Succinct Data Structures for Families of Interval Graphs

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Abstract

We consider the problem of designing succinct data structures for *interval graphs* with n vertices while supporting degree, adjacency, neighborhood and shortest path queries in optimal time in the $\Theta(\log n)$-bit[1] word RAM model. The degree query reports the number of incident edges to a given vertex in constant time, the adjacency query returns true if there is an edge between two vertices in constant time, the neighborhood query reports the set of all adjacent vertices in time proportional to the degree of the queried vertex, and the shortest path query returns a shortest path in time proportional to its length, thus the running times of these queries are optimal. Towards showing succinctness, we first show that at least $n \log n - 2n \log\log n - O(n)$ bits are necessary to represent any unlabeled interval graph G with n vertices, answering an open problem of Yang and Pippenger [Proc. Amer. Math. Soc. 2017]. This is augmented by a data structure of size $n \log n + O(n)$ bits while supporting not only the aforementioned queries optimally but also capable of executing various combinatorial algorithms (like proper coloring, maximum independent set etc.) on the input interval graph efficiently. Finally, we extend our ideas to other variants of interval graphs, for example, *proper/unit interval graphs, k-proper and k-improper interval graphs, and circular-arc graphs*, and design succinct/compact data structures for these graph classes as well along with supporting queries on them efficiently.

1 Introduction

A simple undirected graph G is called an *interval graph* if its vertices can be assigned to intervals on the real line so that two vertices are adjacent in G if and only if their assigned intervals intersect. The set of intervals assigned to the vertices of G is called a *realization* of G. These graphs were first introduced by Hajós [25] who also asked for the characterization of them. The same problem was also asked, independently, by Benser [4] while studying the structure of genes. Interval graphs naturally appear in a variety of contexts, for example, operations research and scheduling theory [3], biology especially in physical mapping of DNA [35], temporal reasoning [21] and many more. We refer the reader to [19, 20] for a thorough treatment of interval graphs and its applications. Eventually answering the question of Hajós [25], several researchers came up with different characterizations of interval graphs, including linear time algorithms for recognizing them; see, for example, [20, Chapter 8] for characterizations, and [5] and [24] for linear time algorithms. Moreover, exploiting the special structure of interval graphs, many otherwise NP-hard problems in general graphs are also shown to have polynomial time algorithms for interval graphs [19]. These include computing maximum independent set, reporting a proper coloring, returning a maximum clique etc. In spite of having many applications in practically motivated problems, we are not aware of, to the best of our knowledge, any study of interval graphs from the point of view of *succinct data structures* where the goal is to store a set Z of objects using the information theoretic minimum $\log(|Z|) + o(\log(|Z|))$ bits of space while still being able to support the relevant set of queries efficiently, and which is what we focus on in this paper. We also assume the usual model of computation, namely a $\Theta(\log n)$-bit word RAM model where n is the size of the input.

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[1]throughout the paper, we use log to denote the logarithm to the base 2

1.1 Related Work

There already exists a large body of work on representing various classes of graphs succinctly. This is partly motivated by theoretical curiosity and partly by the practical needs as these combinatorial structures do arise quite often in various applications. A partial list of such special graph classes would be trees [28], planar graphs [1], chordal graphs [29], partial k-tree [15] among others, while succinct encoding for arbitrary graphs is also considered [16] in the literature. For interval graphs, other than the algorithmic works mentioned earlier, there are plenty of attempts in exactly counting the number of unlabeled interval graphs [26, 27], and the state-of-the-art result is due to [34], which is what we improve in this work. For the variants of the interval graphs that we study in this paper, there exists also a fairly large number of algorithmic results on them as well as structural results. See [19, 20] for details.

1.2 Our Results and Paper Organization

Given an unlabeled interval graph G with n vertices, in Section 3 we first show that at least $n \log n - 2n \log\log n - O(n)$ bits are necessary to represent G, answering an open problem of Yang and Pippenger [34]. More specifically, Yang and Pippenger [34] showed a lower bound of $(n \log n)/3 + O(n)$-bit for representing any unlabeled interval graph and asked whether this lower bound can be further improved. Augmenting this lower bound, in Section 4 we also propose a succinct representation of G using $n \log n + O(n)$ bits while still being able to support the relevant queries optimally, where the queries are defined as follows. For any two vertices $u, v \in G$,

- $\mathsf{degree}(v)$: returns the number of vertices that are adjacent to v in G,
- $\mathsf{adjacent}(u, v)$: returns true if u and v are adjacent in G, and false otherwise,
- $\mathsf{neighborhood}(v)$: returns all the vertices that are adjacent to v in G, and
- $\mathsf{spath}(u, v)$: returns the shortest path between u and v in G.

We show that all these queries can be supported optimally using our succinct data structure for interval graphs. More precisely, for any two vertices $v, u \in G$, we can answer $\mathsf{degree}(v)$ and $\mathsf{adjacent}(u, v)$ queries in $O(1)$ time, $\mathsf{neighborhood}(v)$ queries in $O(\mathsf{degree}(v))$ time, and $\mathsf{spath}(u, v)$ queries in $O(|\mathsf{spath}(u, v)|)$ time. Furthermore, we also show how one can implement various fundamental graph algorithms in interval graphs, for example depth-first search (DFS), breadth-first search (BFS), computing maximum independent set, determining a maximum clique etc, both time and space efficiently using our succinct representation for interval graphs. In Section 5, we extend our ideas to other variants of interval graphs, for example, *proper/unit interval graphs, k-proper and k-improper interval graphs, and circular-arc graphs*, and design succinct data structures for these graph classes as well along with supporting queries on them efficiently. For definitions of these graphs, see Section 5. Finally we conclude in Section 6 with some remarks on possible future directions for exploring. We list all the preliminary data structures and graph theoretic terminologies that will be used throughout this paper, in Section 2.

2 Preliminaries

We will use the following data structures in the rest of this paper.

Rank and Select queries: Let $S = s_1, \ldots, s_n$ be a sequence of size n over an alphabet $\Sigma = \{0, 1, \ldots, \sigma - 1\}$. Then for $1 \le i \le n$, and $\alpha \in \Sigma$, one can define rank and select queries as follows.

- $\mathsf{rank}_\alpha(S, i)$ = the number of occurrences of α in $s_1 \ldots s_i$.
- $\mathsf{select}_\alpha(S, i)$ = the position j where s_j is the i-th α in S.

The following lemma shows that these operations can be supported efficiently using optimal space.

