Irrationality Measure of Pi

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Abstract: The first estimate of the upper bound $\mu(\pi) \leq 42$ of the irrationality measure of the number π was computed by Mahler in 1953, and more recently it was reduced to $\mu(\pi) \leq 7.6063$ by Salikhov in 2008. Here, it is shown that π has the same irrationality measure $\mu(\pi) = \mu(\alpha) = 2$ as almost every irrational number $\alpha > 0$.

1 Introduction

The irrationality measure $\mu(\alpha)$ of a real number $\alpha \in \mathbb{R}$ is the infimum of the subset of real numbers $\mu(\alpha) \geq 1$ for which the Diophantine inequality

$$\left| \alpha - \frac{p}{q} \right| \ll \frac{1}{q^{\mu(\alpha)}} \tag{1}$$

has finitely many rational solutions p and q, see [5, Chapter 11]. The map $\mu : \mathbb{R} \longrightarrow [2, \infty) \cup \{1\}$ is surjective. Any number in the set $[2, \infty) \cup \{1\}$ is the irrationality measure of some irrational number, confer [7, Theorem 2]. More precisely,

(1) A rational number has an irrationality measure of $\mu(\alpha) = 1$, see [12, Theorem 186].

(2) An algebraic irrational number has an irrationality measure of $\mu(\alpha) = 2$, confer Roth Theorem.

(3) Any irrational number has an irrationality measure of $\mu(\alpha) \geq 2$.

(4) A Champernowne number $\kappa_b = 0.123 \cdots b-1 \cdot b \cdot b+1 \cdot b+2 \cdots$ in base $b \geq 2$, concatenation of the b-base integers, has an irrationality measure of $\mu(\kappa_b) = b$.

(5) A Mahler number $\psi_b = \sum_{n \geq 1} b^{-[\tau]^n}$ in base $b \geq 3$ has an irrationality measure of $\mu(\psi_b) = \tau$, for any real number $\tau \geq 2$, see [7, Theorem 2].

(6) A Liouville number $\ell_b = \sum_{n \geq 1} b^{-n!}$ parametized by $b \geq 2$ has an irrationality measure of $\mu(\ell_b) = \infty$, see [12, p. 208].

The analysis of the irrationality measure $\mu(\pi) \geq 2$ was initiated by Mahler in 1953, who proved that

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{42}} \tag{2}$$

for all rational solutions p and q, see [13], et alii. Over the last seven decades, the efforts of several authors have improved this estimate significantly, see Table 1. More recently, it was reduced to $\mu(\pi) \leq 7.6063$, see [16].

Theorem 1.1. *For any number $\varepsilon > 0$, the Diophantine inequality*

$$\left| \pi - \frac{p}{q} \right| \ll \frac{1}{q^{2+\varepsilon}} \tag{3}$$

has finitely many rational solutions p and q. In particular, the irrationality measure $\mu(\pi) = 2$.

After some preliminary preparations, the proof of Theorem 1.1 is assembled in Section 6.

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Table 1: Historical Data For $\mu(\pi)$

Irrationality Measure Upper Bound	Reference	Year
$\mu(\pi) \leq 42$	Mahler, [13]	1953
$\mu(\pi) \leq 20.6$	Mignotte, [14]	1974
$\mu(\pi) \leq 14.65$	Chudnovsky, [8]	1982
$\mu(\pi) \leq 13.398$	Hata, [11]	1993
$\mu(\pi) \leq 7.6063$	Salikhov, [16]	2008

2 Asymptotic Expansions Of The Sine Function

Let $f \geq 0$ and $g \geq 0$ be real valued functions. The proportional symbol $f \asymp g$ is defined by $c_0 g \leq f \leq c_1 g$, where $c_0, c_1 \in \mathbb{R}$ are constants.

Lemma 2.1. *Let p_n/q_n be the sequence of convergents of the irrational number $\pi = [a_0; a_1, a_2, \ldots]$. Then, the followings hold.*

(i) $\sin p_n = \sin(p_n - \pi q_n)$ *for all* p_n *and* q_n.

