

ON THE NUMBER OF DISTINCT EXPONENTS IN THE PRIME FACTORIZATION OF AN INTEGER

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ABSTRACT. Let $f(n)$ be the number of distinct exponents in the prime factorization of the natural number n . We prove some results about the distribution of $f(n)$. In particular, for any positive integer k , we obtain that

$$\#\{n \leq x : f(n) = k\} \sim A_k x$$

and

$$\#\{n \leq x : f(n) = \omega(n) - k\} \sim \frac{Bx(\log \log x)^k}{k! \log x},$$

as $x \rightarrow +\infty$, where $\omega(n)$ is the number of prime factors of n and $A_k, B > 0$ are some explicit constants. The latter asymptotic extends a result of Aktaş and Ram Murty about numbers having mutually distinct exponents in their prime factorization.

1. INTRODUCTION

Let $n = p_1^{a_1} \cdots p_s^{a_s}$ be the factorization of the natural number $n > 1$, where $p_1 < \cdots < p_s$ are prime numbers and a_1, \dots, a_s are positive integers. Several functions of the exponents a_1, \dots, a_s have been studied, including: their product [17], their arithmetic mean [2, 4, 5, 7], and their maximum and minimum [11, 13, 15, 18]. See also [3, 8] for more general functions.

Let f be the arithmetic function defined by $f(1) := 0$ and $f(n) := \#\{a_1, \dots, a_s\}$ for all natural numbers $n > 1$. In other words, $f(n)$ is the number of distinct exponents in the prime factorization of n . The first values of $f(n)$ are listed in sequence A071625 of OEIS [16].

Our first contribution is a quite precise result about the distribution of $f(n)$.

Theorem 1.1. *There exists a sequence of positive real numbers $(A_k)_{k \geq 1}$ such that, given any arithmetic function ϕ satisfying $|\phi(k)| < a^k$ for some fixed $a > 1$, we have that the series*

$$(1) \quad M_\phi := \sum_{k=1}^{\infty} A_k \phi(k)$$

converges and

$$\sum_{n \leq x} \phi(f(n)) = M_\phi x + O_{a,\varepsilon}(x^{1/2+\varepsilon}),$$

for all $x \geq 1$ and $\varepsilon > 0$.

From Theorem 1.1 it follows immediately that all the moments of f are finite and that f has a limiting distribution. In particular, we highlight the following corollary:

Corollary 1.1. *For each positive integer k , we have*

$$\#\{n \leq x : f(n) = k\} = A_k x + O_\varepsilon(x^{1/2+\varepsilon}),$$

for all $x \geq 1$ and $\varepsilon > 0$.

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We provide also a formula for A_k . Before stating it, we need to introduce some notation. Let ψ be the Dedekind function, defined by

$$\psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

for each positive integer n , and let $(\rho_k)_{k \geq 1}$ be the family of arithmetic functions supported on squarefree numbers and satisfying

$$\rho_1(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad \rho_{k+1}(n) = \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{n-1} \sum_{\substack{d|n \\ d < n}} \rho_k(d) & \text{if } n > 1, \end{cases}$$

for all squarefree numbers n and positive integers k .

Theorem 1.2. *We have*

$$A_k = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{\rho_k(n)}{\psi(n)}$$

for each positive integer k .

Clearly, $f(n) \leq \omega(n)$ for all positive integers n , where $\omega(n)$ denotes the number of prime factors of n . Motivated by a question of Recamán Santos [14], Aktaş and Ram Murty [1] studied the natural numbers n such that all the exponents in their prime factorization are distinct, that is, $f(n) = \omega(n)$. They called such numbers *special numbers* (sequence A130091 of OEIS [16]) and they proved the following:

Theorem 1.3. *The number of special numbers not exceeding x is*

$$\frac{Bx}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

for all $x \geq 2$, where

$$B := \sum_{\ell} \frac{1}{\ell}$$

and the sum is over natural numbers ℓ that are powerful and special.

