# A generating polynomial for the two-bridge knot with Conway's notation C(n, r)

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#### Abstract

We construct an integer polynomial whose coefficients enumerate the Kauffman states of the two-bridge knot with Conway's notation C(n, r).

Keywords: generating polynomial, shadow diagram, Kauffman state.

#### 1 Introduction

A *state* of a knot shadow diagram is a choice of splitting its crossings [2, Section 1]. There are two ways of splitting a crossing:

$$(A) \hspace{-0.2cm} \hspace{-0.2cm} \Longrightarrow \hspace{-0.2cm} \hspace{-0.2cm$$

By state of a crossing we understand either of the split of type (A) or (B). An example for the figure-eight knot is shown in Figure 1.

Let K be a knot diagram. If m denotes the initial number of crossings, then the final states form a collection of  $2^m$  diagrams of nonintersecting curves. We can enumerate those states with respect to the number of their components – called circles – by introducing the sum

$$K(x) := \sum_{S} x^{|S|},\tag{1}$$

where S browses the collection of states, and |S| gives the number of circles in S. Here, K(x) is an integer polynomial which we referred to as generating polynomial [6, 7] (in fact, it is a simplified formulation of what Kauffman calls "state polynomial" [2, Section 1–2] or

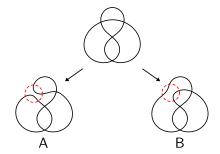


Figure 1: The states of a crossing.

"bracket polynomial" [3]). For instance, if K is the figure-eight knot diagram, then we have  $K(x) = 5x + 8x^2 + 3x^3$  (the states are illustrated in Figure 2).

In this note, we establish the generating polynomial for the two-bridge knot with Conway's notation C(n,r) [4, 5]. We refer to the associated knot diagram as  $B_{n,r}$ , where n and r denote the number of half-twists. For example, the figure-eight knot has Conway's notation C(2,2). Owing to the property of the shadow diagram which we draw on the sphere [1], we can continuously deform the diagram  $B_{n,r}$  into  $B_{r,n}$  without altering the crossings configuration. We let  $B_{n,r} \rightleftharpoons B_{r,n}$  express such transformation (see Figure 3 (a)). Besides, we let  $B_{n,0}$  and  $B_{n,\infty}$  denote the diagrams in Figure 3 (b) and (c), respectively. Here, "0" and " $\infty$ " are symbolic notations – borrowed from tangle theory [2, p. 88] – that express the absence of half-twists. If  $r = \infty$  and  $n \ge 1$ , then  $B_{n,\infty}$  represents the diagram of a (2,n)-torus knot ( $\rightleftharpoons B_{n-1,1}$ ). Correspondingly, we let  $B_{0,r}$  and  $B_{\infty,r}$  denote the diagrams pictured in Figure 3 (d) and (e), respectively.

## 2 Generating polynomial

Let K, K' and  $\bigcirc$  be knot diagrams, where  $\bigcirc$  is the trivial knot, and let # and  $\sqcup$  denote the connected sum and the disjoint union, respectively. The generating polynomial defined in (1) verifies the following basic properties:

- (i)  $\bigcirc(x) = x$ ;
- (ii)  $(K \sqcup K')(x) = K(x)K'(x);$
- (iii)  $(K \# K')(x) = \frac{1}{x} K(x) K'(x)$ .

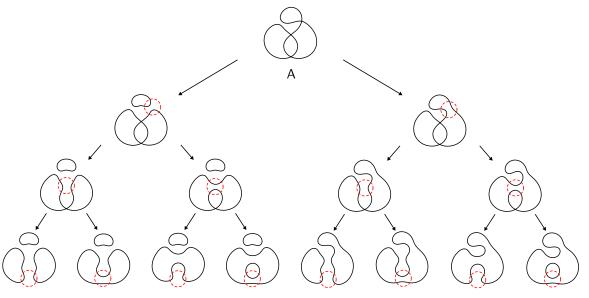
Furthermore, if  $K \rightleftharpoons K'$ , then K(x) = K'(x) [6].

