mori-Ueyama's condition of maximal pertinency is not easy to check in general, we only obtain some special examples of graded isolated singularities in high GKdimension [BHZ2]. It would be nice to understand exactly when the pertinency is maximal, but it seems extremely difficult to achieve this goal. The main object of this paper is to calculate a family of pertinencies all together, using induction. As a consequence, we obtain new examples of graded isolated singularities in arbitrarily large GKdimension.

We now fix some notation. Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let n be an integer ≥ 2. The algebra that we are interested in is the (-1)-skew polynomial ring

$$\Bbbk_{-1}[\mathbf{x}] := \Bbbk_{-1}[x_0, \ldots, x_{n-1}]$$

that is generated by $\{x_0, \ldots, x_{n-1}\}$ and subject to the relations

$$x_i x_j = (-1) x_j x_i$$

for all $i \neq j$. Let C_n be the cyclic group of order n acting on $\Bbbk_{-1}[\mathbf{x}]$ by permuting the generators of the algebra cyclically, namely, C_n is generated by $\sigma = (012 \cdots n-1)$ of order n that acts on the generators by

$$\sigma * x_i = x_{i+1}$$

for all $i \in \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. We have two results which establish a partial dichotomy.

Theorem 0.2. *Let $A := \Bbbk_{-1}[\mathbf{x}]$ and $G := C_n$. If $n = 2^a p^b$ for some prime $p \geq 7$ and integers $a, b \geq 0$, then $\mathsf{p}(A, G) = \operatorname{GKdim}(A) = n$. As a consequence, A^G is a graded isolated singularity.*

Remark 0.3.
(1) Theorem 0.2 is a generalization of [Ue, Examples 5.4] (when $n = 2$) and [BHZ1, Theorem 5.7(4)] (when $n = 2^a$ for some $a \geq 1$).
(2) Although [Ue, Examples 5.4] and [BHZ1, Theorem 5.7(4)] have already provided examples of non-conventional graded isolated singularities of a similar type, Theorem 0.2 is still quite surprising and counter-intuitive.
Note that $\sigma \mid_V$ (where $V = \oplus_{i=0}^{n-1} \Bbbk x_i$) has eigenvalues $\{1, \xi, \xi^2, \ldots, \xi^{n-1}\}$ where ξ is a primitive nth root of unity. In particular, there is an eigenvalue of σ on V that is 1 (which is not a primitive nth root of unity) with eigenvector $\sum_{i=0}^{n-1} x_i$ in V, or equivalently, the isolated singularity is non-conventional.
(3) In fact, almost all graded isolated singularities considered in this paper will be non-conventional. One aim of this paper is to show that non-conventional graded isolated singularities are common in the noncommutative setting.
(4) The proof of Theorem 0.2 is very involved, using several steps of reduction and induction. We hope to have a more conceptual proof in the future.

When $p = 3$ or 5, Theorem 0.2 fails.

Theorem 0.4. *Let $A := \Bbbk_{-1}[\mathbf{x}]$ and $G := C_n$. If either 3 or 5 divides n, then $\mathsf{p}(A, G) < \operatorname{GKdim}(A) = n$. Consequently, A^G is not a graded isolated singularity.*

Combining the above two theorems, if $n = 2^a p^b$ for some prime number p, then A^{C_n} is a graded isolated singularity if and only if $p \neq 3, 5$. It is not obvious to us why the primes 3 and 5 are different from other primes in this situation. Based on the above two results we make a conjecture.

Conjecture 0.5. *Let $A := \Bbbk_{-1}[\mathbf{x}]$ and $G := C_n$. Then A^G is a graded isolated singularity if and only if n is not divisible by 3 and 5.*

The above conjecture holds for n less than 77 following Theorems 0.2 and 0.4.

Corollary 0.6. *If $n < 77$, then Conjecture 0.5 holds.*

Theorem 8.7 provides further evidence for Conjecture 0.5. For general n we have the following lower bound. Let

(E0.6.1) $$\phi_2(n) = \{k \mid 0 \leq k \leq n-1 \text{ with } \gcd(k, n) = 2^w \text{ for some } w \geq 0\}.$$

Theorem 0.7. *Let $A := \Bbbk_{-1}[\mathbf{x}]$ and $G := C_n$. Then $\mathsf{p}(A, G) \geq |\phi_2(n)|$.*

Note that Theorem 0.7 is an improvement of [BHZ1, Theorem 5.7] when n is even. Combining Theorems 0.2, 0.4, 0.7 and further analysis, we have the following table of pertinencies.

Proposition 0.8. *Let $p = \mathsf{p}(A, C_n)$. Then*

n	2	3	4	5	6	7	8	9	10	11	12	13	14
p	2	2	4	4	5	7	8	8	9	11	$\in [8, 11]$	13	14

where the notation $\in [8, 11]$ means that $8 \leq p \leq 11$.

By Proposition 0.8, the integer 12 is the smallest n where that the exact value of $\mathsf{p}(A, C_n)$ is unknown. It would be nice to have exact values of $\mathsf{p}(A, C_n)$ for all n. In particular, we ask:

Question 0.9. Retain the above notation.

(1) If n is divisible by either 3 or 5, what is the exact value of $\mathsf{p}(A, C_n)$?
(2) Does the sequence, from Proposition 0.8,

$$2, 2, 4, 4, 5, 7, 8, 8, 9, 11... \tag{E0.9.1}$$

match up with any other sequences in literature? The On-Line Encyclopedia of Integer Sequences website
https://oeis.org/
does not give any sequences that match up with (E0.9.1)

Graded isolated singularities have various special properties. Ueyama and Mori-Ueyama investigated certain properties of graded isolated singularities from the viewpoint of derived categories and representation theory. As an immediate consequence of [Ue, MU1, MU2], we have the following. We refer to [Ue, MU1, MU2] for undefined terms in the next corollary.

Corollary 0.10. *Suppose $n = 2^a p^b$ for some prime $p \geq 7$ and integers $a, b \geq 0$. Then the following hold.*

(1) *tails $A^G \cong$ tails $A\#G$.*
(2) *A is a $(n-1)$-cluster tilting object in the category of graded maximal Cohen-Macaulay modules over A^G.*
(3) *The derived category $D^b(\text{tails } A^G)$ has a tilting object.*
(4) *The derived category $D^b(\text{tails } A^G)$ has a Serre functor.*

This paper is organized as follows. We provide background material in Section 1. Theorem 0.7 is proven in Section 2. In Section 3, we give some preliminary results and Theorem 0.4 is proven in Section 4. We continue some preparation in Sections 5 and 6. The main result, Theorem 0.2, is proven in Section 7. In Section 8, we discuss some partial results when $n = p_1 p_2$. Proposition 0.8 is proven in Section 9. In Section 10, we construct more examples of non-conventional graded isolated singularities. The final section contains some questions and comments.

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1. Preliminaries

Throughout let $\Bbbk$ be a base field that is algebraically closed of characteristic zero. All objects are $\Bbbk$-linear.

An algebra R is called *connected graded* if $R = \bigoplus_{n\geq 0} R_n$ satisfying $R_i R_j \subseteq R_{i+j}$ for all i, j and $1 \in R_0 = \Bbbk$. We say R is *locally finite* if $\dim_{\Bbbk} R_n < \infty$ for all n. In this paper all connected graded algebras will be locally finite.

We refer to [KL, MR] for the definition of the *Gelfand-Kirillov dimension* (or *GKdimension*) of an algebra or a module. When R is connected graded and finitely generated, its GKdimension is equal to

$$\text{(E1.0.1)} \qquad \operatorname{GKdim}(R) = \limsup_{n\to\infty} \log_n\Big(\sum_{i=0}^{n} \dim_{\Bbbk} R_i\Big).$$

Observe that $\operatorname{GKdim}(R) = 0$ if and only if $\dim_{\Bbbk} R < \infty$. For $q \in \Bbbk^{\times}$, the q-polynomial ring

$$\Bbbk_q[x_1, \ldots, x_m] := \Bbbk\langle x_1, \ldots, x_m\rangle/(x_i x_j - q x_j x_i \mid i < j)$$

has GKdimension m (equal to the number of generators). If B is either a subalgebra or a homomorphic image of an algebra R, then $\operatorname{GKdim}(B) \leq \operatorname{GKdim}(R)$.

Let B be a noetherian connected graded algebra. If M is a finitely generated graded right B-module, then we have a formula similar to (E1.0.1), see [SZ, p.1594],

$$\text{(E1.0.2)} \qquad \operatorname{GKdim}(M) = \limsup_{n\to\infty} \log_n\Big(\sum_{i\leq n} \dim_{\Bbbk} M_i\Big).$$

Let c be a homogenous central element of B of positive degree. If M is a finitely generated left graded B-module, it follows from (E1.0.2) that

$$\text{(E1.0.3)} \qquad \operatorname{GKdim} M \geq \operatorname{GKdim} M/cM \geq \operatorname{GKdim} M - 1.$$

Definitions of other standard concepts such as Artin-Schelter regularity, Auslander regularity, Cohen-Macaulay property are omitted as these can be found in many papers such as [Le, CKWZ1, MSm].

For the first nine sections we consider noncommutative cyclic singularities arising from the action of the cyclic group on the (-1)-skew polynomial ring as follows.

Let n be a fixed integer ≥ 2. Let $\underline{n} := \{0, \ldots, n-1\}$. Note that $\underline{n}$ can be identified with the additive group $\mathbb{Z}_n$. Let $\mathbf{x}$ be the set $\{x_0, \ldots, x_{n-1}\}$ or $\{x_i \mid i \in \mathbb{Z}_n\}$ and A be the (-1)-skew polynomial ring $\Bbbk_{-1}[\mathbf{x}]$ as defined in the introduction. Then A is a graded $\Bbbk$-algebra with $\deg(x_i) = 1$ for each i and we denote A_j the $\Bbbk$-subspace of degree j elements of A. It is well-known that A is noetherian, Artin-Schelter regular, Auslander regular and Cohen-Macaulay of global dimension and GKdimension n. Let σ be the cycle $(012\cdots n-1)$ which generates the cyclic group C_n of order n as a subgroup of the symmetric group S_n (considering S_n as a set of bijections of $\underline{n} := \{0, \ldots, n-1\}$). As abstract groups, we have $\mathbb{Z}_n \cong C_n$. The action of C_n on A is determined by its action on generators

$$\sigma * x_i = x_{i+1}, \quad \forall\, i \in \underline{n} = \mathbb{Z}_n.$$

The skew group algebra $A\#C_n$ with respect to this action consists of all linear combinations of elements $a\#g$ with $a \in A$ and $g \in C_n$, with multiplication given by

$$(a\#g)(a'\#g') = ag(a')\#gg',$$

extended linearly to all of $A\#C_n$. We omit $\#$ if no confusion occurs.

The skew group algebra can be presented in the standard way,

$$A\#C_n \cong \frac{\Bbbk\langle \mathbf{x}, \sigma\rangle}{(x_ix_j + x_jx_i, \sigma^n, \sigma x_i - x_{i+1}\sigma)}.$$

We now describe a different presentation of the above skew group algebra, using eigenvectors of the σ-action, which we will use for the rest of the paper.

Since the action of C_n on A is graded, the generating subspace A_1 is a C_n-module. Let ω be a primitive nth root of unity and M_{ω^j} be the simple (hence 1-dimensional) C_n-module where σ acts by multiplication by ω^j. The σ-action on A_1 has minimal polynomial $p(X) = X^n - 1$, so we can decompose A_1 as a C_n-module as follows

$$A_1 \cong \bigoplus_{\gamma=0}^{n-1} M_{\omega^\gamma} \tag{E1.0.4}$$

For $\gamma = 0, \dots, n-1$, define the following elements of $A_1 \subseteq A\#C_n$

$$b_\gamma \;:=\; \frac{1}{n}\sum_{i=0}^{n-1} \omega^{i\gamma} x_i. \tag{E1.0.5}$$

The following calculation shows that b_γ is a $\omega^{-\gamma}$-eigenvector of σ,

$$\sigma * b_\gamma = \frac{1}{n}\sum_{i=0}^{n-1} \omega^{i\gamma} x_{i+1} = \omega^{-\gamma} b_\gamma. \tag{E1.0.6}$$

In other words, we have $\Bbbk b_\gamma \cong M_{-\gamma}$ as C_n-modules, so the basis $\{b_0, \dots, b_{n-1}\}$ gives the C_n-module decomposition of A_1 in (E1.0.4). We also define the following idempotent elements

$$e_\alpha \;:=\; \frac{1}{n}\sum_{i=0}^{n-1} (\omega^\alpha \sigma)^i$$

in $\Bbbk C_n \subseteq A\#C_n$. Let $\mathbf{b} := (b_0, \dots, b_{n-1})$ and $\mathbf{e} := (e_0, \dots, e_{n-1})$. Define the *graded commutator*, denoted by $[\cdot,\cdot]$, for any homogeneous elements $u, v \in A\#C_n$ (or u, v in another graded algebra) by

$$[u, v] = uv - (-1)^{\deg(u)\deg(v)} vu.$$

We have the following Lemma.

Lemma 1.1. *Suppose* $\deg(b_i) = 1$ *and* $\deg(e_i) = 0$ *for all* $i \in \mathbb{Z}_n$. *The graded algebra* $A\#C_n$ *can be presented as follows*

$$A\#C_n \cong \frac{\Bbbk\langle \mathbf{b}, \mathbf{e}\rangle}{(e_\alpha b_\gamma - b_\gamma e_{\alpha-\gamma}, e_ie_j - \delta_{ij}e_i, [b_0, b_k] - [b_l, b_{k-l}])}$$

where δ_{ij} *is the Kronecker delta and indices are taken modulo* n.

Proof. Let b_i be defined as in (E1.0.5) and let

$$r_{kl} = [b_0, b_k] - [b_l, b_{k-l}].$$

We first show that the map

$$\iota : \Bbbk\langle \mathbf{b}\rangle/(r_{kl}) \to \Bbbk_{-1}[\mathbf{x}]$$

is well-defined and is an isomorphism. By (E1.0.6) the elements $b_0, \dots, b_{n-1}$ are eigenvectors for the σ-action on A_1 with distinct eigenvalues, hence this is a basis for A_1, so ι is an isomorphism in degree 1. To see that ι is well-defined as an algebra map, note that the graded commutator of b_γ and b_δ depends only on the sum of γ and δ,

$$[b_\gamma, b_\delta] = \frac{1}{n^2}\sum_{i,j=0}^{n-1}\omega^{i\gamma+j\delta}[x_i, x_j] = \frac{2}{n^2}\sum_{i=0}^{n-1}\omega^{i(\gamma+\delta)}x_i^2 \tag{E1.1.1}$$

So the relations r_{kl} go to zero in $\Bbbk_{-1}[\mathbf{x}]$. To show that ι is an algebra isomorphism, we count the number of independent quadratic relations in $\mathbf{b}$ and show that this number is equal to $\binom{n}{2}$.

For any fixed k, the only linear relations among $R_k := \{r_{k0}, r_{k1}, \dots, r_{k,n-1}\}$ are $r_{k0} = r_{kk} = 0$ and $r_{kl} = r_{k,k-l}$. Define a C_2-action on R_k by $r_{kl} \mapsto r_{k,k-l}$. Then the number of independent relations in R_k is equal to $|R_k/C_2| - 1$.

Case 1: For odd n, the C_2-action has exactly one fixed point r_{kl} where $2l = k \mod n$. Therefore $|R_k/C_2| = (n+1)/2$. The relations in R_k are independent from the relations in $R_{k'}$ for distinct k, k'. Since k ranges from 0 to $n-1$, the total number of independent relations is equal to

$$n(|R_k/C_2| - 1) = \binom{n}{2}.$$

Case 2: Let n be even. For odd k, the C_2-action has no fixed points. Therefore $|R_k/C_2| = n/2$. If k is even, then the C_2-action has two fixed points, coming from the two solutions of $2l = k \mod n$. Therefore $|R_k/C_2| = n/2+1$. By considering the odd and even cases separately, we get that the total number of independent relations is equal to

$$\sum_{k \text{ odd}}(|R_k/C_2| - 1) + \sum_{k \text{ even}}(|R_k/C_2| - 1) = \frac{n}{2}\left(\frac{n}{2} - 1\right) + \frac{n}{2}\left(\frac{n}{2}\right) = \binom{n}{2}.$$

Therefore ι is an algebra isomorphism.