Let g be the arithmetic function defined by $g(n) := \omega(n) - f(n)$ for all positive integers n . Hence, by the previous observation, g is a nonnegative function and $g(n) = 0$ if and only if n is a special number. We prove the following result about g , which extends Theorem 1.3 and it is somehow dual to Corollary 1.1.

Theorem 1.4. *For each nonnegative integer k , we have*

$$\#\{n \leq x : g(n) = k\} = \frac{Bx(\log \log x)^k}{k! \log x} \left(1 + O_k\left(\frac{1}{\log \log x}\right)\right),$$

for all $x \geq 3$.

Notation. We employ the Landau–Bachmann “Big Oh” notation O , as well as the associated Vinogradov symbol \ll , with their usual meanings. Any dependence of the implied constants is explicitly stated. We reserve the letter p for prime numbers.

2. PRELIMINARIES

Recall that a natural number n is called *powerful* if $p | n$ implies $p^2 | n$, for all primes p . For all $x \geq 1$, let $\mathcal{P}(x)$ be the set of powerful numbers not exceeding x .

Lemma 2.1. *We have $\#\mathcal{P}(x) \ll x^{1/2}$ for every $x \geq 1$.*

Proof. See [9]. □

Lemma 2.2. *We have*

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} \ll \frac{1}{y^{1/2}}, \quad \sum_{\ell \in \mathcal{P}(y)} \frac{1}{\ell^{1/2}} \ll \log y,$$

for all $y \geq 2$.

Proof. By Lemma 2.1 and by partial summation, we have

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} = \frac{\#\mathcal{P}(t)}{t} \Big|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{\#\mathcal{P}(t)}{t^2} dt \ll \int_y^{+\infty} \frac{dt}{t^{1+1/2}} \ll \frac{1}{y^{1/2}}.$$

The proof of the second claim is similar. \square

We need the following upper bound for the number of prime factors of a natural number.

Lemma 2.3. *We have*

$$\omega(n) \ll \frac{\log n}{\log \log n}$$

for all integers $n \geq 3$.

Proof. See, e.g., [6, Proposition 7.10]. \square

For every $x \geq 1$ and every positive integer h , let $Q(x; h)$ denote the number of squarefree numbers not exceeding x and relatively prime with h .

Lemma 2.4. *We have*

$$Q(x; h) = \frac{6}{\pi^2} \frac{h}{\psi(h)} x + O\left(4^{\omega(h)}(x^{1/2} + 1)\right)$$

for all $x \geq 1$ and all positive integers h .

Proof. It follows easily from [10, Eq. 8]. \square

For every $x \geq 1$ and every positive integers s, h , let $Q_s(x; h)$ denote the number of squarefree numbers not exceeding x , having exactly s prime factors, and relatively prime with h .

Lemma 2.5. *We have*

$$Q_s(x; h) = \frac{x(\log \log x)^{s-1}}{(s-1)! \log x} \left(1 + O_{\delta, s} \left(\frac{\log \log(h+2)}{\log \log x}\right)\right)$$

for all $x \geq 3$, $0 < \delta < 1$, and for all integers $1 \leq h \leq x^\delta$ and $s \geq 1$.

Proof. For $s = 1$ the claim follows from the Prime Number Theorem, while for $h = 1$ the claim is a classic result of Landau [12]. Hence, suppose $s, h > 1$. Also, we can assume $x \geq 3^{1/(1-\delta)}$. If $n \leq x$ is a squarefree number having exactly s prime factors and such that $(n, h) > 1$, then $n = pn'$ where p is a prime number dividing h and $n' \leq x/p$ is a squarefree number having exactly $s-1$ prime factor. Therefore,

$$\begin{aligned} 0 \leq Q_s(x; 1) - Q_s(x; h) &\leq \sum_{p|h} Q_{s-1}\left(\frac{x}{p}, 1\right) \ll_s \sum_{p|h} \frac{x(\log \log(x/p))^{s-2}}{p \log(x/p)} \\ &\ll_\delta \frac{x(\log \log x)^{s-2}}{\log x} \sum_{p|h} \frac{1}{p} \ll \frac{x(\log \log x)^{s-1} \log \log(h+2)}{\log x \log \log x}, \end{aligned}$$