**Lemma 1.** The generating polynomial for the knots  $B_{n,0}$  and  $B_{n,\infty}$  are given by

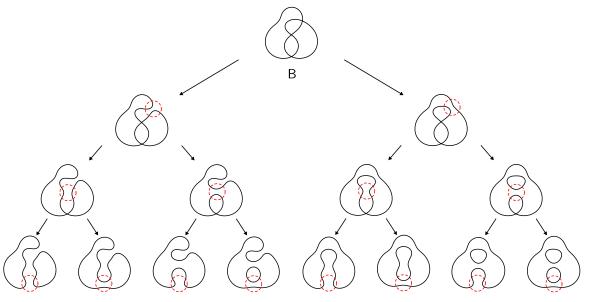
$$B_{n,0}(x) = x(x+1)^n (2)$$

and

$$B_{n,\infty}(x) = (x+1)^n + x^2 - 1.$$
(3)



(a) The states of the figure-eight knot following the initial "A" split.



(b) The states of the figure-eight knot following the initial "B" split.

Figure 2: The states of the figure-eight knot.

The key ingredient for establishing (2) and (3) consists of the states of specific crossings which produce the recurrences

$$B_{n,0}(x) = (\bigcirc \sqcup B_{n-1,0})(x) + B_{n-1,0}(x)$$

and

$$B_{n,\infty}(x) = B_{n-1,0}(x) + B_{n-1,\infty}(x),$$

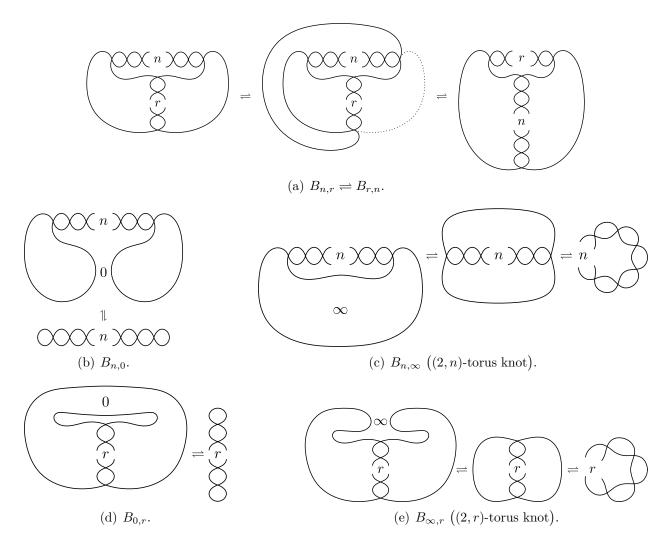


Figure 3: Two-bridge knots with Conway's notation C(n, r).

respectively, with initial values  $B_{0,0}(x) = x$  and  $B_{0,\infty}(x) = x^2$  [6]. Note that the lemma still holds if we replace index n by r.

**Proposition 2.** The generating polynomial for the two-bridge knot  $B_{n,r}$  is given by the recurrence

$$B_{n,r}(x) = B_{n-1,r}(x) + (x+1)^{n-1} B_{\infty,r}(x), \tag{4}$$

and has the following closed form:

$$B_{n,r}(x) = \left(\frac{(x+1)^r + x^2 - 1}{x}\right)(x+1)^n + \left(x^2 - 1\right)\left(\frac{(x+1)^r - 1}{x}\right).$$
 (5)

*Proof.* By Figure 4 we have

$$B_{n,r}(x) = B_{n-1,r}(x) + (B_{n-1,0} \# B_{\infty,r})(x)$$
  
=  $B_{n-1,r}(x) + (x+1)^{n-1} B_{\infty,r}(x),$ 

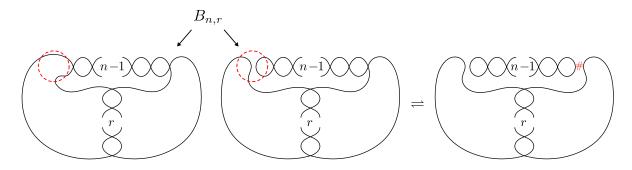


Figure 4: The splits at a crossing allow us to capture  $B_{n-1,r}$ ,  $B_{n-1,0}$  and  $B_{\infty,r}$ .

where the last relation follows from property (iii). Solving the recurrence for n yields

$$B_{n,r}(x) = B_{0,r}(x) + B_{\infty,r}(x) \left( \frac{(x+1)^n - 1}{x} \right).$$

We conclude by the closed forms in Lemma 2.