The isomorphism

$$\Bbbk C_n \cong \Bbbk\langle \mathbf{e}\rangle/(e_ie_j - \delta_{ij}e_i)$$

is well-known. The relations between $\mathbf{b}$ and $\mathbf{e}$ are obtained as follows

$$e_\alpha b_\gamma = \frac{1}{n}\sum_{i=0}^{n-1}(\omega^\alpha\sigma)^i b_\gamma = \frac{b_\gamma}{n}\sum_{i=0}^{n-1}\omega^{(\alpha-\gamma)i}\sigma^i = b_\gamma e_{\alpha-\gamma}.$$

By using the facts

$$\sigma^i = \sum_\alpha \omega^{-\alpha i}e_\alpha$$

and

$$x_j = \sum_\gamma \omega^{-\gamma j}b_\gamma$$

for $i, j \in \mathbb{Z}_n$, it is easy to check that the set of relations

$$\{e_\alpha b_\gamma = b_\gamma e_{\alpha-\gamma} \mid \alpha, \gamma \in \mathbb{Z}_n\}$$

is equivalent to the set of relations

$$\{\sigma^i x_j = x_{j+i}\sigma^i \mid i, j \in \mathbb{Z}_n\}.$$

This completes the proof. $\square$

We define the elements

$$c_j := [b_k, b_{j-k}] \tag{E1.1.2}$$

for all $j \in \mathbb{Z}_n$. Equation (E1.1.1) shows that the definition of c_j does not depend on k, and while they are central elements of A, they are not central in $A\#C_n$. By the relations in Lemma 1.1, we have

$$e_\alpha c_j = c_j e_{\alpha - j}$$

for all $\alpha, j \in \mathbb{Z}_n$. As above, we denote by $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$. Recall that, for a vector $\mathbf{i} = (i_0, i_1, \ldots, i_{n-1}) \in \mathbb{N}^n$,

$$|\mathbf{i}|_1 = |i_0| + |i_1| + \cdots + |i_{n-1}|.$$

We will use the following notation

$$\mathbf{b}^{\mathbf{i}} := b_0^{i_1} b_1^{i_1} \cdots b_{n-1}^{i_{n-1}},$$
$$\mathbf{c}^{\mathbf{i}} := c_0^{i_1} c_1^{i_1} \cdots c_{n-1}^{i_{n-1}}.$$

Proposition 1.2. *For each $r \geq 0$, the set*

$$\mathcal{B}_r = \{\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}} \mid \mathbf{i} \in \{0,1\}^n, \mathbf{j} \in \mathbb{N}^n, |\mathbf{i}|_1 + 2|\mathbf{j}|_1 = r\}$$

is a $\Bbbk$-linear basis for A_r.

Proof. The generating function for $\mathcal{B}_r$, namely, $g(t) = \sum_{r \geq 0} |\mathcal{B}_r| t^r$ is

$$(1+t)^n \frac{1}{(1-t^2)^n} = \frac{1}{(1-t)^n},$$

which agrees with the Hilbert series of A. It remains to show that $\mathcal{B}_r$ spans A_r for each r.

Since $\mathbf{b}$ generates A, the set $\{b_{t_1} \cdots b_{t_r} \mid \text{ for different } t_s\}$ spans A_r. Using the relation $c_j = [b_k, b_{j-k}]$ and the fact that c_j are central, we can ensure that $b_{t_1} \cdots b_{t_r}$ is in the linear span of $\mathcal{B}_r$, as required. □

We can extend the above basis for A_r to a basis for $(A\#C_n)_r$ by adjoining the n idempotent elements coming from $\Bbbk C_n$. Therefore

$$\mathcal{B}_r \times \mathbf{e} = \{z e_j \mid z \in \mathcal{B}_r,\ j = 0, \ldots n-1\}$$

and

$$\mathbf{e} \times \mathcal{B}_r = \{e_j z \mid z \in \mathcal{B}_r,\ j = 0, \ldots n-1\}$$

are both $\Bbbk$-linear bases for $(A\#C_n)_r$. The following is an immediate consequence of Proposition 1.2.

Corollary 1.3. *Retain the above notation.*

1. *The union $\mathcal{B} = \bigcup_{r \in \mathbb{N}} \mathcal{B}_r$ is a $\Bbbk$-linear basis for A.*
2. *Both $\mathbf{e} \times \mathcal{B}$ and $\mathcal{B} \times \mathbf{e}$ are $\Bbbk$-linear bases for $A\#C_n$.*
3. *$A\#C_n$ is a finitely generated left and right module over the commutative subring $\Bbbk[\mathbf{c}] \subseteq A$.*

Let $(e_0) \subset A\#C_n$ denote the two sided ideal generated by the idempotent e_0. We will be concerned with computing the GKdimension of the quotient algebra

$$E := (A\#C_n)/(e_0).$$

Since e_0 is the integral of the group algebra $\Bbbk C_n$, we obtain that

$$\mathsf{p}(A, C_n) = \operatorname{GKdim} A - \operatorname{GKdim} E.$$

Let

$$\Phi_n := \{k \mid c_k^{N_k} \in (e_0) \text{ for some } N_k \geq 0\}. \tag{E1.3.1}$$

The following lemma is easy.

Lemma 1.4. *Retain the above notation.*

(1) *Let $\overline{C}$ be the quotient ring $\Bbbk[\mathbf{c}]/(c_k^{N_k}; k \in \Phi_n)$. Then $\operatorname{GKdim} \overline{C} \leq n - |\Phi_n|$.*
(2) *The algebra E is a finitely generated right module over $\overline{C}$. As a consequence,*

$$\operatorname{GKdim} E \leq \operatorname{GKdim} \overline{C} \leq n - |\Phi_n|.$$

(3) *$k \in \Phi_n$ if and only if, for each α, $e_\alpha c_k^N \in (e_0)$ for $N \gg 0$.*

Proof. (1) This is true because $\{c_k^{N_k} \mid k \in \Phi_n\}$ is a regular sequence of $\Bbbk[\mathbf{c}]$.

(2) The first assertion follows from Proposition 1.2 (or Corollary 1.3(2)). The consequence follows from [MR, Proposition 8.3.2].

(3) If $c_k^N \in (e_0)$, then clearly $e_\alpha c_k^N \in (e_0)$ for all α. The converse follows from the fact $1 = \sum_\alpha e_\alpha$. □

In the next few sections we provide upper and lower estimates for $\operatorname{GKdim} E$.

2. An upper bound on GKdim E

This section is a warm-up for more complicated computations to be done in later sections. Fix $n \in \mathbb{N}$, define the following functions on $\mathbb{Z}_n$. Let k be in $\mathbb{Z}_n$. For every $\alpha \in \mathbb{Z}_n$,

$$\begin{aligned} f_k(\alpha) &:= \alpha - k, \\ g_k(\alpha) &:= 2\alpha - k. \end{aligned}$$

Let S_k be the multiplicative semigroup of $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_n)$ generated by f_k and g_k.

Lemma 2.1. *For each $s \in S_k$ we have $e_\alpha c_k^N \in (e_0) + (e_{s(\alpha)})$ for $N \gg 0$.*

Proof. We have two simple calculations

$$\begin{aligned} e_\alpha c_k^N &= c_k e_{\alpha-k} c_k^{N-1} = c_k e_{f_k(\alpha)} c_k^{N-1}, \qquad \text{and} \\ e_\alpha c_k^N &= e_\alpha (b_\alpha b_{k-\alpha} + b_{k-\alpha} b_\alpha) c_k^{N-1} \\ &= b_\alpha e_0 c_k^{N-1} b_{k-\alpha} + b_{k-\alpha} e_{2\alpha-k} c_k^{N-1} b_\alpha \\ &= b_\alpha e_0 c_k^{N-1} b_{k-\alpha} + b_{k-\alpha} e_{g_k(\alpha)} c_k^{N-1} b_\alpha, \end{aligned}$$

which imply that $e_\alpha c_k^N \in (e_{f_k(\alpha)})$ and that $e_\alpha c_k^N \in (e_0) + (e_{g_k(\alpha)})$. Since s is generated by f_k and g_k, the claim follows. □

For fixed α, it is easy to see that

$$S_k(\alpha) = \{2^s \alpha + tk \mod n \mid s, t \geq 0\} \subseteq \mathbb{Z}_n.$$

Lemma 2.2. *Let $k \in \mathbb{Z}_n$ be fixed. If $0 \in S_k(\alpha)$ for every α, then $k \in \Phi_n$.*

Proof. For each α, pick $s \in S_k$ so that $s(\alpha) = 0$. By Lemma 2.1, $e_\alpha c_k^N \in (e_0) + (e_{s(\alpha)}) = (e_0)$. The assertion follows by Lemma 1.4(3). □

Recall from (E0.6.1) that

$$\phi_2(n) = \{k \mid 0 \leq k \leq n-1, \gcd(k,n) = 2^w \text{ for some } w \geq 0\}.$$

Proposition 2.3. *Retain the above notation.*

(1) *If* $k = 2^w q < n$ *such that* q *is odd and* $(n, q) = 1$*, then* $k \in \Phi_n$*. Equivalently,* $\phi_2(n) \subseteq \Phi_n$*.*
(2) $|\Phi_n| \geq |\phi_2(n)|$.
(3) $\operatorname{GKdim}(E) \leq n - |\phi_2(n)|$*. As a consequence,*
  (3a) *If* $n = 2^j$*, then* $\operatorname{GKdim} E = 0$*.*
  (3b) *If* n *is an odd prime, then* $\operatorname{GKdim} E \leq 1$*.*

Proof. Let n be a positive integer such that $n = 2^m p$ where p is odd. Then let $|n|_2 = m$.

(1) By Lemma 2.2, we need to show that, for every α, there is an $s \in S_k$ such that $s(\alpha) = 0$. Write $\alpha = 2^r \beta$ where $r = |\alpha|_2$. Recall that $k = 2^w q < n$ such that $(p, q) = 1$ where $w = |k|_2$. We have two cases, depending on the relative magnitudes of r and w.

Case 1: If $r \geq w$, then there exists j such that $\alpha = jk$ in $\mathbb{Z}_n$ ($j = 2^{r-w} q^{-1} \beta$ where q^{-1} exists in $\mathbb{Z}_n$), so

$$f_k^j(\alpha) = \alpha - jk = 0 \qquad \text{in } \mathbb{Z}_n.$$

So we take $s = f_k^j$.

Case 2: If $r < w$, then

$$g_k^{w-r}(\alpha) = 2^{w-r}\alpha - (2^{w-r} - 1)k,$$

hence $|g_k^{w-r}(\alpha)|_2 \geq w$. This reduces to the first case.

Hence, in both cases, there is an $s \in S_k$ such that $s(\alpha) = 0$ as required.

(2) This is an immediate consequence of part (1).

(3) The main assertion follows from part (2) and Lemma 1.4(2). Two consequences are special cases of the main assertion. □

It is easy to see that Theorem 0.7 is equivalent to Proposition 2.3(3).

3. Preparation, part one

Recall that $E = (A\#C_n)/(e_0)$. In this section, we reduce the problem of computing $\operatorname{GKdim} E$ to that of a right quotient module of A. Let $\bar{e}_k$ denote the image of the idempotent e_k in E. This gives a right module decomposition

$$E = \bar{e}_1 E \oplus \cdots \oplus \bar{e}_{n-1} E,$$

and it follows that

$$\operatorname{GKdim}(E) = \max_{1 \leq j \leq n-1} \operatorname{GKdim}(\bar{e}_j E). \tag{E3.0.1}$$

For each j, we have the following isomorphism of right $A\#C_n$-modules

$$\bar{e}_j E \cong \frac{e_j(A\#C_n)}{e_j(A\#C_n) \cap (e_0)}. \tag{E3.0.2}$$

Using the basis $\mathbf{e} \times \mathcal{B}$ for $A\#C_n$ we obtain immediately the right A-module isomorphism $e_j(A\#C_n) \cong A$ by $e_j a \mapsto a$ with inverse given by $a \mapsto e_j a$. We will use this isomorphism to identify $e_j(A\#C_n)$ with A below.

In the following, it will be useful to decompose A according to the characters of the C_n-action, or equivalently, as modules over the invariant subring A^{C_n}. Let R_j be the $\Bbbk$-subspace of A spanned by the basis consisting of the elements $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}}$ where $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{v} = j \mod n$ and $\mathbf{v} := (0, 1, \ldots, n-1)$. Since $\mathcal{B}$ is an eigenbasis with respect to the σ-action, we have that R_0 is the invariant subring A^{C_n} and R_j

is the $M_{\omega^{-j}}$-isotypic component of the C_n-action on A. This gives an R_0-module decomposition

$$A \cong R_0 \oplus R_1 \oplus \cdots \oplus R_{n-1}.$$

We next find a finite generating set for R_j.

Lemma 3.1. *For each $j = 1, \ldots, n-1$, define B_j to be the set of elements $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}}$ satisfying*

(i) $(\mathbf{i}+\mathbf{j}) \cdot \mathbf{v} = j \mod n$, *and*

(ii) *for each nontrivial $\mathbf{b}^{\mathbf{i}'}\mathbf{c}^{\mathbf{j}'}$ with $\mathbf{i}' \leq \mathbf{i}$ and $\mathbf{j}' \leq \mathbf{j}$ we have $\mathbf{b}^{\mathbf{i}'}\mathbf{c}^{\mathbf{j}'} \notin R_0$.*

Then B_j generates R_j as a right R_0-submodule of A.

Proof. By definition, the elements $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}}$ satisfying (i) generate R_j. Now suppose $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}}$ satisfies (i) but not (ii), that is, there exist some nontrivial $\mathbf{i}' \leq \mathbf{i}$ and $\mathbf{j}' \leq \mathbf{j}$ such that $\mathbf{b}^{\mathbf{i}'}\mathbf{c}^{\mathbf{j}'} \in R_0$. If $\mathbf{i}' = 0$, then we can write it as $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}-\mathbf{j}'}\mathbf{c}^{\mathbf{j}'}$. If $\mathbf{i}' \neq 0$, then using the commutation relations (E1.1.2), we can move the $\mathbf{b}^{\mathbf{i}'}$ terms, one at a time, to the right side of the expression so that

$$\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}} = \mathbf{b}^{\mathbf{i}-\mathbf{i}'}\mathbf{c}^{\mathbf{j}-\mathbf{j}'}\mathbf{b}^{\mathbf{i}'}\mathbf{c}^{\mathbf{j}'} + \sum_{\mathbf{k},\mathbf{l}} \lambda_{\mathbf{k},\mathbf{l}}\mathbf{b}^{\mathbf{k}}\mathbf{c}^{\mathbf{l}}$$

where each $\mathbf{k}$ in the summation above satisfies $\mathbf{k} < \mathbf{i}$ and $\lambda_{\mathbf{k},\mathbf{l}} \in \Bbbk$. In particular, we have expressed $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}}$ as an R_0-linear combination of terms in $\mathcal{B}$ whose $\mathbf{b}$-exponent vector is strictly less than $\mathbf{i}$. By induction on the $\mathbf{b}$-exponent vector, we obtain the result. □

Lemma 3.2. *Retain the above notation. Suppose $1 \leq j \leq n-1$ and $0 \leq k \leq n-1$.*

(1) *The intersection $e_j(A\#C_n) \cap (e_0)$ considered as a right ideal in A is generated by B_j.*

(2) *For $N \geq 0$, $c_k^N \in B_jA$ if and only if $e_jc_k^N \in (e_0)$.*

(3) *$k \in \Phi_n$ if and only if, for each j, $e_jc_k^N \in (e_0)$ for some $N \gg 0$; and if and only if, for each j, $c_k^N \in B_jA$ for some $N \gg 0$.*

Proof. (1) Using the fact $A\#C_n = \sum_i Ae_i = \sum_i e_iA$, one sees that every element $f \in (e_0) := (A\#C_n)e_0(A\#C_n)$ can be written as a linear combination of terms ue_0v where $u, v \in \mathcal{B}$. Without loss of generality let $f = ue_0v$ where $u, v \in \mathcal{B}$. If, in addition, $f \in e_j(A\#C_n)$, then

$$f = e_jf = e_jue_0v = ue_{j-\gamma}e_0v = \begin{cases} ue_0v & j = \gamma \\ 0 & j \neq \gamma \end{cases}$$

where $u \in R_\gamma$. Hence we can assume that $j = \gamma$ and $u \in R_j = B_jR_0$ by Lemma 3.1. Since elements of R_0 commute with e_0, we can actually assume that $u \in B_j$. Finally, $ue_0v = e_juv$ since $u \in B_j$.

(2) This follows from part (1).

(3) This follows from part (2) and Lemma 1.4(3). □

Identify $e_j(A\#C_n)$ with A and combining Lemma 3.2 and (E3.0.2), we get

(E3.2.1) $$\bar{e}_jE \cong A/B_jA = A/R_jA.$$

We can say more: Lemma 3.4 below finds a sufficient condition for when these quotients are isomorphic.

Definition 3.3. Let $\lambda \in \mathbb{Z}$ be an integer with $\gcd(\lambda, n) = 1$. Let $f_\lambda : A \longrightarrow A$ be the algebra map determined by

$$f_\lambda(b_i) = b_{\lambda i}$$

for all $i \in \mathbb{Z}_n$. To see this is an algebra homomorphism, note that

$$f_\lambda([b_0, b_j] - [b_r, b_{j-r}]) = [b_0, b_{\lambda j}] - [b_{\lambda r}, b_{\lambda j - \lambda r}].$$

Since λ is invertible in $\mathbb{Z}_n$, f_λ is an algebra automorphism of A. It is easy to check that $f_\lambda(x_i) = x_{ai}$ where $a = \lambda^{-1}$ in $\mathbb{Z}_n$.

Lemma 3.4. *For any positive integer λ with $\gcd(\lambda, n) = 1$, we have the following isomorphism of $\Bbbk$-vector spaces*

$$\bar{e}_j E \cong \bar{e}_{\lambda j} E.$$

In particular, if n is prime, then for each $j = 2, \ldots, n-1$, we have $\bar{e}_1 E \cong \bar{e}_j E$.

