**A Generalization of the "Raboter" operation.**

*By Yonah BIERS-ARIEL*

# 1 Introduction

In a recent talk at Rutgers' Experimental Math Seminar, Neil Sloane described the "raboter" operation for the base two representation of a number [1]. From this representation, one reduces by one the length of each run of consecutive 1s and 0s. Denote this operation by $r(n)$; so, for example, $r(12) = 2$ because 12 is represented in binary as 1100, and reducing the length of each run by one yields 10.

Sloane also defined $L(k) = \sum_{n=2^k}^{2^{k+1}-1} r(n)$ and conjectured that $L(k) = 2 \cdot 3^{k-1} - 2^{k-1}$, a fact which was quickly proven by Doron Zeilberger [3] and Chai Wah Wu [2].

In Section 2, we generalize this theorem to bases other than 2. Let $r(b, n)$ be the the number whose base-$b$ representation is generated by taking the base-$b$ representation of $n$ and shortening each run of consecutive identical elements by one. Further, let $L(b, n) = \sum_{n=b^k}^{b^{k+1}-1} r(b, n)$. We will prove that

$$L(b, k) = \frac{b(b-1)}{2b-1}(2b-1)^k - \frac{b-1}{2}b^k.$$

In Section 3, we raise $r(b, n)$ to various powers. Define $L(p, b, k) = \sum_{n=b^k}^{b^{k+1}-1} r(b, n)^p$; we develop an algorithm in Maple to rigorously compute $L(p, b, k)$ as an expression in terms of $k$ for any fixed $p, b$. In addition, for any fixed $p$, we can conjecture an expression for $L(p, b, k)$ in terms of $b$ and $k$.

# 2 More General Bases

Following the example of Zeilberger, we find a recurrence satisfied by $L(b, k)$ and then find a closed form expression satisfying the same recurrence.

**Theorem 2.1.** $L(b, k) = (2b-1) \cdot L(b, k-1) + b^{k-1}\dfrac{(b-1)^2}{2}$ *for* $k \geq 2$.

*Proof.* There are $b^{k+1} - b^k = b^k(b-1)$ numbers which contribute to $L(b, k)$ are exactly those numbers whose base-$b$ representations use $k+1$ digits, so each can be written as $Ab_1b_2$ where $A \in \{1, \ldots, b-1\} \times \{0, \ldots, b-1\}^{k-2}$ and $b_1, b_2 \in \{0, \ldots, b-1\}$. If $b_1 \neq b_2$, then $b_2$ is a run of just one element, so the raboter operation eliminates it and $r(b, Ab_1b_2) = r(b, Ab_1)$. Numbers with representations $Ab_1$ are exactly those which were counted in the calculation of $L(b, k-1)$, and each is counted $b-1$ times here, once for each $b_2 \neq b_1$.

If $b_2 = b_1$, then the base-$b$ representation of $r(b, Ab_1b_2)$ is the representation of $r(b, Ab_1)$ with $b_2$ appended to the end, and so $r(b, Ab_1b_2) = b \cdot r(b, Ab_1) + b_2$. Thus,

$$\begin{aligned} L(b,k) &= \sum_A \sum_{b_1} r(b, Ab_1b_1) + \sum_{b_2 \neq b_1} r(b, Ab_1b_2) \\ &= \sum_A \sum_{b_1} b \cdot r(b, Ab_1) + b_1 + \sum_{b_2 \neq b_1} r(b, Ab_1) \\ &= \sum_A \sum_{b_1} (2b-1) r(b, Ab_1) + \sum_A \sum_{b_1} b_1 \\ &= (2b-1)L(b, k-1) + (b-1)b^{k-2}\frac{b(b-1)}{2} \\ &= (2b-1)L(b, k-1) + b^{k-1}\frac{(b-1)^2}{2}. \end{aligned}$$

□

Together with initial condition $L(b, 1) = \frac{b(b-1)}{2}$, this determines the sequence $(L(b,k))_{k=1}^{\infty}$. Finding an explicit formula for $L(b,k)$ is now just a matter of finding a formula which obeys this same recurrence.

**Corollary 2.2.** $L(b,k) = \frac{b(b-1)}{2b-1}(2b-1)^k - \frac{b-1}{2}b^k$.

*Proof.* With some help from Doron Zeilberger's Maple package Cfinite, we conjecture that the formula for $L(b,k)$ has the form $\alpha_1(2b-1)^k + \alpha_2 b^k$, so we solve the system of equations

$$\begin{aligned} \alpha_1(2b-1) + \alpha_2 b &= \frac{b(b-1)}{2} \\ \alpha_1(2b-1)^2 + \alpha_2 b^2 &= (2b-1)\frac{b(b-1)}{2} + b\frac{(b-1)^2}{2} \end{aligned}$$

for $\alpha_1, \alpha_2$ and find $\alpha_1 = \frac{b(b-1)}{2b-1}$ and $\alpha_2 = -\frac{b}{2} + \frac{1}{2}$. Let $L'(b,k) = \dfrac{b(b-1)}{2b-1}(2b-1)^k - \dfrac{b-1}{2}b^k$. Proving that $L(b,k) = L'(b,k)$ is simply a matter of verifying that $L'(b,1) = \frac{b(b-1)}{2}$ and $L'(b,k) = (2b-1) \cdot L'(b,k-1) + b^{k-1}\dfrac{(b-1)^2}{2}$ for $k \geq 2$, which can easily be done using Maple or any other computer algebra system. □

# 3 Higher Moments

With a formula for $L(b,k)$ found, we consider the following additional generalization:

$$L(p,b,k) = \sum_{n=2^k}^{2^{k+1}-1} r(b,n)^p;$$

that is the sum of $r(b,n)^p$ taken over all numbers $n$ whose base-$b$ representation has $k+1$-digits. The trick in this case is to work inductively beginning with the (solved) $p = 1$ case, and, along the way compute $L(l,p,b,k)$ which we define to be the sum of $r(b,n)^p$ taken over all numbers $n$ whose base-$b$ representation has $k+1$-digits, the last of which is $l$.