Remark 3. We can write

$$B_{n,0}(x) = x^2 \alpha_n(x) + x \tag{6}$$

and

$$B_{n,\infty}(x) = x\alpha_n(x) + x^2,\tag{7}$$

where  $\alpha_n(x) := \frac{(x+1)^n - 1}{x}$ , so that identity (5) becomes

$$B_{n,r}(x) = \left(x^2 \alpha_n(x) + x\right) + \left(x^2 \alpha_r(x) + x \alpha_n(x) \alpha_r(x)\right). \tag{8}$$

Since the coefficients of  $\alpha_n(x)$  are all nonnegative, it is clear, by (6), that the polynomial  $x^2\alpha_n(x)$  counts the states of  $B_{n,0}$  that have at least 2 circles. This is illustrated in Figure 5 (a). Likewise, we have an interpretation of (7) in Figure 5 (b). In Figure 5 and 6, the dashed diagrams represent all possible disjoint union of  $\ell - 1$  circles ( $\ell = n$  or r, depending on the context), counted by  $\alpha_{\ell}(x)$  and eventually empty.

Therefore, for  $n, r \notin \{0, \infty\}$ , identity (8) means that we can classify the states into 4 subset as shown in Figure 6. In these illustrations, there are  $2^n - 1$  and  $2^r - 1$  states of (a) and (b) kind, respectively, and  $\binom{n}{1} \times \binom{r}{1} + 1$  one-component states of (c) and (d) kind. The remaining states are of (c) kind, bringing the total number of states to  $2^{n+r}$ .

### 3 Particular values

Let  $\sum_{k\geq 0} b(n,r;k)x^k := B_{n,r}(x)$ , or  $b(n,r;k) := [x^k] B_{n,r}(x)$ . Then

$$b(n,r;k) = \binom{n+r}{k+1} + \binom{n}{k-1} + \binom{r}{k-1} - \binom{n}{k+1} - \binom{r}{k+1} - \delta_{1,k},$$

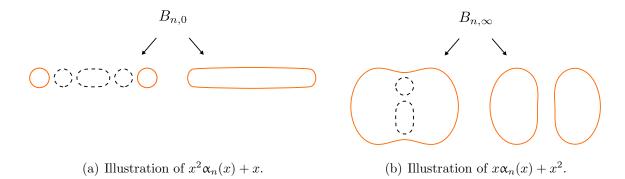


Figure 5: Illustrations of  $B_{n,0}(x)$  and  $B_{n,\infty}(x)$  as functions of  $\alpha_n(x)$ .

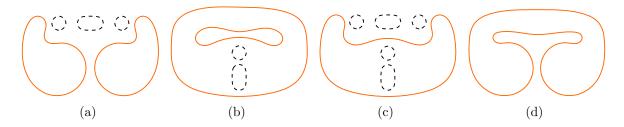


Figure 6: The states of  $B_{n,r}$ : states in (a) are counted by  $x^2 \alpha_n(x)$ , those in (b) by  $x^2 \alpha_r(x)$ , those in (c) by  $x \alpha_n(x) \alpha_r(x)$ , and state in (d) is simply counted by x.

where  $\delta_{1,k}$  is the Kronecker symbol. By (1), we recognize b(n,r;k) as the cardinal of the set  $\{|S| = k : S \text{ is a state of } B_{n,r}\}$ , i.e., the number of states having k circles. In this section, the coefficients b(n,r;k) are tabulated for some values of n, r and k. We give as well the corresponding A-numbers in the On-Line Encyclopedia of Integer Sequences [8].

•  $b(n,0;k) = [x^k] x(x+1)^n$ , essentially giving entries in Pascal's triangle A007318 (see Table 1).

$n \setminus k$	0	1	2	3	4	5	6	7	8
0 1 2 3 4 5 6 7	0	1							
1	0	1	1						
2	0	1	2	1					
3	0	1	3	3	1				
4	0	1	4	6	4	1			
5	0	1	5	10	10	5	1		
6	0	1	6	15	20	15	6	1	
7	0	1	7	21	35	35	21	7	1

Table 1: Values of b(n, 0; k) for  $0 \le n \le 7$  and  $0 \le k \le 8$ .

•  $b(n,1;k) = [x^k]((x+1)^{n+1} + x^2 - 1)$ , generating a subtriangle in A300453 (see Table 2).