where we used the fact that $p \leq x^\delta$ and Mertens' second theorem [6, Theorem 4.5]. Consequently,

$$\begin{aligned} Q_s(x; h) &= Q_s(x; 1) + O_{\delta, s} \left(\frac{x(\log \log x)^{s-1} \log \log(h+2)}{\log x \log \log x}\right) \\ &= \frac{x(\log \log x)^{s-1}}{(s-1)! \log x} + O_{\delta, s} \left(\frac{x(\log \log x)^{s-1} \log \log(h+2)}{\log x \log \log x}\right), \end{aligned}$$

as claimed. \square

Finally, we need a lemma about certain sums of powers.

Lemma 2.6. *Let a_0 be an integer. For all $x_1, \dots, x_k > 1$ we have*

$$\sum_{a_0 < a_1 < \dots < a_k} \frac{1}{x_1^{a_1} \dots x_k^{a_k}} = \frac{1}{(x_1 \dots x_k)^{a_0}} \prod_{j=1}^k \frac{1}{x_j \dots x_k - 1},$$

where the sum is over all integers a_1, \dots, a_k satisfying $a_0 < a_1 < \dots < a_k$.

Proof. We proceed by induction on k . For $k = 1$, we have

$$(2) \quad \sum_{a_0 < a_1} \frac{1}{x_1^{a_1}} = \frac{1}{x_1^{a_0+1}} \sum_{d=0}^{\infty} \frac{1}{x_1^d} = \frac{1}{x_1^{a_0}} \frac{1}{x_1 - 1},$$

as claimed. Supposing that the claim is true for k , we shall prove it for $k + 1$. We have

$$\begin{aligned} \sum_{a_0 < \dots < a_{k+1}} \frac{1}{x_1^{a_1} \dots x_{k+1}^{a_{k+1}}} &= \sum_{a_0 < \dots < a_k} \frac{1}{x_1^{a_1} \dots x_k^{a_k}} \sum_{a_k < a_{k+1}} \frac{1}{x_{k+1}^{a_{k+1}}} \\ &= \sum_{a_0 < \dots < a_{k+1}} \frac{1}{x_1^{a_1} \dots x_{k-1}^{a_{k-1}} (x_k x_{k+1})^{a_k}} \frac{1}{x_{k+1} - 1} \\ &= \frac{1}{(x_1 \dots x_{k+1})^{a_0}} \prod_{j=1}^k \frac{1}{x_j \dots x_{k+1} - 1} \frac{1}{x_{k+1} - 1} \\ &= \frac{1}{(x_1 \dots x_{k+1})^{a_0}} \prod_{j=1}^{k+1} \frac{1}{x_j \dots x_{k+1} - 1}, \end{aligned}$$

where we used (2), with a_0 and x_1 replaced respectively by a_k and x_{k+1} , and the induction hypothesis. \square

3. PROOF OF THEOREM 1.1

We begin by proving that for each positive integer k there exists $A_k > 0$ such that

$$(3) \quad N_k(x) := \#\{n \leq x : f(n) = k\} = A_k x + O_\varepsilon(x^{1/2+\varepsilon/2}),$$

for all $x \geq 1$ and $\varepsilon > 0$. Clearly, every natural number n can be written in a unique way as $n = m\ell$, where m is a squarefree number, ℓ is a powerful number, and $(m, \ell) = 1$. If $m = 1$ then $n = \ell$ is powerful and, by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If $m > 1$ then $f(n) = k$ is equivalent to $f(\ell) = k - 1$. Also, for each ℓ there are exactly $Q(x/\ell; \ell) - 1$ choices for $m > 1$. Therefore, we have

$$(4) \quad N_k(x) = \sum_{\substack{\ell \in \mathcal{P}(x) \\ f(\ell) = k-1}} \left(Q\left(\frac{x}{\ell}; \ell\right) - 1 \right) + O(x^{1/2}),$$

for all $x \geq 1$. For each positive integer $\ell \leq x$, Lemma 2.3 gives $4^{\omega(\ell)} \ll_\varepsilon x^{\varepsilon/4}$. Consequently, by Lemma 2.4, we obtain

