# EXTENDED CONGRUENCES FOR HARMONIC NUMBERS 

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#### Abstract

We derive $p$-adic expansions for the generalized Harmonic numbers $H_{p-1}^{(j)}$ and $H_{\frac{p-1}{2}}^{(j)}$ involving the Bernoulli numbers $B_{j}$ and the the base-2 Fermat quotient $q_{p}$. While most of our results are not new, we obtain them elementarily, without resorting to the theory of $p$-adic L-functions as was the case previously. Moreover, we show that


$$
\sum_{j=0}^{n-1}\left(\frac{\left(2^{j+1}-1\right)}{(j+1)} \frac{\left(2^{j+2}-1\right)}{(j+2)} \frac{B_{j+2}}{2^{j}} H_{\frac{p-1}{2}}^{(j+1)}+2(-1)^{j} \frac{q_{p}^{j+1}}{j+1}\right) p^{j} \equiv 0 \quad\left(\bmod p^{n}\right)
$$

holds under the condition that $p>\frac{n+1}{2}$. This is another generalization, modulo any prime power, of the old $p$-congruence $H_{\frac{p-1}{2}}+2 q_{p} \equiv 0 \bmod p$ attributed to Eisenstein, which is stronger than the one which has been published recently [6].

## 1. Introduction

It is well-known that the Wolstenholme theorem can be stated as a congruence for the Harmonic number $H_{p-1}$ where $p \geq 5$ is a prime number:

$$
\begin{equation*}
H_{p-1}:=\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \quad\left(\bmod p^{2}\right) \tag{1.1}
\end{equation*}
$$

There also exists ([7], Lemma 14) a refinement of (1.1) valid for $p \geq 7$ :

$$
\begin{equation*}
H_{p-1}+\frac{p}{2} H_{p-1}^{(2)} \equiv 0 \quad\left(\bmod p^{4}\right) \tag{1.2}
\end{equation*}
$$

where $H_{n}^{(m)}:=\sum_{j=1}^{n} \frac{1}{j^{m}}$ is the generalized Harmonic number, $\left(H_{n}^{(1)}=H_{n}\right)$.
Another well-known congruence, due to Eisenstein [2], reads

$$
\begin{equation*}
H_{\frac{p-1}{2}}+2 \cdot q_{p} \equiv 0 \quad(\bmod p) \tag{1.3}
\end{equation*}
$$

where $p \geq 3$ is a prime number and $q_{p}=\frac{2^{p-1}-1}{p}$ is the base- 2 Fermat quotient. It happens that this congruence may also be refined [5] in the following way:

$$
\begin{equation*}
H_{\frac{p-1}{2}}+2 \cdot q_{p}-p \cdot q_{p}^{2} \equiv 0 \quad\left(\bmod p^{2}\right) \tag{1.4}
\end{equation*}
$$

Generalizations of these congruences modulo $p^{n}$ for arbitrary large $n$ can be obtained from the theory of $p$-adic L-function (11, 6]. In the present paper, we will show how such generalizations are also obtained more elementarily and we will provide sufficient conditions on $p$ ensuring their validity for even higher powers of $p$. Notations and useful Lemmas are presented in Section 2. Section 3 is devoted to the cases of $H_{p-1}^{(j)}$ and $H_{\frac{p-1}{2}}^{(2 j)}$. Section 4 deals with the more difficult case of $H_{\frac{p-1}{2}}^{(j)}$ for odd $j$.

## 2. Notations and useful preliminaries

In this paper, $g, h, i, j, k, n, m$ are non-negative integers, $p$ is a prime number and $x$ the argument of a generating function. In addition to the notations already given in the introduction for the Harmonic numbers and the base-2 Fermat quotient, we let $\delta_{i}^{j}=1$ when $i=j$ and $\delta_{i}^{j}=0$ otherwise, we let $\binom{n}{k}$ be the usual binomial coefficient, $B_{n}$ a Bernouilli number $\left(B_{0}=1, B_{1}=-1, B_{2}=\frac{1}{6}, ..\right)$, and $\mathbf{i}^{2}=-1$. The following classical results will be used and are stated without proof.
(i) We have the generalized Binomial Theorem:

$$
\begin{equation*}
\frac{1}{(1-x)^{k}}=\sum_{j \geq 0}\binom{j+k-1}{j} x^{j} \tag{2.1}
\end{equation*}
$$

(ii) Let $n \geq 1$, and $1 \leq j \leq p-1$, we have the Euler Theorem:

$$
\begin{equation*}
j^{p^{n-1}(p-1)} \equiv 1 \quad\left(\bmod p^{n}\right) \tag{2.2}
\end{equation*}
$$

(iii) We will use the Legendre formula which states that the highest power of $p$ which divides $j$ ! is $\frac{j-s_{p}(j)}{p-1}$, where $s_{p}(j)$ is the sum of the base- $p$ digits of $j$.
(iv) We shall also make use of some classical properties of the Bernoulli numbers. They have the following exponential generating function:

$$
\begin{equation*}
\frac{x}{e^{x}-1}=\sum_{n \geq 0} B_{n} \frac{x^{n}}{n!} \tag{2.3}
\end{equation*}
$$

and they obey the following recurrence relation:

$$
\begin{equation*}
\sum_{0 \leq k \leq n} B_{k}\binom{n}{k}=(-1)^{n} B_{n} \tag{2.4}
\end{equation*}
$$

They vanish at odd indices, larger than 1 , so that

$$
\begin{equation*}
B_{2 n+1}=0 \quad \text { when } n \geq 1 \tag{2.5}
\end{equation*}
$$

and we have the Faulhaber formula for the sum of consecutive $i$-th powers:

$$
\begin{equation*}
\sum_{j=1}^{n} j^{i}=\frac{1}{i+1} \sum_{h=0}^{i}(-1)^{h}\binom{i+1}{h} B_{h} n^{i+1-h} . \tag{2.6}
\end{equation*}
$$

We will also need the Kummer congruence: let $h, k$ not divisible by $p-1$, such that $h \equiv k(\bmod p-1)$, we have

$$
\begin{equation*}
\frac{B_{h}}{h}-\frac{B_{k}}{k} \equiv 0 \quad(\bmod p) \tag{2.7}
\end{equation*}
$$

We will make use of the Von Staudt-Clausen theorem: let $D\left(B_{2 j}\right)$ be the denominator of $B_{2 j}$ in the reduced form, we have

$$
\begin{equation*}
D\left(B_{2 j}\right)=\prod_{p-1 \mid 2 j} p \tag{2.8}
\end{equation*}
$$

Finally, recall that $(p, 2 k)$ is an irregular pair when $p \geq 2 k+3$ and $B_{2 k} \equiv 0(\bmod p)$.
In addition to the above classical results stated without proof, we shall make use of some lemmas, given hereafter with their proofs for the sake of self-containment.

Lemma 2.1. Let $k$ be a positive natural integer, we have the following identities:

$$
\begin{align*}
& \sum_{j=1}^{k}\binom{2 k-1}{2 j-1} B_{2 j}=\frac{1}{2}+B_{2 k}+B_{2 k-1}  \tag{2.9}\\
& \sum_{j=1}^{k}\binom{2 k}{2 j-1} B_{2 j}=\frac{1}{2}-B_{2 k}  \tag{2.10}\\
& \sum_{j=1}^{k}\binom{2 k+1}{2 j-1} B_{2 j}=\frac{1}{2}  \tag{2.11}\\
& \sum_{j=1}^{k}\binom{2 k+2}{2 j-1} B_{2 j}=\frac{1}{2}-(2 k+3) B_{2 k+2} \tag{2.12}
\end{align*}
$$

Proof. The proofs only make use of (2.4). We have

$$
\begin{aligned}
& \sum_{j=1}^{k}\binom{2 k-1}{2 j-1} B_{2 j}=\sum_{j=1}^{k}\binom{2 k}{2 j} B_{2 j}-\sum_{j=1}^{k}\binom{2 k-1}{2 j} B_{2 j} \\
&= \sum_{j=0}^{2 k}\binom{2 k}{j} B_{j}-1+k-\sum_{j=0}^{2 k-1}\binom{2 k-1}{j} B_{j}+1-\frac{1}{2}(2 k-1) \\
&= B_{2 k}+B_{2 k-1}+\frac{1}{2}, \\
& \begin{aligned}
\sum_{j=1}^{k}\binom{2 k}{2 j-1} B_{2 j} & =\sum_{j=1}^{k}\binom{2 k+1}{2 j} B_{2 j}-\sum_{j=1}^{k}\binom{2 k}{2 j} B_{2 j} \\
= & \sum_{j=0}^{2 k+1}\binom{2 k+1}{j} B_{j}-1+\frac{2 k+1}{2}-B_{2 k+1} \\
& -\sum_{j=0}^{2 k}\binom{2 k}{j} B_{j}+1-(2 k) \frac{1}{2}=\frac{1}{2}-B_{2 k}, \\
\sum_{j=1}^{k}\binom{2 k+1}{2 j-1} B_{2 j}= & \sum_{j=1}^{k+1}\binom{2 k+1}{2 j-1} B_{2 j}-B_{2 k+2} \\
= & B_{2 k+2}+B_{2 k+1}+\frac{1}{2}-B_{2 k+2}=\frac{1}{2}, \\
\sum_{j=1}^{k}\binom{2 k+2}{2 j-1} B_{2 j} & =\sum_{j=1}^{k+1}\binom{2 k+2}{2 j-1} B_{2 j}-(2 k+2) B_{2 k+2} \\
= & \frac{1}{2}-B_{2 k+2}-(2 k+2) B_{2 k+2}=\frac{1}{2}-(2 k+3) B_{2 k+2} .
\end{aligned}
\end{aligned}
$$

Lemma 2.2. Let $k \geq 0$ be an integer, it holds that

$$
\begin{align*}
\sum_{0 \leq j} B_{j}\left(2^{j}-1\right)\binom{k}{j} & =(-1)^{k} B_{k}\left(1-2^{k}\right)  \tag{2.13}\\
\sum_{0 \leq j} B_{j} 2^{j}\binom{k}{j} & =2 B_{k}\left(1-2^{k-1}\right) \tag{2.14}
\end{align*}
$$

Proof. Via the generating function (2.3), we have

$$
\sum_{0 \leq k} B_{k}\left(2^{k}-1\right) \frac{x^{k}}{k!}=\sum_{0 \leq k} B_{k} \frac{(2 x)^{k}}{k!}-\sum_{0 \leq k} B_{k} \frac{x^{k}}{k!}=\frac{2 x}{e^{2 x}-1}-\frac{x}{e^{x}-1}=\frac{-x}{e^{x}+1}
$$

Recall 12 that when $f(x)$ is the exponential generating function of the sequence $a_{k}$, then $e^{x} f(x)$ is the exponential generating function of the sequence $b_{k}=\sum_{0 \leq j}\binom{k}{j} a_{j}$. Then

$$
\sum_{0 \leq k} \sum_{0 \leq j}\binom{k}{j} B_{j}\left(2^{j}-1\right) \frac{x^{k}}{k!}=e^{x} \frac{-x}{e^{x}+1}=\frac{-x}{e^{-x}+1}
$$

and, on the other hand, from the the first line, we also have

$$
-\sum_{0 \leq k} B_{k}\left(2^{k}-1\right) \frac{(-x)^{k}}{k!}=\frac{-x}{e^{-x}+1}
$$

and (2.13) follows from the identification of the coefficients in both series expansions for $\frac{-x}{e^{-x}+1}$. For the proof of (2.14), it is easy to directly check its validity for $k=0$ and $k=1$, and for $k>1$ it follows from (2.13) and (2.5).