Proof. Let $f_\lambda : A \longrightarrow A$ be the algebra isomorphism defined in Definition 3.3. Now

$$f_\lambda(R_j) = R_{\lambda j},$$

hence

$$\bar{e}_j E \cong A/R_j A \cong A/R_{\lambda j} A \cong \bar{e}_{\lambda j} E$$

as $\Bbbk$-vector spaces. □

Since GKdimension of a finitely generated A-module is only dependent on its Hilbert series (E1.0.2), we have the following immediate consequences.

Corollary 3.5. *Retain the above notation.*

(1) *For any $0 < j < n$, we have the following lower bound for* $\operatorname{GKdim}(E)$

$$\operatorname{GKdim}(E) \geq \operatorname{GKdim}(A/B_j A).$$

(2) *We have*

$$\operatorname{GKdim}(E) = \max_j \operatorname{GKdim}(A/B_j A)$$

where j ranges over positive integers less than n that divide n.

(3) *If n is prime, then*

$$\operatorname{GKdim}(E) = \operatorname{GKdim}(A/B_1 A).$$

For the rest of this section we will consider two distinct values of n, with one a factor of the other, and arguments will involve two particular natural algebra homomorphisms between the (-1)-skew polynomial rings of these different dimensions.

We fix two integers m and n such that m divides n. Let A (resp. $\tilde{A}$) denote the (-1)-skew polynomial ring of dimension n (respectively, m). Usually we use $\tilde{}$ to denote the corresponding notation for the algebra $\tilde{A}$. For example, since we use $\mathbf{b}$ for the generating set for A (see (E1.0.5)), then we use $\tilde{\mathbf{b}}$ to denote the corresponding generating set for $\tilde{A}$. Recall from the proof of Lemma 1.1, the algebra A is determined completely by the set of relations of the form

$$r_{kl} : [b_0, b_k] - [b_l, b_{k-l}] = 0 \tag{E3.5.1}$$

for all $k, l \in \mathbb{Z}_n$. Similarly for the algebra $\tilde{A}$.

Definition 3.6. Suppose m divides n. There is a surjective homomorphism

$$\pi_{n,m} : A \to \tilde{A}$$

determined by sending $b_j \mapsto \tilde{b}_j$, where in the $\tilde{\mathbf{b}}$ variables the indices are taken modulo m. Since m divides n, $\pi_{n,m}$ maps any relation of A of the form (E3.5.1) to a relation of $\tilde{A}$. Therefore $\pi_{n,m}$ is an algebra homomorphism. The surjectivity of $\pi_{n,m}$ follows from the fact that it is surjective in degree 1.

Lemma 3.7. *Suppose that m divides n. Then* $\operatorname{GKdim}(E) \geq \operatorname{GKdim}(\tilde{E})$.

Proof. Let $\tilde{R}_j \subset \tilde{A}$, for $j = 0, \ldots, m-1$, be defined as in the beginning of Section 3 for the algebra $\tilde{A}$ with m variables. Since $\pi_{n,m}(R_j) \subset \tilde{R}_j$, for $j = 0, \ldots, m-1$, we get a surjective homomorphism $A/R_jA \to \tilde{A}/\tilde{R}_j\tilde{A}$. Then

$$\operatorname{GKdim}(\tilde{E}) = \max_{j=1}^{m-1}\{\operatorname{GKdim} \tilde{A}/\tilde{R}_j\tilde{A}\} \leq \max_{j=1}^{m-1}\{\operatorname{GKdim} A/R_jA\} \leq \operatorname{GKdim}(E).$$

□

Lemma 3.7 will be used in the proof of Theorem 0.4.

For the proof the main result (Theorem 0.2), we need to consider another homomorphism. As before, let m and n be two integers such that m divides n. Write $q = n/m$.

Definition 3.8. Suppose m divides n and write $q = n/m$. Let

$$\theta_{m,n} : \tilde{A} \to A$$

be an algebra homomorphism determined by $\theta_{m,n}(\tilde{b}_i) = b_{qi}$ for all $i \in \mathbb{Z}_m$. Since $\theta_{m,n}$ maps the relation $\tilde{r}_{kl}$ of $\tilde{A}$ of the form (E3.5.1) to $r_{qk,ql}$ of A, $\theta_{m,n}$ is an algebra homomorphism.

We have the following easy lemma.

Lemma 3.9. *Suppose that m divides n and write $q = n/m$. Let N be a positive integer. Then*

(1) $\theta_{m,n}(\tilde{R}_j) \subseteq R_{qj}$ *for all* $j \in \mathbb{Z}_m$.
(2) $\theta_{m,n}(\tilde{c}_j) = c_{qj}$ *for all* $j \in \mathbb{Z}_m$.
(3) *If* $\tilde{c}_i^N \in \tilde{R}_j\tilde{A}$ *for some* $i, j \in \mathbb{Z}_m$ *and* $N \geq 0$, *then* $c_{qi}^N \in R_{qj}A$.

4. Proof of Theorem 0.4

We first show that for $n = 3, 5$ the GKdimension of A/B_1A is equal to 1. Hence the GKdimension of E is also equal to 1 by Corollary 3.5(3). It turns out that we can use these two cases to infer that the GKdimension of E is positive whenever 3 or 5 divides n.

Proposition 4.1. *Let $n = 3$.*

(1) $B_1A \cong (b_1A + c_1A)$ *as right A-modules.*
(2) $\operatorname{GKdim}(A/B_1A) = 1$.
(3) $\operatorname{GKdim}(E) = 1$.

Proof. (1) Recall that the definition of B_j is given in Lemma 3.1. By definition one can easily check that $B_1 = \{b_1, c_1, c_2^2, b_2c_2\}$. Note that $c_2 = 2b_1^2$ so $c_2^2 = 4b_1^4$ and $b_2c_2 = 2b_1^2b_2$, hence $B_1A = b_1A + c_1A$.

(2) Since c_1 is central in A the quotient $W = A/(c_1)$ has the structure of a $\Bbbk$-algebra with $\mathrm{GKdim}(W) = \mathrm{GKdim}(A) - 1 = 2$. Then $A/B_1A \cong W/b_1W$, so

$$\mathrm{GKdim}\, A/B_1A = \mathrm{GKdim}(W/b_1W) \geq \mathrm{GKdim}(W) - 1 = 1.$$

On the other hand, $b_1W \supseteq c_2W$. Then

$$\mathrm{GKdim}\, A/B_1A = \mathrm{GKdim}(W/b_1W) \leq \mathrm{GKdim}(W/c_2W) = \mathrm{GKdim}(W) - 1 = 1.$$

The assertion follows.

(3) The assertion follows from Corollary 3.5(3) and part (2). □

Proposition 4.2. *Let $n = 5$.*

1. $B_1A \subseteq I$ *where* $I = (b_1A + b_2A + c_1A + c_2A + c_3A + c_4A)$.
2. $\mathrm{GK}(A/I) = 1$.
3. $\mathrm{GKdim}(E) = \mathrm{GKdim}(A/B_1A) = 1$.

Proof. (1) An element $\mathbf{b}^{\mathbf{i}}\mathbf{c}^{\mathbf{j}} \in B_1$ with $\mathbf{j} \neq 0$ is clearly in I. To verify the inclusion, it suffices to show that if $\mathbf{b}^{\mathbf{i}} \in B_1$ with $\mathbf{i} \in \{0,1\}^n$ then $\mathbf{b}^{\mathbf{i}} \in I$. There are two such elements b_1 and b_2b_4 and these are both in I.

(2) Let J be the two sided ideal of A generated by central elements c_1, c_2, c_3, c_4. Let β_j (resp. γ_0) denote the image of b_j (resp. c_0) in A/J. Then A/J is a finitely generated left $\Bbbk[b_0]$-module. Since A/J has no β_0-torsion, it is actually a free module over $\Bbbk[\beta_0]$. Moreover, β_0 skew-commutes with the other β_i's, so a $\Bbbk[\beta_0]$-basis for A/J is given by squarefree monomials (with respect to the lexicographical ordering) in $\beta_1, \ldots, \beta_4$. Using this basis, we see that

$$A/I \cong \frac{A/J}{\beta_1(A/J) + \beta_2(A/J)} \cong \Bbbk[\gamma_0] \oplus \Bbbk[\gamma_0]\beta_3 \oplus \Bbbk[\gamma_0]\beta_4 \oplus \Bbbk[\gamma_0]\beta_3\beta_4.$$

Hence $\mathrm{GK}(A/I) = 1$.

(3) By part (1), the map $A/B_1A \to A/I$ is surjective, so $\mathrm{GKdim}(A/B_1A) \geq 1$. By Proposition 2.3(3b), $\mathrm{GKdim}(E) \leq 1$. Combining with Corollary 3.5(3), we have $\mathrm{GKdim}(E) = \mathrm{GKdim}(A/B_1A) = 1$. □

Now we are ready to prove Theorem 0.4.

Proof of Theorem 0.4. Retain notation as in Lemma 3.7. Take $m = 3$ or 5 (two different cases). By Proposition 4.1(3) and 4.2(3), $\mathrm{GKdim}(\tilde{E}) = 1$. By Lemma 3.7, $\mathrm{GKdim}(E) \geq 1$. Hence $\mathsf{p}(A, G) \leq n - 1$. By [MU1, Theorem 3.10], A^G is not a graded isolated singularity. □

5. Preparation, part two

In Section 7 we will prove Theorem 0.2. We need to do several reduction steps, some of which are given in this section. First we fix some convention throughout the rest of the paper.

Convention 5.1. Let n denote a fixed integer ≥ 2. Letters such as i, j, k denote elements in $\mathbb{Z}_n$. Usually these take values in $[0, 1, 2, \ldots, n-1]$. However 0 is identified with n. If we use induction, the induction process starts with 1 and ends with n (then n is identified with 0). So, when we use induction on the integer i it will take values in $[1, 2, \ldots, n]$.

In Section 5, we only use i and j.

Some ideas in this section have appeared in previous sections, but we will do finer analysis. In order to prove Theorem 0.2, we seek to show that for every j with $1 \leq j < n$, the $\Bbbk$-vector space A/B_jA is finite dimensional. It is necessary and sufficient to show that for every i, the element $c_i^N \in B_jA$ for some $N \geq 0$.

Definition 5.2. Retain notation above.

(1) We say c_i is *nilpotent* in A/B_jA if $c_i^N \in B_jA$ for some $N \geq 0$. In this case we write $i \in \Psi_j^{[n]}$.
(2) We say n is *admissible* if, for every all i and j, $i \in \Psi_j^{[n]}$, or equivalently, $\operatorname{GKdim} E = 0$, see Lemma 5.3(1) below.

Note that it is automatic that $i \in \Psi_0^{[n]}$. Therefore usually we only consider the case when $1 \leq j \leq n-1$. We start with some initial analysis and easy reductions.

Lemma 5.3. *Retain notation above.*

(1) *n is admissible if and only if* $\operatorname{GKdim}(E) = 0$. *In this case,* $\mathsf{p}(A, G) = n$.
(2) *If $i \in \Psi_j^{[n]}$ for all i and all divisors $j \mid n$ with $1 \leq j < n$, then n is admissible.*
(3) *If m is a factor of n and n is admissible, then m is admissible.*

Proof. (1) The assertion follows from (E3.0.1), (E3.2.1) and the fact that

$$\operatorname{GKdim}(A/B_jA) = 0 \text{ if and only if } i \in \Psi_j^{[n]} \text{ for all } i.$$

(2) This is Corollary 3.5(2).
(3) This follows from part (1) and Lemma 3.7. $\square$

Lemma 5.4. *Retain notation above.*

(1) *If $n = mq$ and $i \in \Psi_j^{[m]}$, then $iq \in \Psi_{jq}^{[n]}$.*
(2) *Let j be a divisor of n. If* $\gcd(i, n) = \gcd(i, j)$, *or* $\gcd(i, n)|j$, *then $i \in \Psi_j^{[n]}$.*

Proof. (1) This is Lemma 3.9(3).

(2) Let $q = \gcd(i, n)$. Then $q = \gcd(i, j) = \gcd(i, j, n)$. By part (1), we might assume that $q = 1$. In this case, i is invertible in $\mathbb{Z}_n$. Let s be the inverse of i in $\mathbb{Z}_n$. Then there is a $t := js$ such that $j = ti$ in $\mathbb{Z}_n$. In this case $c_i^t \in R_j = B_jA$ as desired. $\square$

Parts (1) to (3) of the next lemma are in fact a slightly different version of Lemma 2.1.

Lemma 5.5. *Retain notation above.*

(1) *If $i \in \Psi_{2j-i}^{[n]}$, then $i \in \Psi_j^{[n]}$.*
(2) *If $i \in \Psi_{j-i}^{[n]}$, then $i \in \Psi_j^{[n]}$.*
(3) *If $i \in \Psi_{2^s j+ti}^{[n]}$ for some integers $s, t \geq 0$, then $i \in \Psi_j^{[n]}$.*
(4) *Suppose that every proper divisor of n is admissible. If* $\gcd(i, n)$ *is even, then $i \in \Psi_j^{[n]}$.*

Proof. (1) If $c_i^N \in B_{2j-i}A$, then we can show $c_i^{N+1} \in B_jA$ as follows:

$$\begin{aligned} c_i^{N+1} &= c_i c_i^N = (b_j b_{i-j} + b_{i-j} b_j) c_i^N \\ &= b_j(b_{i-j}c_i^N) + (b_{i-j}c_i^N)b_j \\ &\in B_j(b_{i-j}c_i^N) + (b_{i-j}B_{2j-i}A)b_j \\ &\subseteq R_jA = B_jA. \end{aligned}$$

(2) By definition, we have $c_i^N \in B_{j-i}A$ for some $N > 0$. Then

$$c_i^{N+1} = c_i c_i^N \in c_i B_{j-i} A \subseteq B_j A.$$

The assertion follows.

(3) Applying the statement of (2) multiple times, we have that $i \in \Psi^{[n]}_{2^s j-i}$. By part (1), we have $i \in \Psi^{[n]}_{2^{s-1}j}$. The assertion follows by induction on s.

(4) Let $i = 2i'$ and $n = 2n'$. Since n' is admissible, $i' \in \Psi^{[n']}_{j-i'}$. By Lemma 5.4(1), $i \in \Psi^{[n]}_{2j-i}$. The assertion follows from part (1). $\square$

6. Preparation, part three

Recall from (E1.3.1) that

$$\Phi_n := \{k \mid c_k^{N_k} \in (e_0) \text{ for some } N_k \geq 0\}.$$

For each $k \in \Phi_n$, there exists $N_k \geq 0$ such that $c_k^{N_k} = 0$ in $E = (A\#C_n)/(e_0)$. It is easy to see that the set Φ_n satisfies the condition in the following definition.

Definition 6.1. A subset of $\Phi \subseteq \mathbb{Z}_n$ is called *special* if $k \in \Phi$ if and only if $\lambda k \in \Phi$ for all invertible elements $\lambda \in \mathbb{Z}_n$. In this case, the ideal $c_\Phi := \langle c_k \mid k \in \Phi\rangle$ of A is called the *special ideal* of A associated to Φ.

Here are some examples of special subsets:

(1) $\Phi = \emptyset$ (in which case, $c_\Phi = 0$).
(2) $\Phi = \Phi_n$ as in (E1.3.1).
(3) $\Phi = \phi_2(n)$ as in (E0.6.1).
(4) $\Phi = \{1, 2, \ldots, n-1\}$.
(5) $\Phi = \{0, 1, 2, \ldots, n-1\}$ (in which case, $c_\Phi = \{c_k \mid 0 \leq k \leq n-1\}$).

Fix one special ideal c_Φ of A, and write $\overline{A} = A/c_\Phi$. Clearly, C_n acts on $\overline{A}$. Let $\overline{E}$ be the algebra $(\overline{A}\#C_n)/(e_0)$. The following lemma shows that it is useful to pass into the quotient rings.

Lemma 6.2. *Retain the notation above and suppose that $\Phi = \Phi_n$. Then*

$$\operatorname{GKdim} E = \operatorname{GKdim} \overline{E}.$$

Proof. Since E is noetherian,

$$\operatorname{GKdim} E = \max_{\mathfrak{p}} \operatorname{GKdim} E/\mathfrak{p}$$

where the max runs over all prime ideals $\mathfrak{p}$ of E. Since c_k, for each $k \in \Phi_n$, is normal and nilpotent in E, we have $c_k \in \mathfrak{p}$ for each prime $\mathfrak{p}$. Hence $E/\mathfrak{p}$ is annihilated by the ideal c_Φ. As a consequence,

$$\operatorname{GKdim} E/\mathfrak{p} = \operatorname{GKdim} E/\mathfrak{p} \otimes A/c_\Phi \leq \operatorname{GKdim} E \otimes A/c_\Phi = \operatorname{GKdim} \overline{E}.$$

This implies that $\operatorname{GKdim} E \leq \operatorname{GKdim} \overline{E}$. It is clear that $\operatorname{GKdim} E \geq \operatorname{GKdim} \overline{E}$. The assertion follows. $\square$

Next we repeat some arguments in Section 3. Going back to a general fixed special ideal (not necessarily associated to Φ_n), by abuse of notation, let $\bar{e}_k$ also denote the image of the idempotent e_k in $\overline{E}$. Then we have a right $\overline{E}$-module decomposition

$$\overline{E} = \bar{e}_1\overline{E} \oplus \cdots \oplus \bar{e}_{n-1}\overline{E}$$

and it follows that

$$\mathrm{GKdim}(\overline{E}) = \max_{1\le j\le n-1} \mathrm{GKdim}(\bar{e}_j\overline{E}).$$

For any j, we have the following isomorphism of right $\overline{A}\#C_n$-modules

$$\bar{e}_j\overline{E} \cong \frac{e_j(\overline{A}\#C_n)}{e_j(\overline{A}\#C_n)\cap(e_0)}.$$

We recycle the letters x_i, b_i, c_i for $\overline{A}$ (with some of $c_i = 0$ in $\overline{A}$). There is a right $\overline{A}$-module isomorphism $e_j(\overline{A}\#C_n) \cong \overline{A}$ by $e_j a \mapsto a$ with inverse given by $a \mapsto e_j a$. So we will identify $e_j(\overline{A}\#C_n)$ with $\overline{A}$ below.