$n \setminus k$	0	1	2	3	4	5	6	7	8
0 1 2 3 4 5 6 7	0	1	1						
1	0	2	2						
2	0	3	4	1					
3	0	4	7	4	1				
4	0	5	11	10	5	1			
5	0	6	16	20	15	6	1		
6	0	7	22	35	35	21	7	1	
7	0	8	29	56	70	56	28	8	1

Table 2: Values of b(n, 1; k) for  $0 \le n \le 7$  and  $0 \le k \le 8$ .

•  $b(n,2;k) = [x^k]((2x+2)(x+1)^n + (x^2-1)(x+2))$ , giving triangle in A300454 (see Table 3).

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	0	1	2	1	2 10 30 70 140				
1	0	3	4	1					
2	0	5	8	3					
3	0	7	14	9	2				
4	0	9	22	21	10	2			
5	0	11	32	41	30	12	2		
6	0	13	44	71	70	42	14	2	
7	0	15	58	113	140	112	56	16	2

Table 3: Values of b(n, 2; k) for  $0 \le n \le 7$  and  $0 \le k \le 8$ .

- $b(n, n; k) = \left[x^k\right] \left(\frac{(x+1)^{2n} + \left(x^2 1\right)\left(2(x+1)^n 1\right)}{x}\right)$ , giving triangle in <u>A321127</u> (see Table 4).
- b(n, r; 1) = nr + 1, giving <u>A077028</u>, and displayed as square array in Table 5. In Kauffman's language, b(n, r; 1) is, for a fixed choice of star region, the number of ways of placing *state markers* at the crossings of the diagram  $B_{n,r}$ , i.e., of the forms

$$X$$
,  $X$ ,  $X$ ,

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1												
1	0	2	2											
2	0	5	8	3										
3	0	10	24	21	8	1								
4	0	17	56	80	64	30	8	1						
5	0	26	110	220	270	220	122	45	10	1				
6	0	37	192	495	820	952	804	497	220	66	12	1		
7	0	50	308	973	2030	3059	3472	3017	2004	1001	364	91	14	1

Table 4: Values of b(n, n; k) for  $0 \le n \le 7$  and  $0 \le k \le 13$ .

$n \setminus r$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
2	1	3	5	7	9	11	13	15
3	1	4	7	10	13	16	19	22
4	1	5	9	13	17	21	25	29
5	1	6	11	16	21	26	31	36
6	1	7	13	19	25	31	37	43
7	1	8	15	22	29	36	43	50

Table 5: Values of b(n, r; 1) for  $0 \le n \le 7$  and  $0 \le r \le 7$ .

so that the resulting states are "Jordan trails" [2, Section 1–2]. Note that a state marker is interpreted as an instruction to split a crossing as shown below:

$$\nearrow$$
  $\Rightarrow$   $\nearrow$  and  $\Rightarrow$   $\nearrow$ .

The process is illustrated in Figure 7 for the figure-eight knot.

• 
$$b(n, r; 2) = n\left(\binom{r}{2} + 1\right) + r\left(\binom{n}{2} + 1\right)$$
, giving square array in A300401 (see Table 6).

We paid a special attention to the case k = 2 because, surprisingly, columns  $(b(n, 1; 2))_n$  and  $(b(n, 2; 2))_n$  match sequences A000124 and A014206, respectively [6]. The former gives the maximal number of regions into which the plane is divided by n lines, and the latter the maximal number of regions into which the plane is divided by (n + 1) circles.

•  $b(n,r;d(n,r)) = leading coefficient of <math>B_{n,r}(x)$ , giving square array in A321125 (see Table 7). Here,  $d(n,r) = \max(n+1,r+1,n+r-1)$  denotes the degree of  $B_{n,r}(x)$ , and gives entries in A321126. We have Table 8 giving the numbers d(n,r) for  $0 \le n \le 7$  and  $0 \le r \le 7$ .

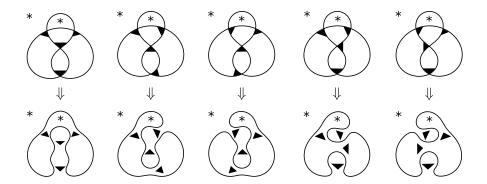


Figure 7: Illustration of b(2, 2; 1): mark two adjacent regions by stars (\*), then assign a state marker at each crossing so that no region of  $B_{2,2}$  contains more than one state marker, and regions with stars do not have any.