Lemma 2.3. Let $k \geq 1$ be an integer, we have

$$
\begin{equation*}
\sum_{j=0}^{2 k-1}\binom{2 k-1}{j}\left(2^{j}-1\right)\left(2^{j+1}-1\right) \frac{B_{j+1}}{j+1}=\left(2^{2 k}-1\right) \frac{B_{2 k}}{2 k} \tag{2.15}
\end{equation*}
$$

Proof. We start from

$$
\begin{aligned}
\frac{x}{2} \tan \frac{x}{2} & =\frac{\mathbf{i} x}{2} \frac{e^{-\frac{\mathbf{i} x}{2}}-e^{\frac{\mathbf{i} x}{2}}}{e^{-\frac{\mathbf{i} x}{2}}+e^{\frac{\mathbf{i} x}{2}}}=\frac{\mathbf{i} x}{2} \frac{1-e^{\mathbf{i} x}}{1+e^{\mathbf{i} x}}=\frac{\mathbf{i} x}{e^{\mathbf{i} x}+1}-\frac{1}{2} \mathbf{i} x \\
& =-\sum_{0 \leq k} B_{k}\left(2^{k}-1\right) \frac{(\mathbf{i} x)^{k}}{k!}-\frac{1}{2} \mathbf{i} x \quad \text { from the proof of Lemma } 2.2 \\
& =-\sum_{2 \leq k} B_{k}\left(2^{k}-1\right) \frac{(\mathbf{i} x)^{k}}{k!} \\
& =-\sum_{1 \leq k} B_{2 k}\left(2^{2 k}-1\right) \frac{(\mathbf{i} x)^{2 k}}{(2 k)!} \quad \text { since } B_{k+1}=0 \text { for even } k \geq 2 \text { and then } \\
\tan \frac{x}{2} & =2 \sum_{1 \leq k}(-1)^{k-1}\left(2^{2 k}-1\right) \frac{B_{2 k}}{2 k} \frac{x^{2 k-1}}{(2 k-1)!}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\tan x-\tan \frac{x}{2} & =2 \sum_{1 \leq k} \mathbf{i}^{k-1}\left(2^{k}-1\right)\left(2^{k+1}-1\right) \frac{B_{k+1}}{k+1} \frac{x^{k}}{k!}, \\
\mathbf{i} \frac{\tan x-\tan \frac{x}{2}}{2} & =\sum_{0 \leq k}\left(2^{k}-1\right)\left(2^{k+1}-1\right) \frac{B_{k+1}}{k+1} \frac{(\mathbf{i} x)^{k}}{k!}, \\
\mathbf{i} e^{\mathbf{i} x \frac{\tan x-\tan \frac{x}{2}}{2}} & =\sum_{0 \leq k} \sum_{0 \leq j}\binom{k}{j}\left(2^{j}-1\right)\left(2^{j+1}-1\right) \frac{B_{j+1}}{j+1} \frac{(\mathbf{i} x)^{k}}{k!} .
\end{aligned}
$$

We take the imaginary part, so that

$$
\cos x \frac{\tan \frac{x}{2}-\tan x}{2}=\sum_{1 \leq k} \sum_{0 \leq j}\binom{2 k-1}{j}\left(2^{j}-1\right)\left(2^{j+1}-1\right) \frac{B_{j+1}}{j+1}(-1)^{k} \frac{x^{2 k-1}}{(2 k-1)!}
$$

which is

$$
\tan \frac{x}{2}=2 \sum_{1 \leq k} \sum_{0 \leq j}\binom{2 k-1}{j}\left(2^{j}-1\right)\left(2^{j+1}-1\right) \frac{B_{j+1}}{j+1}(-1)^{k-1} \frac{x^{2 k-1}}{(2 k-1)!}
$$

The desired result is obtained by comparison of the coefficients in both series expansions for $\tan \frac{x}{2}$ which have been obtained in the above derivations.

Lemma 2.4. Let $p$ be prime and $n \geq 1, i, j \geq 0$ integers, we have

$$
j!\binom{p^{n-1}(p-1)-i}{j} \equiv(-1)^{j} j!\binom{i+j-1}{j} \quad\left(\bmod p^{n-1}\right)
$$

Proof. We write the factors of the binomial coefficient on the left hand side and reduce modulo $p^{n-1}$, so that

$$
\begin{aligned}
j!\binom{p^{n-1}(p-1)-i}{j} & =\left(p^{n-1}(p-1)-i\right) \cdots\left(p^{n-1}(p-1)-i-j+1\right) \\
& \equiv(-1)^{j} j!\binom{i+j-1}{j} \quad\left(\bmod p^{n-1}\right) .
\end{aligned}
$$

Lemma 2.5. Let $p$ be prime, $j \geq 1$ an integer and $v_{p}(k)$ the higest power of $p$ which divides the integer $k$, we have

$$
v_{p}\left(\frac{p^{j-1}}{j!}\right) \geq(j-1) \frac{p-2}{p-1}
$$

In particular, $p$ divides $\frac{p^{j-1}}{j!}$ when $p$ is odd and $j \geq 2$.
Proof. We reproduce the proof from [4]. We have $v_{p}\left(\frac{p^{j-1}}{j!}\right)=j-1-v_{p}(j!)=j-$ $1-\frac{j-s_{p}(j)}{p-1}$, by Legendre formula. But clearly, $s_{p}(j) \geq 1$, and the claim follows.

Lemma 2.6. Let $p$ be an odd prime and $n \geq 1, h \geq 0$ integers. We have the two following congruences:

$$
\begin{gather*}
p B_{p^{n-1}(p-1)} \equiv p-1 \quad\left(\bmod p^{n}\right)  \tag{2.16}\\
p B_{p^{n-1}(p-1)-2 h} \equiv H_{p-1}^{(2 h)} \quad(\bmod p) \tag{2.17}
\end{gather*}
$$

Proof. We make use of the Faulhaber formula (2.6) with $i=p^{n-1}(p-1)-2 h=2 m$ an even exponent. After some manipulation, accounting for the vanishing of the odd-indexed Bernoulli numbers from $B_{3}$, we have

$$
\sum_{j=1}^{p} j^{2 m}=p B_{2 m}+2 m \sum_{g=1}^{m} \frac{p B_{2 m-2 g}\binom{2 m-1}{2 g-1}}{2 g(2 g+1)} p^{2 g}+\frac{1}{2} p^{2 m} .
$$

From Euler theorem, $\sum_{j=1}^{p} j^{2 m} \equiv H_{p-1}^{(2 h)} \bmod p^{n}$, then

$$
p B_{2 m} \equiv H_{p-1}^{(2 h)}-\left(p^{n-1}(p-1)-2 h\right) \sum_{g=1}^{m} \frac{p B_{2 m-2 g}\binom{2 m-1}{2 g-1}}{2 g(2 g+1)} p^{2 g}+\frac{1}{2} p^{2 m} \quad\left(\bmod p^{n}\right)
$$

But, when $n \geq 2 h, 2 m=p^{n-1}(p-1)-2 h \geq p^{n-1}(p-1)-n \geq n$ as soon as $p \geq 3$. Also $p B_{2 m-2 g}\binom{2 m-1}{2 g-1}$ is $p$-integral and, for $g \geq 1, \frac{p^{2 g}}{2 g(2 g+1)}$ is divisible by $p$, by Lemma 2.5. Then

$$
p B_{p^{n-1}(p-1)-2 h} \equiv H_{p-1}^{(2 h)}+2 h \sum_{g=1}^{m} \frac{p B_{2 m-2 g}\binom{2 m-1}{2 g-1}}{2 g(2 g+1)} p^{2 g} \quad\left(\bmod p^{n}\right)
$$

When $h=0$, this readily establishes (2.16), and when $h \neq 0$, we have (2.17).
Lemma 2.7. Let $n \geq 1$ be an integer and $p$ an odd prime such that $p>\frac{n+1}{2}$. We have the following congruence:

$$
\begin{equation*}
\frac{2^{p^{n-1}(p-1)}-1}{p^{n}} \equiv \sum_{j=0}^{n-1}(-1)^{j} \frac{q_{p}^{j+1}}{j+1} p^{j}+\delta_{p}^{n+1} q_{p} p^{n-1} \quad\left(\bmod p^{n}\right) \tag{2.18}
\end{equation*}
$$

Proof. This is mainly taken from Lemma 2.8 in 10, but our condition on $p$ is less restrictive than in [10]. We start with

$$
\begin{aligned}
\frac{2^{p^{n-1}(p-1)}-1}{p^{n}} & =\frac{\left(1+p \cdot q_{p}\right)^{p^{n-1}}-1}{p^{n}}=\frac{1}{p^{n}} \sum_{j=1}^{p^{n-1}}\binom{p^{n-1}}{j} p^{j} q_{p}^{j} \\
& =q_{p}+\sum_{j=2}^{p^{n-1}}\left(p^{n-1}-1\right) \cdots\left(p^{n-1}-j+1\right) \frac{p^{j-1}}{j!} q_{p}^{j}
\end{aligned}
$$

By Lemma 2.5, we have $\frac{p^{j-1}}{j!} \equiv 0(\bmod p)$, when $j \geq 2$, then

$$
\begin{aligned}
\frac{2^{p^{n-1}(p-1)}-1}{p^{n}} & \equiv q_{p}+\sum_{j=2}^{p^{n-1}}(-1) \cdots(-j+1) \frac{p^{j-1}}{j!} q_{p}^{j} \quad\left(\bmod p^{n}\right) \\
& \equiv \sum_{j=1}^{p^{n-1}}(-1)^{j-1} \frac{p^{j-1}}{j} q_{p}^{j} \\
& \equiv \sum_{j=0}^{n-1}(-1)^{j} \frac{p^{j}}{j+1} q_{p}^{j+1}+p^{n} \sum_{j=n+1}^{p^{n-1}}(-1)^{j-1} \frac{p^{j-(n+1)}}{j} q_{p}^{j} \\
& \equiv \sum_{j=0}^{n-1}(-1)^{j} \frac{p^{j}}{j+1} q_{p}^{j+1}+p^{n} \sum_{\substack{j=n+1 \\
j \equiv 0 \bmod p}}^{p^{n-1}}(-1)^{j-1} \frac{p^{j-(n+1)}}{j} q_{p}^{j} \quad\left(\bmod p^{n}\right)
\end{aligned}
$$

To finish the proof, we consider now the summands in the second term of the latter congruence and we show that they are all $p$-integral, except when $j=n+1=p$. Since $j \equiv 0 \bmod p$, let $m>0$ be the highest integer such that $j=k \cdot p^{m}$. We have $j-(n+1)-m=k \cdot p^{m}-(n+1)-m \geq p^{m}-(n+1)-m$. Suppose first that $m \geq 2$. Then we have $p^{m}-m-(n+1) \geq 2 p-1-(n+1) \geq 0$ since $p^{m}-m \geq 2 p-1$ when $m \geq 2, p \geq 3$, and since $p \geq \frac{n+2}{2}$ by hypothesis. Then the corresponding summands in second term on the right hand side in the latter congruence are $p$-integral. Suppose now that $m=1$. Then, we have

$$
\begin{aligned}
j-(n+1)-m=k \cdot p-(n+1)-1 & =(k-1) p+p-(n+1)-1 \\
& \geq(k-1) \frac{n+2}{2}-\frac{n+2}{2} \geq 0, \text { if } k \geq 2
\end{aligned}
$$

Then, when $m=1, k \geq 2$, the corresponding summands in second term on the right hand side in the latter congruence are also $p$-integral. There remains the case $m=k=1$. Then, $j=p$ and $j-(n+1)-m=j-(n+2) \geq 0$ if $j \geq n+2$, and then the only case which is left is $j=n+1=p$. Finally, the last term on the right hand side of (2.18) is justified, since $q_{p}^{p} \equiv q_{p} \bmod p$.

## 3. Extended congruences for $H_{p-1}^{(j)}$ and $H_{\frac{p-1}{2}}^{(2 j)}(j \geq 1)$

We present now a first kind of $p$-adic expansions for $H_{p-1}^{(j)}$ and $H_{\frac{p-1}{2}}^{(2 j)}(j \geq 1)$ in terms of higher order Harmonic numbers and a second one, with higher order Harmonic numbers as well, but also involving Bernoulli numbers. These results are not new, except possibly for Theorem 3.4. Hereafter, they are obtained with elementary
algebric and arithmetic arguments and with the classical Falhauber formula (2.6) for sums of consecutive powers in terms of Bernoulli numbers.