Let $\overline{\mathcal{B}}_j$ (respectively, $\overline{\mathcal{B}}$, $\overline{B}_j$) be defined as in Proposition 1.2 (respectively, Corollary 1.3, Lemma 3.1) after removing all $\{c_k \mid k \in \Phi\}$. Let $\overline{R}_j$ be the $M_{\omega^{-j}}$-isotypic component of the C_n-action on $\overline{A}$. We have an R_0-module decomposition

$$\overline{A} \cong \overline{R}_0 \oplus \overline{R}_1 \oplus \cdots \oplus \overline{R}_{n-1}$$

where $\overline{R}_0 = (\overline{A})^{C_n}$. The following is an $\overline{A}$-version of Lemma 3.2.

Lemma 6.3. *Retain the notation above. We are working in the algebra $\overline{A}\#C_n$.*

1. *The intersection $e_j(\overline{A}\#C_n)\cap(e_0)$ considered as a right ideal in $\overline{A}$ is generated by $\overline{B}_j$.*
2. *For $N \ge 0$, we have $c_k^N \in \overline{B}_j\overline{A}$ if and only if $e_j c_k^N \in (e_0)$.*
3. *If there is an integer $N \ge 0$ such that, for each j, we have $e_j c_k^N \in (e_0)$, then $c_k^N = 0$ in $\overline{E}$. If, in addition, we have $\Phi = \Phi_n$, then $c_k^N = 0$ in E, or equivalently, $c_k = 0$ in $\overline{E}$.*

Proof. For (1) and (2), see the proof of Lemma 3.2.

(3) Since $1 = \sum e_j$, we have $c_k^N \in (e_0)$. This means that $c_k^N = 0$ in $\overline{E}$.

Now assume $\Phi = \Phi_n$. Since c_k is normal, $c_k \in \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of $\overline{E}$. By the proof of Lemma 6.2, every prime quotient $E/\mathfrak{p}$ of E is isomorphic to $\overline{E}/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ of $\overline{E}$. This implies that c_k is zero in $E/\mathfrak{p}$, consequently, c_k is nilpotent in E, or $c_i^{N'} = 0$ in E for some N'. The assertion follows. □

We also have the $\overline{A}$-versions of Lemma 3.4 and Corollary 3.5. The statements are the following and proofs are omitted.

Lemma 6.4. *For any positive integer λ with $\gcd(\lambda, n) = 1$, we have the following isomorphism of $\Bbbk$-vector spaces*

$$\bar{e}_j\overline{E} \cong \bar{e}_{\lambda j}\overline{E}.$$

In particular, if n is prime, then for each $j = 2, \ldots, n-1$, we have $\bar{e}_1\overline{E} \cong \bar{e}_j\overline{E}$.

Lemma 6.5. *Retain the above notation.*

1. *For any $0 < j < n$, we have the following lower bound for $\mathrm{GKdim}\,\overline{E}$*

$$\mathrm{GKdim}\,\overline{E} \ge \mathrm{GKdim}(\overline{A}/\overline{B}_j\overline{A}).$$

(2) *We have*

$$\operatorname{GKdim} \overline{E} = \max_j \operatorname{GKdim}(\overline{A}/\overline{B}_j\overline{A})$$

where j ranges over positive integers less than n that divide n.

(3) *If n is prime, then*

$$\operatorname{GKdim} \overline{E} = \operatorname{GKdim}(\overline{A}/\overline{B}_1\overline{A}).$$

One advantage of working with $\overline{A}$ is that

$$b_i b_{k-i} = b_i b_{k-i} + b_{k-i} b_i - b_{k-i} b_i = c_k - b_{k-i} b_i = -b_{k-i} b_i \tag{E6.6.1}$$

for all $k \in \Phi$.

Similar to Definition 5.2(1), we say $c_i \in \overline{A}$ is *nilpotent* in $\overline{A}/\overline{B}_j\overline{A}$ if $c_i^N \in \overline{B}_j\overline{A}$ for some $N \geq 0$. In this case we write $i \in \overline{\Psi}_j^{[n]}$. Now we are ready to take care of Theorem 0.2 when n is prime.

Proposition 6.6. *Suppose $n \geq 2$ is neither 3 nor 5.*

(1) *Suppose $\Phi \supseteq \{1, \ldots, n-1\}$. Then $0 \in \overline{\Psi}_1^{[n]}$.*

(2) *If n is prime, then $\operatorname{GKdim} E = 0$. Consequently, n is admissible.*

Proof. (1) We start with the trivial observation that if $u, v \in \overline{B}_1\overline{A}$, then $[u, v] \in \overline{B}_1\overline{A}$. The strategy is to start with $b_1 \in \overline{B}_1\overline{A}$ and apply a sequence of graded commutations with selected elements of $\overline{B}_1\overline{A}$ to obtain $c_0^{n-1} \in \overline{B}_1\overline{A}$. Note that in $\overline{A}$, we have $[b_i, b_j] = 0$ unless $i + j = 0 \mod n$.

Claim 1: Suppose that $0 \leq j \leq n-1$ and $2j+1 \neq 0 \mod n$ (or $2j+1 \neq n$). If $c_0^s b_j \in \overline{B}_1\overline{A}$ for some s, then $c_0^{s+1} b_{j+1} \in \overline{B}_1\overline{A}$

Proof of Claim 1: First of all, $b_{j+1} b_{n-j} \in \overline{B}_1\overline{A}$ (which is not central). Since $2j+1 \neq 0 \mod n$, $b_j b_{j+1} = -b_{j+1} b_j$ in $\overline{A}$ (E6.6.1). We obtain

$$\begin{aligned} -[c_0^s b_j, b_{j+1} b_{n-j}] &= -c_0^s b_j b_{j+1} b_{n-j} + c_0^s b_{j+1} b_{n-j} b_j \\ &= c_0^s (b_{j+1} b_j b_{n-j} + b_{j+1} b_{n-j} b_j) \\ &= c_0^s b_{j+1} c_0 = c_0^{s+1} b_{j+1}. \end{aligned}$$

The assertion follows.

Claim 2: Suppose otherwise that $2j+1 = 0 \mod n$, so that $2j+1 = n$. If $c_0^s b_j \in \overline{B}_1\overline{A}$, then $c_0^{s+2} b_{j+2} \in \overline{B}_1\overline{A}$.

Proof of Claim 2: Under the hypothesis of j, we have that n is odd and that $b_{j+1} b_{j+2} b_{n-1} \in \overline{B}_1\overline{A}$. Given that $n \neq 3, 5$, we have $n \geq 7$, and consequently,

$$j + 2 = (n+3)/2 < n - 1,$$

so that the indices in $b_{j+1} b_{j+2} b_{n-1}$ are strictly increasing. This gives the following commutator computation in $\overline{B}_1\overline{A}$,

$$\begin{aligned} [c_0^s b_j, b_{j+1} b_{j+2} b_{n-1}] &= c_0^s (b_j b_{j+1} b_{j+2} b_{n-1} + b_{j+1} b_{j+2} b_{n-1} b_j) \\ &= c_0^s (b_j b_{j+1} b_{j+2} b_{n-1} + b_{j+1} b_j b_{j+2} b_{n-1}) \\ &= c_0^{s+1} b_{j+2} b_{n-1}. \end{aligned}$$

We apply the one additional commutator to get

$$[[c_0^s b_j, b_{j+1} b_{j+2} b_{n-1}], b_1] = [c_0^{s+1} b_{j+2} b_{n-1}, b_1] = c_0^{s+2} b_{j+2}.$$

Therefore, if $c_0^s b_j \in \overline{B}_1\overline{A}$ and $2j+1 = n$, then $c_0^{s+2} b_{j+2} \in \overline{B}_1\overline{A}$.

Claim 3: $c_0^{n-1} \in \overline{B}_1\overline{A}$.

Proof of Claim 3: Starting with b_1, we can apply **Claim 1** $(n-2)$-times to get $c_0^{n-2}b_{n-1} \in \overline{B}_1\overline{A}$ whenever n is even. Hence $c_0^{n-1} = [c_0^{n-2}b_{n-1}, b_1] \in \overline{B}_1\overline{A}$ as required.

If n is odd, we apply **Claim 1** (j_0-1)-times to get $c_0^{j_0-1}b_{j_0} \in \overline{B}_1\overline{A}$ where $j_0 = \frac{n-1}{2}$. Next we apply **Claim 2** to get $c_0^{j_0+1}b_{j_0+2} \in \overline{B}_1\overline{A}$. Then apply **Claim 1** again (j_0-2)-times to get $c_0^{n-2}b_{n-1} \in \overline{B}_1\overline{A}$. Finally we have

$$c_0^{n-1} = [c_0^{n-2}b_{n-1}, b_1] \in \overline{B}_1\overline{A}$$

as desired.

The assertion follows from **Claim 3**.

(2) Now n is a prime integer $\neq 3, 5$. By Proposition 2.3(1), $\{1, \ldots, n-1\} \subseteq \Phi_n$, so the hypothesis of part (1) holds when taking $\Phi = \Phi_n$. By part (1), $0 \in \overline{\Psi}_1^{[n]}$. Since $c_k = 0$ in $\overline{A}$ for all $k \neq 0 \mod n$, we have that $c_k^N = 0$ in $\overline{B}_1\overline{A}$ for all k. This implies that $\operatorname{GKdim}(\overline{A}/\overline{B}_1\overline{A}) = 0$. By Lemma 6.5(2), $\operatorname{GKdim} \overline{E} = 0$. The assertion follows from Lemma 6.2. $\square$

Proposition 6.6(2) is one of the initial steps in the proof of Theorem 0.2 and Proposition 6.6(1) is a step of reduction. The following technical lemma is needed for the proof of the proposition below.

Lemma 6.7. *Suppose i and j satisfy the following conditions:*

(1) *i is an (odd) integer with $0 \leq i \leq n-1$ and $\gcd(i, n) > 1$,*
(2) *$2 \leq j \leq n-1$ such that $\gcd(i, j, n) = 1$.*

Then there is an integer $t \geq 0$ such that $\gcd(j + ti, n) = 1$.

Proof. Let $n = p_1^{n_1} \cdots p_s^{n_s} \cdots p_r^{n_r}$ for some $1 \leq s \leq r$, where $\{p_i\}$ are the prime factors of n and $n_u \geq 1$ for all $1 \leq u \leq r$. The ordering of the prime and the integer s are chosen so that $j = p_1^{j_1} \cdots p_{s-1}^{j_{s-1}} j'$ where $\gcd(j', n) = 1$ and $j_w \geq 1$ for all $1 \leq w \leq s-1$. Since we assume that $\gcd(i, j, n) = 1$, we can write $i = p_s^{i_s} \cdots p_r^{i_r} i'$ where $\gcd(i', n) = 1$ with $i_v \geq 0$ for all $s \leq v \leq r$. Let $t = p_s \cdots p_r$. Then it is easy to see that each p_u, for $1 \leq u \leq r$, does not divide $j+ti$. Thus $\gcd(j+ti, n) = 1$. $\square$

Proposition 6.8. *Let $n \geq 2$ and denote $\Phi = \Phi_n$. Suppose that*

(a) *every proper factor of n is admissible, and that*
(b) *for each $0 \leq i \leq n-1$, $i \in \overline{\Psi}_1^{[n]}$.*

Then n is admissible.

Proof. By Lemmas 6.2 and 6.3, and the ideas in Lemma 3.4, it suffices to show that $i \in \overline{\Psi}_j^{[n]}$ for all $0 \leq i \leq n-1$ and $1 \leq j \leq n-1$ with $j \mid n$. We use induction on j and then on i. The minimal possible j is 1, in which the assertion follows from hypothesis (b). Now assume that $j > 1$.

If $\gcd(i, n) = 1$, or $i = 2^w q$ with q odd and $\gcd(q, n) = 1$, then the assertion follows from Proposition 2.3(1). This shows that the assertion holds for $i = 1$. So we can assume that $i \geq 2$ and proceed with induction on i.

Suppose i is even and write $i = 2i'$. If n is even, then $i' \in \Psi_j^{[n/2]}$ by hypothesis (a). By Lemma 3.9(3), $i \in \Psi_{2j}^{[n]}$. Consequently, $i \in \overline{\Psi}_{2j}^{[n]}$. By the $\overline{A}$-version of Lemma 5.5(3), $i \in \overline{\Psi}_j^{[n]}$. If n is odd, then by the induction hypothesis, $i' \in \overline{\Psi}_j^{[n]}$.

Applying the automorphism f_2 in Definition 3.3, we obtain that $i \in \overline{\Psi}^{[n]}_{2j}$. By the $\overline{A}$-version of Lemma 5.5(3), we have $i \in \overline{\Psi}^{[n]}_{j}$.

For the rest of proof we assume that i is odd and $\gcd(i, n) > 1$. If $\gcd(i, j, n) =: q > 1$, write $n = qn'$, $i = qi'$ and $j = qj'$. By hypothesis (a), $i' \in \Psi^{[n']}_{j'}$. By Lemma 3.9(3), $i \in \Psi^{[n]}_{j}$. Consequently, $i \in \overline{\Psi}^{[n]}_{j}$. The other alternative is $\gcd(i, j, n) = 1$. By Lemma 6.7, there is a $t \geq 0$ such that $\gcd(j + ti, n) = 1$. By hypothesis (b), $i' \in \overline{\Psi}^{[n]}_{1}$ for all i'. Let λ be $j + ti$, which is invertible in $\mathbb{Z}_n$ by Lemma 6.7, and let f_λ be the ($\overline{A}$-version of the) algebra automorphism defined as in Definition 3.3. Pick $i' = i\lambda^{-1}$ in $\mathbb{Z}_n$. Then, after applying f_λ, $i' \in \overline{\Psi}^{[n]}_{1}$ becomes $i \in \overline{\Psi}^{[n]}_{j+ti}$. By the $\overline{A}$-version of Lemma 5.5(3), $i \in \overline{\Psi}^{[n]}_{j}$ for all i. Thus we have finished the inductive step and the whole proof. $\square$

7. Proof of Theorem 0.2

The proof of Theorem 0.2 follows the strategy of Proposition 6.6. Let us recall the argument for showing that some power of c_0 is in $\overline{B}_1\overline{A}$. Given an element $c_0^s b_j \in \overline{B}_1\overline{A}$, depending on whether j satisfies a certain congruence, we applied commutators to $c_0^s b_j$ to conclude that $c_0^{s+1} b_{j+1}$ or $c_0^{s+2} b_{j+2}$ is in $\overline{B}_1\overline{A}$. Starting with b_1 and continuing in this way, we eventually reach $c_0^{n-2} b_{n-1} \in \overline{B}_1\overline{A}$. Applying $[-, b_1]$ to this gives $c_0^{n-1} \in \overline{B}_1\overline{A}$.

In the more general situation of Theorem 0.2, we have to show that for each divisor i of n, some power of c_i is in $\overline{B}_{i_0}\overline{A}$ for all i_0. Given an element $c_i^s b_j \in \overline{B}_{i_0}\overline{A}$ we apply certain commutators to $c_i^s b_j$ depending on congruences satisfied by i, j, i_0 to conclude that other elements of the form $c_i^{s'} b_{j'}$ are in $\overline{B}_{i_0}\overline{A}$ (see Lemmas 7.4 and 7.5). These congruences are described in Definition 7.1. Then we show that there are indeed integers satisfying Definition 7.1 (see Lemma 7.2) and that we eventually reach $c_0^{s'} b_{i-i_0}$ (see Lemma 7.6), so that applying $[-, b_{i_0}]$ gives what we want. The final induction steps needed for the proof of Theorem 0.2 are given in Proposition 7.7 and Corollary 7.8.

To simplify notation, let

$$\Lambda_{i,i_0} := \{j \mid b_j c_i^t (= c_i^t b_j) \in \overline{B}_{i_0}\overline{A}, \quad \text{for some } t \geq 0\}.$$

It is clear that $i_0 \in \Lambda_{i,i_0}$. Write

$$i_\xi = i_0 + \xi(i_0 - i)$$

and

$$\bar{i}_\xi = (-\xi)(i_0 - i)$$

for all integers ξ. Since i_0 will be a fixed integer in most of proofs below, we hope that the probability of serious confusion is not high. Let

$$\Xi_{i,i_0} := \{\xi \mid i_\xi \in \Lambda_{i,i_0} \quad \text{and} \quad 0 \leq \xi \leq \frac{1}{2}(\operatorname{mop}(n) - 3)\}.$$

where for $n \geq 2$

$$\operatorname{mop}(n) := \text{the minimal odd prime factor of } n.$$

It is clear that $0 \in \Xi_{i,i_0}$.