(ii) $\sin\left(\frac{1}{p_n}\right) = \frac{1}{p_n}\left(1 - \frac{1}{3!}\frac{1}{p_n^2} + \frac{1}{5!}\frac{1}{p_n^4} - \cdots\right)$ *as* $p_n \to \infty$.

(iii) $\sin\left(\frac{1}{p_n}\right) \asymp \sin p_n$ *as* $p_n \to \infty$.

Proof. (iii) Let $A = p_n - \pi q_n - 1/p_n$ and $B = 1/p_n$. The addition formula $\sin(A+B) = \cos A \sin B + \cos B \sin A$ leads to

$$\left|\sin\left(p_n - \pi q_n - \frac{1}{p_n} + \frac{1}{p_n}\right)\right| = \left|\cos A \sin\left(\frac{1}{p_n}\right) + \cos\left(\frac{1}{p_n}\right)\sin A\right|. \tag{4}$$

For the irrational number π, the relations $|p_n - \pi q_n| \leq 1/q_n$ and $p_n \approx \pi q_n$ as $n \to \infty$ are valid. Thus,

$$\frac{c_0}{p_n} \leq |A| = \left|p_n - \pi q_n - \frac{1}{p_n}\right| \leq \frac{c_1}{p_n}, \tag{5}$$

where $c_0 > 0$ and c_1 are constants. Now, using Lemma 3.1 returns

$$\cos A = 1 + O\left(A^2\right) = 1 + O\left(\frac{1}{q_n^2}\right) = 1 + O\left(\frac{1}{p_n^2}\right), \tag{6}$$

and

$$\sin A = A + O\left(A^3\right) = O\left(\frac{1}{q_n}\right) = O\left(\frac{1}{p_n}\right) \tag{7}$$

since $p_n \approx \pi q_n$ as $n \to \infty$. Substituting these estimates into (4) produces

$$\begin{aligned} \sin p_n &= \left|\sin\left(p_n - \pi q_n - \frac{1}{p_n} + \frac{1}{p_n}\right)\right| \qquad (8) \\ &= \left|\cos A \sin\left(\frac{1}{p_n}\right) + \cos\left(\frac{1}{p_n}\right)\sin A\right| \\ &= \left|\left(1 + O\left(\frac{1}{p_n^2}\right)\right)\sin\left(\frac{1}{p_n}\right) + \cos\left(\frac{1}{p_n}\right) O\left(\frac{1}{p_n}\right)\right| \\ &= \sin\left(\frac{1}{p_n}\right) + O\left(\frac{1}{p_n}\right) \\ &\asymp \sin\left(\frac{1}{p_n}\right). \end{aligned}$$

This completes the proof. ■

Lemma 2.2. *Let $z \in \mathbb{C}$ be complex number such that $|z| < 1$. Then,*

(i) $1 - \dfrac{|z|^2}{2} \leq \cos|z| \leq 1$.

(ii) $|z| - \dfrac{|z|^3}{6} \leq \sin|z| \leq |z|$.

Lemma 2.3. *Let p_n/q_n be the sequence of convergents of the real number $\pi = [a_0; a_1, a_2, \ldots]$. Then, as $n \to \infty$,*

(i) $\dfrac{1}{2q_n} \leq \left| p_n - \pi q_n - \dfrac{1}{q_n} \right| \leq \dfrac{2}{q_n}$.

(ii) $\dfrac{1}{2q_n} \leq \left| p_{n+1} - \pi q_{n+1} - \dfrac{1}{q_n} \right| \leq \dfrac{2}{q_n}$.