Theorem 3.1. Let $k \geq 1$ an integer and $p$ a prime number, the generalized Harmonic numbers $H_{p-1}^{(k)}$ is expanded in the following $p$-adically converging sum of powers of $p$ :

$$
\begin{equation*}
H_{p-1}^{(k)}=(-1)^{k} \sum_{j \geq 0}\binom{j+k-1}{j} H_{p-1}^{(k+j)} p^{j} \tag{3.1}
\end{equation*}
$$

Moreover, when $p$ is odd, we have

$$
\begin{align*}
H_{p-1}^{(2 k)} & =2 H_{\frac{p-1}{2}}^{(2 k)}+\sum_{j \geq 1}\binom{j+2 k-1}{j} H_{\frac{p-1}{2}}^{(2 k+j)} p^{j},  \tag{3.2}\\
H_{p-1}^{(k)} & =\frac{1+(-1)^{k}}{2^{k}} H_{\frac{p-1}{2}}^{(k)}+(-1)^{k} \sum_{j \geq 1}\binom{j+k-1}{j} \frac{1}{2^{j+k}} H_{\frac{p-1}{2}}^{(k+j)} p^{j},  \tag{3.3}\\
2\left(2^{2 k}-1\right) H_{\frac{p-1}{2}}^{(2 k)} & =-\sum_{j \geq 1}\binom{j+2 k-1}{j} \frac{2^{2 k+j}-1}{2^{j}} H_{\frac{p-1}{2}}^{(2 k+j)} p^{j} \tag{3.4}
\end{align*}
$$

Corollary 3.1.1. From these series, one readily obtains the following congruences, valid but for odd $p$ :

$$
\begin{align*}
& H_{p-1}^{(2 k-1)} \equiv 0  \tag{3.5}\\
&(\bmod p),  \tag{3.6}\\
& H_{p-1}^{(2 k-1)}+p(2 k-1) H_{\frac{p-1}{2}}^{(2 k)} \equiv 0  \tag{3.7}\\
&\left(\bmod p^{2}\right),  \tag{3.8}\\
& H_{p-1}^{(2 k-1)}+\frac{1}{2} p(2 k-1) H_{p-1}^{(2 k)} \equiv 0 \quad\left(\bmod p^{2}\right),  \tag{3.9}\\
& H_{p-1}^{(2 k)}-2 H_{\frac{p-1}{2}}^{(2 k)}-p \cdot 2 k H_{\frac{p-1}{2}}^{(2 k+1)} \equiv 0  \tag{3.10}\\
&\left(\bmod p^{2}\right), \\
& H_{\frac{p-1}{2}}^{(2 k)}\left(2^{2 k}-1\right)+p \cdot \frac{k}{2}\left(2^{2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 k+1)} \equiv 0 \quad\left(\bmod p^{2}\right), \\
& 2 H_{\frac{p-1}{2}}^{(2 k)}-\left(2^{2 k+1}-1\right) H_{p-1}^{(2 k)} \equiv 0 \quad\left(\bmod p^{2}\right) .
\end{align*}
$$

Remark. In the case $k=1$ (3.1) and (3.4) reads, respectively,

$$
\begin{align*}
H_{p-1}+\frac{1}{2} \sum_{j \geq 1} H_{p-1}^{(j+1)} p^{j} & =0  \tag{3.11}\\
H_{\frac{p-1}{2}}^{(2)}+\sum_{j \geq 1} \frac{j+1}{6 \cdot 2^{j}}\left(2^{j+2}-1\right) H_{\frac{p-1}{2}}^{(j+2)} p^{j} & =0 \tag{3.12}
\end{align*}
$$

Proof of Theorem 3.1. We have

$$
\begin{aligned}
& H_{p-1}^{(k)}=\sum_{i=1}^{p-1} \frac{1}{i^{k}}=\sum_{1 \leq i \leq p-1} \frac{1}{(p-i)^{k}}=(-1)^{k} \sum_{i=1}^{p-1} \frac{1}{i^{k}\left(1-\frac{p}{i}\right)^{k}} \\
& =(-1)^{k} \sum_{i=1}^{p-1} \sum_{j \geq 0}\binom{j+k-1}{j} \frac{p^{j}}{i^{k+j}} \quad \text { from (2.1) } \\
& =(-1)^{k} \sum_{j \geq 0} p^{j} \sum_{i=1}^{p-1}\binom{j+k-1}{j} \frac{1}{i^{k+j}} \\
& =(-1)^{k} \sum_{j \geq 0}\binom{j+k-1}{j} H_{p-1}^{(k+j)} p^{j} . \\
& H_{p-1}^{(2 k)}=\sum_{i=1}^{p-1} \frac{1}{i^{2 k}}=\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^{2 k}}+\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(p-i)^{2 k}}=\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^{2 k}}+\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^{2 k}\left(1-\frac{p}{i}\right)^{2 k}} \\
& =2 H_{\frac{p-1}{2}}^{(2 k)}+\sum_{i=1}^{\frac{p-1}{2}} \sum_{j \geq 1}\binom{j+2 k-1}{j} \frac{p^{j}}{i^{2 k+j}} \quad \text { from (2.1) } \\
& =2 H_{\frac{p-1}{2}}^{(2 k)}+\sum_{j \geq 1}\binom{j+2 k-1}{j} H_{\frac{p-1}{2}}^{(2 k+j)} p^{j} . \\
& H_{\frac{p-1}{2}}^{(k)}=\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^{k}}=\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{\left(\frac{p+1}{2}-i\right)^{k}}=(-2)^{k} \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2 i-1)^{k}\left(1-\frac{p}{2 i-1}\right)^{k}} \\
& =(-2)^{k} \sum_{i=1}^{\frac{p-1}{2}} \sum_{j \geq 0}\binom{j+k-1}{j} \frac{p^{j}}{(2 i-1)^{k+j}} \quad \text { from (2.1) } \\
& =(-2)^{k} \sum_{j \geq 0} p^{j}\binom{j+k-1}{j} \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2 i-1)^{k+j}} \\
& =(-2)^{k} \sum_{j \geq 0}\binom{j+k-1}{j}\left(\sum_{i=1}^{p-1} \frac{1}{i^{k+j}}-\sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2 i)^{k+j}}\right) p^{j} \\
& =(-2)^{k} \sum_{j \geq 0}\binom{j+k-1}{j}\left(H_{p-1}^{(k+j)}-\frac{1}{2^{k+j}} H_{\frac{p-1}{2}}^{(k+j)}\right) p^{j} \\
& =2^{k} H_{p-1}^{(k)}-(-1)^{k} \sum_{j \geq 0}\binom{j+k-1}{j} \frac{1}{2^{j}} H_{\frac{p-1}{2}}^{(k+j)} p^{j} \text { by using (3.1) } \\
& =2^{k} H_{p-1}^{(k)}-(-1)^{k} H_{\frac{p-1}{2}}^{(k)}+\sum_{j \geq 1}\binom{j+k-1}{j} \frac{1}{2^{j}} H_{\frac{p-1}{2}}^{(k+j)} p^{j} .
\end{aligned}
$$

After rearragement, this gives (3.3) and then (3.4) is obtained by comparing (3.2) to (3.3) in the even case, and eliminating $H_{p-1}^{(2 k)}$.

Theorem 3.2. Let $k \geq 1$ a natural integer, and $p$ a prime. When $p \geq 2 k+3$, the following congruences hold:

$$
\begin{align*}
H_{p-1}^{(2 k)} & \equiv p \frac{2 k}{2 k+1} B_{p-1-2 k} \quad\left(\bmod p^{2}\right)  \tag{3.13}\\
H_{\frac{p-1}{2}}^{(2 k)} & \equiv p \frac{k\left(2^{2 k+1}-1\right)}{2 k+1} B_{p-1-2 k} \quad\left(\bmod p^{2}\right),  \tag{3.14}\\
H_{p-1}^{(2 k-1)} & \equiv-p^{2} \frac{k(2 k-1)}{2 k+1} B_{p-1-2 k} \quad\left(\bmod p^{3}\right)  \tag{3.15}\\
H_{\frac{p-1}{2}}^{(2 k+1)} & \equiv \frac{2\left(1-4^{k}\right)}{2 k+1} B_{p-1-2 k} \quad(\bmod p) \tag{3.16}
\end{align*}
$$

Moreover, the last one is also valid when $p=2 k+1$.
Proof. All this is already well-known: see for example [3, [5, [1, 9]. From Fermat little theorem, for $1 \leq i \leq p-1$, we have $\frac{1}{i^{k}}-\frac{i^{p-1}}{i^{k}} \equiv 0(\bmod p)$, hence, by squaring, we have $\left(\frac{1}{i^{k}}-i^{p-1-k}\right)^{2} \equiv 0\left(\bmod p^{2}\right)$, that is $\frac{1}{i^{2 k}} \equiv 2 i^{p-1-2 k}-i^{2 p-2-2 k}\left(\bmod p^{2}\right)$ and then

$$
H_{p-1}^{(2 k)} \equiv 2 \sum_{i=1}^{p-1} i^{p-1-2 k}-\sum_{i=1}^{p-1} i^{2 p-2-2 k} \quad\left(\bmod p^{2}\right)
$$

but from (2.6) we have

$$
\sum_{i=1}^{p} i^{h} \equiv p(-1)^{h} B_{h}+p^{2}(-1)^{h-1} \frac{h}{2} B_{h-1} \quad\left(\bmod p^{3}\right)
$$

that is, if $h \geq 3$,

$$
\sum_{i=1}^{p-1} i^{h} \equiv p(-1)^{h} B_{h}+p^{2}(-1)^{h-1} \frac{h}{2} B_{h-1} \quad\left(\bmod p^{3}\right)
$$

Hence, as the odd-index Bernoulli numbers are zero, we have, for $h \geq 2$,

$$
\sum_{1 \leq i \leq p-1} i^{2 h} \equiv p B_{2 h} \quad\left(\bmod p^{3}\right)
$$

And then

$$
H_{p-1}^{(2 k)} \equiv 2 \sum_{i=1}^{p-1} i^{p-1-2 k}-\sum_{i=1}^{p-1} i^{2 p-2-2 k} \equiv p\left(2 B_{p-1-2 k}-B_{2 p-2-2 k}\right) \quad\left(\bmod p^{2}\right)
$$

But, since $p \geq 2 k+3, p-1-2 k$ is not divisible by $p-1$ then, from Kummer congruence (2.7), we see that $B_{2 p-2-2 k} \equiv \frac{2 k+2}{2 k+1} B_{p-1-2 k} \bmod p$ and therefore

$$
H_{p-1}^{(2 k)} \equiv p \frac{2 k}{2 k+1} B_{p-1-2 k} \quad\left(\bmod p^{2}\right)
$$

which is (3.13). Then from (3.10), we have

$$
H_{\frac{p-1}{2}}^{(2 k)} \equiv p \frac{k\left(2^{2 k+1}-1\right)}{2 k+1} B_{p-1-2 k} \quad\left(\bmod p^{2}\right)
$$

which is (3.14). We also have

$$
\begin{aligned}
H_{p-1}^{(2 k-1)} & =\sum_{i=1}^{p-1} \frac{1}{i^{2 k-1}}=\sum_{i=1}^{p-1} \frac{1}{(p-i)^{2 k-1}}=-\sum_{i=1}^{p-1} \frac{1}{i^{2 k-1}\left(1-\frac{p}{i}\right)^{2 k-1}} \\
& \equiv-H_{p-1}^{(2 k-1)}-p(2 k-1) H_{p-1}^{(2 k)}-p^{2} \frac{(2 k)(2 k-1)}{2} H_{p-1}^{(2 k+1)} \quad\left(\bmod p^{3}\right)
\end{aligned}
$$

Then, rearranging and accounting for (3.13), we obtain (3.15) as

$$
\begin{aligned}
H_{p-1}^{(2 k-1)} & \equiv-p \frac{2 k-1}{2} H_{p-1}^{(2 k)} \quad\left(\bmod p^{3}\right) \\
& \equiv-p^{2} \frac{k(2 k-1)}{2 k+1} B_{p-1-2 k} \quad\left(\bmod p^{3}\right)
\end{aligned}
$$

For the case $p \geq 2 k+3$, (3.16) is obtained starting from (3.8) and taking (3.13) and (3.14) into account

$$
\begin{aligned}
H_{p-1}^{(2 k)}-2 H_{\frac{p-1}{2}}^{(2 k)} & \equiv p \cdot 2 k H_{\frac{p-1}{2}}^{(2 k+1)} \\
p \frac{2 k}{2 k+1} B_{p-1-2 k}-p \frac{2 k\left(2^{2 k+1}-1\right)}{2 k+1} B_{p-1-2 k} & \equiv p \cdot 2 k H_{\frac{p-1}{2}}^{(2 k+1)} \quad\left(\bmod p^{2}\right), \\
\left.\frac{2\left(1-4^{k}\right)}{2 k+1} B_{p-1-2 k}\right) & \equiv H_{\frac{p-1}{2}}^{(2 k+1)} \quad(\bmod p)
\end{aligned}
$$

Finally, for the case $p=2 k+1$, (3.16) is obtained by Fermat little theorem, and Eisenstein congruence (1.3) as

$$
H_{\frac{p-1}{2}}^{(p)} \equiv H_{\frac{p-1}{2}} \equiv-2 \cdot q_{p} \equiv \frac{2\left(1-4^{\frac{p-1}{2}}\right)}{p} B_{0} \quad(\bmod p)
$$

Corollary 3.2.1. For $p \geq 2 k+3$, we have

$$
\begin{align*}
H_{\frac{p-1}{2}}^{(2 k)} & \equiv 0 \quad(\bmod p)  \tag{3.17}\\
H_{p-1}^{(2 k)} & \equiv 0 \quad(\bmod p)  \tag{3.18}\\
H_{p-1}^{(2 k-1)} & \equiv 0 \quad\left(\bmod p^{2}\right) . \tag{3.19}
\end{align*}
$$

Moreover, when $B_{p-2 k-1} \equiv 0(\bmod p)$, in other words when $(p, p-2 k-1)$ is an irregular pair, we have

$$
\begin{align*}
& H_{p-1}^{(2 k)} \equiv 0 \quad\left(\bmod p^{2}\right),  \tag{3.20}\\
& H_{\frac{p-1}{2}}^{(2 k)} \equiv 0 \quad\left(\bmod p^{2}\right),  \tag{3.21}\\
& H_{p-1}^{(2 k-1)} \equiv 0 \quad\left(\bmod p^{3}\right),  \tag{3.22}\\
& H_{\frac{p-1}{2}}^{(2 k+1)} \equiv 0 \quad(\bmod p) . \tag{3.23}
\end{align*}
$$

We will now derive congruences modulo arbitrary prime powers for the Harmonic numbers, involving the Bernoulli numbers. The next theorem is essentially already known: it was originally obtained from quite advanced mathematics, involving $p$ adic L-functions ([11], Theorem 1). Our derivation will be more elementary.