Lemma 2.1 ([11, 22]). *Given a sequence $S = s_1, \ldots, s_n$ of size n over an alphabet $\Sigma = \{0, 1, \ldots, \sigma - 1\}$, for any $\alpha \in \Sigma$, there exists data structures as follows.*

- *when $\sigma = 2$, $n + o(n)$-bit data structure which answers rank_α and select_α queries on S in $O(1)$ time.*
- *when $\sigma > 2$, $n \log \sigma + o(n \log \sigma)$-bit data structure which answers rank_α queries on S in $O(\log \log \sigma)$ time and select_α queries on S in $O(1)$ time.*

Note that the one can access the any element of the input sequence (at a given index) in $O(1)$ (resp. $O(\log \log \sigma)$) time with the $n + o(n)$ (resp. $n \log \sigma + o(n \log \sigma)$)-bit data structure of Lemma 2.1

Range Maximum Queries: Given a sequence $S = s_1, \ldots, s_n$ of size n, for $1 \leq i, j \leq n$, the *Range Maximum Query* on range $[i, j]$ (denoted by $\mathsf{RMax}_S(i, j)$) returns the position $i \leq k \leq j$ such that s_k is a maximum value in $s_i \ldots s_j$ (if there is a tie, we return the leftmost such position). One can define the *Range Minimum Queries* on range $[i, j]$ ($\mathsf{RMin}_S(i, j)$) analogously. The following lemma shows that there exist data structures which can answer these queries efficiently using optimal space.

Lemma 2.2 ([7, 18]). *Given a sequence S of size n and for any $1 \leq c \leq n$,*

1. *there exists a data structure of size $O(n/c)$ bits, in addition to storing the sequence S, which supports RMax_S and RMin_S queries in $O(c)$ time while supporting access on S in $O(1)$ time.*
2. *there exists a data structure of size $2n + o(n)$ bits (that does not store the sequence S) which supports RMax_S or RMin_S queries in $O(1)$ time.*

Graph Terminology and Input Representation: We will assume the knowledge of basic graph theoretic terminology as given in [13] and basic graph algorithms as given in [12]. Throughout this paper, $G = (V, E)$ will denote a simple undirected graph with the vertex set V of cardinality n and the edge set E having cardinality m. We call G an *interval graph* if (a) with every vertex we can associate a closed interval on the real line, and (b) two vertices share an edge if and only if the corresponding intervals are not disjoint (see Figure 1 for an example). It is well known that given an interval graph with n vertices, one can assign intervals to vertices such that every end point is a distinct integer from 1 to $2n$ using $O(n \log n)$ time [26], and in the rest of this paper, we deal exclusively with such representation. Moreover, for vertex $v \in V$, we refer to I_v as the interval corresponding to v.

3 Counting the number of unlabeled interval graphs

This section deals with counting unlabeled interval graphs on n vertices, and let $\mathcal{I}_n$ denote this quantity. (This is the sequence with id A005975 in the On–Line Encyclopedia of Integer Sequences [32].) Initial values of this sequence are given by Hanlon [26] but he did not prove an asymptotic form for enumerating the sequence. Answering a question posed by Hanlon [26], Yang and Pippenger [34] proved that the generating function $\mathcal{I}(x) = \sum_{n \geq 1} \mathcal{I}_n x^n$ diverges for any $x \neq 0$ and they established the bounds

$$\frac{n \log n}{3} + O(n) \leq \log \mathcal{I}_n \leq n \log n + O(n). \tag{1}$$

The upper bound in (1) follows from $\mathcal{I}_n \leq (2n-1)!! = \prod_{j=1}^{n}(2j-1)$, where the right hand side is the number of matchings on $2n$ points on a line. For the lower bound, the authors showed $\mathcal{I}_{3k} \geq k!/3^{3k}$ by finding an injection from S_k, the set of permutations of length k, to three-colored interval graphs of size $3k$. Furthermore, they left it open whether the leading terms of the lower and upper bounds in (1) can be matched, which is what show in affirmative by improving the lower bound. In other words, we find the asymptotic value of $\log \mathcal{I}_n$. In what follows, for a set S, we denote by $\binom{S}{k}$ the set of k-subsets of S.

Theorem 1. *Let $\mathcal{I}_n$ be the number of unlabeled interval graphs with n vertices. As $n \to \infty$, we have*

$$\log \mathcal{I}_n \geq n \log n - 2n \log\log n - O(n). \tag{2}$$

Proof. We consider certain interval graphs on n vertices with colored vertices. Let k be a positive integer smaller than $n/2$ and ε a positive constant smaller than $1/2$. For $1 \leq j \leq k$, let B_j and R_j denote the intervals $[-j-\varepsilon, -j+\varepsilon]$ and $[j-\varepsilon, j+\varepsilon]$, respectively. These $2k$ pairwise-disjoint intervals will make up $2k$ vertices in the graphs we consider. Now let $\mathcal{W}$ denote the set of k^2 closed intervals with one endpoint in $\{-k, \dots, -1\}$ and the other in $\{1, \dots, k\}$. We color $B_1, \dots, B_k$ with blue, $R_1, \dots, R_k$ with red, and the k^2 intervals in $\mathcal{W}$ with white.

Together with $\mathcal{S} := \{B_1, \dots, B_k, R_1, \dots, R_k\}$, each $\{J_1, \dots, J_{n-2k}\} \in \binom{\mathcal{W}}{n-2k}$ gives an n-vertex, three-colored interval graph. For a given $\mathcal{J} = \{J_1, \dots, J_{n-2k}\}$, let $G_{\mathcal{J}}$ denote the colored interval graph whose vertices correspond to n intervals in $\mathcal{S} \cup \mathcal{J}$, and let $\mathcal{G}$ denote the set of all $G_{\mathcal{J}}$.