Proof. (ii) Using the triangle inequality and Dirichlet approximation theorem yield the upper bound

$$\begin{aligned} \left| p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} \right| &\leq |p_{n+1} - \pi q_{n+1}| + \frac{1}{q_n} \\ &\ll \frac{c_0}{q_{n+1}} + \frac{1}{q_n} \\ &\leq \frac{2}{q_n}, \end{aligned} \tag{9}$$

since $q_{n+1} = a_n q_n + q_{n-1}$, and the lower bound

$$\begin{aligned} \left| \frac{1}{q_n} - p_{n+1} - \pi q_{n+1} \right| &\geq \frac{1}{q_n} - |p_{n+1} - \pi q_{n+1}| \\ &\geq \frac{1}{q_n} - \frac{c_1}{q_{n+1}} \\ &\geq \frac{1}{2q_n}, \end{aligned} \tag{10}$$

where $c_0 > 0$ and $c_1 > 0$ are constants, as $n \to \infty$ as claimed. ■

3 Bounded Partial Quotients

Lemma 3.1. *Let $\pi = [a_0; a_1, a_2, \ldots]$ be the continued fraction of the real number $\pi \in \mathbb{R}$. Then, the quotients $a_n \in \mathbb{N}$ are bounded. Specifically, $a_n = O(1)$.*

Proof. Let p_n/q_n be the sequence of convergents of the irrational number $\pi = [a_0; a_1, a_2, \ldots]$. For all sufficiently large integers $n \geq 1$, the Dirichlet approximation theorem and the addition formula $\sin(C + D) = \cos C \sin D + \cos D \sin C$ lead to

$$\begin{aligned} \frac{1}{q_{n+1}} &\gg |p_{n+1} - \pi q_{n+1}| \\ &\geq |\sin(p_{n+1} - \pi q_{n+1})| \\ &= \left| \sin\left(p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} + \frac{1}{q_n} \right) \right| \\ &= \left| \cos C \sin\left(\frac{1}{q_n}\right) + \cos\left(\frac{1}{q_n}\right) \sin C \right| \\ &\geq \left| \left(\frac{1}{q_n} - \frac{1}{6q_n^3} \right) \cos C + \left(1 - \frac{1}{2q_n^2} \right) \sin C \right|. \end{aligned} \tag{11}$$

By Lemma 2.2,

$$\cos C \geq 1 - \frac{C^2}{2} \geq 1 - \frac{2}{q_n^2}, \tag{12}$$

and

$$\sin C \geq C - \frac{C^3}{6} \geq \frac{1}{2q_n^2} - \frac{8}{6q_n^3} \tag{13}$$

Substituting these estimates, and using the reverse triangle inequality $|A+B| \geq ||A|-|B||$, produce

$$\begin{aligned} \frac{1}{q_{n+1}} &\gg \left| \left(\frac{1}{q_n} - \frac{1}{6q_n^3} \right) \cos C + \left(1 - \frac{1}{2q_n^2} \right) \sin C \right| \\ &\geq \left| \left| \left(\frac{1}{q_n} - \frac{1}{6q_n^3} \right) \left(1 - \frac{2}{q_n^2} \right) \right| - \left| \left(1 - \frac{1}{2q_n^2} \right) \left(\frac{1}{2q_n} - \frac{8}{6q_n^3} \right) \right| \right| \\ &\geq \left| \left| \frac{1}{q_n} - \frac{1}{6q_n^3} - \frac{2}{q_n^3} + \frac{1}{3q_n^5} \right| - \left| \frac{1}{2q_n} - \frac{8}{6q_n^3} - \frac{1}{4q_n^3} + \frac{2}{3q_n^5} \right| \right| \\ &\geq \frac{c_0}{q_n}, \end{aligned} \tag{14}$$

where $c_0 > 0$ is a constant. These show that

$$\frac{1}{q_{n+1}} \gg \frac{1}{q_n} \tag{15}$$

as $n \to \infty$. Furthermore, (15) imply that $q_n \gg q_{n+1} = a_n q_n + q_{n-1}$ as claimed. ■

Example 3.1. A short survey of the partial quotients $\pi = [a_0; a_1, a_2, \ldots]$ is provided here to sample this phenomenon, most computer algebra systems can generate a few thousand terms within minutes. A limited numerical experiment established that $a_n \leq 21000$ for $n \leq 10000$, here $a_{432} = 20776$ is the only unusual partial quotient.