Theorem 3.3. Let $n$, $i$ be natural integers, $p$ a prime. The congruence

$$
\begin{equation*}
\sum_{j=0}^{2 n+1}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} \equiv 0 \quad\left(\bmod p^{2 n+m}\right) \tag{3.24}
\end{equation*}
$$

holds with the following values of $m$ under the corresponding restrictions on $p$ :

$$
\begin{aligned}
& m=1 \text { when } p \geq 2 \\
& m=2 \text { when } p \geq 3 \\
& m=3 \text { when } p \geq 5 \text { or } p=3 \text { and } n \equiv 0 \quad(\bmod 3) \\
& m=4 \text { when } p \geq 2 n+2 i+7, \\
& m=5 \text { when }(p, p-2 n-2 i-5) \text { is an irregular pair. }
\end{aligned}
$$

Remark. Congruence (1.2) is a particular case, $n=i=0$, $m=4$, of (3.24). Similar congruences are

$$
\begin{aligned}
H_{p-1}+\frac{p}{2} H_{p-1}^{(2)}+\frac{p^{2}}{6} H_{p-1}^{(3)} \equiv 0 \quad\left(\bmod p^{6}\right) \quad \text { when } p \geq 9 \\
H_{p-1}+\frac{p}{2} H_{p-1}^{(2)}+\frac{p^{2}}{6} H_{p-1}^{(3)}-\frac{p^{4}}{30} H_{p-1}^{(5)} \equiv 0 \quad\left(\bmod p^{8}\right) \quad \text { when } p \geq 11
\end{aligned}
$$

etc.
Remark. Also in particular when $n=0$, we have

$$
\begin{equation*}
H_{p-1}^{(2 i+1)}+p \frac{2 i+1}{2} H_{p-1}^{(2 i+2)} \equiv 0 \quad\left(\bmod p^{m}\right) \tag{3.25}
\end{equation*}
$$

with:

$$
\begin{aligned}
& m=1 \text { when } p \geq 2 \\
& m=2 \text { when } p \geq 3, \\
& m=3 \text { when } p \geq 5 \text { or } p=3 \text { and } n \equiv 0 \quad(\bmod 3), \\
& m=4 \text { when } p \geq 2 i+7, \\
& m=5 \text { when }(p, p-2 i-5) \text { is an irregular pair. }
\end{aligned}
$$

Example. When $m=5$, since $(37,32)$ is an irregular pair, we have $i=0$, and

$$
\sum_{j=1}^{36} \frac{1}{j}+\frac{37}{2} \sum_{j=1}^{36} \frac{1}{j^{2}}=\frac{1422091936194747472864459922257}{41704772176589465865841920000}=\frac{N}{D}
$$

and $N=37^{5} \cdot 1123 \cdot 9133 \cdot 1999520400972139$ is divisible by $37^{5}$, as expected.

Corollary 3.3.1. For any odd prime p, we also have the following weaker p-adically converging series involving generalized Harmonic numbers:

$$
\begin{equation*}
\sum_{j=0}^{k-1}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} \equiv 0 \quad\left(\bmod p^{k}\right) \tag{3.26}
\end{equation*}
$$

In particular, when $i=0$, we have

$$
\begin{equation*}
\sum_{0 \leq j} B_{j} H_{p-1}^{(j+1)}(-p)^{j}=0 \tag{3.27}
\end{equation*}
$$

Remark. Note that (3.27) coincide with (3.11) only up to $j \leq 1$.

Proof of Theorem 3.3. We first show how (3.26) is obtained from (3.24). Suppose $p \geq 3$. In the case $k=1$, (3.26) reduces to (3.5). Suppose now $k>1$. If $k$ is even, then $k=2 n+2(n \geq 0)$ and by (3.24), we have

$$
\begin{aligned}
\sum_{j=0}^{k-1}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} & =\sum_{j=0}^{2 n+1}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} \\
& \equiv 0 \quad\left(\bmod p^{2 n+2}=p^{k}\right)
\end{aligned}
$$

If $k$ is odd, then $k=2 n+3(n \geq 0)$ and $B_{2 n+3}=0$, then

$$
\begin{aligned}
\sum_{j=0}^{k-1}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} & =\sum_{j=0}^{2 n+2}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} \\
& =\sum_{j=0}^{2 n+3}\binom{j+2 i}{2 i} B_{j} H_{p-1}^{(j+2 i+1)}(-p)^{j} \\
& \equiv 0 \quad\left(\bmod p^{2 n+4}\right)(\text { by }(3.24)) \\
& \equiv 0 \quad\left(\bmod p^{2 n+3}=p^{k}\right)
\end{aligned}
$$

Now, for the proof of (3.24), let $S$ be the left hand side in (3.24). From (3.1) we have

$$
H_{p-1}^{(j+2 i+1)}=(-1)^{j+2 i+1} \sum_{h \geq 0}\binom{h+j+2 i}{h} H_{p-1}^{(j+2 i+h+1)} p^{h}
$$

then

$$
S=-\sum_{j=0}^{2 n+1}\binom{j+2 i}{2 i} B_{j} \sum_{h \geq 0}\binom{h+j+2 i}{h} H_{p-1}^{(j+2 i+h+1)} p^{h+j}
$$

Let $k=h+j, j \leq 2 n+1$ and $j \leq k$, hence $j \leq \min (2 n+1, k)$ and we can derive the following series expansion of $S$ in terms of $p^{k}$ :

$$
\begin{aligned}
S & =-\sum_{k \geq 0} p^{k} H_{p-1}^{(k+2 i+1)} \sum_{j=0}^{\min (2 n+1, k)} B_{j}\binom{j+2 i}{2 i}\binom{k+2 i}{j+2 i} \\
& =-\sum_{k \geq 0} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{\min (2 n+1, k)} B_{j}\binom{k}{j} \\
& =-\sum_{k=0}^{2 n+1} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{k} B_{j}\binom{k}{j} \\
& -\sum_{k \geq 2 n+2} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{2 n+1} B_{j}\binom{k}{j} \\
& =-\sum_{k=0}^{2 n+1} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i}(-1)^{k} B_{k} \\
& -\sum_{k \geq 2 n+2} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{2 n+1} B_{j}\binom{k}{j} \\
& =-S-\sum_{k \geq 2 n+2} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{2 n+1} B_{j}\binom{k}{j} .
\end{aligned}
$$

Hence

$$
2 S=-\sum_{k \geq 2 n+2} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{2 n+1} B_{j}\binom{k}{j}
$$

Note that, in the above derivation, it is made use of o(2.4) and also of the identity $\binom{j+2 i}{2 i}\binom{k+2 i}{j+2 i}=\binom{k+2 i}{2 i}\binom{k}{j}$. Moreover, from (2.4), we also have $\sum_{j=0}^{2 n+1} B_{j}\binom{2 n+2}{j}=0$. Hence

$$
\begin{equation*}
2 S=-\sum_{k \geq 2 n+3} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{2 n+1} B_{j}\binom{k}{j} \tag{3.28}
\end{equation*}
$$

By the Von Staudt-Clausen theorem (2.8), we know that $p$ may divide the denominator of any $B_{j}$ only once at most, then we must have

$$
2 S \equiv 0 \quad\left(\bmod p^{2 n+2}\right)
$$

and this makes a proof of the above claim for the cases $m=1$ and $m=2$. Also, again from (2.4), $\sum_{j=0}^{2 n+1} B_{j}\binom{2 n+3}{j}=-(2 n+3) B_{2 n+2}$, then (3.28) becomes

$$
\begin{align*}
2 S & =p^{2 n+3} H_{p-1}^{(2 n+2 i+4)}\binom{2 n+2 i+3}{2 i}(2 n+3) B_{2 n+2} \\
& -\sum_{k \geq 2 n+4} p^{k} H_{p-1}^{(k+2 i+1)}\binom{k+2 i}{2 i} \sum_{j=0}^{2 n+1} B_{j}\binom{k}{j} . \tag{3.29}
\end{align*}
$$

The second term on the right hand side of (3.29) is unconditionnally zero modulo $p^{2 n+3}$, by Von Staudt-Clausen. The first term on the right hand side of (3.29) is also zero modulo $p^{2 n+3}$ when $2 n+2$ is not a multiple of $p-1$, also by Von StaudtClausen. Then the congruence modulo $p^{2 n+3}$ might fail only when $p=3$ or when $n+1$ is a multiple of $\frac{p-1}{2}$. But in the latter case, we have

$$
\begin{aligned}
H_{p-1}^{(2 n+4)}=\sum_{i=1}^{p-1} \frac{1}{i^{2} i^{2(n+1)}} \equiv \sum_{i=1}^{p-1} \frac{1}{i^{2}} & =H_{p-1}^{(2)} \\
& \equiv 0 \quad(\bmod p) \quad \text { when } p \geq 5, \quad \text { by }
\end{aligned}
$$

and this makes a compensation. Note that in the case where $p=3$ there is also a compensation when 3 divides $2 n+3$, that is when $n$ is a multiple of 3 . This remark completes the proof in the case $m=3$.
Now, if $p \geq 2 n+2 i+7$, from Theorem $3.2\left(\sqrt{3.18)}, H_{p-1}^{(2 n+2 i+4)} \equiv 0(\bmod p)\right.$ and, again from the Von Staudt-Clausen theorem, $p$ is large enough so that it never divides the denominators of $B_{j}$ when $0 \leq j \leq 2 n+1$ then when $p \geq 2 n+2 i+7$, we have $2 S \equiv 0\left(\bmod p^{2 n+4}\right)$, this completes the proof for the case $m=4$.
Finally, if $B_{p-2 n-2 i-5} \equiv 0(\bmod p)$, then $H_{p-1}^{(2 n+2 i+4)} \equiv 0\left(\bmod p^{2}\right)$, from Theorem 3.2, and since as soon as $p \geq 3, H_{p-1}^{(2 n+2 i+5)} \equiv 0(\bmod p)$, by Theorem 3.1 and this completes the proof for the case $m=5$.

Now, we give a $p$-adic expansion for $H_{\frac{p-1}{2}}^{(2 i)}$ which seems to be new. It is in the same spirit as the previous one, though.