Now let $G \in \mathcal{G}$. For a white vertex $w \in G$, the pair $(d_B(w), d_R(w))$, which represents the numbers of blue and red neighbors of w, uniquely determine the interval corresponding to w; this is the interval $[-d_B(w), d_R(w)]$. In other words, $\mathcal{J}$ can be recovered from $G_{\mathcal{J}}$ uniquely. Thus $|\mathcal{G}| = \binom{k^2}{n-2k}$. Since there are at most 3^n ways to color the vertices of an interval graph with blue, red, and white, we have

$$\mathcal{I}_n \cdot 3^n \geq |\mathcal{G}| = \binom{k^2}{n-2k} \geq \left(\frac{k^2}{n-2k}\right)^{n-2k} \geq \left(\frac{k^2}{n}\right)^{n-2k}$$

for any $k < n/2$. Setting $k = \lfloor n/\log n \rfloor$ and taking the logarithms, we get

$$\log \mathcal{I}_n \geq (n-2k)\log(k^2/n) - O(n) = n\log n - 2n\log\log n - O(n).$$

□

Remark. Yang and Pippenger [34] also posed the question whether $\log \mathcal{I}_n = Cn \log n + O(n)$ for some C or not. According to Theorem 1, this boils down to getting rid of the $2n \log\log n$ term in (2). Such a result would imply that the exponential generating function $J(x) = \sum_{n \geq 1} I_n x^n/n!$ has a finite radius of convergence. (As noted in [34], the bound $\mathcal{I}_n \leq (2n-1)!!$ implies that the radius of convergence of $J(x)$ is at least 1/2).

4 Succinct representation of interval graphs

In this section, we introduce a succinct $n \log n + (2+\epsilon)n + o(n)$-bit representation of unlabeled interval graph G on n vertices with constant $\epsilon > 0$, and show that the navigational queries (degree, adjacent, neighborhood, and spath queries) and some basic graph algorithms (BFS, DFS, PEO traversals, proper coloring, computing the size of maximum clique and maximum independent set etc.) on G can be answered/executed efficiently using our representation of G.

4.1 Succinct Representation of G

We first label the vertices of G using the integers from 1 to n, as described in the following. By the assumption in Section 2, the vertices in G can be represented by n intervals $I = \{I_1 = [l_1, r_1], I_2 = [l_2, r_2], \dots, I_n = [l_n, r_n]\}$ where all the endpoints in I are distinct integers in the range $[1, 2n]$. Since there are $2n$ distinct endpoints for the n intervals in I, every integer in $[1, 2n]$ corresponds to a unique l_i or r_i for some $1 \leq i \leq n$. We assign the labels to the vertices in G based on the sorted order of left endpoints of their corresponding intervals, i.e., for any two vertices $a, b \in G$, $a < b$ if and only if $l_a < l_b$.

Now we describe the representation of G. Let $S = s_1 \dots s_{2n}$ be the binary sequence of length $2n$ such that for $1 \leq i \leq 2n$, $s_i = 0$ if $i \in \{l_1, l_2, \dots, l_n\}$ (i.e., if i corresponds to the left end point of an interval in I), and $s_i = 1$ otherwise. If $i = l_k$ or $i = r_k$, we say that s_i corresponds to the interval I_k. We represent

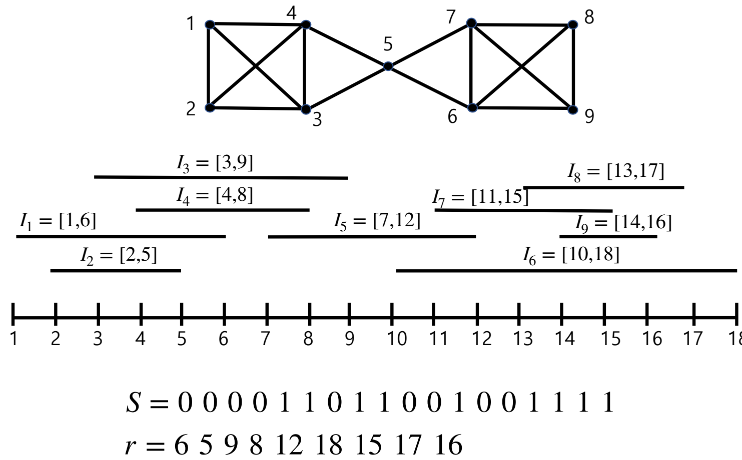


Figure 1: Example of the interval graph and its representation.

the the sequence S using the data structure of Lemma 2.1, using a $2n + o(n)$ bits to support rank and select queries on S in $O(1)$ time. Next, we store the sequence $r = r_1 \ldots r_n$, and for some fixed constant $\epsilon > 0$, we also store an ϵn-bit data structure of Lemma 2.2(1) (with $c = 1/\epsilon$) to support RMax and RMin queries on r in $O(1)$ time. Using the representations of S and r, it is easy to show that for any vertex $v \in G$, we can return its corresponding interval $I_v = [l_v, r_v]$ in $O(1)$ time by computing $l_v = \mathsf{select}_0(S, v)$, and r_v can be accessed from the sequence r. Thus, the total space usage of our representation is $n \log n + (3 + \epsilon)n + o(n)$ bits. See Figure 1 for an example.

4.2 Supporting Navigational Queries

In this section, we show that degree, $\mathsf{adjacent}$, $\mathsf{neighborhood}$, and spath queries on G can be answered in asymptotically optimal time using the representation described in the Section 4.1.

degree(v) query: We count the number of vertices in G which are not adjacent to v, which is a disjoint union of the two sets: (i) the set of intervals that end before the starting point l_v, and (ii) the set of intervals that start after the end point r_v. Using our representation the cardinalities of these two sets can be computed as follows. The number of intervals u with $r_u < l_v$ is given by $\mathsf{rank}_1(S, l_v)$. Similarly, the number of intervals u with $r_v < l_u$ is given by $n - \mathsf{rank}_0(S, r_v)$. Therefore, we can answer $\mathsf{degree}(v)$ query in $O(1)$ time by returning $n - \mathsf{rank}_1(S, l_v) - (n - \mathsf{rank}_0(S, r_v)) = \mathsf{rank}_0(S, r_v) - \mathsf{rank}_1(S, l_v)$.

adjacent(u, v) query: Since we can compute the intervals I_u and I_v in $O(1)$ time, $\mathsf{adjacent}(u, v)$ query can be answered in $O(1)$ by checking $r_u < l_v$ or $r_v < l_u$ (u and v are not adjacent if and only if one of these conditions is satisfied).

neighborhood(v) query: The set of all neighbors of a vertex v can be reported by considering all the intervals I_u whose left end points are in within the range $[1, \ldots, r_v]$ and returning all such u's with $r_u > l_v$ (i.e., which start to the left of r_v and end after l_v). With our data structure, this query can be supported by returning the set $\{u \mid 1 \leq u \leq rank_0(S, r_v) \text{ and } r_u > l_v\}$. Using the RMax structure stored on r, this can be supported in $O(\mathsf{degree}(v))$ time. Note that, given a threshold value t and a query range $[a, b]$ of the sequence r, the range max data structure can be used to report all the elements r_u within the range $[a, b]$ such that $r_u > t$, in $O(1)$ time per element, using the following recursive procedure. Compute the position $c = \mathsf{RMax}_r(a, b)$. If $r_c > r_v$, then return r_c, and recurse on the subintervals $[a, c-1]$ and $[c+1, b]$; else stop.