$$\begin{aligned} \pi = [&3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, \\ &6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1, 1, 12, 1, 1, 1, 3, 1, 1, 8, 1, 1, \ldots]. \end{aligned} \tag{16}$$

4 Convergence Of The Flint Hills Series

This analysis of the the convergence of the Flint Hills series is based on the asymptotic relation relation

$$\sin(p_n) \asymp \sin\left(\frac{1}{p_n}\right), \tag{17}$$

where $p_n/q_n \in \mathbb{Q}$ is the sequence of convergents of the real number $\pi \in \mathbb{R}$. In some way, the representation (17) removes any reference to the difficult problem of estimating the maximal value of the function $1/\sin n$ as $n \to \infty$.

Theorem 4.1. *If the real numbers $u > 0$ and $v > 0$ satisfy the relation $u - v \geq 1$, then the Flint Hills series*

$$\sum_{n \geq 1} \frac{1}{n^u \sin^v n} \tag{18}$$

is absolutely convergent.

Proof. The sequence of convergents p_n/q_n of the real number $\pi = [a_0, a_1, a_2, \ldots]$ minimizes the sine function $\sin z \geq \sin p_n$ over the integers $z \in \mathbb{Z}$. Hence, it is sufficient to consider the infinite series over the subsequence $\{p_n : n \geq 1\}$. Substituting the sequence of convergents p_n/q_n of the irrational number $\pi = [a_0; a_1, a_2, \ldots]$ yields

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{n^u \sin^v n} &\ll \sum_{n \geq 1} \frac{1}{p_n^u (\sin p_n)^v} \\ &\ll \sum_{n \geq 1} \frac{1}{p_n^u \sin\left(\frac{1}{p_n}\right)^v}, \end{aligned} \tag{19}$$

see Lemma 2.1. Substituting the Taylor series at infinity, again in Lemma 2.1, return

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{p_n^u\sin\left(\frac{1}{p_n}\right)^v} &= \sum_{n\geq 1}\frac{1}{p_n^u\left(\frac{1}{p_n}\left(1-\frac{1}{3!}\frac{1}{p_n^2}+\frac{1}{5!}\frac{1}{p_n^4}-\cdots\right)\right)^v}\\ &\ll \sum_{n\geq 1}\frac{1}{p_n^{u-v}}.\end{aligned} \tag{20}$$

By the Binet formula, the sequence $p_n = a_n p_{n-1} + p_{n-1}$, $a_n \geq 1$, has exponential growth, namely,

$$p_n \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \tag{21}$$

for $n \geq 2$. Hence, it immediately follows that the infinite series converges whenever $u - v \geq 1$. ■

Example 4.1. The infinite series

$$\sum_{n\geq 1}\frac{1}{n^3\sin^2 n} \tag{22}$$

has $u - v = 1$. Hence, by Theorem 4.1, it is convergent.

Example 4.2. The infinite series

$$\sum_{n\geq 1}\frac{1}{n^2\sin n} \tag{23}$$

has $u - v = 1$. Hence, by Theorem 4.1, it is absolutely convergent.

5 The Sine Function And The Number Pi

The argument linking the sequence of convergents $\{p_n/q_n : n \geq 1\}$ of the real number π and the sine function to the irrationality measure of the number π is well known in the number theory literature, refer to [3, Section 8], and [1] for some details.

Lemma 5.1. *Let $u \geq 1$ and $v \geq 1$ be a pair of fixed parameters, and let $\mu(\pi) \geq 2$ be the irrationality measure of the number π. Then, the sequence of real numbers $\{n^{-u}\sin(n)^{-v} : n \geq 1\}$ converges if and only if*

$$\frac{u}{v} + 1 > \mu(\pi). \tag{24}$$

Proof. From the Taylor series $\sin x = x - x^3/3! + x^5/5! - \cdots$, it follows that $|x|/2 \leq |\sin x| \leq |x|$ for any number $|x| < \pi/2$.