Theorem 3.4. Let $p$ be an odd prime, $n, i$ positive integers and $j$ a non-negative integer. Let $C_{j}:=2 B_{j+2}+(-1)^{j} B_{j+1}+\frac{1}{2}$. The congruence

$$
\begin{equation*}
\sum_{j=0}^{2 n-1}\binom{j+2 i-1}{j+1} \frac{2^{j+2 i}-1}{2^{j}} C_{j} H_{\frac{p-1}{2}}^{(j+2 i)} p^{j} \equiv 0 \quad\left(\bmod p^{2 n+m}\right) \tag{3.30}
\end{equation*}
$$

holds with the following value for $m$ and the corresponding restrictions on $p$ : we have $m=0$ when $p \geq 3, m=1$ when $p>2 n+1$, and $m=2$ when

$$
\binom{2 n+2 i}{2 n+2}\left(2^{2 n+2 i+1}-1\right) H_{\frac{p-1}{2}}^{(2 n+2 i+1)}\left((2 n+3) B_{2 n+2}+\frac{n}{2}\right) \equiv 0 \quad(\bmod p)
$$

Proof. We start from a reformulation of (3.4), with $j \geq 1, g \geq 0$. We have

$$
\begin{align*}
H_{\frac{p-1}{2}}^{(2 j+2 g)}= & -\sum_{h \geq 0}\binom{2 h+2 j+2 g-1}{2 h} \frac{2^{2 h+2 g+2 j}-1}{2^{2 j+2 g}-1} H_{\frac{p-1}{2}}^{(2 h+2 g+2 j)}\left(\frac{p}{2}\right)^{2 h} \\
& -\sum_{h \geq 0}\binom{2 h+2 j+2 g}{2 h+1} \frac{2^{2 h+2 g+2 j+1}-1}{2^{2 j+2 g}-1} H_{\frac{p-1}{2}}^{(2 h+2 g+2 j+1)}\left(\frac{p}{2}\right)^{2 h+1} \tag{3.31}
\end{align*}
$$

Let

$$
S=\sum_{j=1}^{n}\binom{2 j+2 g-1}{2 g} B_{2 j}\left(2^{2 g+2 j}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 j)}\left(\frac{p}{2}\right)^{2 j}
$$

Substituting $H_{\frac{p-1}{2}}^{(2 j+2 g)}$ from (3.31) in the above expression for $S$, we obtain $S=-\sum_{j=1}^{n}\binom{2 j+2 g-1}{2 g} B_{2 j} \sum_{h \geq 0}\binom{2 h+2 j+2 g-1}{2 h}\left(2^{2 h+2 g+2 j}-1\right) H_{\frac{p-1}{2}}^{(2 h+2 g+2 j)}\left(\frac{p}{2}\right)^{2 j+2 h}$ $-\sum_{j=1}^{n}\binom{2 j+2 g-1}{2 g} B_{2 j} \sum_{h \geq 0}\binom{2 h+2 j+2 g}{2 h+1}\left(2^{2 h+2 g+2 j+1}-1\right) H_{\frac{p-1}{2}}^{(2 h+2 g+2 j+1)}\left(\frac{p}{2}\right)^{2 j+2 h+1}$.

Now, we let $2 k=2 h+2 j$ and we will replace the summation index $h$ by $k$. The new index $k$ runs from 1 , with no upper bound and the index $j$ runs from 1 to $\min (n, k)$, since $1 \leq j \leq n$ and $2 j=2 k-2 h \leq 2 k$. Then, with the new summation indices and after inverting the sums, we have

$$
\begin{aligned}
S & =-\sum_{1 \leq k}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k} \sum_{j=1}^{\min (n, k)}\binom{2 j+2 g-1}{2 g}\binom{2 g+2 k-1}{2 j+2 g-1} B_{2 j} \\
& -\sum_{1 \leq k}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1} \sum_{j=1}^{\min (n, k)}\binom{2 j+2 g-1}{2 g}\binom{2 g+2 k}{2 j+2 g-1} B_{2 j} .
\end{aligned}
$$

But

$$
\binom{2 j+2 g-1}{2 g}\binom{2 g+2 k-1}{2 j+2 g-1}=\binom{2 k+2 g-1}{2 g}\binom{2 k-1}{2 j-1}
$$

and

$$
\binom{2 j+2 g-1}{2 g}\binom{2 g+2 k}{2 j+2 g-1}=\binom{2 k+2 g}{2 g}\binom{2 k}{2 j-1}
$$

Then

$$
\begin{aligned}
S= & -\sum_{1 \leq k}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k} \sum_{j=1}^{\min (n, k)}\binom{2 k-1}{2 j-1} B_{2 j} \\
& -\sum_{1 \leq k}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1} \sum_{j=1}^{\min (n, k)}\binom{2 k}{2 j-1} B_{2 j} . \\
S= & -\sum_{k=1}^{n}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k} \sum_{j=1}^{k}\binom{2 k-1}{2 j-1} B_{2 j} \\
& -\sum_{k=1}^{n}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1} \sum_{j=1}^{k}\binom{2 k}{2 j-1} B_{2 j} \\
& -\binom{2 n+2 g+1}{2 g}\left(2^{2 g+2 n+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+2)}\left(\frac{p}{2}\right)^{2 n+2} \sum_{j=1}^{n}\binom{2 n+1}{2 j-1} B_{2 j} \\
& -\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+3} \sum_{j=1}^{n}\binom{2 n+2}{2 j-1} B_{2 j} \\
& -\sum_{n+2 \leq k}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k} \sum_{j=1}^{n}\binom{2 k-1}{2 j-1} B_{2 j} \\
& -\sum_{n+2 \leq k}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1} \sum_{j=1}^{n}\binom{2 k}{2 j-1} B_{2 j} .
\end{aligned}
$$

We now make use of Lemma 2.1, so that

$$
\begin{aligned}
S= & -\sum_{k=1}^{n}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k}\left(B_{2 k}+B_{2 k-1}+\frac{1}{2}\right) \\
& -\sum_{k=1}^{n}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1}\left(\frac{1}{2}-B_{2 k}\right) \\
& -\binom{2 n+2 g+1}{2 g}\left(2^{2 g+2 n+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+2)}\left(\frac{p}{2}\right)^{2 n+2} \frac{1}{2} \\
& -\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+3}\left(\frac{1}{2}-(2 n+3) B_{2 n+2}\right) \\
& -\sum_{n+2 \leq k}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k} \sum_{j=1}^{n}\binom{2 k-1}{2 j-1} B_{2 j} \\
& -\sum_{n+2 \leq k}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1} \sum_{j=1}^{n}\binom{2 k}{2 j-1} B_{2 j} .
\end{aligned}
$$

By the Von Staudt Clausen theorem, $p$ may divide the inner sum in the two last lines only once at most, and then the last line in the above expression for $S$ is $0 \bmod p^{2 n+4}$, and, since by $(3.9)\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)} \equiv 0 \bmod p$, the fifth line is also $0 \bmod p^{2 n+4}$. Then, we obtain

$$
\begin{aligned}
S \equiv & -\sum_{k=1}^{n}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k}\left(B_{2 k}+B_{2 k-1}+\frac{1}{2}\right) \\
& -\sum_{k=1}^{n}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1}\left(\frac{1}{2}-B_{2 k}\right) \\
& -\binom{2 n+2 g+1}{2 g}\left(2^{2 g+2 n+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+2)}\left(\frac{p}{2}\right)^{2 n+2} \frac{1}{2} \\
& -\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+3}\left(\frac{1}{2}-(2 n+3) B_{2 n+2}\right) \\
& \left(\bmod p^{2 n+4}\right) .
\end{aligned}
$$

Now by (3.9), we have

$$
\left(2^{2 g+2 n+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+2)} \equiv-p \frac{n+g+1}{2}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)} \bmod p^{2}
$$

and after some calculations the third and forth lines in the above expression for $S$
can be merged so that:

$$
\begin{aligned}
S \equiv & -\sum_{k=1}^{n}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k}\left(B_{2 k}+B_{2 k-1}+\frac{1}{2}\right) \\
& -\sum_{k=1}^{n}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k+1}\left(\frac{1}{2}-B_{2 k}\right) \\
& +\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+3}\left(\frac{n}{2}+(2 n+3) B_{2 n+2}\right) \\
& \left(\bmod p^{2 n+4}\right) .
\end{aligned}
$$

Recall that $S=\sum_{k=1}^{n}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k} B_{2 k}$, then after rearranging and dividing throughout by $p^{2}$, we obtain the following congruence $\left(\bmod p^{2 n+2}\right)$ :

$$
\begin{aligned}
0 \equiv & -\sum_{k=1}^{n}\binom{2 k+2 g-1}{2 g}\left(2^{2 g+2 k}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k)}\left(\frac{p}{2}\right)^{2 k-2}\left(2 B_{2 k}+B_{2 k-1}+\frac{1}{2}\right) \\
& -\sum_{k=1}^{n}\binom{2 k+2 g}{2 g}\left(2^{2 g+2 k+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 k+1)}\left(\frac{p}{2}\right)^{2 k-1}\left(\frac{1}{2}-B_{2 k}\right) \\
& +\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+1}\left(\frac{n}{2}+(2 n+3) B_{2 n+2}\right) \\
& \left(\bmod p^{2 n+2}\right) .
\end{aligned}
$$

That is

$$
\begin{aligned}
0 \equiv & -\sum_{\substack{k=0 \\
k \text { even }}}^{2 n-2}\binom{k+2 g+1}{2 g}\left(2^{2 g+k+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+k+2)}\left(\frac{p}{2}\right)^{k}\left(2 B_{k+2}+B_{k+1}+\frac{1}{2}\right) \\
& -\sum_{\substack{k=1 \\
k o d d}}^{2 n-1}\binom{k+2 g+1}{2 g}\left(2^{2 g+k+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+k+2)}\left(\frac{p}{2}\right)^{k}\left(\frac{1}{2}-B_{k+1}\right) \\
& +\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+1}\left(\frac{n}{2}+(2 n+3) B_{2 n+2}\right) \\
& \left(\bmod p^{2 n+2}\right) .
\end{aligned}
$$

But again, since from $B_{3}$, the odd-indexed Bernoulli number are 0 , we have

$$
\begin{aligned}
0 \equiv & -\sum_{k=0}^{2 n-2}\binom{k+2 g+1}{2 g}\left(2^{2 g+k+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+k+2)}\left(\frac{p}{2}\right)^{k}\left(2 B_{k+2}+\frac{1}{2}\right) \\
& -\sum_{\substack{k=0 \\
k \text { even }}}^{2 n-2}\binom{k+2 g+1}{2 g}\left(2^{2 g+k+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+k+2)}\left(-\frac{p}{2}\right)^{k} B_{k+1} \\
& -\sum_{\substack{k=1 \\
k \text { odd }}}^{2 n-1}\binom{k+2 g+1}{2 g}\left(2^{2 g+k+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+k+2)}\left(\frac{p}{2}\right)^{k}(-1)^{k} B_{k+1} \\
& +\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+1}\left(\frac{n}{2}+(2 n+3) B_{2 n+2}\right) \\
\equiv & \sum_{k=0}^{2 n-2}\binom{k+2 g+1}{2 g}\left(2^{2 g+k+2}-1\right) H_{\frac{p-1}{2}}^{(2 g+k+2)}\left(\frac{p}{2}\right)^{k}\left(2 B_{k+2}+(-1)^{k} B_{k+1}+\frac{1}{2}\right) \\
& -\binom{2 n+2 g+2}{2 g}\left(2^{2 g+2 n+3}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+3)}\left(\frac{p}{2}\right)^{2 n+1}\left(\frac{n}{2}+(2 n+3) B_{2 n+2}\right) \\
& \left(\bmod p^{2 n+2}\right) .
\end{aligned}
$$

Introducing $C_{k}$, and shifting $g$, so that $g \geq 1$, we obtain

$$
\begin{aligned}
0 \equiv & \sum_{k=0}^{2 n-1}\binom{k+2 g-1}{k+1}\left(2^{2 g+k}-1\right) C_{k} H_{\frac{p-1}{2}}^{(2 g+k)}\left(\frac{p}{2}\right)^{k} \\
& -\binom{2 n+2 g}{2 n+2}\left(2^{2 g+2 n+1}-1\right) H_{\frac{p-1}{2}}^{(2 g+2 n+1)}\left(\frac{p}{2}\right)^{2 n+1}\left(\frac{n}{2}+(2 n+3) B_{2 n+2}\right) \\
& \left(\bmod p^{2 n+2}\right) .
\end{aligned}
$$

The latter congruence obviously proves the theorem in the case $m=2$, but also in the case $m=1$ since when $p>2 n+1, \frac{n}{2}+(2 n+3) B_{2 n+2}$ is $p$-integral: this is true for $p>2 n+3$ by Von Staudt-Clausen, and also for $p=2 n+3$ as $p B_{p-1}$ is also p-integral; and in the case $m=0$, under the only condition that $p$ be odd.

Corollary 3.4.1. When $i=1, m=0$, we have the following $p$-adically converging series, for $p$ an odd prime:

$$
\begin{equation*}
\sum_{0 \leq j}\left(2 B_{j+2}+(-1)^{j} B_{j+1}+\frac{1}{2}\right)\left(2^{j+2}-1\right) H_{\frac{p-1}{2}}^{(j+2)}\left(\frac{p}{2}\right)^{j}=0 \tag{3.32}
\end{equation*}
$$

Remark. Note that (3.32) coincide with (3.12) up to $j \leq 1$ only.