$n \setminus r$	0	1	2	3	4	5	6	7
0	0	1	2	3	4 11 22 37 56 79 106 137	5	6	7
1	1	2	4	7	11	16	22	29
2	2	4	8	14	22	32	44	58
3	3	7	14	24	37	53	72	94
4	4	11	22	37	56	79	106	137
5	5	16	32	53	79	110	146	187
6	6	22	44	72	106	146	192	244
7	7	29	58	94	137	187	244	308

Table 6: Values of b(n, r; 2) for  $0 \le n \le 7$  and  $0 \le r \le 7$ .

$n \setminus r$	0	1	2	3	4	5	6	7
0 1 2 3 4 5 6 7	1	1	1	1	1	1	1	1
1	1	2	1	1	1	1	1	1
2	1	1	3	2	2	2	2	1
3	1	1	2	1	1	1	1	1
4	1	1	2	1	1	1	1	1
5	1	1	2	1	1	1	1	1
6	1	1	2	1	1	1	1	1
7	1	1	2	1	1	1	1	1

Table 7: Leading coefficients of  $B_{n,r}(x)$  for  $0 \le n \le 7$  and  $0 \le r \le 7$ .

We have the following properties:

- d(n,r) = d(r,n);
- if r = 0, then d(n, r) = n + 1;
- if  $r=\infty$ , then sequence  $\left(d(n,r)\right)_n$  begins:  $2,2,2,3,4,5,6,7,8,\ldots$  (A233583) with

$n \setminus r$	0	1	2	3	4	5	6	7
0	1	2	3	4	5 5 5 6 7 8 9 10	6	7	8
1	2	2	3	4	5	6	7	8
2	3	3	3	4	5	6	7	8
3	4	4	4	5	6	7	8	9
4	5	5	5	6	7	8	9	10
5	6	6	6	7	8	9	10	11
6	7	7	7	8	9	10	11	12
7	8	8	8	9	10	11	12	13

Table 8: Values of d(n,r) for  $0 \le n \le 7$  and  $0 \le r \le 7$ .

offset 1).

Diagramatically, we give the corresponding illustration for some values of n and r in Figure 8.

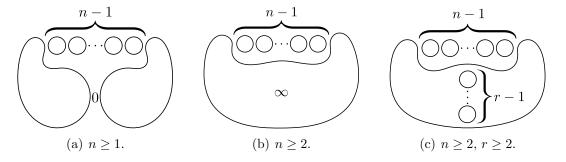


Figure 8: Illustration of the numbers d(n, r).

Correspondingly, we have

- -b(n,r;d(n,r)) = b(r,n;d(r,n));
- if r = 0, then b(n, r; d(n, r)) = 1;
- if  $r = \infty$ , then sequence  $\left(b(n, r; d(n, r))\right)_n$  begins:  $1, 1, 2, 1, 1, 1, 1, \dots$  (A294619) with initial term equals to 0).

Remarquable values in Table 7 correspond to knots  $B_{1,1}$  ("Hopf link", see Figure 9),  $B_{2,2}$  (figure-eight knot, see Figure 1, 2) and  $B_{n,2}$  ("twist knot" [6]) for  $n \geq 3$ . The latter case can be observed from identity (8) for which the leading coefficient is larger than 1 when n+1=n+r-1 is satisfied. Also, considere the identity below:

$$B_{n,2}(x) = B_{n,0}(x) + B_{n,\infty}(x) + B_{n,\infty}(x) + (\bigcirc \sqcup B_{n,\infty})(x).$$

We notice that the leading coefficient is deduced from the contribution of the summands  $B_{n,0}(x)$  and  $(\bigcirc \sqcup B_{n,\infty})(x)$  [6].

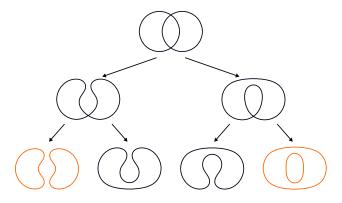


Figure 9: The states of the knot  $B_{1,1}$ : d(1,1) = 2 and b(1,1,d(1,1)) = 2.

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(Concerned with sequences <u>A000124</u>, <u>A007318</u>, <u>A014206</u>, <u>A077028</u>, <u>A233583</u>, <u>A294619</u>, <u>A300401</u>, <u>A300453</u>, <u>A300454</u>, <u>A321125</u>, <u>A321126</u> and <u>A321127</u>.)