Let $\varepsilon > 0$ be an arbitrary small number, and let p_k/q_k be a large convergent of the irrational number π. For any pair of integers $m \leq q_k$ and $n \leq p_k$, it follows that

$$|n - \pi m| \geq |\pi q_k - p_k| \gg \frac{1}{q_k^{\mu(\pi)-1+\varepsilon}}, \tag{25}$$

where $\mu(\pi) \geq 2$ is the irrationality measure of the number π.

Thus, for any pair of integers $m, n \in \mathbb{N}$ such that $|n - \pi m| < \pi/2$, the inequality

$$\begin{aligned}|\sin(n)| &= |\sin(n - m\pi)|\\ &\gg |n - \pi m|\\ &\gg \frac{1}{q_k^{\mu(\pi)-1+\varepsilon}}\end{aligned} \tag{26}$$

holds for all large convergents p_k/q_k. Equivalently,

$$\frac{1}{|\sin n|} \ll q_k^{\mu(\pi)-1+\varepsilon}. \tag{27}$$

Substituting (27) and $n = q_k \approx q_k$ to maximize the kth term of the sequence return

$$\left|\frac{1}{n^u \sin(n)^v}\right| \ll \frac{1}{q_k^{u-(\mu(\pi)-1+\varepsilon)v}}. \tag{28}$$

This implies that the sequence $n^{-u}\sin(n)^{-v} \longrightarrow 0$ converges as $n \longrightarrow \infty$ if and only if

$$\frac{u}{v} + 1 > \mu(\pi). \tag{29}$$

■

6 Main Result

This analysis of the irrationality measure $\mu(\pi)$ of the number π is completely independent of the prime number theorem, and it is not related to the earlier analysis used by many authors, based on the Laplace integral of a factorial like function

$$\frac{1}{i2\pi}\int_C \left(\frac{n!}{z(z-1)(z-2)(z-3)\cdots(z-n)}\right)^{k+1} e^{-tz}dz, \tag{30}$$

where C is a curve around the simple poles of the integrand, and $k \geq 0$, see [13], [14], [8], [11], [16], and [6] for an introduction to the rational approximations of π and the various proofs.

Proof. (Theorem 1.1) Let $v \geq 1$ be a large number, and let $u = v + 1$.

1. By Theorem 4.1, the infinite series

$$\sum_{n\geq 1} \frac{1}{n^{v+1}\sin^v n} \tag{31}$$

converges since $u - v = v + 1 - v \geq 1$. This implies that the sequence of terms

$$\frac{1}{n^{v+1}\sin^v n} \longrightarrow 0 \tag{32}$$

converges as $n \longrightarrow \infty$.

2. By Lemma 5.1, the sequence $n^{-u}\sin(n)^{-v} \longrightarrow 0$ converges as $n \longrightarrow \infty$ if and only if

$$\frac{u}{v} + 1 > \mu(\pi). \tag{33}$$

Therefore, comparing items (31) and (32) return

$$\frac{u}{v} + 1 = \frac{v+1}{v} + 1 = 2 + \frac{1}{v} > \mu(\pi). \tag{34}$$

In particular, the inequality

$$\left|\pi - \frac{p}{q}\right| \gg \frac{1}{q^{2+1/v}} \tag{35}$$

holds for all large convergents p/q, where $v \geq 1$ is any large number. Quod erat demontrandum. ■