$\mathsf{spath}(\mathbf{u},\mathbf{v})$ query: We first define the SUCC query as described in [10]. For an interval I_u, $\mathsf{SUCC}(I_u)$ returns the interval $I_{u'}$ such that $I_u \cap I_{u'} \neq \emptyset$ and there is no $I_{u''}$ with $I_u \cap I_{u''} \neq \emptyset$ and $r_{u'} < r_{u''}$. (For example in Figure 1, $\mathsf{SUCC}(I_2) = I_3$ and $\mathsf{SUCC}(I_5) = I_6$.) To answer the $\mathsf{spath}(u, v)$ query, let P_{uv} be the shortest path from u to v initialized with $\emptyset$ (without loss of generality, we assume that $u \leq v$). If u and v are identical, we simply add u to P_{uv} and return P_{uv}. If not, we first add u to P_{uv} and consider two cases as follows [10].

- If u is adjacent to v, add v to P_{uv} and return P_{uv}.
- If I_u is not adjacent to I_v, we perform $\mathsf{spath}(\mathsf{SUCC}(u), v)$ query recursively.

Since we can answer adjacent queries in $O(1)$ time, it is enough to show how to answer the SUCC queries in $O(1)$ time. Let k be the number of vertices v which satisfies $l_v < r_u$, which can be answered in $O(1)$ time by $k = rank_0(S, r_u)$). Then by the definition of SUCC query, I_i with $i = \mathsf{RMax}_r(1, k)$ gives an answer of $\mathsf{SUCC}(I_u)$ if $r_i > l_u$ (if not, there is no vertex in G adjacent to u). Therefore we can answer the SUCC query in $O(1)$ time, which implies $\mathsf{spath}(u, v)$ query can be answered in $O(|spath(u, v)|)$ time.

In Appendix A we discuss how to support some basic graph algorithms (BFS, DFS, PEO traversals, proper coloring, computing the size of maximum clique, maximum independent set and minimum vertex cover) efficiently on G with the above set of operations along with the representation of Section 4.1.

5 Representation of some related families of interval graphs

In this section, we propose space-efficient representations for proper interval graphs, k-proper and k-improper interval graphs, and circular arc graphs. Since these graphs are restrictions or extensions (i.e., sub/super-classes) of interval graphs, we can represent them by modifying the representation in Section 4.1 (to make the representation asymptotically optimal in terms of space). We also show that navigation queries on these graph classes can be answered efficiently with the modified representation.

5.1 Proper interval graphs

An interval graph G is *proper* if there exists an interval representation of G such that for any two vertices $u, v \in G$, $I_u \not\subset I_v$ and $I_v \not\subset I_u$ (let such interval representation of G be *proper representation* of G). Also it is known that proper interval graphs are equivalent to the *unit interval graphs*, which have an interval representation such that every interval has the same length [31].

Now we consider how to represent a proper interval graph G with n vertices while support navigational queries efficiently on G. We first obtain an interval representation of the graph G where the intervals satisfy the property of proper interval graph. We then assign labels to vertices of G based on the sorted order left end points of their corresponding intervals, as described in Section 4.1. Let S be the bit sequence obtained from this representation, as defined in the Section 4.1. Then by the definition of G, there are no two vertices $u, v \in G$ with $l_u < l_v$ and $r_u > r_v$ (if so, $I_v \subset I_u$). Thus by the Lemma 2.1, for any vertex $i \in G$ we can compute l_i and r_i in $O(1)$ time by $\mathsf{select}_0(S, i)$ and $\mathsf{select}_1(S, i)$ respectively using $2n + o(n)$ bits. Also note that r is strictly increasing sequence when G is a proper interval graph, and hence one can support the RMax queries on $r = r_1 \ldots r_n$ in $O(1)$ time without maintaining any data structure, by simply returning the rightmost position of the query range. Thus, we obtain a following theorem.

Theorem 2. *Given a proper interval graph or unit interval graph G with n vertices, there exists a $2n+o(n)$-bit representation of G which answers $\mathsf{degree}(v)$ and $\mathsf{adjacent}(u, v)$ queries in $O(1)$ time, $\mathsf{neighborhood}(v)$ queries in $O(\mathsf{degree}(v))$ time, and $\mathsf{spath}(u, v)$ queries in $O(|\mathsf{spath}(u, v)|)$ time, for any vertices $u, v \in G$.*

It is known that there are asymptotically $\frac{1}{8\kappa\sqrt{\pi}} n^{-3/2} 4^n$ non-isomorphic unlabeled unit interval graphs with n vertices, for some constant $\kappa > 0$ [17], and hence $2n - O(\log n)$ bits is an information-theoretic lower bound on representing an arbitrary proper interval graph. Thus our representation in Theorem 2 gives a succinct representation for proper interval graphs.