Let $\mathbb{Z}_n^\times$ be the invertible elements in $\mathbb{Z}_n$ and S be the set of odd integers between 1 and n which are not in $\phi_2(n)$ (E0.6.1). Then define

$$\Omega_2(n) := \bigcap_{s\in S} \big(\mathbb{Z}_n^\times + s\big) \tag{E7.0.1}$$

It is not hard to show that $\Omega_2(n) \subseteq \phi_2(n)$.

Definition 7.1. Let $n \geq 2$ be an integer. An integer i_0 with $1 \leq i_0 \leq n-1$ is called n**-special** if

(1) $i_0 \in \phi_2(n)$. (This is also a consequence of (2) below.)

For all odd integers $i \notin \phi_2(n)$ with $1 \leq i \leq n$, the following hold.

(2) $i_0 - i$ is invertible in $\mathbb{Z}_n$.

Part (2) is just that $i_0 \in \Omega_2(n)$. Now fix any i as in part (2). For every $1 \leq j \leq n-1$, either

(3) $2j + (i_0 - i) \in \phi_2(n)$,

or

(4) (If $2j + (i_0 - i) \notin \phi_2(n)$, then) there is a $\xi \in \Xi_{i,i_0}$ such that
(4(ξ)i) $2j + (\xi + 2)(i_0 - i) \in \phi_2(n)$,
(4(ξ)ii) $j - (\xi + 1)(i_0 - i) \in \phi_2(n)$, and
(4(ξ)iii) $j + i_0 + 2(\xi + 1)(i_0 - i) \in \phi_2(n)$.
For any $\xi \in \Xi_{i,i_0}$, conditions (4(ξ)i), (4(ξ)ii) and (4(ξ)iii) all together are denoted by (4(ξ)).

Let $Spl(n)$ denote the set of integers i_0 that are n-special.

Lemma 7.2. *Let $n = 2^a p^b$ where p is a prime ≥ 3.*

(1) *$2 \in \Omega_2(n)$.*
(2) *If $p \geq 7$, then $i_0 = 2$ is n-special.*

Proof. (1) For every odd integer $i \notin \phi_2(n)$, we have $p \mid i$. Therefore 2 and p do not divide $i_0 - i = 2 - i$, so Definition 7.1(2) holds, and the assertion follows.

(2) Now assume $p \geq 7$. Note that Definition 7.1(1) is obvious. Definition 7.1(2) holds by part (1). For Definition 7.1(3,4), note that $0 \in \Xi_{i,i_0}$. Note that, for every i given in Definition 7.1(2), $i \notin \phi_2(n)$. Hence i is divisible by p. When Definition 7.1(3) fails, namely, $2j + (i_0 - i)$ (or equivalently, $2j + i_0$) is divisible by p, then $j + 1 = \frac{1}{2}(2j + i_0)$ is divisible by p, that is, $j = -1 \mod p$. By taking $\xi = 0$, we have

$$2j + 2(2 - i) = -2 + 4 = 2 \neq 0 \mod p,$$
$$j - (1)(2 - i) = -1 - 2 = -3 \neq 0 \mod p,$$
$$j + 2 + 2(1)(2 - i) = -1 + 2 + 4 = 5 \neq 0 \mod p.$$

This means that (4(0)i), (4(0)ii) and (4(0)iii) hold. Therefore $i_0 = 2$ is n-special. □

We have another case when $\Omega_2(n)$ is non-empty. The following lemma is not needed for the proof of Theorem 0.2. It will be used in §8 (see Theorem 8.7).

Lemma 7.3. *Let p_1 and p_2 be two distinct odd primes.*

(1) *If $n = p_1 p_2$, then $\Omega_2(n) \neq \emptyset$.*
(2) *If n is either $p_1 p_2^2 n'$ or $2p_1 p_2 n'$ for some $n' \geq 1$, then $\Omega_2(n) = \emptyset$. As a consequence, $Spl(n) = \emptyset$.*

Proof. (1) Let $p_1 < p_2$. Every integer m can be written uniquely as $m = ap_1 + bp_2$ where $0 \le a < p_2$. In particular, $1 = a_1 p_1 + b_1 p_2$. If a_1 is odd, we claim that $i_0 = -1 \in \Omega_2(n)$. If a_1 is even, we claim that $i_0 = 1 \in \Omega_2(n)$. Since the proofs are very similar, we only consider the first case.

Suppose that i is an odd integer $1 \le i \le n$ that is not in $\phi_2(n)$ such that $i - i_0 = i + 1$ is not invertible in $\mathbb{Z}_n$. Then i and $i+1$ are divisible by different prime factors. We need to consider two cases.

Case 1: $p_1 \mid i$ and $p_2 \mid i+1$. Write $i = i'p_1$ (where i' is odd as i is odd) and $i + 1 = jp_2$. Then

$$-1 = i - (i+1) = i'p_1 - jp_2$$

where i' is odd, which implies that

$$1 = (p_2 - i')p_1 + (j - p_1)p_2.$$

Note that $p_2 - i'$ is even, which contradicts the fact that a_1 is odd.

Case 2: $p_2 \mid i$ and $p_1 \mid i+1$. Write $i = i'p_2$ (where i' is odd as i is odd) and $i + 1 = jp_1$ (where j is even as $i+1$ is even). Then

$$-1 = i - (i+1) = i'p_2 - jp_1,$$

which implies that

$$1 = jp_1 - i'p_2.$$

Note that j is even, which contradicts the fact that a_1 is odd.

(2) Since p_1 and p_2 are distinct, $1 = ap_1 + bp_2$. For every i_0, one can write it as

$$i_0 = cp_1 + dp_2$$

for some c, d with $0 \le c < p_2$. If c is odd, take $i = cp_1 < n$, which is odd and not in $\phi_2(n)$. Then $i_0 - i = dp_2$ is not invertible in $\mathbb{Z}_n$. If c is even, take $i = (c+p_2)p_1 < n$, which is odd and not in $\phi_2(n)$. Then $i_0 - i = (d - p_1)p_2$ is not invertible in $\mathbb{Z}_n$. This means that $i_0 \notin \Omega_2(n)$ for every i_0. $\square$

The next two lemmas describe a family of partially defined maps $\Lambda_{i,i_0} \dashrightarrow \Lambda_{i,i_0}$, where i_0 and i satisfy hypotheses (1) and (2) of Definition 7.1.

Lemma 7.4. *Retain the above notation.*

1. *Let $j \in \Lambda_{i,i_0}$. If j satisfies hypothesis (3) of Definition 7.1, that is, $2j + (i_0 - i) \in \phi_2(n)$, then $j + (i_0 - i) \in \Lambda_{i,i_0}$. In particular, we get a partially defined map $\omega_0 : \Lambda_{i,i_0} \dashrightarrow \Lambda_{i,i_0}$ given by $\omega_0(j) := j + (i_0 - i)$.*
2. *If $i_\xi \in \Lambda_{i,i_0}$ and $2i_\xi + (i_0 - i) = 2i_0 + (2\xi + 1)(i_0 - i) \in \phi_2(n)$, then $i_{\xi+1} \in \Lambda_{i,i_0}$.*
3. *If $2i_\xi + (i_0 - i) = 2i_0 + (2\xi+1)(i_0 - i) \in \phi_2(n)$ for all $0 \le \xi < \frac{1}{2}(\mathrm{mop}(n) - 3)$, then $\Xi_{i,i_0} = [0, 1, \ldots, \frac{1}{2}(\mathrm{mop}(n) - 3)]$.*

Proof. Part (2) is a special case of part (1) by taking $j = i_\xi$. Part (3) follows from part (2) and induction. So we only prove part (1) below.

Let $s = i - j$ and $r = j - (i - i_0)$. Then $j + r = 2j - i + i_0$ is in $\phi_2(n)$ by the hypothesis. By Proposition 2.3(1), $j + r \in \Phi_n$ and $b_j b_r = -b_r b_j$ in $\overline{A}$. We start with $b_j c_i^t \in \overline{B}_{i_0} \overline{A}$ for some $t \ge 0$ (as $j \in \Lambda_{i,i_0}$). By the choice of r, s, we have $b_r b_s = b_{j-i+i_0} b_{i-j} \in \overline{B}_{i_0} \overline{A}$.

Consider the commutator $[b_j c_i^t, b_r b_s]$, we have the following elements in $\overline{B}_{i_0}\overline{A}$

$$\begin{aligned} -[b_j c_i^t, b_r b_s] &= c_i^t(-b_j b_r b_s + b_r b_s b_j) \\ &= c_i^t(b_r b_j b_s + b_r b_s b_j) \\ &= c_i^t b_r c_{j+s} \\ &= b_r c_i^{t+1} = b_{j+(i_0-i)} c_i^{t+1}. \end{aligned}$$

The assertion follows. □

Lemma 7.5. *Let $\xi \in \Xi_{i,i_0}$ and $j \in \Lambda_{i,i_0}$. Suppose that ξ and j satisfy the hypotheses (4(ξ)) in Definition 7.1. Then $j + (\xi+2)(i_0 - i) \in \Lambda_{i,i_0}$. In particular, we get a partially defined map $\omega_{\xi+1} : \Lambda_{i,i_0} \dashrightarrow \Lambda_{i,i_0}$ given by $\omega_{\xi+1}(j) := j + (\xi+2)(i_0 - i)$.*

Proof. Note that $\xi \in \Xi_{i,i_0}$ means that $i_\xi \in \Lambda_{i,i_0}$ where $i_\xi = i_0 + \xi(i_0 - i)$.

Let $a = i - j$, $c = j + (\xi+2)(i_0 - i)$ and $d = -(\xi+1)(i_0 - i)$. By hypotheses (4(ξ)i)-(4(ξ)iii), we have that $c + j$, $d + j$ and $i_\xi + c$ are in $\phi_2(n)$. This means that $[b_c, b_j] = 0 = [b_d, b_j] = [b_c, b_{i_\xi}]$ in $\overline{A}$.

Starting with $b_j c_i^t \in \overline{B}_{i_0}\overline{A}$ for some $t \geq 0$ (as $j \in \Lambda_{i,i_0}$), we have the two sets of equations in $\overline{B}_{i_0}\overline{A}$. The first set is

$$\begin{aligned} [b_j c_i^t, b_a b_c b_d] &= c_i^t(b_j b_a b_c b_d + b_a b_c b_d b_j) \\ &= c_i^t(b_j b_a b_c b_d + b_a b_j b_c b_d) \\ &= c_i^t c_{a+j} b_c b_d \\ &= c_i^{t+1} b_c b_d. \end{aligned}$$

In the above computation, note that $b_a b_c b_d = b_{i-j} b_{j+(\xi+2)(i_0-i)} b_{-(\xi+1)(i_0-i)} \in \overline{B}_{i_0}$. We also need b_j to skew commute with b_d and b_c, and $a + j = i$. Since $\xi \in \Xi_{i,i_0}$, there is a $t' \geq 0$ such that $b_{i_\xi} c_i^{t'} \in \overline{B}_{i_0}\overline{A}$. The second set of equations is

$$\begin{aligned} [c_i^{t+1} b_c b_d, b_{i_\xi} c_i^{t'}] &= c_i^{t+1+t'}(b_c b_d b_{i_\xi} - b_{i_\xi} b_c b_d) \\ &= c_i^{t+t'+1}(b_c b_d b_{i_\xi} + b_c b_{i_\xi} b_d) \\ &= c_i^{t+t'+1} b_c c_{d+i_\xi} \\ &= c_i^{t+t'+2} b_c = c_i^{t+t'+2} b_{j+(\xi+2)(i_0-i)}. \end{aligned}$$

Therefore $j + (\xi+2)(i_0 - i) \in \Lambda_{i,i_0}$ and the assertion follows. □

We usually apply the above lemma with the additional hypothesis $2j + (i_0 - i) \notin \phi_2(n)$.

If i_0 is n-special and $i_0 - i \in \mathbb{Z}_n^\times$, then by Definition 7.1, the unions of the domains of definition of ω_0 and $\omega_{\xi(i)+1}$ for $\xi \in \Xi_{i,i_0}$ is equal to Λ_{i,i_0}. In other words, for any $j \in \Lambda_{i,i_0}$ there is some ω_t which can be applied to j.

Lemma 7.6. *Suppose that i_0 is n-special. Then $-(i_0 - i) \in \Lambda_{i,i_0}$. Consequently $c_i^N \in \overline{B}_{i_0}\overline{A}$ for some $N \geq 0$.*

Proof. Suppose that $\bar{i}_\xi \in \Lambda_{i,i_0}$ for some $\xi \in [0, M]$, where $M = \frac{1}{2}(\mathrm{mop}(n) - 3)$. Then $2\bar{i}_\xi + (i_0 - i) = (1 - 2\xi)(i_0 - i) \in \phi_2(n)$ since $2\xi - 1$ is invertible if $0 \leq \xi \leq M$.

By Lemma 7.4(1) we have $\bar{i}_\xi$ is in the domain of definition of ω_0, so $\omega_0(\bar{i}_\xi) = \bar{i}_{\xi-1} \in \Lambda_{i,i_0}$. Repeating the argument gives $\bar{i}_{-1} \in \Lambda_{i,i_0}$.

Now, let r be the maximal integer such that $\bar{i}_\xi \notin \Lambda_{i,i_0}$ for every $\xi \in \{-1, 0, \ldots, r\}$. In other words, $\bar{i}_{r+1} \in \Lambda_{i,i_0}$. Since i_0 is n-special, either Lemma 7.4 or 7.5 applies. That is, $\bar{i}_{r+1}$ is in the domain of definition of ω_ζ for some $\zeta \in [0, M+1]$. Applying any such ω_ζ gives $\omega_\zeta(\bar{i}_{r+1}) = \bar{i}_{r-\zeta} \in \Lambda_{i,i_0}$. By maximality of r, we have $r - \zeta < -1$ so $r < \zeta - 1 \leq M$. Thus we may apply the argument in the first paragraph to conclude that $\bar{i}_{-1} \in \Lambda_{i,i_0}$.

The above shows that we always have $\bar{i}_{-1} \in \Lambda_{i,i_0}$ under the hypotheses that i_0 is n-special. Equivalently, $b_{i-i_0} c_i^t \in \overline{B}_{i_0}\overline{A}$. Finally, $c_i^{t+1} = [b_{i-i_0} c_i^t, b_{i_0}] \in \overline{B}_{i_0}\overline{A}$. $\square$

Here is one of the main results of this section, which leads to Theorem 0.2.

Theorem 7.7. *Let $n \geq 2$. Suppose that*

(1) *every proper factor of n is admissible,*
(2) *there is an n-special integer i_0.*

Then, for each $0 \leq i \leq n-1$, $i \in \overline{\Psi}_1^{[n]}$.

Proof. By Definition 7.1(1), we can express the n-special integer i_0 as the product $i_0 = 2^w g$ where $w \geq 0$ with g odd and $\gcd(n, g) = 1$. Since g is invertible in $\mathbb{Z}_n$, by using the automorphism f_g of A defined in Definition 3.3, the assertion is equivalent to $i \in \overline{\Psi}_g^{[n]}$ for all $0 \leq i \leq n-1$. By the $\overline{A}$-version of Lemma 5.5(3), it suffices to show the following claim.

Claim: for each $0 \leq i \leq n-1$, $i \in \overline{\Psi}_{i_0}^{[n]}$.
Proof of Claim: We prove the **Claim** by induction on i starting at $i = 1$ and ending at $i = n$ (which is also 0 in $\mathbb{Z}_n$). We consider several cases.

Case 1: $i = 1$.

The assertion follows from Proposition 2.3(1). For the inductive step, we assume that $i \geq 2$ and that $i' \in \overline{\Psi}_{i_0}^{[n]}$ for all $i' < i$.

Case 2: i is even.

If n is also even, it follows from Lemma 5.5(4) that $i \in \Psi_{i_0}^{[n]}$. Passing to the quotient ring, we have $i \in \overline{\Psi}_{i_0}^{[n]}$ as desired.

If n is not even, then f_2 in Definition 3.3 is an automorphism. Write $i = 2i'$ for some $1 \leq i' < i$. By the induction hypothesis, $i' \in \overline{\Psi}_{i_0}^{[n]}$. Applying f_2, we obtain that $i = 2i' \in \overline{\Psi}_{2i_0}^{[n]}$. By the $\overline{A}$-version of Lemma 5.5(3), we have $i \in \overline{\Psi}_{i_0}^{[n]}$. This takes care of the case when i is even. For cases 3 and 4 below, we assume that i is odd.

Case 3: i is odd and $i \in \phi_2(n)$.

In this case, the assertion follows from Proposition 2.3(1) as $c_i = 0$ in $\overline{A}$.

The remaining case to consider is

Case 4: i is odd and $i \notin \phi_2(n)$. By hypothesis, i_0 is n-special. By Lemma 7.6, $i \in \overline{\Psi}_{i_0}^{[n]}$

Hence we finished the inductive step and therefore we complete the proof of the **Claim**. $\square$

Corollary 7.8. *Let $n \geq 2$. Suppose that*

(1) *every proper factor of n is admissible,*
(2) *$Spl(n) \neq \emptyset$, namely, there is an n-special integer i_0.*

Then n is admissible.

Proof. By Theorem 7.7, for each $0 \leq i \leq n-1$, we have $i \in \overline{\Psi}_1^{[n]}$. The assertion then follows from Proposition 6.8. □

Now we are ready to show Theorem 0.2.