Remark. When $i=1, m=1$, Theorem 3.4 gives

$$
\begin{array}{rlrl}
H_{\frac{p-1}{2}}^{(2)}+\frac{7}{6} H_{\frac{p-1}{2}}^{(3)} p & \equiv 0 & \left(\bmod p^{3}\right) & \text { when } p>3 \\
H_{\frac{p-1}{2}}^{(2)}+\frac{7}{6} H_{\frac{p-1}{2}}^{(3)} p+\frac{13}{8} H_{\frac{p-1}{2}}^{(4)} p^{2}+\frac{31}{15} H_{\frac{p-1}{2}}^{(5)} p^{3} & \equiv 0 & \left(\bmod p^{5}\right) & \text { when } p>5 \\
\text { etc. } &
\end{array}
$$

Remark. Note also how these congruences differ from what can be obtained from Theorem 3.1 and its Corollary. They read differently from $j>1$ and they don't need the same restrictions on $p$. For example, (3.4) limited at order $j \leq 4$, combined with (3.9) leads to

$$
H_{\frac{p-1}{2}}^{(2)}+\frac{7}{6} H_{\frac{p-1}{2}}^{(3)} p+\frac{15}{8} H_{\frac{p-1}{2}}^{(4)} p^{2}+\frac{31}{12} H_{\frac{p-1}{2}}^{(5)} p^{3} \equiv 0 \quad\left(\bmod p^{5}\right) \quad \text { when } p>3
$$

Corollary 3.4.2. There are quite many cases where the congruence from Theorem 3.4 holds modulo $p^{2 n+2}$. Most notably:

$$
\begin{aligned}
& m=2 \text { when } i \geq 2 \text { and } p=2 n+3, \\
& m=2 \text { when }(p, p-2 n-2 i-1) \text { is an irregular pair, } \\
& m=2 \text { when } p=2^{2 n+2 i+1}-1(p \text { is a Mersenne prime }), \\
& m=2 \text { when } p=2 n+2 i+1 \text { is a Wieferich prime }(p=1093,3511, . .[8]) .
\end{aligned}
$$

Proof. The first case is clear because when $p=2 n+3,(2 n+3) B_{2 n+2}+\frac{n}{2}$ is $p$ integral, and $2 n+3$ divides $\binom{2 n+2 i}{2 n+2}$ as soon as $i \geq 2$. The second case follows from (3.23). The third case is clear because $2^{2 n+2 i+1}-2>2 n+2$. The last case derives from (3.16) : $H_{\frac{p-1}{2}}^{(p)} \equiv \frac{2\left(1-2^{p-1}\right)}{p} B_{0} \bmod p$ because by definition a Wieferich prime $p$ satisfies $2^{p-1}-1 \equiv 0 \bmod p^{2}$ 。

Examples. (i) When $n=1=i$ and $p=37,(p, p-2 i-2 n-1)$ is an irregular pair, so $m=2$. We check:

$$
\sum_{j=1}^{18} \frac{1}{j^{2}}+37 \frac{7}{6} \sum_{j=1}^{18} \frac{1}{j^{3}}=\frac{9356942544006649495921}{175168974229337088000}
$$

and $9356942544006649495921=19 \cdot 37^{4} \cdot 262768598968219$ is divisible by $37^{4}$.
(ii) When $n=1=i$ and $p=31=2^{2 n+2 i+1}-1$ so $m=2$ as well. We check that

$$
\sum_{j=1}^{15} \frac{1}{j^{2}}+31 \frac{7}{6} \sum_{j=1}^{15} \frac{1}{j^{3}}=\frac{1804176116127398723}{40110949726848000}
$$

and $1804176116127398723=19 \cdot 31^{4} \cdot 619 \cdot 809 \cdot 3901153$ is indeed divisible by $31^{4}$. (iii) And at last, when $n=1, i=2, p=2 n+3=5$, we also have $m=2$ as well. We check that

$$
\begin{aligned}
\binom{3}{1} \frac{2^{4}-1}{2^{0}} C_{0} H_{\frac{5-1}{2}}^{(4)} 5^{0}+\binom{4}{2} \frac{2^{5}-1}{2^{1}} C_{1} H_{\frac{5-1}{2}}^{(5)} 5^{1} & =15\left(1+\frac{1}{16}\right)+5 \cdot 31\left(1+\frac{1}{32}\right) \\
& =\frac{3^{2} \cdot 5^{4}}{2^{5}} \text { is indeed divisible by } 5^{4} .
\end{aligned}
$$

## 4. Extended congruences for $H_{\frac{p-1}{2}}^{(2 j-1)}(j \geq 1)$.

The case of $H_{\frac{p-1}{2}}^{(2 j-1)}(j \geq 1)$ has not been adressed in Section 3. This is a more complicated case as it does not involve only Bernoulli numbers, but also the base- 2 Fermat quotient. We begin with two propositions.

Proposition 4.1. Let $p$ be an odd prime and $n$ an integer, such that $p>\frac{n+1}{2} \geq 1$. We have the following congruence:

$$
\begin{align*}
H_{\frac{p-1}{2}} & +2 \sum_{j=0}^{n-1}(-1)^{j} \frac{q_{p}^{j+1}}{j+1} p^{j}+2 \delta_{p}^{n+1} q_{p} p^{n-1} \\
& +\sum_{1 \leq i<\frac{n+1}{2}} \frac{B_{p^{n-1}(p-1)-2 i}^{2 i}}{2 i}\left(2^{2 i+1}-1\right)\left(\frac{p}{2}\right)^{2 i} \equiv 0 \quad\left(\bmod p^{n}\right) \tag{4.1}
\end{align*}
$$

Remark. This proposition is very close to a particular case of Corollary 4.3 in [6] which was obtained from the theory of $p$-adic L-functions. Our statement here is stronger: at the small cost of the additional $2 \delta_{p}^{n+1} q_{p} p^{n-1}$ term, we can be less restrictive on $p$.

Proof of Proposition 4.1. The proof starts from Faulhaber formula (2.6). We have

$$
\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} j^{i} & =\frac{1}{i+1} \sum_{g=0}^{i}(-1)^{g}\binom{i+1}{g} B_{g}\left(\frac{p-1}{2}\right)^{i+1-g} \\
& =\frac{1}{i+1} \sum_{g=1}^{i+1}(-1)^{i+1-g}\binom{i+1}{g} B_{i+1-g}\left(\frac{p-1}{2}\right)^{g} \\
& =\frac{1}{i+1} \sum_{g=0}^{i}(-1)^{i-g} \frac{\binom{i+1}{g+1} B_{i-g}}{2^{g+1}}(p-1)^{g+1} \\
& =\frac{1}{i+1} \sum_{g=0}^{i}(-1)^{i-g} \frac{\binom{i+1}{g+1} B_{i-g}}{2^{g+1}} \sum_{m \geq 0} p^{m}(-1)^{g+1-m}\binom{g+1}{m}
\end{aligned}
$$

After inversion of the summations, with $\binom{i+1}{g+1}\binom{g+1}{m}=\binom{i+1}{m}\binom{i+1-m}{i-g}$ and some rearrangement, we obtain

$$
\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} j^{i} & =\frac{(-1)^{i+1}}{(i+1) 2^{i+1}} \sum_{0 \leq m}(-1)^{m} p^{m}\binom{i+1}{m} \sum_{g=0}^{i} 2^{g}\binom{i+1-m}{g} B_{g} \\
& =\frac{(-1)^{i+1}}{(i+1) 2^{i+1}} \sum_{g=0}^{i} 2^{g}\binom{i+1}{g} B_{g} \\
& +\frac{(-1)^{i+1}}{(i+1) 2^{i+1}} \sum_{1 \leq m}(-1)^{m} p^{m}\binom{i+1}{m} \sum_{g=0}^{i} 2^{g}\binom{i+1-m}{g} B_{g} \\
& =\frac{(-1)^{i+1}}{(i+1) 2^{i+1}} \sum_{g=0}^{i+1} 2^{g}\binom{i+1}{g} B_{g}-\frac{(-1)^{i+1}}{(i+1)} B_{i+1} \\
& -\frac{(-1)^{i+1}}{(i+1) 2^{i+1}} \sum_{0 \leq m}(-1)^{m} p^{m+1}\binom{i+1}{m+1} \sum_{g=0}^{i-m} 2^{g}\binom{i-m}{g} B_{g} .
\end{aligned}
$$

Then, accounting for (2.14) from Lemma 2.2 and rearranging, we have

$$
\begin{equation*}
\sum_{j=1}^{\frac{p-1}{2}} j^{i}=\frac{(-1)^{i+1}}{(i+1) 2^{i}} B_{i+1}\left(1-2^{i+1}\right)+\sum_{m \geq 0} \frac{(-1)^{m+i} p^{m+1}}{(m+1)!} m!\binom{i}{m} B_{i-m}\left(\frac{1}{2^{i}}-\frac{1}{2^{m+1}}\right) . \tag{4.2}
\end{equation*}
$$

Let $i=p^{n-1}(p-1)-1$.
By Euler theorem, $2^{i} \equiv \frac{1}{2} \bmod p^{n}$ and $H_{\frac{p-1}{2}} \equiv \sum_{j=1}^{\frac{p-1}{2}} j^{p^{n-1}(p-1)-1} \bmod p^{n}$, so that

$$
\begin{aligned}
H_{\frac{p-1}{2}} & \equiv \frac{2 p B_{p^{n-1}(p-1)}}{p-1} \frac{1-2^{p^{n-1}(p-1)}}{p^{n}} \\
& -\sum_{m \geq 1} \frac{(-1)^{m} p^{m}}{(m+1)!} m!\binom{p^{n-1}(p-1)-1}{m} p B_{p^{n-1}(p-1)-1-m}\left(2-\frac{1}{2^{m+1}}\right)
\end{aligned}
$$

$\left(\bmod p^{n}\right)$.
Note that the sum now starts from $m=1$, because $B_{p^{n-1}(p-1)-1}=0$. Now, we simplify the sum by accounting for Lemma 2.5 and Lemma 2.4 and we also make use of Lemma 2.6 and Lemma 2.7, so that

$$
\begin{aligned}
H_{\frac{p-1}{2}} & \equiv-2 \sum_{j=0}^{n-1}(-1)^{j} \frac{q_{p}^{j+1}}{j+1} p^{j}-2 \delta_{p}^{n+1} q_{p} p^{n-1} \\
& -\sum_{m \geq 1} \frac{p^{m+1}}{m+1} B_{p^{n-1}(p-1)-1-m}\left(2-\frac{1}{2^{m+1}}\right) \quad\left(\bmod p^{n}\right)
\end{aligned}
$$

Accounting for the vanishing Bernoulli numbers, when $m$ is even in the above congruence (nota: when $m=p^{n-1}(p-1)-2$, the Bernoulli number does not vanish: it is $-\frac{1}{2}$ which is $p$-integral, but since $p^{n-1}(p-1)-1 \geq n$ for $p \geq 3$, the corresponding term in the sum is clearly zero $\bmod p^{n}$ ), we can then write

$$
\begin{aligned}
H_{\frac{p-1}{2}} \equiv & -2 \sum_{j=0}^{n-1} \frac{(-p)^{j} q_{p}^{j+1}}{j+1}-2 \delta_{p}^{n+1} q_{p} p^{n-1}-\sum_{m \geq 1} \frac{p^{2 m}}{2 m} B_{p^{n-1}(p-1)-2 m}\left(2-\frac{1}{2^{2 m}}\right) \\
& \left(\bmod p^{n}\right)
\end{aligned}
$$

To complete the proof of (4.1), we argue that the second sum on the right hand side of the above congruence may be limited to $m<\frac{n+1}{2}$, because when $m \geq \frac{n+1}{2}, \frac{p^{2 m}}{2 m}$ is divisible by $p^{n+1}$ : this is obviously the case when $p$ does not divides $m$. When $p$ divides $m$, let $p^{k}$ be the largest power of $p$ which divides $m$. Suppose $2 m-k<n+1$ then $2 p^{k}-k<n+1$ but by hypothesis $n+1<2 p$, then $2 p-2 p^{k}+k \geq 2$ and this is not possible when $k>0$.