$$(5) \quad Q\left(\frac{x}{\ell}; \ell\right) = \frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_\varepsilon\left(\frac{x^{1/2+\varepsilon/4}}{\ell^{1/2}}\right),$$

for all positive integers $\ell \leq x$. By Lemma 2.2, we have

$$(6) \quad \sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{1}{\psi(\ell)} < \sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{1}{\ell} \ll \frac{1}{x^{1/2}},$$

for all $x \geq 1$. In particular, the series

$$(7) \quad A_k := \frac{6}{\pi^2} \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\psi(\ell)}$$

converges. Also, again by Lemma 2.2, we have

$$(8) \quad \sum_{\ell \in \mathcal{P}(x)} \frac{1}{\ell^{1/2}} \ll \log x \ll_{\varepsilon} x^{\varepsilon/4}.$$

At this point, putting together (4) and (5), and using (6) and (8), we obtain

$$\begin{aligned} N_k(x) &= \sum_{\substack{\ell \in \mathcal{P}(x) \\ f(\ell) = k-1}} \left(\frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_{\varepsilon} \left(\frac{x^{1/2+\varepsilon/4}}{\ell^{1/2}} \right) \right) + O(x^{1/2}) \\ &= A_k x + O \left(\sum_{\substack{\ell \in \mathcal{P} \\ \ell > x}} \frac{x}{\psi(\ell)} \right) + O_{\varepsilon} \left(\sum_{\ell \in \mathcal{P}(x)} \frac{x^{1/2+\varepsilon/4}}{\ell^{1/2}} \right) + O(x^{1/2}) \\ &= A_k x + O_{\varepsilon}(x^{1/2+\varepsilon/2}), \end{aligned}$$

as desired. Thus (3) is proved.

Now we shall show that

$$(9) \quad A_k \leq \frac{6}{\pi^2} \frac{1}{(k-1)!}$$

for all positive integers k . For $k = 1$ the claim is obvious since $A_1 = 6/\pi^2$. Hence, assume $k \geq 2$. If ℓ is a powerful number such that $f(\ell) = k - 1$, then $\ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}$ for some integers $m_1, \dots, m_{k-1} \geq 2$ and $2 \leq a_1 < \cdots < a_{k-1}$. Consequently,

$$\begin{aligned} \frac{\pi^2}{6} A_k &= \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\psi(\ell)} < \sum_{\substack{\ell \in \mathcal{P} \\ f(\ell) = k-1}} \frac{1}{\ell} < \prod_{j=1}^{k-1} \sum_{m=2}^{\infty} \sum_{a=j+1}^{\infty} \frac{1}{m^a} \\ &= \prod_{j=1}^{k-1} \sum_{m=2}^{\infty} \frac{1}{m^j(m-1)} \leq \prod_{j=1}^{k-1} \frac{1}{j} = \frac{1}{(k-1)!}, \end{aligned}$$

where we used the facts that

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 1$$

and

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{1}{m^j(m-1)} &< \frac{1}{2^j} + \frac{1}{3^j \cdot 2} + \sum_{n=3}^{\infty} \frac{1}{n^{j+1}} \\ &< \frac{1}{2^j} + \frac{1}{3^j \cdot 2} + \int_2^{+\infty} \frac{dt}{t^{j+1}} = \frac{1}{2^j} + \frac{1}{3^j \cdot 2} + \frac{1}{j2^j} < \frac{1}{j}, \end{aligned}$$

for all integers $j \geq 2$. Thus (9) is proved.

Now let ϕ be an arithmetic function satisfying $|\phi(k)| < a^k$ for all positive integers k , where $a > 1$ is some constant. From (9) it follows that series (1) converges. Define

$$y := 2a + \lfloor C \log x / \log \log(x+2) \rfloor,$$

where $C > 0$ is some absolute constant. Since $f(n) \leq \omega(n)$ for all positive integers n , by Lemma 2.3, we can choose C sufficiently large so that $f(n) \leq y$ for all natural numbers $n \leq x$.