Proof of Theorem 0.2. In this case $n = 2^a p^b$ for some prime $p \geq 7$. If $(a,b) = (0,1)$ or $(1,0)$, the assertion follows from Proposition 6.6(2). This takes care of the initial step for induction.

By Lemma 7.2, $i_0 = 2$ is n-special, which is hypothesis (2) in Corollary 7.8. Hypothesis (1) in Corollary 7.8 follows by induction. Hence we can conclude from Corollary 7.8 that n is admissible. By definition, $\operatorname{GKdim}(E) = 0$. Hence $\mathsf{p}(A,G) = n$ and, by [MU1, Theorem 3.10], A^G is a graded isolated singularity. □

Proof of Corollary 0.6. For each $n < 77$, n is either divisible by 3 or 5, or n is of the form $2^a p^b$ for some prime $p \geq 7$. Hence the assertion follows by Theorem 0.2 and 0.4. □

Proof of Corollary 0.10. By Theorem 0.2 and [BHZ1, Theorem 5.7(1)], A^G is a graded isolated singularity. By [KKZ1, Theorem 1.5], A^G is Gorenstein.

(1) This follows from [MU1, Corollary 2.6].
(2) This follows from [MU1, Theorem 3.15].
(3) This follows from [MU1, Theorem 3.14].
(4) This follows from [Ue, Theorem 1.3]. □

Below is a slightly more general result than Theorem 0.2.

Theorem 7.9. *Let S be a set of integers $n \geq 2$. Suppose that*

(1) *each n in S is not divisible by 3 or 5,*
(2) *every proper factor of $n \in S$ is still in S.*
(3) *for each $n \in S$, $Spl(n) \neq \emptyset$.*

Then every $n \in S$ is admissible.

Proof. The assertion follows by induction on $n \in S$. Since each n in S is not divisible by 3 or 5, the initial step follows from Proposition 6.6(2). Now we assume that the assertion holds for all proper factors of n. The induction step follows from hypothesis (3) and Corollary 7.8. □

8. Partial results when $n = p_1 p_2$

In this section we give some partial answer to the case when $n = p_1 p_2$ for p_i being distinct primes. Some lemmas works for the case when $n = 2^a p_1^b p_2^c$.

Another way of defining $\Omega_2(n)$ (E7.0.1) is the following

$$\Omega_2(n) := \{i_0 \in \mathbb{Z}_n \mid \text{if } 1 \leq i \leq n \text{ is odd and } i \notin \phi_2(n), \text{ then } i_0 - i \in \mathbb{Z}_n \text{ is invertible}\}.$$

As noted in Section 7, $\Omega_2(n) \subseteq \phi_2(n)$.

In the rest of this section, **let n be $2^a p_1^b p_2^c$ where p_1 and p_2 are distinct odd primes ≥ 7 and $b, c \geq 1$.** We start with a linear algebra fact.

Lemma 8.1. *Let $i_0 \in \phi_2(n)$ and $i \in [1,\dots,n]$ such that $p_1 \mid i$. If p_2 divides both $c_{11}i_0 + c_{12}i$ and $c_{21}i_0 + c_{22}i$, then p_2 divides*

$$\mathrm{Det} := \det \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = c_{11}c_{22} - c_{12}c_{21}.$$

Proof. By easy linear algebra, p_2 divides both $\mathrm{Det}\, i_0$ and $\mathrm{Det}\, i$. Since p_2 and i_0 are coprime, p_2 divides Det. □

Lemma 8.2. *Let $i_0 \in \phi_2(n)$ and $i \in [1,\dots,n]$ be an odd integer not in $\phi_2(n)$. There is at most one integer*

$$\xi \in [0, 1, \dots, -1 + \frac{1}{2}(\mathrm{mop}(n) - 3)] \tag{E8.2.1}$$

such that $2i_\xi + (i_0 - i) \notin \phi_2(n)$.

Proof. Without loss of generality, we can assume that p_1 divides i. For each ξ in (E8.2.1), p_1 does not divide $2i_\xi + (i_0 - i) = (2\xi + 3)i_0 - (2\xi + 1)i$, as

$$2\xi + 3 < \mathrm{mop}(n) := \min\{p_1, p_2\}.$$

If there are ξ_1 and ξ_2 in (E8.2.1) such that $2i_{\xi_1} + (i_0 - i)$ and $2i_{\xi_2} + (i_0 - i)$ are not in ϕ_2, then p_2 must divide both $2i_{\xi_1} + (i_0 - i)$ and $2i_{\xi_2} + (i_0 - i)$ (or equivalently, divide both $(2\xi_1 + 3)i_0 - (2\xi_1 + 1)i$ and $(2\xi_2 + 3)i_0 - (2\xi_2 + 1)i$). By Lemma 8.1, p_2 divides Det, and an easy computation shows that

$$\mathrm{Det} = \det \begin{pmatrix} 2\xi_1 + 3 & 2\xi_1 + 1 \\ 2\xi_2 + 3 & 2\xi_2 + 1 \end{pmatrix} = 4(\xi_2 - \xi_1).$$

By the choices of ξ_1, ξ_2 in (E8.2.1), p_2 does not divide $4(\xi_2 - \xi_1)$, a contradiction. The assertion follows. □

Lemma 8.3. *Retain the hypothesis as in Lemma* 8.2. *Suppose that ξ in* (E8.2.1) *is such that $2i_\xi + (i_0 - i) \notin \phi_2(n)$, namely, Definition* 7.1(3) *fails for $j = i_\xi$. Then Definition* 7.1(4(0)) *holds for $j = i_\xi$.*

Proof. By Lemma 8.2, ξ is unique. It suffices to show that the following elements in (4(0)i), (4(0)ii) and (4(0)iii) of Definition 7.1(4), when $j = i_\xi$, are in ϕ_2:

$$2i_\xi + 2(i_0 - i) = (2\xi + 4)i_0 - (2\xi + 2)i = 2((\xi + 2)i_0 - (\xi + 1)i), \tag{E8.3.1}$$

$$i_\xi - (i_0 - i) = \xi i_0 - (\xi - 1)i, \tag{E8.3.2}$$

$$i_\xi + i_0 + 2(i_0 - i) = (\xi + 4)i_0 - (\xi + 2)i. \tag{E8.3.3}$$

Since $i \notin \phi_2(n)$, either p_1 or p_2 divides i. Without loss of generality, we say p_1 divides i. By the proof of Lemma 8.2, we have p_2 divides $2i_\xi + (i_0 - i) = (2\xi + 3)i_0 - (2\xi + 1)i$. Since $\mathrm{mop}(n) \geq 7$, all of $\xi + 2, \xi, \xi + 4$ are strictly less than $\mod(n)$. Hence p_1 does not divide elements in (E8.3.1)-(E8.3.3). We claim that p_2 does not divide elements in (E8.3.1)-(E8.3.3). If this is false, say, the element in (E8.3.3) is divisible by p_2, then p_2 divides both $(2\xi + 3)i_0 - (2\xi + 1)i$ and $(\xi + 4)i_0 - (\xi + 2)i$. By Lemma 8.1, p_2 divides Det, where

$$\mathrm{Det} = \det \begin{pmatrix} 2\xi + 3 & 2\xi + 1 \\ \xi + 4 & \xi + 2 \end{pmatrix} = -2(\xi - 1).$$

However, by the choice of ξ in (E8.2.1), p_2 does not divide Det, a contradiction. The claim is proved.

Finally, by the above, elements in (E8.3.1)-(E8.3.3) are not divisible by either p_1 or p_2. Hence these elements are in $\phi_2(n)$. The assertion follows. □

Lemma 8.4. *Retain the hypothesis as in Lemma* 8.2*. Then either*

$$\Xi_{i,i_0} = [0, 1, 2, \ldots, \frac{1}{2}(\mathrm{mop}(n) - 3)]$$

or

$$\Xi_{i,i_0} = [0, 1, 2, \ldots, \xi, \widehat{\xi + 1}, \xi + 2, \ldots, \frac{1}{2}(\mathrm{mop}(n) - 3)].$$

The second case can happen only when there is a ξ *in* (E8.2.1) *such that* $2i_\xi + (i_0 - i) \notin \phi_2(n)$*.*

By Lemma 8.2, ξ is Lemma 8.4 above is unique if it exists.

Proof of Lemma 8.4. If, for each ξ in (E8.2.1), $2i_\xi + (i_0 - i)$ is in $\phi_2(n)$, then, by Lemma 7.4(3),

$$\Xi_{i,i_0} = [0, 1, 2, \ldots, \frac{1}{2}(\mathrm{mop}(n) - 3)]$$

which is the first case.

To prove the lemma, we may assume that

$$\Xi_{i,i_0} \neq [0, 1, 2, \ldots, \frac{1}{2}(\mathrm{mop}(n) - 3)]$$

and that there is a ξ in (E8.2.1) such that $2i_\xi + (i_0 - i) \notin \phi_2(n)$. By Lemma 8.2, ξ is unique, namely, for all $\xi' \neq \xi$ in (E8.2.1), $2i_{\xi'} + (i_0 - i)$ is in $\phi_2(n)$. By Lemma 7.4(2) and induction, $0, 1, 2, \ldots, \xi \in \Xi_{i,i_0}$. By Lemma 8.3, Definition 7.1(4(0)) holds for $j = i_\xi$. By Lemma 7.5 (for a different $\xi = 0 \in \Xi_{i,i_0}$ in the setting of Lemma 7.5),

$$j + 2(i_0 - i) = i_\xi + 2(i_0 - i) = i_{\xi+2}$$

is in Λ_{i,i_0}. If $\xi + 2 \leq \frac{1}{2}(\mathrm{mop}(n) - 3)$, then $\xi + 2 \in \Xi_{i,i_0}$ by definition. By Lemma 8.2, $2i_{\xi'} + (i_0 - i) \in \phi_2(n)$ for all

$$\xi + 2 \leq \xi' \leq -1 + \frac{1}{2}(\mathrm{mop}(n) - 3).$$

Using Lemma 7.4(2) and induction again, $\xi+3, \ldots, \frac{1}{2}(\mathrm{mop}(n)-3) \in \Xi_{i,i_0}$. Therefore

$$\Xi_{i,i_0} = [0, 1, 2, \ldots, \xi, \widehat{\xi + 1}, \xi + 2, \ldots, \frac{1}{2}(\mathrm{mop}(n) - 3)].$$

This finishes the proof. □

Corollary 8.5. *If* $\mathrm{mop}(n) \geq 11$*, then* Ξ_{i,i_0} *contains one of the following subsets*

$$[0, 1, 2, 3], \quad [0, 2, 3, 4], \quad [0, 1, 3, 4], \quad [0, 1, 2, 4].$$

Proof. The assertion follows from Lemma 8.4 and the fact $\frac{1}{2}(\mathrm{mop}(n) - 3) \geq 4$. □

Proposition 8.6. *If* $\mathrm{mop}(n) \geq 17$*, then Definition* 7.1(4) *holds automatically. As a consequence,* $\Omega_2(n) = Spl(n)$*.*

Proof. We start with the assumption that $2j + (i_0 - i) \notin \phi_2(n)$. Without loss of generality, we can assume that p_1 divides $2j + (i_0 - i)$. For simplicity, let $j_0 = i_0 - i$. So j_0 is not divisible by either p_1 or p_2. The assumption is that $2j + j_0$ is divisible by p_1. By Lemma 8.1 (replacing (i, i_0) by (j, j_0)), elements of the forms in Definition 7.1(4(ξ)i) and (4(ξ)ii) for $\xi = 0, 1, 2, 3, 4$ are not divisible by p_1 since the corresponding Det is not divisible by $p_1 \geq 17$. Further, any two distinct elements of the form in Definition 7.1(4(ξ)i) and (4(ξ)ii) can not be divided by p_2 either (using the fact $p_2 \geq 17$). This implies that there is only one ξ, say $\xi_0 \in [0,1,2,3,4]$, such that either (4(ξ_0)i) or (4(ξ_0)ii) fails. Removing ξ_0 from the list Ξ_{i,i_0}, we still have three integers $\{\xi_1,\xi_2,\xi_3\} \subseteq [0,1,2,3,4] \cap \Xi_{i,i_0}$ such that Definition 7.1(4(ξ_s)i) and (4(ξ_s)ii) hold for all $s = 1,2,3$. It remains to show that Definition 7.1(4(ξ_s)iii) holds for one of s. Suppose on the contrary that Definition 7.1(4(ξ_s)iii) fails for all three s. Then there are two s such that $j + i_0 + 2(\xi_s + 1)(i_0 - i)$ is divisible by the same prime factor, say p_2. Applying Lemma 8.1 to these two element with $(j', j_0) = (j + i_0, i_0 - i)$, we obtain that p_2 divides $|\operatorname{Det}| = 2|\xi_{s_1} - \xi_{s_2}| < \operatorname{mop}(n)$. This is impossible. Therefore Definition 7.1(4(ξ_s)iii) holds for one of s. Thus we show that Definition 7.1(4) holds automatically.

The consequence is clear. □

Theorem 8.7. *Suppose $n = p_1 p_2$ where p_s are prime ≥ 17. Then n is admissible. As a consequence A^G has a graded isolated singularity.*

Proof. Since every proper factor of n is admissible by Theorem 0.2, hypothesis of Corollary 7.8(1) holds. By Lemma 7.3(1), $\Omega_2(n) \neq \emptyset$. By Proposition 8.6, $Spl(n) \neq \emptyset$. Hence hypothesis of Corollary 7.8(2) holds. The assertion now follows from Corollary 7.8. □

9. Proof of Proposition 0.8

We start with $n = 6$ and 10.

Lemma 9.1. *Retain the notation as in Theorem 0.4. If $n = 6$, then $\mathsf{p}(A, C_n) = 5$.*

Proof. First let $\Phi := \Phi_6 = \{1,2,4,5\}$. By Lemma 6.2, $\operatorname{GKdim} E = \operatorname{GKdim} \overline{E}$. It suffices to show that $\operatorname{GKdim} \overline{E} = 1$. By Theorem 0.4, it is enough to show that $\operatorname{GKdim} \overline{E} \leq 1$. By Lemma 6.5(2),

$$\operatorname{GKdim} \overline{E} = \max_j \operatorname{GKdim}(\overline{A}/\overline{B}_j\overline{A})$$

where j ranges over $\{1,2,3\}$ (all positive integers less than 6 that divide 6).

Case 1: $j = 3$. Since $c_3 \in \overline{B}_3\overline{A}$, $c_3 = 0$ in $\overline{A}/\overline{B}_3\overline{A}$. By the definition of $\overline{A}$, $c_1 = c_2 = c_4 = c_5 = 0$. Therefore $c_i = 0$ for $i = 1,2,3,4,5$ in $\overline{A}/\overline{B}_3\overline{A}$. Therefore $\overline{A}/\overline{B}_3\overline{A}$ is a finitely generated module over $\Bbbk[c_0]$, which implies that $\operatorname{GKdim}(\overline{A}/\overline{B}_3\overline{A}) \leq 1$.

Case 2: $j = 2$. We need to show that $\operatorname{GKdim}(\overline{A}/\overline{B}_2\overline{A}) \leq 1$. By (E1.0.3), it is enough to show the claim that

$$\operatorname{GKdim}(\overline{A}/(\overline{B}_2\overline{A} + c_3 A)) = 0. \tag{E9.1.1}$$

Now we change Φ from $\{1,2,4,5\}$ to $\{1,2,3,4,5\}$. Re-cycle all notation such as $\overline{A}$, $\overline{B}_i$, etc, for the new Φ, claim (E9.1.1) becomes

$$\operatorname{GKdim}(\overline{A}/\overline{B}_2\overline{A}) = 0 \tag{E9.1.2}$$

with $\Phi = \{1,2,3,4,5\}$. For the rest of the proof in Case 2, let $\Phi = \{1,2,3,4,5\}$. Note that we have the following elements in $\overline{B}_2\overline{A}$:

$$b_2, \quad b_3 b_5, \quad b_1 b_3 b_4.$$

Taking commutators in $\overline{B}_2\overline{A}$, we have the following computations in $\overline{B}_2\overline{A}$:

$$\begin{aligned}[b_3b_5, b_1b_3b_4] &= b_3b_5b_1b_3b_4 + b_1b_3b_4b_3b_5 \\ &= b_3^2(b_5b_1b_4 - b_1b_4b_5) \\ &= \frac{1}{2}c_0^2(b_5b_1 + b_1b_5)b_4 \\ &= \frac{1}{2}c_0^3b_4, \\ [c_0^3b_4, b_2] &= c_0^4.\end{aligned}$$

Therefore $c_0^4 = 0$ in $\overline{A}/\overline{B}_2\overline{A}$, and consequently, $0 \in \overline{\Psi}_2$. By definition, $c_i = 0$ in $\overline{A}/\overline{B}_2\overline{A}$ for all $i = 1, 2, 3, 4, 5$. Therefore (E9.1.2) holds.

Case 3: $j = 1$. We need to show that $\operatorname{GKdim}(\overline{A}/\overline{B}_1\overline{A}) \leq 1$ for $\Phi = \{1, 2, 4, 5\}$. Similar to the proof of Case 2, it is sufficient to show

$$\operatorname{GKdim}(\overline{A}/\overline{B}_1\overline{A}) = 0$$

with new $\Phi = \{1, 2, 3, 4, 5\}$. But this is Proposition 6.6(1).