5.2 k-proper and k-improper interval graphs

One can generalize the proper interval graph to the following two sub-classes of interval graphs. For an interval graph G with n vertices, G is *k-proper interval graph* (resp. *k-improper interval graph*) if there exists an interval representation G such that for any vertex $v \in G$, I_v *is* contained by (resp., contains) at most $k \leq n$ intervals in G other than I_v. We call such an interval representation of G as the *k-proper representation* (resp. *k-improper representation*) of G. Note that, every proper interval graph is both a 0-proper and a 0-improper graph. The graph in Figure 1 is a 2-proper, and a 3-improper graph. Now we consider how to represent a k-proper interval graph G with n vertices and support navigation queries efficiently on G. We first represent G k-properly into n intervals, and assign the labels to vertices of G based on the sorted order of their left end points, as described in Section 4.1. Same as the representation in Section 4.1, we first maintain the data structure for supporting rank and select queries on S in $O(1)$ time, using $2n+o(n)$ bits in total. Also we maintain the $2n+o(n)$-bit data structure of Lemma 2.2 on $r = r_1, \ldots, r_n$ for supporting RMax queries on r in $O(1)$ time. Next, to access r without using $n \log n$ bits, we define the sequence $T = t_1 \ldots t_{2n}$ of size $2n$ over the alphabet $\{0, \ldots, 2k+1\}$ such that $t_i = 2k'$ (resp. $t_i = 2k'+1$) if $s_i = 0$ (resp. $s_i = 1$) and its corresponding interval is contained by $k' \leq k$ intervals in $I = \{I_1 \ldots I_n\}$. Now for any $0 \leq i \leq k$, let $R_i \subset I$ be the set of all intervals such that for any $[a,b] \in R_i$, $t_a = 2i$ and $t_b = 2i+1$. It is easy to show that each R_i corresponds to the proper interval graph. For example the graph in Figure 1 is 2-proper interval graph, and $T = 0\ 2\ 0\ 2\ 3\ 1\ 0\ 3\ 1\ 0\ 2\ 1\ 2\ 4\ 3\ 5\ 3\ 1$, $R_0 = \{I_1, I_3, I_5, I_6\}$, $R_1 = \{I_2, I_4, I_7, I_8\}$, and $R_2 = \{I_9\}$. By Lemma 2.1, we can maintain T using $2n \log(2k+2) + o(n \log k) = 2n \log k + 2n + o(n \log k)$ bits with supporting rank and select queries in $O(\log \log k)$ and $O(1)$ time respectively. Then for any vertex $v \in G$, we can answer its corresponding interval $I_v = [l_v, r_v]$ in $O(\log \log k)$ time by $l_v = \mathsf{select}_0(S, v)$ and $r_v = \mathsf{select}_{(t_{l_v}+1)}(T, \mathsf{rank}_{t_{l_v}}(T, l_v))$. Thus, we obtain a following theorem.

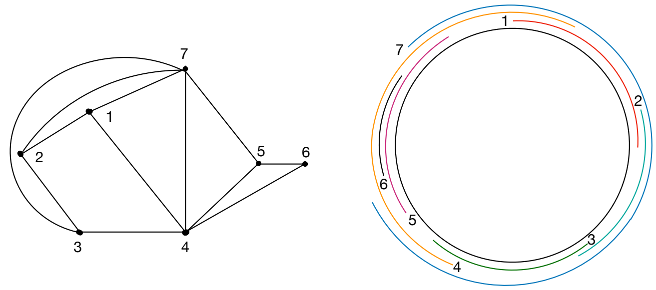
Theorem 3. *Given a k-proper interval graph G with n vertices, there exists a $(2n \log k + 6n + o(n \log k))$-bit representation of G which answers $\mathsf{degree}(v)$ and $\mathsf{adjacent}(u,v)$ queries in $O(\log \log k)$ time, $\mathsf{neighborhood}(v)$ queries in $O(\log \log k \cdot \mathsf{degree}(v))$ time, and $\mathsf{spath}(u,v)$ queries in $O(\log \log k \cdot |\mathsf{spath}(u,v)|)$ time, for any vertices $u, v \in G$.*

Note that we can represent k-improper interval graphs in same space with same query time as in Theorem 3 by changing the definition of T to be $t_i = 2k'$ (resp. $t_i = 2k'+1$) if $s_i = 0$ (resp. $s_i = 1$) and its corresponding interval contains $k' \leq k$ intervals in $\{I_1 \ldots I_n\}$.

5.3 Circular-arc graphs

In this section, we propose a succinct representation for circular-arc graphs, and show how to support navigation queries efficiently on the representation. A *circular-arc graph G* is a graph whose vertices can be assigned to arcs on a circle so that two vertices are adjacent in G if and only if their assigned arcs intersect. It is easy to see that every interval graph is a circular-arc graph. Thus, by the Lemma 3, we need at least $n \log n - 2n \log \log n - O(n)$ bits to represent an arbitrary circular-arc graph G.

Suppose that G is represented by the circle C together with n arcs of C. For an arc, we define its start point to be the unique point on it such that the arc continues from that point in the clock-wise direction but stops in the anti-clockwise direction; and similarly define its end point to be the unique point on it such that the arc stops in the clockwise direction but continues in the anti-clockwise direction. As in the case of interval graphs, we assume, without loss of generality, that all the start and end points of all the arcs are distinct. We label the vertices of G with the integers form 1 to n as described below. We first select an arbitrary arc, and label the vertex (and the arc) corresponding to this arc by 1. We then traverse the circle from the starting point of that arc in the clockwise direction, and label the remaining vertices and arcs in the order in which their starting points are encountered during the traversal, and finish the traversal when we return to the starting point of the first arc. We also map all the start and end points of all arcs, in the order in which they are encountered in the above traversal, into the range $[1, \ldots, 2n]$ (since the start and end points of all the n arcs are distinct). With the above defined labeling of the arcs, and the numbering of their start and end points, let l_i and r_i start and end points of the arc labeled i, for $1 \leq i \leq n$. Now the arcs can be thought of as two types of intervals in the range $[1, \ldots, 2n]$; we call an interval i as *normal* if $l_i < r_i$ (i.e., we traverse l_i prior to r_i), and *reversed* otherwise. A normal interval i corresponds to the interval $[l_i, r_i]$, while a reversed interval i actually corresponds to the union of the two intervals $[1, \ldots, r_i]$ and $[l_i, \ldots, 2n]$. See Figure 2 for an example; intervals numbered 4 and 7 are reversed, while the others are normal. Our representation of G consists of the following substructures.



$S = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1$

$S' = 0\ 3\ 0\ 1\ 0\ 1\ 2\ 1\ 0\ 3\ 0\ 1\ 2\ 1$

$R_1 = \{(1,1), (2,2), (3,3), (4,5), (5,4)\}$

$R_2 = \{(1,1), (2,2)\}$

Figure 2: Example of the circular graph and its representation.

1. Define a binary sequence $S = s_1, \ldots, s_{2n}$ of length $2n$ such that for $1 \le i \le 2n$, $s_i = 0$ (resp. $s_i = 1$) if i-th end point encountered during the traversal of C is in $\{l_1, \ldots, l_n\}$ (resp. $\{r_1, \ldots, r_n\}$). Now, construct a sequence $S' = s'_1, \ldots, s'_{2n}$ of size $2n$ over an alphabet $\{0, 1, 2, 3\}$ such that for all $1 \le i \le 2n$, $s'_i = s_i + 2$ if the position s_i corresponds to the end point of a reversed interval, and $s'_i = s_i$ otherwise (i.e., if s_i corresponds to a normal interval). We represent S' using the structure of Lemma 2.1, using $4n + o(n)$ bits, so that we can answer rank and select queries on S' in $O(1)$ time. In addition, we also store auxiliary structures (of $o(n)$ bits) on top of S' to support rank and select queries on S (without explicitly storing S – note that, one can efficiently reconstruct any subsequence of S from S').