Moreover, from (9) and $y \geq 2a$, we get that

$$(10) \quad \sum_{k>y} A_k \phi(k) \ll \sum_{k>y} \frac{a^k}{(k-1)!} < \frac{a^{y+1}}{y!} \sum_{j=0}^{\infty} \left(\frac{a}{y}\right)^j \ll_a \frac{a^y}{y!} \ll_a \frac{1}{x^{1/2}},$$

and

$$(11) \quad a^y y \ll_{a,\varepsilon} x^{\varepsilon/2},$$

for all $x \geq 1$. Therefore, putting together (3), (10), and (11), we have

$$\begin{aligned} \sum_{n \leq x} \phi(f(n)) &= \sum_{k \leq y} N_k(x) \phi(k) = \sum_{k \leq y} \left(A_k \phi(k) x + O_\varepsilon(\phi(k) x^{1/2+\varepsilon/2}) \right) \\ &= M_\phi x + O\left(\sum_{k>y} A_k \phi(k) x \right) + O_\varepsilon(a^y y x^{1/2+\varepsilon/2}) = M_\phi x + O_{a,\varepsilon}(x^{1/2+\varepsilon}), \end{aligned}$$

for all $x \geq 1$ and $\varepsilon > 0$. The proof is complete.

4. PROOF OF THEOREM 1.2

Recall that A_k is defined by (7). For $k = 1$ the claim is obvious, since $f(\ell) = 0$ if and only if $\ell = 1$. Hence, assume $k \geq 2$. If ℓ is a powerful number such that $f(\ell) = k - 1$, then ℓ can be written in a unique way as $\ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}$, where $1 < a_1 < \cdots < a_{k-1}$ are integers and $m_1, \dots, m_{k-1} > 1$ are pairwise coprime squarefree numbers. Therefore, from (7) and Lemma 2.6 we obtain

$$\begin{aligned} \frac{\pi^2}{6} A_k &= \sum_{m_1, \dots, m_{k-1}} \sum_{1 < a_1 < \cdots < a_{k-1}} \frac{1}{\psi(m_1^{a_1} \cdots m_{k-1}^{a_{k-1}})} \\ &= \sum_{m_1, \dots, m_{k-1}} \frac{m_1 \cdots m_{k-1}}{\psi(m_1 \cdots m_{k-1})} \sum_{1 < a_1 < \cdots < a_{k-1}} \frac{1}{m_1^{a_1} \cdots m_{k-1}^{a_{k-1}}} \\ &= \sum_{m_1, \dots, m_{k-1}} \frac{1}{\psi(m_1 \cdots m_{k-1})} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1}, \end{aligned}$$

where, here and for the rest of the proof, in summation subscripts m_1, \dots, m_{k-1} are meant to be pairwise coprime, squarefree, and greater than 1. At this point, it is enough to prove that

$$\sum_{n = m_1 \cdots m_{k-1}} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1} = \rho_k(n)$$

for all squarefree numbers $n > 1$. We proceed by induction on k . For $k = 2$, the claim is true since

$$\frac{1}{n-1} = \frac{\rho_1(1)}{n-1} = \frac{1}{n-1} \sum_{\substack{d|n \\ d < n}} \rho_1(d) = \rho_2(n),$$

for all squarefree numbers $n > 1$. Assuming that the claim is true for k , we shall prove it for $k + 1$. We have

$$\begin{aligned} \sum_{n = m_1 \cdots m_k} \prod_{j=1}^k \frac{1}{m_j \cdots m_k - 1} &= \frac{1}{n-1} \sum_{m_1 | n} \sum_{n/m_1 = m_2 \cdots m_k} \prod_{j=2}^k \frac{1}{m_j \cdots m_k - 1} \\ &= \frac{1}{n-1} \sum_{m_1 | n} \rho_k(n/m_1) = \frac{1}{n-1} \sum_{\substack{d|n \\ d < n}} \rho_k(d) = \rho_{k+1}(n), \end{aligned}$$

for all squarefree numbers $n > 1$, as desired. The proof is complete.