Combining these three cases, we finish the proof. □

Lemma 9.2. *Retain the notation as in Theorem 0.4. If $n = 10$, then $\mathsf{p}(A, C_n) = 9$.*

Proof. This proof is very similar to the proof of Lemma 9.1

First we let $\Phi := \Phi_{10} = \{1, 2, 3, 4, 6, 7, 8, 9\}$. By Lemma 6.2, $\operatorname{GKdim} E = \operatorname{GKdim} \overline{E}$. It suffices to show that $\operatorname{GKdim} \overline{E} = 1$. By Lemma 6.5(2),

$$\operatorname{GKdim} \overline{E} = \max_j \operatorname{GKdim}(\overline{A}/\overline{B}_j\overline{A})$$

where j ranges over $\{1, 2, 5\}$ (all positive integers less than 6 that divide 10).

Case 1: $j = 5$. The proof of Case 1 in Lemma 9.1 can be easily modified by replacing $j = 3$ to $j = 5$.

Case 2: $j = 2$. We need to show that $\operatorname{GKdim}(\overline{A}/\overline{B}_2\overline{A}) \leq 1$. By (E1.0.3), it is enough to show the claim that

$$\operatorname{GKdim}(\overline{A}/(\overline{B}_2\overline{A} + c_5A)) = 0. \tag{E9.2.1}$$

Now we change Φ from $\{1, 2, 3, 4, 6, 7, 8, 9\}$ to $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Recycle all notation such as $\overline{A}$, $\overline{B}_i$, etc, for the new Φ, claim (E9.2.1) becomes

$$\operatorname{GKdim}(\overline{A}/\overline{B}_2\overline{A}) = 0 \tag{E9.2.2}$$

with new $\Phi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For the rest of the proof in Case 2, we use this new Φ. Note that we have the following elements in $\overline{B}_2\overline{A}$:

$$b_2, \quad b_4b_8, \quad b_1b_5b_6, \quad b_5b_8b_9.$$

Taking commutators in $\overline{B}_2\overline{A}$, we have the following computations in $\overline{B}_2\overline{A}$:

$$\begin{aligned}[b_4b_8, b_2] &= c_0b_4, \\ [b_1b_5b_6, c_0b_4] &= c_0^2b_1b_5, \\ [b_5b_8b_9, c_0^2b_1b_5] &= -\frac{1}{2}c_0^4b_8, \\ [b_2, c_0^4b_8] &= c_0^5.\end{aligned}$$

Therefore $c_0^5 = 0$ in $\overline{A}/\overline{B}_2\overline{A}$, and consequently, $0 \in \overline{\Psi}_2$. By definition, $c_i = 0$ in $\overline{A}/\overline{B}_2\overline{A}$ for all $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$. Therefore (E9.2.2) holds.

Case 3: $j = 1$. The proof of Case 3 in Lemma 9.1 works.

Combining these three cases with Theorem 0.4, we finish the proof. $\square$

Next we consider $n = 9$.

Lemma 9.3. *Retain the notation as in Theorem 0.4. If $n = 9$, then $\mathsf{p}(A, C_n) = 8$.*

Proof. First we let $\Phi := \Phi_9 = \{1, 2, 4, 5, 7, 8\}$. By Lemma 6.2, $\operatorname{GKdim} E = \operatorname{GKdim} \overline{E}$. It suffices to show that $\operatorname{GKdim} \overline{E} = 1$. By Lemma 6.5(2),

$$\operatorname{GKdim} \overline{E} = \max_j \operatorname{GKdim}(\overline{A}/\overline{B}_j\overline{A})$$

where j ranges over $\{1, 3\}$. So we need to consider two cases.

Case 1: $j = 3$. Since $b_3, c_3 \in \overline{B}_3\overline{A}$, $c_6, c_3 \in \overline{B}_3\overline{A}$. This shows that $c_i = 0$ in $\overline{A}/\overline{B}_3\overline{A}$ for all $i = 1, 2, 3, 4, 5, 6, 7, 8$. So $\operatorname{GKdim}(\overline{A}/\overline{B}_3\overline{A}) \leq 1$.

Case 2: $j = 1$. We need to show that $\operatorname{GKdim}(\overline{A}/\overline{B}_1\overline{A}) \leq 1$. Note that we have the following elements in $\overline{B}_1\overline{A}$:

$$b_1, \quad c_6b_5b_8, \quad c_6^2b_7.$$

Taking commutators in $\overline{B}_1\overline{A}$, we have the following computations inside $\overline{B}_1\overline{A}$:

$$\begin{aligned}
[b_1, c_6b_5b_8] &= b_1c_6b_5b_8 - c_6b_5b_8b_1 \\
&= c_6[b_1b_5]b_8 - c_6b_5b_8b_1 \\
&= c_6[c_6b_8 - b_5b_1b_8] - c_6b_5b_8b_1 \\
&= c_6^2b_8 - c_0c_6b_5, \\
[c_6^2b_7, c_6^2b_8 - c_0c_6b_5] &= c_6^5 - c_0c_3c_6^3.
\end{aligned}$$

Similarly we have the following elements in $\overline{B}_1\overline{A}$:

$$c_3b_7, \quad c_3b_5b_8, \quad c_3^2b_4.$$

Taking commutators in $\overline{B}_1\overline{A}$, we have the following computations inside $\overline{B}_1\overline{A}$:

$$\begin{aligned}
[c_3b_7, c_3^2b_5b_8] &= c_3^3(b_7b_5b_8 - b_5b_8b_7) \\
&= c_3^3(c_3b_8 - b_5b_7b_8 - b_5b_8b_7) \\
&= c_3^4b_8 - c_3^3c_6b_5, \\
[c_3^2b_4, c_3^4b_8 - c_3^3c_6b_5] &= c_3^8 - c_3^6c_0c_6.
\end{aligned}$$

It is easy to see that the quotient algebra

$$D := \frac{\overline{A}}{(c_6^5 - c_0c_3c_6^3, c_3^8 - c_3^6c_0c_6)}$$

has GKdimension 1. Since $\overline{A}/\overline{B}_1\overline{A}$ is a quotient of D by the above computation. Therefore $\operatorname{GKdim} \overline{A}/\overline{B}_1\overline{A} \leq 1$ as desired.

Combining these two cases with Theorem 0.4, we finish the proof. $\square$

Now we are ready to prove Proposition 0.8.

Proof of Proposition 0.8. When $n = 6, 10, 9$, the p is $5, 9, 8$ by Lemmas 9.1, 9.2 and 9.3 respectively. For $n = 3, 5$, the assertion follows by Propositions 4.1 and 4.2 respectively. For $n = 2, 4, 7, 8, 11, 13, 14$, the assertion follows from Theorem 0.2. The statement for $n = 12$ follows by combining Theorems 0.4 and 0.7. $\square$

10. More examples of graded isolated singularities

To save space, we will omit some non-essential details in Sections 10 and 11.

In this section, we give more examples of graded isolated singularities. Some nice results of He-Y.H. Zhang [HZ] and Gaddis-Kirkman-Moore-Won [GKMW] will be reviewed and used in this section. First we recall some definitions from [HZ].

Let R be a noetherian algebra and G be a finite group acting on R. We say that two sequences $(a_1, \ldots, a_w)$ and $(b_1, \ldots, b_w)$ of elements of R are *pertinent under the G-action*, if

$$\sum_{i=1}^{w} a_i(g \cdot b_i) = 0$$

for all $1 \neq g \in G$. In this case we write $(a_1, \ldots, a_w) \sim (b_1, \ldots, b_w)$. The *radical* of the G-action on R is defined to be

$$\mathfrak{r}(R, G) := \left\{ \sum_{i=1}^{w} a_i b_i \in R \mid (a_1, \ldots, a_w) \sim (b_1, \ldots, b_w) \right\}.$$

By [HZ, Section 1], $\mathfrak{r}(R, G)$ is a 2-sided ideal of R.

Let e_0 be the element $1\#(\frac{1}{|G|}\sum_{g\in G} g)$ in $R\#G$. By the proof of [HZ, Proposition 2.4], $\mathfrak{r}(R, G) = R \cap (e_0)$. Therefore we have [HZ, (3.1.1)],

$$\mathsf{p}(R, G) = \operatorname{GKdim} R - \operatorname{GKdim} R/\mathfrak{r}(R, G).$$

If R is noetherian and Artin-Schelter regular, then R^G is a graded isolated singularity if and only if $R/\mathfrak{r}(R, G)$ is finite dimensional over the base field $\Bbbk$.

As said in introduction, almost all graded isolated singularities studied in this paper are non-conventional in the following sense.

Definition 10.1. Let R be a noetherian Artin-Schelter regular algebra with graded maximal ideal $\mathfrak{m} := A_{\geq 1}$. Let G be a finite subgroup of $\operatorname{Aut}_{gr}(R)$ such that R^G is a graded isolated singularity. We say the graded isolated singularity R^G is *non-conventional* if there is an element $1 \neq \sigma \in G$ such that at least one of the eigenvalues of σ restricted to the $\Bbbk$-vector space $\mathfrak{m}/\mathfrak{m}^2$ is 1. Otherwise, we say R^G is *conventional*.

If R is the commutative polynomial ring $\Bbbk[V]$, then every graded isolated singularity R^G is conventional, see [MU1, Corollary 3.11]. A similar statement holds for skew polynomial rings. Let $\{p_{ij} \mid 1 \leq i < j \leq n-1\}$ be a set of nonzero scalars in $\Bbbk^{\times}$. The skew polynomial ring $\Bbbk_{p_{ij}}[x_0, x_1, \ldots, x_{n-1}]$ is generated by $\{x_0, \ldots, x_{n-1}\}$, with $\deg x_i > 0$ for each i, and subject to the relations $x_j x_i = p_{ij} x_i x_j$ for all $i < j$. Let $V = \bigoplus_{i=0}^{n-1} \Bbbk x_i$.

Lemma 10.2. *Let R be a skew polynomial ring $\Bbbk_{p_{ij}}[x_0, x_1, \ldots, x_{n-1}]$ and let G be a finite group acting on R linearly and diagonally, namely, each x_i is an eigenvector of G. Then R^G is a graded isolated singularity if and only if the G-action on $V \setminus \{0\}$ is free.*

Proof. Let $d = |G|$.

$\Longleftarrow$: Assume that the G-action on $V \setminus \{0\}$ is free. In this setting, for each i, the G-action on $\Bbbk x_i \setminus \{0\}$ is also free. This implies that there is an $\sigma \in G$ and a $\xi \in \Bbbk$ being a primitive dth root of unity such that $\sigma(x_i) = \xi x_i$. As a consequence,

G is generated by σ and $\sigma^w(x_i) = \xi^w x_i$ for all $w \in \mathbb{Z}_d$. By [HZ, Lemma 3.4], $x_i^d \in \mathfrak{r}(R, G)$. Therefore $R/\mathfrak{r}(R, G)$ is finite dimensional. As a consequence, R^G is a graded isolated singularity.

$\Longrightarrow$: We prove the statement by contradiction and assume that the G-action on $V \setminus \{0\}$ is not free. Pick an element $1 \neq \sigma \in G$ so that σ has a fixed point in $V \setminus \{0\}$. This implies that σ fixes one x_i. Replacing G by the subgroup $\langle\sigma\rangle$, we can assume that $G = \langle\sigma\rangle$ following [GKMW, Theorem 3.4]. Since σ fixes x_i, one can show that x_i^N is not in $\mathfrak{r}(R, G)$ for all $N \geq 0$ (which also follows from Lemma 10.4(6) in an appropriate setting). Therefore $R/\mathfrak{r}(R, G)$ is not finite dimensional, whence R^G is not a graded isolated singularity. $\square$

As a consequence of [GKMW, Theorem 3.4], if R^G is a graded isolated singularity, then so is R^H for all subgroups $1 \subsetneq H \subseteq G$. The graded isolated singularities in the above lemma are all conventional. One nice example of non-conventional graded isolated singularities is given by Gaddis-Kirkman-Moore-Won [GKMW].

Example 10.3. [GKMW, Theorem 5.2] Let R be a generic 3-dimensional Sklyanin algebra $S(a, b, c)$ generated by $\{x, y, z\}$ with standard relations, see [GKMW, Introduction]. Let G be the cyclic group of order 3 acting on R by permuting the standard generators $\{x, y, z\}$. Then R^G is a graded isolated singularity by [GKMW, Theorem 5.2]. Since G has a fixed point $x + y + z$ in $R_1 \setminus \{0\}$, we obtain that R^G is non-conventional.

We will use a few more lemmas. In Lemma 10.4 below we do not assume that the G-actions is inner-faithful.

Lemma 10.4. *Let R and S be two connected graded algebra with G-action where G is a finite group. Let $e_0 = 1\#(\frac{1}{|G|}\sum_{g\in G} g)$. Suppose that $f : R \to S$ be a graded algebra homomorphism that is compatible with G-action.*

1. *There is an induced algebra homomorphism $f\#G : R\#G \to S\#G$ such that $f\#G(r\#g) = f(r)\#g$ for all $r \in R$ and $g \in G$.*
2. *$f\#G$ maps $e_0 \in R\#G$ to $e_0 \in S\#G$. As a consequence, there is an induced algebra homomorphism $\overline{f\#G} : R\#G/(e_0) \to S\#G/(e_0)$.*
3. *If $x \in R$ such that $x := x\#1 \in (e_0)$ in $R\#G$, then $f(x) := f(x)\#1 \in (e_0)$ in $S\#G$.*
4. *If f is surjective, so is $f\#G$. If, further, $S\#G/(e_0)$ is infinite dimensional, so is $R\#G/(e_0)$.*
5. *f maps $\mathfrak{r}(R, G)$ to $\mathfrak{r}(S, G)$. As a consequence, f induces an algebra homomorphism from $R/\mathfrak{r}(R, G)$ to $S/\mathfrak{r}(S, G)$.*
6. *Suppose f is surjective. If $S/\mathfrak{r}(S, G)$ is infinite dimensional, then so is $R/\mathfrak{r}(R, G)$.*

The proof of Lemma 10.4 is easy and omitted.

Lemma 10.5. *Let $A = \Bbbk_{-1}[\mathbf{x}]$ with $n \geq 2$.*

1. *Let p be a prime number such that $p \neq 3, 5$ and $p \leq n$. Then there is a group $G \subseteq \mathrm{Aut}_{gr}(A)$ of order p such that A^G is a non-conventional graded isolated singularity.*
2. *Let $p = 3^a 5^b$ for some $a, b \geq 0$. If G is a subgroup of $\mathrm{Aut}_{gr}(A)$ of order p such that A^G is a graded isolated singularity, then it is conventional.*

Proof. We omit the proof of part (2). For part (1), we show give a proof when $p = 2$.

We construct the group $G = \langle\sigma\rangle$ as follows. If n is even, let $\sigma \in \mathrm{Aut}_{gr}(A)$ be defined by

$$\sigma : x_i \to x_{n-1-i}$$

for all $i \in \mathbb{Z}_n$. If n is odd, let $\sigma \in \mathrm{Aut}_{gr}(A)$ be defined by

$$\sigma : x_i \to x_{n-1-i}, \quad \text{and} \quad x_{\frac{n-1}{2}} \to -x_{\frac{n-1}{2}}$$

for all $i \in \mathbb{Z}_n$ not equal to $\frac{n-1}{2}$. By [HZ, Example 1.6(ii)] and [HZ, Lemma 3.4], $x_i^2 \in \mathfrak{r}(A, G)$ for all i. (Some details are omitted.) Therefore $A/\mathfrak{r}(A, G)$ is finite dimensional and A^G is a graded isolated singularity. Since G preserves $x_0 + x_{n-1}$, it is non-conventional. □

The next lemma is due to Jason Bell. We thank him for sharing his result with us. We say an algebra B is PI if it satisfies a polynomial identity.

Lemma 10.6 (Jason Bell)**.** *Let B be a noetherian connected graded PI algebra generated in degree 1. If every linear combination of homogenous elements of odd degrees is nilpotent, then B is finite dimensional.*

Proof. Suppose on the contrary that B is infinite dimensional. Let W be the set of graded ideals I of B such that B/I is infinite dimensional. Since B is noetherian, there is a maximal element J in W. Replacing B by B/J, we may assume that every nonzero ideal of B has finite codimension. Since B is graded, every minimal prime of B is graded. As a consequence, the nilradical N of B is graded. Since B is noetherian, B is infinite dimensional if and only if B/N is infinite dimensional. This implies that $N = 0$. As a consequence, a product of minimal prime ideals is zero. This in turn implies that one of minimal prime is zero, or B is prime.

Since B is PI, there is a nonzero central element in B. We can further assume that this element, say z, is homogeneous and a nonzerodivisor (or regular element). By the last paragraph, $B/(z)$ is finite dimensional. Then $\mathrm{GKdim}\, B = 1$ by (E1.0.3).

By Small-Warfield's theorem [SW], the center $Z(B)$ of B is a finitely generated graded algebra of GKdimension one and B is a finite module over $Z(B)$. Note that every nonzero element in $Z(B)$ is regular. Hence $Z(B)$ is contained in the second Veronese subring of B since all odd degree elements are nilpotent.