2. To store the interval end points efficiently, we introduce two 2-dimensional grids of points, R_1 and R_2, defined as follows. Suppose there are $q \le n$ vertices in G which correspond to normal intervals (and $n - q$ vertices correspond to reversed intervals). Then let R_1 be a set of q points on the 2-dimensional grid $[1, q] \times [1, q]$ which consist of $(\mathsf{rank}_0(S', l_i), \mathsf{rank}_1(S', r_i))$, for all $1 \le i \le n$ with $l_i < r_i$. Similarly let R_2 be a set of $n - q$ points on the 2-dimensional grid $[1, n - q] \times [1, n - q]$ which consist of $(\mathsf{rank}_2(S', l_i), \mathsf{rank}_3(S', r_i))$, for all $1 \le i \le n$ with $r_i < l_i$. Given a set of points P on 2-dimensional grid, we define the following queries

   - $Y(R, x)$: returns y with $(x, y) \in R$.
   - *count*(R, A): returns number of points in R within the rectangular range A.

   We represent R_1 and R_2 using $n \log n + o(n \log n)$ bits in total, such that Y and *count* queries can be supported in $O(\log n / \log\log n)$ time [6].

Using these data structures, when the vertex $1 \le i \le n$ is given, we can answer l_i and r_i in $O(\log n / \log\log n)$ time by $l_i = \mathsf{select}_0(S, i)$, and $r_i = \mathsf{select}_1(S', Y(R_1, \mathsf{rank}_0(S', l_i)))$ if $S'_{l_i} = 0$ (i.e., if l_i is the left end point of a normal interval), and $r'_i = \mathsf{select}_3(Y(R_2, \mathsf{rank}_2(S', l'_i)))$ otherwise (i.e., if if l_i is the left end point of a reversed interval). Finally, let $r' = r'_1, \ldots, r'_q$ be a sequence such that for $1 \le i \le q$, $r'_i = r_{j_i}$ with $j_i = \mathsf{select}_0(S', i)$.

Similarly, let $r'' = r''_1, \ldots, r''_{n-q}$ be a sequence such that for $1 \le i \le n-q$, $r''_i = r_{j_i}$ with $j_i = \mathsf{select}_2(S', i)$. Then we maintain the data structure of Lemma 2.2 on r' and r'', using a total of $2n + o(n)$ bits, to support RMax queries on each of them. Thus, the overall representation takes $n \log n + o(n \log n)$ bits in total. Now we prove the following theorem (See Appendix B for the proof).

Theorem 4. *Given a circular arc graph G with n vertices, there exists a $(n \log n + o(n \log n))$-bit representation of G which answers $\mathsf{degree}(v)$ and $\mathsf{adjacent}(u, v)$ queries in $O(\log n / \log \log n)$ time, $\mathsf{neighborhood}(v)$ queries in $O(\log n / \log \log n \cdot \mathsf{degree}(v))$ time, and $\mathsf{spath}(u, v)$ queries in $O(|\mathsf{spath}(u, v)| \log n / \log \log n)$ time for any two vertices $u, v \in G$.*

6 Conclusion and Final Remarks

We considered the problem of succinctly encoding an unlabeled interval graph with n vertices so as to support adjacency, degree, neighborhood and shortest path queries. To this end, we designed a succinct data structure that can support these queries optimally. We also showed how one can implement various combinatorial algorithms in interval graphs using our succinct data structure in both time and space efficient manner. Extending these ideas, finally, we also showed succinct/compact data structures for multiple other variants of interval graphs. For some of these variants, the query times of our data structures are super constant, hence non-optimal and we leave them as open problems whether we can design data structures for supporting these queries in constant time.

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A Some graph algorithms on the succinct representation of interval graphs

Depth-first search (DFS) and Breath-first search (BFS) : DFS and BFS are the two most widely known and popular graph search methods because of their versatile usage as the backbone of so many other powerful and important graph algorithms. In what follows, we show that essentially the vertices sorted by its ascending order of the labels i.e., $1, \ldots, n$ gives both DFS and BFS vertex ordering of the G. Note that there may be more than one valid DFS or BFS ordering on G, but here we are interested in any of those valid and correct orderings. Moreover along the lines of recent papers [2, 8, 9], here we are interested only in the ordering of the vertices in DFS and BFS traversals i.e., the order in which the vertices are visited for the first time during the DFS/BFS traversal of the input graph G, not in actually reporting the final DFS/BFS tree. Towards this, we show the following,

Theorem 5. *Given an interval graph G with n vertices, suppose we label the vertices of G to $\{1, \ldots, n\}$ to be for any vertices $a, b \in G$, $a < b$ if and only if $l_a < l_b$. Then ascending order from 1 to n gives a valid* DFS *and* BFS *ordering of G.*

Proof. We only consider the DFS traversal in the proof (the case of BFS traversal can be proved using the similar argument). We prove by induction on the number of visited vertex. Since we can start from arbitrary vertex in G, the theorem statement holds with starting the traversal with the vertex 1. Next, suppose that we already visited the vertices $1 \ldots i$ with $i < n-1$ (the case $i = n-1$ is trivial) and for every valid DFS traversal, there exists a vertex $i' > i+1$ which is visited prior to $i+1$. This implies that there exists at least one vertex $v \in \{1, \ldots, i\}$ such that v is adjacent to i' but not $i+1$, contradicting to the fact that $l_v < l_{i+1} < l_{i'}$. Therefore there exists a valid DFS traversal which visits the vertex $i+1$ after visiting the vertex i. □

Perfect Elimination Ordering (PEO) : PEO of a graph G, if it exists, is defined as an ordering of the vertices of G such that, for each vertex v, v and the neighbors of v that occur before v in the order form a clique [20]. If we order the vertices corresponding to the intervals by sorting based on their left endpoints, then the resulting vertex order is a PEO, as the predecessor set of every vertex forms a clique. Thus, from our representation it is trivial to generate a PEO of the given interval graph.