5. PROOF OF THEOREM 1.4

We have to count the number of positive integers $n \leq x$ such that $g(n) = k$. As in the proof of Theorem 1.1, every n can be written in a unique way as $n = m\ell$, where m is a squarefree number, ℓ is a powerful number, and $(m, \ell) = 1$. If $m = 1$ then $n = \ell$ is powerful and, by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If $m > 1$ then

$$\omega(m) = \omega(n) - \omega(\ell) = g(n) + f(n) - f(\ell) - g(\ell) = k + 1 - g(\ell).$$

In particular, $1 \leq \omega(m) \leq k + 1$. Assume x sufficiently large, and put $y := (\log x)^2$. Then, by Lemma 2.2, the number of $n \leq x$ such that $\ell > y$ is at most

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{x}{\ell} \ll \frac{x}{y^{1/2}} = \frac{x}{\log x}.$$

Therefore,

$$(12) \quad M_k(x) := \#\{n \leq x : g(n) = k\} = \sum_{s=1}^{k+1} \sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = k+1-s}} Q_s\left(\frac{x}{\ell}; \ell\right) + O\left(\frac{x}{\log x}\right).$$

For each nonnegative integer r , put

$$B_r := \sum_{\substack{\ell \in \mathcal{P} \\ g(\ell) = r}} \frac{1}{\ell}.$$

Note that, in light of Lemma 2.2, the series defining B_r converges and, more precisely,

$$(13) \quad \sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = r}} \frac{1}{\ell} = B_r + O\left(\frac{1}{y^{1/2}}\right) = B_r + O\left(\frac{1}{\log x}\right).$$

Clearly, we can assume x sufficiently large so that $x/y \geq 3$ and $y \leq x^{\delta/(1+\delta)}$, for some fixed $0 < \delta < 1$. Hence, applying Lemma 2.5 we obtain

$$\begin{aligned} Q_s\left(\frac{x}{\ell}; \ell\right) &= \frac{x(\log \log(x/\ell))^{s-1}}{\ell(s-1)! \log(x/\ell)} \left(1 + O_k\left(\frac{\log \log(\ell+2)}{\log \log(x/\ell)}\right)\right) \\ &= \frac{x(\log \log x)^{s-1}}{\ell(s-1)! \log x} \left(1 + O_k\left(\frac{\log \ell}{\log x}\right)\right) \left(1 + O_k\left(\frac{\log \log(\ell+2)}{\log \log x}\right)\right) \\ &= \frac{x(\log \log x)^{s-1}}{\ell(s-1)! \log x} \left(1 + O_k\left(\frac{\log(\ell+1)}{\log \log x}\right)\right), \end{aligned}$$

for all positive integers $s \leq k + 1$ and $\ell \leq y$. Consequently,

$$(14) \quad \begin{aligned} \sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = k+1-s}} Q_s\left(\frac{x}{\ell}; \ell\right) &= \frac{x(\log \log x)^{s-1}}{(s-1)! \log x} \sum_{\substack{\ell \in \mathcal{P}(y) \\ g(\ell) = k+1-s}} \frac{1}{\ell} \left(1 + O_k\left(\frac{\log(\ell+1)}{\log \log x}\right)\right) \\ &= \frac{x(\log \log x)^{s-1}}{(s-1)! \log x} \left(B_{k+1-s} + O\left(\frac{1}{\log x}\right) + O_k\left(\frac{1}{\log \log x}\right)\right) \\ &= \frac{x(\log \log x)^{s-1}}{(s-1)! \log x} \left(B_{k+1-s} + O_k\left(\frac{1}{\log \log x}\right)\right), \end{aligned}$$

where we used (13) and the fact that the series

$$\sum_{\ell \in \mathcal{P}} \frac{\log(\ell+1)}{\ell}$$

converges. Thus, putting together (12) and (14), and noting that $B_0 = B$, we obtain

$$M_k(x) = \frac{Bx(\log \log x)^k}{k! \log x} \left(1 + O_k \left(\frac{1}{\log \log x} \right) \right),$$

as desired. The proof is complete.

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