Proposition 4.2. Let $p$ be an odd prime. Let $n, h$ integers and $p>\frac{n+1}{2}>h \geq 1$. The following congruence holds:

$$
\begin{align*}
H_{\frac{p-1}{2}}^{2 h+1} & +\left(2^{2 h+1}-2\right) \frac{B_{p^{n-1}(p-1)-2 h}}{2 h} \\
& +\sum_{i=1}^{\frac{n-1}{2}} p^{2 i}\binom{2 i+2 h}{2 i} \frac{B_{p^{n-1}(p-1)-2(h+i)}}{2(h+i)}\left(2^{2 h+1}-\frac{1}{2^{2 i}}\right) \equiv 0 \quad\left(\bmod p^{n-1}\right) \tag{4.3}
\end{align*}
$$

Proof. We start from (4.2), with $i=p^{n-1}(p-1)-2 h-1$. Again by Euler theorem, we have $2^{i} \equiv \frac{1}{2^{2 h+1}} \bmod p^{n}, H_{\frac{p-1}{2}}^{2 h+1} \equiv \sum_{j=1}^{\frac{p-1}{2}} j^{p^{n-1}(p-1)-2 h-1} \bmod p^{n}$ and $2^{p^{n-1}(p-1)-2 h} \equiv 2^{-2 h} \bmod p^{n}$, so that, by the same argumentation as in the proof of the previous proposition, we have

$$
\begin{aligned}
H_{\frac{p-1}{2}}^{2 h+1} & \equiv \frac{2^{2 h+1}\left(1-2^{-2 h}\right)}{p^{n-1}(p-1)-2 h} B_{p^{n-1}(p-1)-2 h} \\
& -\sum_{m \geq 1} \frac{p^{m+1}}{m+1}\binom{m+2 h}{m} B_{p^{n-1}(p-1)-2 h-1-m}\left(2^{2 h+1}-\frac{1}{2^{m+1}}\right) \quad\left(\bmod p^{n}\right)
\end{aligned}
$$

We now account for the vanishing Bernoulli numbers, when $m$ is even in the above congruence. Nota: when $m=p^{n-1}(p-1)-2 h-2$, the Bernoulli number is $-\frac{1}{2}$ (which is $p$-integral), but in this case, since $n \geq 2 h$ by hypothesis, we have
$p^{n-1}(p-1)-2 h-1 \geq p^{n-1}(p-1)-n-1 \geq n+1$ for $p \geq 3$ and $n \geq 2$. Moreover $m+1=p^{n-1}(p-1)-2 h-1 \equiv-(2 h+1) \bmod p$, therefore, since $h<p$ by hypothesis, $p$ may divide $m+1$ only when $h=\frac{p-1}{2}$, and $p$ is the highest power of $p$ which may divide $m+1$. Then the term of index $m=p^{n-1}(p-1)-2 h-2$ in the sum is zero $\bmod p^{n}$. We can then write

$$
\begin{aligned}
H_{\frac{p-1}{2}}^{2 h+1} & \equiv \frac{2^{2 h+1}\left(1-2^{-2 h}\right)}{p^{n-1}(p-1)-2 h} B_{p^{n-1}(p-1)-2 h} \\
& -\sum_{m \geq 1} \frac{p^{2 m}}{2 m}\binom{2 m+2 h-1}{2 m-1} B_{p^{n-1}(p-1)-2(h+m)}\left(2^{2 h+1}-\frac{1}{2^{2 m}}\right) \quad\left(\bmod p^{n}\right) \\
& \equiv \frac{2^{2 h+1}\left(1-2^{-2 h}\right)}{p^{n-1}(p-1)-2 h} B_{p^{n-1}(p-1)-2 h} \\
& -\sum_{m \geq 1} p^{2 m}\binom{2 m+2 h}{2 m} \frac{B_{p^{n-1}(p-1)-2(h+m)}}{2(h+m)}\left(2^{2 h+1}-\frac{1}{2^{2 m}}\right) \quad\left(\bmod p^{n}\right)
\end{aligned}
$$

Reducing $\bmod p^{n-1}$, we obtain

$$
\begin{aligned}
H_{\frac{p-1}{2}}^{2 h+1} & \equiv \frac{2-2^{2 h+1}}{2 h} B_{p^{n-1}(p-1)-2 h} \\
& -\sum_{m \geq 1} p^{2 m}\binom{2 m+2 h}{2 m} \frac{B_{p^{n-1}(p-1)-2(h+m)}^{2(h+m)}\left(2^{2 h+1}-\frac{1}{2^{2 m}}\right) \quad\left(\bmod p^{n-1}\right)}{} .
\end{aligned}
$$

To complete the proof of (4.3), we argue that the second sum on the right hand side of the above congruence may be limited to $m \leq \frac{n-1}{2}$, because the summands may also be written like

$$
\frac{p^{2 m}}{2 m}\binom{2 m+2 h-1}{2 m-1} B_{p^{n-1}(p-1)-2(h+m)}\left(2^{2 h+1}-\frac{1}{2^{2 m}}\right)
$$

and when $m>\frac{n-1}{2}, \frac{p^{2 m}}{2 m}$ is divisible by $p^{n}$ : this is obviously the case when $p$ does not divides $m$. When $p$ divides $m$, let $p^{k}$ be the largest power of $p$ which divides $m$. Suppose $2 m-k<n$ then $2 p^{k}-k<n$ but by hypothesis $n+1<2 p$, so $2 p-2 p^{k}+k \geq 3$. This is impossible when $k>0$.

We are now ready for our final result:
Theorem 4.1. Let $n \geq 1$ be an integer, $p$ an odd prime, $p>\frac{n+1}{2}$. We have

$$
\begin{equation*}
\sum_{j=0}^{n-1} \frac{\left(2^{j+1}-1\right)}{(j+1)} \frac{\left(2^{j+2}-1\right)}{(j+2)} \frac{B_{j+2}}{2^{j-1}} H_{\frac{p-1}{2}}^{(j+1)} p^{j}+\sum_{j=0}^{n-1}(-1)^{j} \frac{q_{p}^{j+1}}{j+1} p^{j} \equiv 0 \quad\left(\bmod p^{n}\right) \tag{4.4}
\end{equation*}
$$

Remark. Eisenstein and E. Lehmer congruences (1.3) and (1.4) are particular cases of (4.4) for $n=1$ and $n=2$, respectively. Next, for $n=3$, we obtain

$$
H_{\frac{p-1}{2}}-\frac{7}{24} H_{\frac{p-1}{2}}^{(3)} p^{2}+2\left(q_{p}-\frac{q_{p}^{2}}{2} p+\frac{q_{p}^{3}}{3} p^{2}\right) \equiv 0 \quad\left(\bmod p^{3}\right)
$$

By taking (3.16) into account, we recover a congruence originally given by Z.-H. Sun ( 9 , Theorem 5.2(c)):

$$
H_{\frac{p-1}{2}}+\frac{7}{12} B_{p-3} p^{2}+2\left(q_{p}-\frac{q_{p}^{2}}{2} p+\frac{q_{p}^{3}}{3} p^{2}\right) \equiv 0 \quad\left(\bmod p^{3}\right) .
$$

Next

$$
H_{\frac{p-1}{2}}-\frac{7}{24} H_{\frac{p-1}{2}}^{(3)} p^{2}+2\left(q_{p}-\frac{q_{p}^{2}}{2} p+\frac{q_{p}^{3}}{3} p^{2}-\frac{q_{p}^{4}}{4} p^{3}\right) \equiv 0 \quad\left(\bmod p^{4}\right)
$$

Next, under the condition $p \geq 5$, we have

$$
\begin{aligned}
& H_{\frac{p-1}{2}}-\frac{7}{24} H_{\frac{p-1}{2}}^{(3)} p^{2}+\frac{31}{80} H_{\frac{p-1}{2}}^{(5)} p^{4}+2\left(q_{p}-\frac{q_{p}^{2}}{2} p+\frac{q_{p}^{3}}{3} p^{2}-\frac{q_{p}^{4}}{4} p^{3}+\frac{q_{p}^{5}}{5} p^{4}\right) \equiv 0 \\
& H_{\frac{p-1}{2}}-\frac{7}{24} H_{\frac{p-1}{2}}^{(3)} p^{2}-\frac{93}{40} B_{p-5} p^{4}+2\left(q_{p}-\frac{q_{p}^{2}}{2} p+\frac{q_{p}^{3}}{3} p^{2}-\frac{q_{p}^{4}}{4} p^{3}+\frac{q_{p}^{5}}{5} p^{4}\right) \equiv 0
\end{aligned}
$$

$\left(\bmod p^{5}\right)$
etc.
Remark. The condition $p>\frac{n+1}{2}$ is sharp. For instance, for $n=5$ and $p=3$, we have $q_{p}=1, H_{\frac{p-1}{2}}^{(j)}=1$ and the left hand side of (4.4) is not divisible by $3^{5}$. Indeed, we check that $1-\frac{7 \cdot 3^{2}}{24}+\frac{31 \cdot 3^{4}}{80}+2\left(1-\frac{3}{2}+\frac{3^{2}}{3}-\frac{3^{3}}{4}+\frac{3^{4}}{5}\right)=\frac{4293}{80}=\frac{3^{4} \cdot 53}{5 \cdot 2^{4}}$.

Proof of Theorem 4.1. It suffices to prove (4.4) when $n$ is even, because then it is true for $n-1$ since the first sum in (4.4) is actually limited to $n-2$ when $n$ is even $\left(B_{n+1}=0\right)$ and $\frac{q_{p}^{n}}{n} p^{n-1} \equiv 0 \bmod p^{n-1}$ because $p$ does not divide $n$ otherwise $p \leq \frac{n}{2}$ which is excluded by hypothesis. So from now on, it is supposed that $n$ is even. For $h \geq 1$ an integer we define $Z_{n, h}:=\frac{B_{p^{n-1}(p-1)-2 h}}{2 h}$ and $A_{h}:=4 \frac{\left(2^{2 h+2}-1\right) B_{2 h+2}}{(2 h+1)(2 h+2)}$. Congruences (4.1) and (4.3) now read, respectively,

$$
\begin{equation*}
H_{\frac{p-1}{2}}+2 \sum_{j=0}^{n-1}(-1)^{j} \frac{q_{p}^{j+1}}{j+1} p^{j}+2 \delta_{p}^{n+1} q_{p} p^{n-1}+\sum_{i=1}^{\frac{n}{2}} Z_{n, i}\left(2^{2 i+1}-1\right)\left(\frac{p}{2}\right)^{2 i} \equiv 0 \bmod p^{n} \tag{4.5}
\end{equation*}
$$

$$
\begin{equation*}
H_{\frac{p-1}{2}}^{(2 h+1)}+Z_{n, h}\left(2^{2 h+1}-2\right)+\sum_{i=1}^{\frac{n-2}{2}}\binom{2 h+2 i}{2 h} Z_{n, h+i}\left(2^{2(h+i)+1}-1\right)\left(\frac{p}{2}\right)^{2 i} \equiv 0 \bmod p^{n-1} \tag{4.6}
\end{equation*}
$$

Comparing (4.5) to (4.4), eliminating the terms with powers of the Fermat quotient, rearranging and using $h$ as summation index, we see that we need to prove that

$$
\left(2 q_{p} \delta_{p}^{n+1}+p Z_{n, \frac{n}{2}} \frac{2^{n+1}-1}{2^{n}}\right) p^{n-1} \equiv \sum_{h=1}^{\frac{n-2}{2}}\left(2^{2 h+1}-1\right)\left(\frac{p}{2}\right)^{2 h}\left(A_{h} H_{\frac{p-1}{2}}^{(2 h+1)}-Z_{n, h}\right)
$$

$$
\begin{equation*}
\bmod p^{n} \tag{4.7}
\end{equation*}
$$

Now (4.6) may be rearranged, so that

$$
H_{\frac{p-1}{2}}^{(2 h+1)} \equiv Z_{n, h}-\sum_{i=0}^{\frac{n-2}{2}}\binom{2 h+2 i}{2 h} Z_{n, h+i}\left(2^{2(h+i)+1}-1\right)\left(\frac{p}{2}\right)^{2 i} \quad\left(\bmod p^{n-1}\right)
$$

In Congruence (4.7), $H_{\frac{p-1}{2}}^{(2 h+1)}$ may be replaced by the right hand side of the latter congruence because (i) $\left(2^{2 h+2}-1\right) B_{2 h+2}$ is $p$-integral, by the Von StaudtClausen theorem, and (ii) $\frac{p^{2 h}}{(2 h+1)(2 h+2)} \equiv 0 \bmod p$ when $h \geq 1$, since $\frac{p^{2 h}}{(2 h+1)(2 h+2)}=$ $\frac{p^{2 h}}{(2 h+1)!} \frac{(2 h)!}{2(h+1)}$, where the second factor is clearly an integer while the first one is $0 \bmod p$ by Lemma 2.5.