Maximum Independent Set (MIS) and Minimum Vertex Cover (MVC) : To compute an MIS, we simulate the greedy algorithm of [23] which works as follows. Initialize the sets E and M to $\emptyset$. We first find the vertex i such that r_i is the leftmost among all the right endpoints of the intervals in $I - E$. If such an i exists, we add i to M and add $E = E \cup I'$ where $I' \subseteq I$ is the set of all intervals whose corresponding vertices are adjacent to i. We repeat this procedure until no such vertex i exists, and return M. Also MVC can be computed from MIS by returning the complement of MIS, in $O(n)$ time. (For the graph in Figure 1, MIS $= \{2, 5, 9\}$ and MVC $= \{1, 3, 4, 6, 7, 8\}$.)

Now we show how the algorithm can be implemented in time linear in the size of the input, with our representation of G. We first initialize the set M to $\emptyset$ and compute $i = \mathsf{RMin}(1, n)$ (which returns the interval with the smallest right end point among all the intervals), and add vertex i to M. Then the greedy algorithm picks the next interval with the smallest right end point in the range $[\mathsf{rank}_0(S, r_i) + 1, n]$ of the sequence r. In general, suppose $M = \{m_1, m_2 \ldots m_k\}$ and m_k is the last vertex added to M. Then we compute $m_{k+1} = \mathsf{RMin}(\mathsf{rank}_0(S, r_{m_k}) + 1, n)$, and add m_{k+1} to M, if it exists. Thus, we can compute MIS in time linear in the size of MIS.

Computing a Maximum Clique : In order to find a maximum clique in G, we define a sequence $D = d_1, \ldots, d_{2n}$ of length $2n$ where (i) $d_1 = 0$, and (ii) for $1 < i \leq 2n$, $d_i = d_{i-1} + 1$ if $s_i = 0$ and $d_i = d_{i-1} - 1$ otherwise. From the definition of d_i, if $s_i = 0$, there are exactly d_i vertices in G such that all corresponding intervals of these vertices have left endpoints at most i and right endpoints larger then i. Thus all such d_i vertices form a clique. This gives an algorithm for computing a maximum clique in G as follows. While constructing the sequence D in $O(n)$ time, we maintain the index k such that d_k is a largest value in D. We then scan all the intervals and return those intervals whose left end point is at most k and right end point is larger than k. Therefore we can compute the maximum clique in G in $O(n)$ time in total.

Computing a Proper Coloring: It is well-known that the greedy algorithm on G yields the optimal proper coloring if we process the vertices of G in the order of their corresponding intervals' left endpoints [20]. Thus, we simply implement this greedy coloring on G from the vertex 1 to n as follows. We first maintain n values $c_1, \dots, c_n$ such that for $1 \leq i \leq n$, $c_i \leq \mathsf{degree}(i)$ stores the color of vertex i. Since each c_i can be stored using $O(\log(\mathsf{degree}(i)))$ bits, we can maintain all c_i's using $O(n \log(m/n))$ bits in total, where m denotes the number of edges in G. To access the color of a given vertex in $O(1)$ time, we also store a parallel bit vector which stores a 1 at the beginning of each vertex's color, and 0 in all other positions; and store auxiliary data structure to support select queries on it. Now initialize all $c_1, \dots, c_n$ to 0 and scan the vertices from 1 to n. While we visit the vertex i, we perform the $\mathsf{neighborhood}(i)$ query and choose the minimum color in $\{1 \dots \mathsf{degree}(i)\} - \{c_v | v \in \mathsf{neighborhood}(i)\}$. Since we use $O(\mathsf{degree}(i))$ time for each $\mathsf{neighborhood}(i)$ query to assign the color of i, we can assign the color of all vertices in G in $O(n+m)$ time, using $O(n \log(m/n))$ extra bits of space.

Another alternative way to implement the greedy coloring on G is to use a priority queue. In this case, we first compute $\chi(G)$, which is a chromatic number of G. Since G is an interval graph, we can compute $\chi(G)$ in $O(n)$ time on our representation by computing the size of the maximum clique of G. Now we initialize $c_1, \dots, c_n$ to 0 and insert $1, \dots, \chi(G)$ to the priority queue PQ, and scanning S from left to right. Suppose we currently access s_i which corresponds to I_j (we can compute the index j in $O(1)$ time). If $s_i = 0$, we assign the minimum element of PQ to c_j, and delete c_j from PQ. Otherwise, we insert c_j to PQ. Note that we exactly perform $2n$ insert operations and n delete operations on PQ. Therefore we can compute a proper coloring of G in $O(n \log\log \chi(G))$ time using $O(n \log n)$ bits of space, using the integer priority queue structure of [33].

Note that these two solutions use $\Omega(n)$ bits of space, With $O(n)$ bits, we cannot store the colors of all the vertices simultaneously (unless the graph is sparse), and this poses a challenge for the greedy algorithm. We leave open the problem to find a proper coloring of interval graphs using extra $O(n)$ bits.

B Proof of Theorem 4

Theorem 4. *Given a circular arc graph G with n vertices, there exists a $(n \log n + o(n \log n))$-bit representation of G which answers $\mathsf{degree}(v)$ and $\mathsf{adjacent}(u,v)$ queries in $O(\log n/\log\log n)$ time, $\mathsf{neighborhood}(v)$ queries in $O(\log n/\log\log n \cdot \mathsf{degree}(v))$ time, and $\mathsf{spath}(u,v)$ queries in $O(|\mathsf{spath}(u,v)| \log n/\log\log n)$ time for any two vertices $u, v \in G$.*

Proof. Suppose we have the $n \log n + o(n \log n)$-bit representation described in Section 5.3. Now we consider the following queries, which extends the proof in Section 4.2.

$\mathsf{degree}(v)$ query: To answer $\mathsf{degree}(v)$ query, We first compute (i) counting the vertices u with $l_u < r_u$, and (ii) counting the vertices u with $r_u < l_u$ and return the sum of them. Now we consider the two cases based on l_v and r_v as follows.

- $l_v < r_v$: We can count the number of vertices in (i) in $O(\log n/\log\log n)$ time by returning $\mathsf{rank}_0(S, r_v) - \mathsf{rank}_1(S, l_v)$, same as in Section 4.2. Next, we classify the vertices u in (ii) into three cases as 1) $l_u < l_v$, 2) $r_v < r_u$, and 3) $l_v < l_u < r_v$ or $l_v < r_u < r_v$ and return the sum of them. First, number of vertices in case 1) and 2) can be easily answered in $O(\log n/\log\log n)$ time by returning $\mathsf{rank}_2(S', l'_v)$ and $\mathsf{rank}_3(S', r'_v)$ respectively. To count the number of vertices in case 3), we first count the number of start and end points between l_v and r_v by returning $(\mathsf{rank}_2(S', r_v) - \mathsf{rank}_2(S', l_v)) + (\mathsf{rank}_3(S', r_v) - \mathsf{rank}_3(S', l_v))$. After that we subtract the number of vertices whose both start and end points exist between l_v and r_v, which is $count(R_2, R)$ where $R = [\mathsf{rank}_2(S', l_v), \mathsf{rank}_2(S', r_v)] \times$

$[\mathsf{rank}_3(S', l_v), \mathsf{rank}_3(S', r_v)]$. Thus we can count the number of vertices in this case in $O(\log n / \log\log n)$ time.