Table 2: Numerical Data For $\pi/\sin p_n$

n	p_n	$1/\sin(p_n)$	$1/\sin 1/p_n$	$\sin p_n/\sin(1/p_n)$
1	3	7.086178	3.0562	0.431303
2	22	-112.978	22.0076	-0.194796
3	333	-113.364	333.001	-2.93745
4	355	-33173.7	355.0	-0.0107013
5	103993	-52275.7	103993.0	-1.98932
6	104348	-90785.1	104348.0	-1.1494
7	208341	123239.0	208341.	1.69055
8	312689	344744.0	312689.0	0.907017
9	833719	432354.0	833719.0	1.92832
10	1146408	-1.70132×10^6	1.14641×10^6	-0.673836
11	4272943	-1.81957×10^6	4.27294×10^6	-2.34832
12	5419351	-2.61777×10^7	5.41935×10^6	-0.207022
13	80143857	-6.76918×10^7	8.01439×10^7	-1.18395
14	165707065	-1.15543×10^8	1.65707×10^8	-1.43416
15	245850922	1.6345×10^8	2.45851×10^8	1.50413
16	411557987	3.9421×10^8	4.11558×10^8	1.04401
17	1068966896	9.57274×10^8	1.06897×10^9	1.11668
18	2549491779	2.23489×10^9	2.54949×10^9	1.14077
19	6167950454	6.6785×10^9	6.16795×10^9	0.923554
20	14885392687	6.75763×10^9	1.48854×10^{10}	2.20275
21	21053343141	5.70327×10^{11}	2.10533×10^{10}	0.0369145
22	1783366216531	1.43483×10^{12}	1.78337×10^{12}	1.24291
23	3587785776203	2.78176×10^{12}	3.58779×10^{12}	1.28975
24	5371151992734	-2.96328×10^{12}	5.37115×10^{12}	-1.81257
25	8958937768937	-4.54136×10^{13}	8.95894×10^{12}	-0.197274

7 Numerical Data

The maxima of the function $1/\sin x$ occur at the numerators $x = p_n$ of the sequence of convergents $p_n/q_n \longrightarrow \pi$. The first few terms of the sequence p_n, which is cataloged as $A046947$ in [15], are:

$$\mathcal{N}_\pi = \{1, 3, 22, 333, 355, 103993, 104348, 208341, 312689, 833719, 1146408, 4272943, \\ 5419351, 80143857, 165707065, 245850922, 411557987, \ldots\} \tag{36}$$

A few values were computed to illustrate the prediction in Lemma 2.1.

8 Beta and Gamma Functions

Certain properties of the beta function $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, and gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ are used in the main result.

Lemma 8.1. *Let $B(a, b)$ and $\Gamma(z)$ be the beta function and gamma function of the complex numbers $a, b, z \in \mathbb{C} - \{0, -1, -2, -3, \ldots\}$. Then,*

(i) $\Gamma(z + 1) = z\Gamma(z)$, *the functional equation.*

(ii) $B(a, b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$, *the multiplication formula.*

(iii) $\dfrac{B(1 - z, z)}{\pi} = \dfrac{1}{\sin \pi z}$.

(iv) $\dfrac{\Gamma(1 - z)\Gamma(z)}{\pi} = \dfrac{1}{\sin \pi z}$, *the reflection formula.*

Proof. Standard analytic methods on the $B(a, b)$, and gamma function $\Gamma(z)$, see [9, Equation 5.12.1], [2, Chapter 1], et cetera. ■

9 Beta and Gamma Functions

Certain properties of the beta function $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, and gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ are useful in the proof of the main result.

Lemma 9.1. *Let $B(a, b)$ and $\Gamma(z)$ be the beta function and gamma function of the complex numbers $a, b, z \in \mathbb{C} - \{0, -1, -2, -3, \ldots\}$. Then,*

(i) $\Gamma(z + 1) = z\Gamma(z)$, *the functional equation.*

(ii) $B(a, b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$, *the multiplication formula.*

(iii) $\dfrac{B(1 - z, z)}{\pi} = \dfrac{1}{\sin \pi z}$.

(iv) $\dfrac{\Gamma(1 - z)\Gamma(z)}{\pi} = \dfrac{1}{\sin \pi z}$, *the reflection formula.*

Proof. Standard analytic methods on the $B(a, b)$, and gamma function $\Gamma(z)$, see [9, Equation 5.12.1], [2, Chapter 1], et cetera. ■

The current result on the irrationality measure $\mu(\pi) = 7.6063$, see Table 1, of the number π implies that

$$|\Gamma(z + 1)\Gamma(z)| \ll |z|^{7.6063}. \tag{37}$$

A sharper upper bound is computed here.