Let $L$ (resp. $R$ ) be the left (resp. right) hand side of (4.7), then we have

$$
\begin{aligned}
R & \equiv \sum_{h=1}^{\frac{n-2}{2}}\left(2^{2 h+1}-1\right)\left(\frac{p}{2}\right)^{2 h}\left(A_{h}-1\right) Z_{n, h} \\
& -\sum_{h=1}^{\frac{n-2}{2}} \sum_{i=0}^{\frac{n-2}{2}} A_{h}\left(2^{2 h+1}-1\right)\binom{2 h+2 i}{2 i} Z_{n, h+i}\left(2^{2(h+i)+1}-1\right)\left(\frac{p}{2}\right)^{2 i+2 h} \quad\left(\bmod p^{n}\right) \\
& \equiv \sum_{h=1}^{\frac{n-2}{2}}\left(2^{2 h+1}-1\right)\left(\frac{p}{2}\right)^{2 h}\left(A_{h}-1\right) Z_{n, h} \\
& -\sum_{h=1}^{\frac{n-2}{2}} \sum_{i=1}^{\frac{n-2}{2}} A_{h}\left(2^{2 h+1}-1\right)\binom{2 h+2 i}{2 i} Z_{n, h+i}\left(2^{2(h+i)+1}-1\right)\left(\frac{p}{2}\right)^{2 i+2 h} \\
& -\sum_{h=1}^{\frac{n-2}{2}} A_{h}\left(2^{2 h+1}-1\right) Z_{n, h}\left(2^{2 h+1}-1\right)\left(\frac{p}{2}\right)^{2 h}\left(\bmod p^{n}\right)
\end{aligned}
$$

Let $k=i+h$, and we re-index the sums in the second line with $h$ and $k . k$ runs from 2 to $n-2$ and since $1 \leq i, h \leq \frac{n-2}{2}$, $h$ must also satisfy $k-\frac{n-2}{2} \leq h \leq k-1$
so that actually $h$ runs from $\max \left(1, k-\frac{n-2}{2}\right)$ to $\min \left(\frac{n-2}{2}, k-1\right)$, and then

$$
\begin{aligned}
R & \equiv \sum_{h=1}^{\frac{n-2}{2}}\left(2^{2 h+1}-1\right)\left(\frac{p}{2}\right)^{2 h}\left(A_{h}-1\right) Z_{n, h} \\
& -\sum_{k=2}^{n-2} Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k} \sum_{h=\max \left(1, k-\frac{n-2}{2}\right)}^{\min \left(\frac{n-2}{2}, k-1\right)}\binom{2 k}{2 h} A_{h}\left(2^{2 h+1}-1\right) \\
& -\sum_{h=1}^{\frac{n-2}{2}} A_{h}\left(2^{2 h+1}-1\right) Z_{n, h}\left(2^{2 h+1}-1\right)\left(\frac{p}{2}\right)^{2 h} \quad\left(\bmod p^{n}\right) .
\end{aligned}
$$

We now argue that, in the above congruence, $k$ may be limited to $k \leq \frac{n}{2}$. Indeed, $p$ may divide the denominator of $\binom{2 k}{2 h} Z_{n, k}$ once at most, because

$$
\binom{2 k}{2 h} Z_{n, k}=\frac{1}{2 h}\binom{2 k-1}{2 h-1} B_{p^{n-1}(p-1)-2 k}
$$

and $p$ does not divide $h$ because $p>\frac{n+1}{2} \geq h+\frac{3}{2}$. On the other hand, $p$ may also divide the denominator of $4 \frac{\left(2^{2 h+2}-1\right) B_{2 h+2}}{(2 h+1)(2 h+2)}$ once at most, since $\left(2^{2 h+2}-1\right) B_{2 h+2}$ is $p$-integral, $p$ does not divide $h+1$, since $p>h+1$ and if we suppose $p^{g}$ is the largest power of $p$ that divides $2 h+1$, then $p^{g} \leq 2 h+1 \leq n-2+1=n-1<2 p-1-1$ that is $p^{g}<2 p-2$, which is possible only if $g \leq 1$. Then

$$
\begin{aligned}
R & \equiv \sum_{k=1}^{\frac{n-2}{2}}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k}\left(A_{k}-1\right) Z_{n, k} \\
& -\sum_{k=2}^{\frac{n}{2}} Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k} \sum_{h=1}^{k-1}\binom{2 k}{2 h} A_{h}\left(2^{2 h+1}-1\right) \\
& -\sum_{k=1}^{\frac{n-2}{2}} A_{k}\left(2^{2 k+1}-1\right) Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k} \quad\left(\bmod p^{n}\right) .
\end{aligned}
$$

We now evaluate the inner sum in the second line of the above congruence:

$$
\begin{aligned}
\sum_{h=1}^{k} A_{h}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h} & =\sum_{h=1}^{k} 4 \frac{\left(2^{2 h+2}-1\right) B_{2 h+2}}{(2 h+1)(2 h+2)}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h} \\
& =\sum_{h=0}^{k} 4 \frac{\left(2^{2 h+2}-1\right) B_{2 h+2}}{(2 h+1)(2 h+2)}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h}-4 \frac{\left(2^{2}-1\right) B_{2}}{2}
\end{aligned}
$$

That is

$$
\begin{aligned}
\sum_{h=1}^{k} A_{h}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h} & =\sum_{h=0}^{k} 4 \frac{\left(2^{2 h+2}-1\right) B_{2 h+2}}{(2 h+1)(2 h+2)}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h}-1 \\
& =\frac{1}{2 k+1} \sum_{h=0}^{k} 4 \frac{\left(2^{2 h+2}-1\right) B_{2 h+2}}{(2 h+2)}\left(2^{2 h+1}-1\right)\binom{2 k+1}{2 h+1}-1
\end{aligned}
$$

On the right hand side of the last line, let $j=2 h+1 . j$ takes all odd values from 1 to $2 k+1$, but we may let $j$ take all values from 0 to $2 k+1$, because the summands corresponding to even $j$ are zero, either because $2^{0}-1=0$ or because of vanishing Bernoulli numbers.

Then

$$
\begin{aligned}
\sum_{h=1}^{k} A_{h}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h} & =\frac{4}{2 k+1} \sum_{j=0}^{2 k+1} \frac{B_{j+1}}{(j+1)}\left(2^{j}-1\right)\left(2^{j+1}-1\right)\binom{2 k+1}{j}-1 \\
& =\frac{4}{2 k+1} \frac{B_{2 k+2}}{2 k+2}\left(2^{2 k+2}-1\right)-1 \quad \text { by Lemma } 2.3 \\
& =A_{k}-1
\end{aligned}
$$

Then

$$
\sum_{h=1}^{k-1} A_{h}\left(2^{2 h+1}-1\right)\binom{2 k}{2 h}=A_{k}-1-A_{k}\left(2^{2 k+1}-1\right)
$$

And the latter may now be used into $R$, so that

$$
\begin{aligned}
R & \equiv \sum_{k=1}^{\frac{n-2}{2}}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k}\left(A_{k}-1\right) Z_{n, k} \\
& -\sum_{k=2}^{\frac{n}{2}} Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k}\left(A_{k}-1-A_{k}\left(2^{2 k+1}-1\right)\right) \\
& -\sum_{k=1}^{\frac{n-2}{2}} A_{k}\left(2^{2 k+1}-1\right) Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k} \quad\left(\bmod p^{n}\right)
\end{aligned}
$$

That is

$$
\begin{aligned}
R & \equiv \sum_{k=1}^{\frac{n-2}{2}}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k}\left(A_{k}-1\right) Z_{n, k} \\
& -\sum_{k=1}^{\frac{n-2}{2}} Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k}\left(A_{k}-1-A_{k}\left(2^{2 k+1}-1\right)\right) \\
& -\sum_{k=1}^{\frac{n-2}{2}} A_{k}\left(2^{2 k+1}-1\right) Z_{n, k}\left(2^{2 k+1}-1\right)\left(\frac{p}{2}\right)^{2 k} \\
& +7 Z_{n, 1}\left(\frac{p}{2}\right)^{2}\left(-6 A_{1}-1\right)-Z_{n, \frac{n}{2}}\left(2^{n+1}-1\right)\left(\frac{p}{2}\right)^{n}\left(2 A_{\frac{n}{2}}\left(1-2^{n}\right)-1\right) \quad\left(\bmod p^{n}\right)
\end{aligned}
$$

This can be considerably simplified, since all the sums cancel out and $A_{1}=-\frac{1}{6}$, so that we are left with

$$
R \equiv p Z_{n, \frac{n}{2}} \frac{2^{n+1}-1}{2^{n}}\left(2 A_{\frac{n}{2}}\left(2^{n}-1\right)+1\right) p^{n-1} \quad\left(\bmod p^{n}\right)
$$

Recall that in order to prove our theorem we just need to show that $L \equiv R \bmod p^{n}$. So there merely remains to prove that

$$
\begin{equation*}
q_{p} \delta_{p}^{n+1} \equiv p Z_{n, \frac{n}{2}} \frac{\left(2^{n+2}-1\right)\left(2^{n+1}-1\right)\left(2^{n}-1\right) B_{n+2}}{2^{n-2}(n+1)(n+2)} \quad(\bmod p) \tag{4.8}
\end{equation*}
$$

This is obtained as follows. Recall that $\left(2^{n+2}-1\right) B_{n+2}$ is $p$-integral, and first consider the case where $p-1$ does not divide $n$, then $Z_{n, \frac{n}{2}}$ is $p$-integral since $p$ does not divide $n$ otherwise $p \leq \frac{n}{2}$ which is excluded by hypothesis. Moreover, $p$ does not divide $n+2$ otherwise $p \leq \frac{n}{2}+1$ and $\frac{n+1}{2}<p$, then $p=\frac{n}{2}+1$, this is impossible since $p-1$ does not divide $n$; and $p$ does not divide $n+1$ either, because $p \leq n+1<2 p$ would imply $p=n+1$ which is impossible since $p-1$ does not divide $n$. So we have shown that if $p-1$ does not divide $n$, the right hand side of (4.8) is $0 \bmod p$ and obviously so is the left hand side since in this case $\delta_{p}^{n+1}=0$. Now, if $p-1$ divides $n$, then since $p>\frac{n+1}{2}$, either $p=n+1$ or $2 p=n+2$. If $n=2 p-2$, the left hand side of (4.8) is 0 and by Lemma 2.6, $p Z_{n, \frac{n}{2}} \equiv \frac{1}{2} \bmod p$. If $p=3, n=4$, the right hand side of (4.8) is $\frac{1}{2} \frac{\left(2^{6}-1\right)\left(2^{5}-1\right)\left(2^{4}-1\right)}{2^{2}(4+1)(4+2)(42)}=\frac{93}{32} \equiv 0 \bmod 3$. If $p \neq 3$, by Kummer congruence (2.7), $\frac{B_{n+2}}{n+2} \equiv \frac{B_{2}}{2} \bmod p$ then the the right hand side of (4.8) is $\frac{1}{2} \frac{\left(2^{2 p}-1\right)\left(2^{2 p-1}-1\right)\left(4^{p-1}-1\right)}{2^{2 p-4}(2 p-1) 12}$ which is $0 \bmod p$, by Fermat little theorem. If $n=p-1$, the left hand side of (4.8) is $q_{p}$ and by Lemma 2.6] $p Z_{n, \frac{n}{2}} \equiv 1 \bmod p$. If $p=3, n=2$, $q_{3}=1$ and the right hand side of (4.8) is $-\frac{\left(2^{4}-1\right)\left(2^{3}-1\right)\left(2^{2}-1\right)}{(2+1)(2+2)(30)}=\frac{-7}{8} \equiv 1=q_{3} \bmod 3$. If $p \neq 3$, by Kummer congruence (2.7), $\frac{B_{n+2}}{n+2} \equiv \frac{B_{2}}{2} \bmod p$ then the right hand side is $\frac{\left(2^{p+1}-1\right)\left(2^{p}-1\right)\left(2^{p-1}-1\right)}{2^{p-3} \cdot p \cdot 12} \equiv \frac{2^{p-1}-1}{p}=q_{p} \bmod p$.

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