- $l_v > r_v$: We classify the number of vertices in case (i) into three cases as 1) $r_u < r_v$, 2) $l_v < l_u$, and 3) $r_v < l_u < l_v$ or $r_v < r_u < l_v$ separately and return the sum of them. This can be answered in $O(\log n / \log\log n)$ time by the same argument as above. For counting the vertices in (ii), we simply return $\mathsf{rank}_2(S', 2n) - 1$ since all the vertices corresponds to the reverse interval cross l_1 in C, i.e., all such vertices form a clique in G.

adjacent(u, v) query: This can be answered in $O(\log n / \log\log n)$ time by checking l_u, r_u, l_v, and r_v.

neighborhood(v) query: We only describe how to answer the vertices u adjacent to v when the corresponding interval of v is normal. The case when the interval is reverse can be handled similarly. First we can return the all vertices u with $l_u < r_u$ in $O(\log n / \log\log n \cdot \mathsf{degree}(v))$ time using the same argument in Section 4.2. Next, the set of vertices u adjacent to v with $l_u > r_u$, is a disjoint union of the following two sets: 1) the set S_1 of all vertices u with $l_u < r_v$, and 2) the set S_2 of all vertices u with $l_v < r_u$. We can answer all the vertices in S_1 in $O(\mathsf{degree}(v))$ time by returning $\mathsf{rank}_0(S, \mathsf{select}_2(S', 1)), \ldots, \mathsf{rank}_0(S, \mathsf{select}_2(S', \mathsf{rank}_2(S', r_v)))$, which takes $O(1)$ time per each element. Finally vertices in $S_2 - S_1$ is equivalent to the the vertices u in $\{\mathsf{rank}_0(S, r_v) + 1, \ldots, n\}$ with $l_v < r_u$. Using the data structure RMax on r'' with a query range $[\mathsf{rank}_2(S, r_v) + 1, \ldots, n - q]$ on r'', these vertices can be answered in $O(\log n / \log\log n)$ time per element by the same procedure to answer the neighborhood queries on interval graphs. Thus, we can answer $\mathsf{neighborhood}(v)$ query in $O(\log n / \log\log n \cdot \mathsf{degree}(v))$ time in total.

spath(u, v) query: We simulate the algorithm of [10] with our representation of G. We first define $SUCC$ query on circular arc graphs and show how to answer the $SUCC(u)$ query in $O(\log n / \log\log n)$ time. For a set of vertices V of G, let $V_1 = \{u \in V | l_u < r_u\}$ and $V_2 = V - V_1$. Then for vertex $u \in V_1$ we can define $SUCC(u)$ as follows.

- If there exists a vertex $V_2 - V_u$ where $V_u = \{u' | \max(l_u, r_u) < l_{u'}\}$, $SUCC(u)$ returns a vertex $u' \in V_2 - V_u$ with the arc u and u' are intersect, and there is no vertex $u'' \in V_2 - V_u$ with the arc u and u'' are intersect and $r_{u'} < r_{u''} < r_u$. Let this vertex be u_1.

- Otherwise, $SUCC(u)$ returns a vertex $u' \in V_1$ with the arc u and u' are intersect, and there is no vertex $u'' \in V_1$ with the arc u and u'' are intersect and $r_u < r_{u'} < r_{u''}$ let this vertex be u_2.

To answer u_1, we consider two cases as follows. If $u \in V_1$, We can find u_1 by returning $\mathsf{rank}_0(S, \mathsf{select}_2(S', v'))$ where $v' = \mathsf{RMax}_{r''}(1, \mathsf{rank}_2(S', r_u))$, which can be answered in $O(\log n / \log\log n)$ time. Similarly if $u \in V_2$, we can find u_1 in $O(\log n / \log\log n)$ time by returning $\mathsf{rank}_0(S, \mathsf{select}_2(S', v'))$ where $v' = \mathsf{RMax}_{r''}(1, \mathsf{rank}_2(S', l_u))$. Also we can find u_2 in $O(\log n / \log\log n)$ time by the same argument for answering $SUCC$ queries on interval graphs.

To answer the $\mathsf{spath}(u, v)$ query, we do a same procedure for answering $\mathsf{spath}(u, v)$ and $\mathsf{spath}(v, u)$ queries on interval graphs (with the $SUCC$ function defined on circular arc graphs) in parallel, and return one of them which completes the procedure earlier. Since we can answer $SUCC$ query in $\log n / \log\log n$ time, we can answer $\mathsf{spath}(u, v)$ query in $O(\log n / \log\log n \cdot |\mathsf{spath}(u, v)|)$ time. □

It is easy to prove that we can answer Y and *count* queries on R_1 and R_2 in $O(\log n)$ time with $n \log n + o(n \log n)$ bits of space by maintaining the wavelet tree [30] on r' and r'', instead of maintaining the data structure of [6] on R_1 and R_2. This gives a simple succinct representation of G while using the same space and support degree and adjacent queries in $O(\log n)$ time, neighborhood queries in $O(\log n \cdot \mathsf{degree}(v))$ time, and $\mathsf{spath}(u, v)$ queries in $O(|\mathsf{spath}(u, v)| \log n)$ time. Also the difference in query time on interval graphs and circular-arc graphs comes from the fact that when $\mathcal{A}_v$ is given, we need to know the number of arcs which contained in $\mathcal{A}_v$ on circular-arc graphs to answer $\mathsf{degree}(v)$ query. We can improve the query time by maintaining i) use $n \log n + O(n/c)$-bit data structure on r' and r'' to support RMax on them, and ii) for every vertex $v \in G$, store $\mathsf{degree}(v)$ explicitly using $\lceil n \log n \rceil$ bits [14], instead of maintaining the data structures on R_1 and R_2. In this case, we can support degree, adjacent, neighborhood, and spath queries in same time as interval graphs, using $2n \log n + o(n \log n)$ bits of space.