Let $Q := Q_{gr}(B)$ be the graded quotient ring of B. By a graded version of Posner's theorem, this is just the result of inverting the homogeneous nonzero central elements, all of which have even degree. The important point here is that every element of odd degree in Q can be written in the form az^{-1} with a, z homogeneous and $a \in B$ of odd degree and $z \in Z(B)$ of even degree. Let T be the (ungraded) total quotient ring of B (or of Q). Then T can be embedded into a matrix algebra over a field F. With this embedding, we fix a trace map tr (the usual matrix trace). (With a bit more care one can even show that $T \cong M_n(F)$ where F is the fraction field of $Z(B)$.) In particular, $tr(1) \neq 0$.

As a general fact, since B is generated in degree 1, Q is strongly $\mathbb{Z}$-graded in the sense of [NvO, A.I.3]. Let $Q_{odd} := \bigoplus_{i \text{ is odd}} Q_i$ and $Q_{even} := \bigoplus_{i \text{ is even}} Q_i$. Then $Q = Q_{odd} \oplus Q_{even}$ is a strongly $\mathbb{Z}_2$-graded algebra, namely, $Q_{odd}^2 = Q_{even}$. By the last paragraph, every element u in Q_{odd} is of the form az^{-1} where $a \in B$ is a linear combination of homogeneous elements of odd degrees and where $z \in Z(B)$ is of even degree. Therefore u is nilpotent by hypothesis. Let u, v be any two elements in Q_{odd}. Then $u, v, u+v$ are all in Q_{odd}; and consequently, all nilpotent. By [MOR, Lemma 1], $tr(uv) = 0$. Since $Q_{odd}^2 = Q_{even}$, $tr(Q_{even}) = 0$. This contradicts $tr(1) \neq 0$. $\square$

Now we consider twisted tensor products. Let $\{B(i)\}_{i=1}^{w}$ be a family of connected graded algebras. Then the tensor product

$$\bigotimes{}^n B(i) := B(1) \otimes B(2) \otimes \cdots \otimes B(n)$$

is a connected graded and $\mathbb{Z}^{\oplus n}$-graded algebra. Let u_i denote the ith unit element $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{\oplus n}$ where 1 is in the ith position. Let $\{p_{ij} \in \Bbbk^{\times} \mid 1 \leq i < j \leq n\}$ be a set of nonzero scalar. Define f_{u_i} to be the $\mathbb{Z}^{\oplus n}$-graded algebra automorphism of $\bigotimes^n B(i)$ determined by

$$f_{u_i}(1^{\otimes(j-1)} \otimes x_j \otimes 1^{\otimes(n-j)}) = 1^{\otimes(j-1)} \otimes x_j \otimes 1^{\otimes(n-j)}$$

for all $i \geq j$ and $x_j \in B(j)$ and

$$f_{u_i}(1^{\otimes(j-1)} \otimes x_j \otimes 1^{\otimes(n-j)}) = p_{ij}^{-\deg x_j} 1^{\otimes(j-1)} \otimes x_j \otimes 1^{\otimes(n-j)}$$

for all $i < j$ and homogeneous elements $x_j \in B(j)$. Then

$$F := \{f_{u_1^{a_1} \cdots u_n^{a_n}} := f_{u_1}^{a_1} \cdots f_{u_n}^{a_n} \mid u_1^{a_1} \cdots u_n^{a_n} \in \mathbb{Z}^{\oplus n}\}$$

is an twisting system of $\bigotimes^n B(i)$ in the sense of [Zh1, Definition 2.1]. By [Zh1, Proposition and Definition 2.3], one can define a twisted algebra of $\bigotimes^n B(i)$ associated to the twisting system F. This twisted algebra is denoted by $\bigotimes_{\{p_{ij}\}}^n B(i)$. If $B(i) = \Bbbk[x]$ for all i, then $\bigotimes_{\{p_{ij}\}}^n B(i)$ is canonically isomorphic to skew polynomial ring $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$, see [Zh1, p.310]. Note that if $a = 1^{\otimes(i-1)} \otimes x_i \otimes 1^{\otimes(n-i)}$ and $b = 1^{\otimes(j-1)} \otimes x_j \otimes 1^{\otimes(n-j)}$ for two homogeneous elements $x_i \in B(i)$ and $x_j \in B(j)$ for $i < j$. Then one can check that

$$ba = p_{ij}^{\deg x_i \deg x_j}\ ab.$$

Suppose each $B(i)$ is a noetherian PI Artin-Schelter regular algebra (and it is possible that the "PI" hypothesis can be weakened). One can easily check that $\bigotimes_{\{p_{ij}\}}^n B(i)$ is noetherian and Artin-Schelter regular. Further, $\bigotimes_{\{p_{ij}\}}^n B(i)$ has *enough normal elements* in the sense of [Zh2, p.392]. By [Zh2, Theorem 1], it is Auslander regular and Cohen-Macaulay.

Suppose G is a finite group and $\phi_i : G \to \mathrm{Aut}_{gr}(B(i))$ is an injective map for each i. Then there is a unique extension of the G-action on $\bigotimes_{\{p_{ij}\}}^n B(i)$.

Proposition 10.7. *Retain the above notation. Suppose G is a finite group and $\phi_i : G \to \mathrm{Aut}_{gr}(B(i))$ is an injective map for each i. Let $B = \bigotimes_{\{p_{ij}\}}^n B(i)$.*

(1) *B^G is a graded isolated singularity if and only if each $B(i)^G$ is a graded isolated singularity.*
(2) *Assume B^G is a graded isolated singularity. Then B^G is conventional if and only if each $B(i)^G$ is conventional.*

Proof. The proof follows from Lemma 10.4(5,6). Details are omitted. $\square$

Proposition 10.7 provides a lot examples of graded isolated singularities.

Next let $B(i) = B$, for $i = 1, \ldots, n$, be a noetherian PI Artin-Schelter regular algebra generated in degree 1. Let $p_{ij} = -1$ for all $i < j$. We consider (-1)-twisted tensor product $\bigotimes_{\{-1\}}^n B$ and the permutation automorphism $\sigma \in \operatorname{Aut}_{gr}(\bigotimes_{\{-1\}}^n B)$ determined by

(E10.7.1)
$$\sigma : 1^{\otimes(j-1)} \otimes x_j \otimes 1^{\otimes(n-j)} \mapsto 1^{\otimes j} \otimes x_j \otimes 1^{\otimes(n-j-1)}, \quad 1^{\otimes(n-1)} \otimes x_n \mapsto x_n \otimes 1^{\otimes(n-1)}$$

for all $x_j, x_n \in B$.

Proposition 10.8. *Retain the above notation. Assume that $n \geq 2$ is admissible in the sense of Definition 5.2(2). Let B be any noetherian PI Artin-Schelter regular algebra generated in degree 1. Let G be the group $\langle\sigma\rangle$ where σ is defined in* (E10.7.1). *Then $(\bigotimes_{\{-1\}}^n B)^G$ is a non-conventional graded isolated singularity.*

Proof. Let $S = \bigotimes_{\{-1\}}^n B$. It suffices to show that $S/\mathfrak{r}(S,G)$ is finite dimensional.

Let $x \in B$ be a linear combination of homogeneous elements of odd degrees. Let $x_i = 1^{\otimes i} \otimes x \otimes 1^{\otimes(n-i-1)} \in S$, for $i = 0, \ldots, n-1$. Then the subalgebra generated by $\{x_0, x_1, \ldots, x_n\}$ is the (-1)-skew polynomial ring $R := \Bbbk_{-1}[\mathbf{x}]$. So the inclusion $f : R \to S$ is compatible with the G-action. (Note that f is not a graded algebra homomorphism.) Since n is admissible, the quotient $R/\mathfrak{r}(R,G)$ is finite dimensional. Hence, for each x_i, we have $x_i^N \in \mathfrak{r}(R,G)$ for some $N \geq 0$. By Lemma 10.4(5), $x_i^N \in \mathfrak{r}(S,G)$. This is true for all x that is a linear combination of homogeneous elements of odd degrees in B. By Lemma 10.6, the image of the map

$$B \to B \otimes \Bbbk^{\otimes(n-1)} \subset \bigotimes_{\{-1\}}^n B (= S) \to S/\mathfrak{r}(S,G)$$

is finite dimensional. Say this image is $\overline{B}$. By symmetry, $S/\mathfrak{r}(S,G)$ is a quotient ring of $\bigotimes_{\{-1\}}^n \overline{B}$, which is finite dimensional. Therefore $S/\mathfrak{r}(S,G)$ is finite dimensional as desired. □

Proposition 10.8 also provides a lot examples of graded isolated singularities by varying B.

11. Some questions and comments

It is quite reasonable to adapt Ueyama's definition of a graded isolated singularity [Ue, Definition 2.2], at least in the connected graded case. By Remark 0.3(2), the straightforward generalization of the freeness criterion for commutative quotient isolated singularities [MSt, Lemma 2.1] fails badly in the noncommutative case. However the freeness of the G-action on $V \setminus \{0\}$ is one of the easiest and most effective criterions for isolated singularities. Therefore we ask

Question 11.1. What is the analogue of the freeness criterion of isolated singularities in the (connected graded) noncommutative setting?

Let R be a noetherian Artin-Schelter regular algebra and let G be a finite subgroup of $\operatorname{Aut}_{gr}(R)$. By a result of Mori-Ueyama [MU1, Theorem 3.10] together with [HZ], the following are equivalent:

(1) R^G is a graded isolated singularity,
(2) $R/\mathfrak{r}(R,G)$ is finite dimensional,
(3) $R\#G/(e_0)$, where $e_0 = 1\#(\sum_{g\in G} g)$, is finite dimensional,
(4) $\mathsf{p}(R,G) = \operatorname{GKdim} R$.

Mori-Ueyama's criterion of graded isolated singularities is quite convenient. On the other hand, it could be very difficult to verify (2), or (3), or to calculate the exact value of $\mathsf{p}(R,G)$.

One of the key steps in the proof of Theorem 0.2 is to show that the set $Spl(n)$ is non-empty. But we can not prove that $Spl(n)\neq\emptyset$ is necessary. In particular, we do not have answers to the following questions.

Question 11.2. Let $n=p_1p_2$ for two distinct odd primes p_1,p_2.

(1) If $7\leq \mathrm{mop}(n)\leq 17$, is then n admissible?
(2) Is $Spl(77)\neq\emptyset$?
(3) If $Spl(77)=\emptyset$, is 77 admissible?

Hypersurface isolated singularities have been studied extensively, and form a rich topic in algebraic geometry [Mi]. The noncommutative version of a hypersurface was defined in [KKZ2, Definition 1.3(c)].

In the commutative theory, every hypersurface isolated singularity produces a finite dimensional Milnor algebra (as well as the Tjurina algebra). It would be interesting to develop a similar theory for the noncommutative hypersurface isolated singularities. At this point, it is not clear to us what is the best way of defining the noncommutative Jacobian ideal, since there are no canonically defined partial derivatives in the noncommutative case. Here we will like to propose a definition of the Milnor algebra when the hypersurface singularity is defined by "double twisted superpotentials".

Let V be a finite dimensional vector space $\bigoplus_{s=1}^{v}\Bbbk x_i$, or $\{x_s\}_{s=1}^{v}$ be a basis of V. Let F be the free algebra $\Bbbk\langle x_1,\ldots,x_v\rangle=\Bbbk\langle V\rangle$. Let σ denote an element in $\mathrm{GL}(V)$. We define two $\Bbbk$-linear maps from F to F. The first one is ϕ, which is determined by

$$\phi: x_{i_1}\otimes\cdots\otimes x_{i_{n-1}}\otimes x_{i_n}\mapsto x_{i_n}\otimes x_{i_1}\otimes\cdots\otimes x_{i_{n-1}}$$

for all x_{i_s} in the basis of V. The second one $\sigma\otimes 1$, where $\sigma\in\mathrm{GL}(V)$, is determined by

$$\sigma\otimes 1: x_{i_1}\otimes\cdots\otimes x_{i_{n-1}}\otimes x_{i_n}\mapsto \sigma(x_{i_1})\otimes\cdots\otimes x_{i_{n-1}}\otimes x_{i_n}.$$

Following [DV, Definition 1], [BSW, p.1502], [Ka, Definitions 2.1.3 and 2.1.4], [MSm, Definition 2.5] (and taking the quiver with one vertex and v arrows), a *twisted superpotential* in the free algebra F is an element w in F such that

$$w=(\sigma\otimes 1)\phi(w)$$

for some $\sigma\in\mathrm{GL}(V)$. (All papers [DV, BSW, Ka, MSm] use slightly different notation, but one can easily figure out the discrepancies). For every x_i, we define a partial derivation ∂_i as follows

$$\partial_i(x_{i_1}\otimes x_{i_2}\otimes\cdots\otimes x_{i_w})=\begin{cases} x_{i_2}\otimes\cdots\otimes x_{i_w} & i_1=i\\ 0 & i\neq i_1.\end{cases}$$

(This definition of a partial derivative is slightly different from the ordinary partial derivative in calculus. Another possibility is the cyclic, or circular, derivative.) For every w, let $\partial(w)$ be the $\Bbbk$-linear span of $\{\partial_i(w)\}_{i=1}^{v}$. For an integer N, one can define $\partial^N(w)$ inductively by $\partial^N(w)=\partial(\partial^{N-1}(w))$. Given a twisted superpotential w and an integer N, one can define *superpotential algebra* $\mathcal{D}(w,N)$ [Ka, Definition

2.1.6] (which is the same as the derivation-quotient algebra in the sense of [DV, BSW, MSm]) to be

$$\mathcal{D}(w, N) := F/(\partial^N(w)).$$

Dubois-Violette proved a very nice result [DV, Theorem 11]: a Koszul (or higher Koszul) algebra is twisted Calabi-Yau if and only if it is isomorphic to a superpotential algebra for a unique-up-to-scalar-multiples twisted superpotential w.

Definition 11.3. Retain the above notation.

(1) A pair of elements (w_1, w_2) in F are called *double twisted superpotentials* if
   (a) w_1 is a twisted superpotential (with an automorphism $\sigma_1 \in \mathrm{GL}(V)$) such that the superpotential algebra $D := \mathcal{D}(w_1, N)$ is a noetherian Artin-Schelter regular algebra.
   (b) w_2 is a twisted superpotential (with an automorphism $\sigma_2 \in \mathrm{GL}(V)$) such that w_2 is a normal regular element in D.

   Let (w_1, w_2) be double twisted superpotentials in parts (2,3,4).
(2) The algebra $D/(w_2)$ is called the hypersurface singularity associated to (w_1, w_2), and is denoted by $T(w_1, w_2)$.
(3) The *Milnor algebra* associated to (w_1, w_2) is defined to be

$$\mathcal{M}(w_1, w_2) := D/(\partial(w_2)).$$

(4) The *Milnor number* associated to (w_1, w_2) is defined to be

$$m(w_1, w_2) := \dim_{\Bbbk} \mathcal{M}(w_1, w_2).$$

With these definitions, we can ask the following:

Question 11.4. Is $T(w_1, w_2)$ being a graded isolated singularity equivalent to $m(w_1, w_2)$ being finite?

The following example of a hypersurface isolated singularity is non-conventional such that Question 11.4 has an affirmative answer.

Example 11.5. Let $A = \Bbbk_{-1}[x_0, x_1]$ and G be the group of automorphism of A generated by f, where f is determined by

$$f : x_0 \mapsto x_1, \quad x_1 \mapsto x_0.$$

By [KKZ1, Example 3.1], A^G is a hypersurface singularity, which can be written as

$$A^G = D/(w_2)$$

where D is an Artin-Schelter regular algebra of global dimension three and w_2 is a normal element of degree 6 in D. In details, $x = x_0 + x_1$ and $y = x_0^3 + x_1^3$,

$$D = \Bbbk\langle x, y\rangle/(x^2y - yx^2, xy^2 - y^2x)$$

and

$$w_2 = 2x^6 - \frac{3}{2}(x^3y + x^2yx + xyx^2 + yx^3) + 4y^2.$$

By Theorem 0.2, A^G has a non-conventional graded isolated singularity.

Note that D is (-1)-twisted Calabi-Yau [RRZ, Example 1.6]. There is a twisted superpotential

$$w_1 = xy^2x + yx^2y - y^2x^2 - x^2y^2$$

with automorphism σ determined by

$$\sigma : x \mapsto -x, y \mapsto -y,$$

and D is the superpotential algebra associated to w_1. It is easy to check that

(1) w_2 is a regular normal element in D,
(2) w_2 is a superpotential.

The Milnor ring of the hypersurface singularity A^G is

$$D/(\partial w_2) = D/(12x^5 - \frac{3}{2}(x^2y + xyx + yx^2), -\frac{3}{2}x^3 + 4y),$$

which is isomorphic to $\Bbbk[x]/(x^5)$ by an easy calculation. As a consequence, the Milnor number of A^G is 5.

Note that the McKay quiver corresponding to (A, G) is of type $\widetilde{L}_1$, see [CKWZ1, Proposition 7.1 and pp. 249-250]. This is slightly different from the classical $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E}$ types.

Remark 11.6. Some other noncommutative hypersurface graded isolated singularities are given in [CKWZ2, Theorem 5.2] and [CKWZ2, Table 3 in p.537]. These are related to noncommutative McKay correspondence in dimension two. It would be interesting to answer Question 11.4 for these hypersurface singularities.

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