In order to compute $L(l,p,b,k)$, we use the following recurrence:

**Theorem 3.1.** $L(l,p,b,k) = (b^p-1)\cdot L(l,p,b,k-1) + L(p,b,k-1) + \sum_{i=1}^{p} l^i b^{p-i}\binom{p}{i} L(l,p-i,b,k-1)$.

*Proof.* The numbers with length-$(k+1)$ base-$b$ representations ending in $l$ are exactly those which can be written as $Ab_1b_2$ with $A \in \{1,\dots,b-1\}\times\{0,\dots,b-1\}^{k-2}$, $b_1 \in \{0,\dots,b-1\}$, and $b_2 = l$. Therefore,

$$\begin{aligned}
L(l,p,b,k) &= \sum_A \Big(\sum_{b_1\neq l} r(b,Ab_1l)^p + r(b,All)^p\Big)\\
&= \sum_A \Big(\sum_{b_1\neq l} r(b,Ab_1)^p + (b\cdot r(b,Al)+l)^p\Big)\\
&= L(p,b,k-1) - L(l,p,b,k-1) + \sum_A \sum_{i=0}^{p} \binom{p}{i} b^{p-i} r(b,Al)^{p-i} l^i\\
&= L(p,b,k-1) - L(l,p,b,k-1) + b^p L(l,p,b,k-1)\\
&\qquad + \sum_{i=1}^{p} \binom{p}{i} b^{p-i} L(l,p-i,b,k-1)^{p-i} l^i\\
&= (b^p-1)\cdot L(l,p,b,k-1) + L(p,b,k-1) + \sum_{i=1}^{p} l^i b^{p-i}\binom{p}{i} L(l,p-i,b,k-1).
\end{aligned}$$

□

We find a similar recurrence for $L(p,b,k)$.

**Theorem 3.2.** $L(p,b,k) = (b^p+b-1)L(p,b,k-1) + \sum_{l=0}^{b-1}\sum_{i=1}^{p} b^{p-i} l^i \binom{p}{i} L(l,p-i,b,k-1)$.

*Proof.* Again, note that the numbers counted by $L(p,b,k)$ are those which can be written as $Ab_1b_2$ with $A \in \{1,\dots,b-1\}\times\{0,\dots,b-1\}^{k-2}$, and $b_1,b_2 \in \{0,\dots,b-1\}$. Therefore, the following equations hold:

$$\begin{aligned}
L(p,b,k) &= \sum_A \Big(\sum_{b_1\neq b_2} r(b,Ab_1b_2)^p + \sum_{b_1} r(b,Ab_1b_1)^p\Big)\\
&= \sum_A (b-1)\sum_{b_1} r(b,Ab_1)^p + \sum_A\sum_{b_1} (br(Ab_1)+b_1)^p\\
&= (b-1)L(p,b,k-1) + \sum_A\sum_{b_1}\sum_{i=0}^{p}\binom{p}{i} b^{p-i} r(Ab_1)^{p-i} b_1^i\\
&= (b-1)L(p,b,k-1) + \sum_A\sum_{b_1} b^p r(Ab_1)^p + \sum_{b_1}\sum_{i=1}^{p}\sum_A \binom{p}{i} b^{p-i} r(Ab_1)^{p-i} b_1^i\\
&= (b^p+b-1)L(p,b,k-1) + \sum_{b_1}\sum_{i=1}^{p} b_1^i b^{p-i}\binom{p}{i} L(b_1,p-i,b,k-1).
\end{aligned}$$

Change the name of $b_1$ to $l$ to maintain consistent notation, and we have derived the claimed equation. □

## 4 Maple Implementation

The Maple package `raboter.txt` available at `http://sites.math.rutgers.edu/~yb165/raboter.txt` contains functions to implement this recurrence. The most important are `SumPowers(b,k,p)` which finds an expression in terms of $k$ for $L(p, b, k)$ (for fixed $b$ and $p$) and `GuessGeneralForm(b,n,p)` which conjectures an expression in terms of $k$ and $b$ for $L(p, b, k)$ (for fixed $p$).

For example, this package proves that

$$L(2, 2, k) = \frac{2}{3}5^k - \frac{1}{6}2^k - \frac{2}{3}3^k$$

and conjectures that

$$L(2, b, k) = \Big(\frac{1}{6}b^2 - \frac{1}{6}b - \frac{1}{3}\Big)(b-1)^k + \Big(-\frac{1}{6}b^2 + \frac{1}{3}b - \frac{1}{6}\Big)b^k - \frac{b(b-1)}{2b-1}(2b-1)^k + \frac{2b^3 + 3b^2 - 3b - 2}{6(b^2 + b - 1)}(b^2 + b - 1)^k.$$

## References


[1] N.J.A. Sloane, *Coordination Sequences, Planing Numbers, and Other Recent Sequences*, talk given in Rutgers University Experimental Mathematics Seminar, Nov. 15, 2018. Video part 1: `https://vimeo.com/301216222`; video part 2: `https://vimeo.com/301219515`; slides: `http://sites.math.rutgers.edu/~my237/expmath/EMNov2018.pdf`.

[2] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, **Sequence A318921**, `http://oeis.org/A318921`.

[3] Doron Zeilberger, "Proof of a Conjecture of Neil Sloane Concerning Claude Lenormand's "Raboter" Operation (OEIS sequence A318921)" *The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger* (2018). `http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/rabot.pdf`.