Lemma 9.2. *Let $\Gamma(z)$ be the gamma function of a complex number $z \in \mathbb{C}$, but not a negative integer $z \neq 0, -1, -2, \ldots$. Then,*

$$|\Gamma(z + 1)\Gamma(z)| \ll |z|. \tag{38}$$

Proof. Assume $\Re e(z) > 0$. Then, the functional equation

$$\Gamma(z + 1) = z\Gamma(z), \tag{39}$$

see Lemma 9.1, or [9, Equation 5.5.1], provides an analytic continuation expression

$$\frac{\Gamma(1 - z)\Gamma(z)}{\pi} = \frac{1}{\sin \pi z},$$

for any complex number $z \in \mathbb{C}$ such that $z \neq 0, -1, -2, -3, \ldots$. By Lemma 2.1,

$$\begin{aligned} \frac{1}{\sin|\pi z|} &\asymp \frac{1}{\sin\left(\frac{1}{|\pi z|}\right)} \\ &\asymp |\pi z| \end{aligned}$$

as $|\pi z| \to \infty$. ■

The asymptotic expansions of the gamma function

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + O\left(\frac{1}{z}\right) \tag{40}$$

and

$$\Gamma(z) = \sqrt{2\pi e}\left(\frac{z}{e}\right)^{z-1/2}\left(1 + O\left(\frac{1}{z}\right)\right), \tag{41}$$

and several other formulas, see [9, Equation 5.10.1], [2, Chapter 1], et alii, provide other routes to the proof of Lemma 9.2.

Table 3: Numerical Data For $\pi/\sin p_n$

p_n	$\Gamma(1-p_n/\pi)\Gamma(p_n/\pi)$	$\pi^2/p_n \sin p_n$
3	22.2619	23.3126
22	-354.93	-50.6838
333	-356.143	-3.35992
355	-104218.0	-922.286
103993	-164229.0	-4.9613
104348	-285210.0	-8.58678
208341	387167.0	5.83812
312689	1.08305×10^6	10.8814
833719	1.35828×10^6	5.11823
1146408	-5.34484×10^6	-14.6469
4272943	-5.71636×10^6	-4.20283
5419351	-8.22396×10^7	-47.6742
80143857	-2.1266×10^8	-8.33615
165707065	-3.62989×10^8	-6.88181
245850922	5.13494×10^8	6.56166
411557987	1.23845×10^9	9.45359
1068966896	3.00736×10^9	8.83836
2549491779	7.02111×10^9	8.65171
6167950454	2.09811×10^{10}	10.6866
14885392687	2.12297×10^{10}	4.48058
21053343141	1.79173×10^{12}	267.364
1783366216531	4.50764×10^{12}	7.9407
3587785776203	8.73917×10^{12}	7.65233

10 Numerical Data

The maxima of the function $1/\sin x$ occur at the numerators $x = p_n$ of the sequence of convergents $p_n/q_n \longrightarrow \pi$. The first few terms of sequence p_n, which is cataloged as $A046947$ in [15], are:

$$\mathcal{N}_\pi = \{1, 3, 22, 333, 355, 103993, 104348, 208341, 312689, 833719, 1146408, 4272943, \\ 5419351, 80143857, 165707065, 245850922, 411557987, \ldots\} \tag{42}$$

Observe that the substitution $z \longrightarrow p_n/\pi$ in $\Gamma(1-z)\Gamma(z)$ leads to

$$\frac{\Gamma(1-z)\Gamma(z)}{z} = \frac{\pi}{z \sin \pi z} = \frac{\pi^2}{p_n \sin p_n} = O(1). \tag{43}$$

A few values were computed to illustrate the prediction in Lemma **??**. The ratio (43) is tabulated in the third column of Table 3.

11 Problem

11.1 Flint Hill Class Series

Exercise 11.1. Assume that the infinite series is absolutely convergent. Use a Cauchy integral formula to evaluate the integral

$$\frac{1}{i2\pi} \int_C \frac{1}{z^u \sin^v z} dz,$$

where C is a suitable curve.

Exercise 11.2. Determine whether or not there is a complex valued function $f(z)$ for which the Cauchy integral evaluate to

$$\frac{1}{i2\pi} \int_C f(z) dz = \sum_{n \geq 1} \frac{1}{n^u \sin^v n},$$

where C is a suitable curve, and the infinite series is absolutely convergent.

as $|\pi z|\to\infty$.

Exercise 11.3. The infinite series

$$S_1=\sum_{n\geq 1}\frac{(-1)^{n+1}}{n} \qquad \text{and} \qquad S_2=\sum_{n\geq 1}\frac{1}{n^2\sin n}$$

have many asymptotic similarities. The first is conditionally convergent. Is the series S_2 conditionally convergent?

11.2 Complex Analysis

Exercise 11.4. Let $z\in\mathbb{C}$ be a large complex variable. Prove or disprove the asymptotic relation

$$\frac{1}{\sin|\pi z|}\asymp\frac{1}{\sin\left(\frac{1}{|\pi z|}\right)}\asymp|\pi z|.$$

Exercise 11.5. Consider the product

$$\sin\left(\frac{1}{z}\right)\times\sin z=1+O(1/z^2) \tag{44}$$

of a complex variable $z\in\mathbb{C}$. Explain its properties in the unit disk $\{z\in\mathbb{C}:|z|<1\}$ and at infinity.

Exercise 11.6. Let $z\in\mathbb{C}$ be a large complex variable. Use the properties of the gamma function to prove the asymptotic relation

$$|\Gamma(1-z)\Gamma(z)|=O(|z|).$$

11.3 Irrationality Measures

Exercise 11.7. Do the irrationality measures of the numbers π and π^2 satisfy $\mu(\pi)=\mu(\pi^2)=2$? This is supported by the similarity of $\sin(n)=\sin(n-m\pi)$ and

$$\frac{\sin x}{\pi}=\prod_{m\geq 1}\left(1-\frac{x^2}{\pi^2m^2}\right)$$

at $x=n$. More generally, $x=n^k\pi^{-k+1}$

Exercise 11.8. What is the relationship between the irrationality measures of the numbers π and π^k, for example, do these measures satisfy $\mu(\pi)=\mu(\pi^k)=2$ for $k\geq 2$?

11.4 Partial Quotients

Exercise 11.9. Determine an explicit bound $a_n\leq B$ for the continued fraction of the irrational number $\pi=[a_0;a_1,a_2,a_3,\ldots]$ for all partial quotients a_n as $n\to\infty$.

Exercise 11.10. Does $\pi^2=[a_0;a_1,a_2,a_3,\ldots]$ has bounded partial quotients a_n as $n\to\infty$?

11.5 Concatenated Sequences

Exercise 11.11. A Champernowne number $\kappa_b=0.123\cdots b-1\cdot b\cdot b+1\cdot b+2\cdots$ in base $b\geq 2$ is formed by concatenating the sequence of consecutive integers in base b is irrationality. Show that the number $0.F_0F_1F_2F_3\ldots=1/F_{11}$ formed by concatenating the sequence of Fibonacci numbers $F_{n+1}=F_n+F_{n-1}$ is rational.

Exercise 11.12. Let $f(x)\in\mathbb{Z}[x]$ be a polynomial, and let $D_n=|f(n)|$. Show that the number $0.D_0D_1D_2D_3\ldots$ formed by concatenating the sequence of values is irrational.

Exercise 11.13. Let $\{D_n\geq 0:n\geq 0\}$ be an infinite sequence of integers, and let $\alpha=0.D_0D_1D_2D_3\ldots$ formed by concatenating the sequence of integers. Determine a sufficient condition on the sequence of integers to have an irrational number $\alpha>0$.

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