AN AERATED TRIANGULAR ARRAY OF INTEGERS

René Gy
rene.gy@numericable.com

Abstract

Congruences modulo prime powers involving generalized Harmonic numbers are known [16], [7]. While looking for similar congruences, we have encountered simple, but not so well-known identities for the Stirling cycle numbers and a curious triangular array of numbers indexed with positive integers n, k, involving the Bernoulli and Stirling cycle numbers. It is shown that these numbers are all integers and that they vanish when $n-k$ is odd. This triangle has many similarities with the Stirling triangle. In particular, we show how it can be extended to negative indices and how this extension produces a *second kind* of such integers, for which a generating function is easily found. But our knowledge of these integers remains very limited, especially for those of the *first kind*.

1. Introduction

Let n and k be non-negative integers and let the *generalized Harmonic numbers* $H_n^{(k)}$ and $G_n^{(k)}$ be defined as

$$H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k} \quad \text{and} \quad G_n^{(k)} := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \frac{1}{i_1 i_2 \cdots i_k},$$

with $H_n^{(1)} = G_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} = H_n = G_n$; $H_n^{(0)} = n$ and $G_n^{(0)} = 1$. It is known [10] that $\left[{n+1 \atop k+1}\right] = n! G_n^{(k)}$, $\left[{n \atop k}\right]$ being the *Stirling cycle number* (unsigned Stirling number of first kind), so that the Harmonic and Stirling cycle numbers are inter-related by the convolution

$$k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = -\sum_{j=0}^{k-1} (-1)^{k-j} H_n^{(k-j)} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}, \tag{1.1}$$

which is obtained as a direct application of the well-known [9] relation between elementary symetric polynomials and power sums. Extended congruences for the Harmonic numbers $H_{p-1}^{(k)}$, modulo any power of a prime p are known [16], [7]. Our initial motivation for the work reported in the present paper is to look for similar congruences modulo prime powers, involving $G_{p-1}^{(k)}$, or the Stirling cycle numbers $\left[{p \atop k+1}\right]$, instead of $H_{p-1}^{(k)}$. We will show that such similar congruences for $G_{p-1}^{(k)}$ do exist, but that they are just the particular prime instances of not very well-known but elementary identities for the Stirling cycle numbers. This will lead us to introduce a triangular array of integers, involving the Bernoulli and Stirling cycle numbers which we believe is new.

2. Notation and preliminaries

In addition to what was exposed in the previous introduction, further notation that we use throughout this paper is presented in this section, along with classical results which we will need. Most of these results can be found in textbooks like [11] and they are given hereafter without proof. In the following, $g, h, i, j, k, \ell, m, n$ denote integers, p a prime number, and x or t denote the argument in a generating function. Let $f(x)$ be a formal series in powers of x, we denote $[[x^n]](f(x))$ the coefficient of x^n in $f(x)$ and $D^m f(x)$ is the m-order derivative of $f(x)$ with respect to x. If x is a real number, we denote $\lfloor x \rfloor$ the largest integer smaller or equal to x, $\lceil x \rceil$ the smallest integer larger or equal to x. We will use the Iverson bracket notation: $\left[\mathfrak{P}\right] = 1$ when proposition $\mathfrak{P}$ is true, and $\left[\mathfrak{P}\right] = 0$ otherwise.

The *binomial coefficients* $\binom{n}{k}$, are defined by $\sum_k \binom{n}{k} x^k = (1+x)^n$, whatever the sign of integer n. They obviously vanish when $k < 0$. When $n > 0$, we have $\binom{-n}{k} = (-1)^k \binom{n+k-1}{n-1}$. They are easily obtained by the basic recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and they obey the "hockey stick" identity

$$\binom{n}{k+1} = \sum_{j \geq 1} \binom{n-j}{k} \tag{2.1}$$

and the inversion formula

$$\sum_k (-1)^{k-j} \binom{k}{j} \binom{n}{k} = [n = j]. \tag{2.2}$$

The *Stirling cycle numbers* $\left[{n \atop k}\right]$, $n \geq 0$ may be defined by the generating function

$$\sum_k \left[{n \atop k}\right] x^k = \prod_{j=0}^{n-1} (x + j), \tag{2.3}$$

where an empty product is meant to be 1. They obviously vanish when $k < 0$ and $k > n$. They are easily obtained by the basic recurrence $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, valid for $n \geq 1$, with $\left[{0 \atop k}\right] = [k = 0]$. They obey the generalized recurrence relation

$$\left[{n+1 \atop m+1}\right] = \sum_{h \geq 0} \binom{h+m}{m} \left[{n \atop h+m}\right]. \tag{2.4}$$

Let $\left\{ {n \atop k} \right\}$, $n \geq 0$, be the *partition (or second kind) Stirling number*. They also vanish when $k < 0$ and $k > n$. Their basic recurrence is $\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$ for $n \geq 1$, with $\left\{ {0 \atop k} \right\} = [k = 0]$. They also obey a generalized recurrence relation

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{j \geq 0} \binom{n}{j} \left\{ {j \atop m} \right\} \tag{2.5}$$

and they have an explicit expression

$$\left\{ {n \atop k} \right\} = \frac{(-1)^k}{k!} \sum_{j \geq 0} (-1)^j \binom{k}{j} j^n. \tag{2.6}$$

Let B_h be the *Bernoulli number* ($B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots$ and $B_{2h+1} = 0$ for $h > 0$). The Bernoulli numbers have the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!},$$

they obey the following recurrence

$$(-1)^n B_n = \sum_{k=0}^{n} \binom{n}{k} B_k \tag{2.7}$$

and they are related to the Stirling partition numbers by [11]

$$B_k = \sum_{m \geq 0} (-1)^m \frac{m!}{m+1} \left\{ {k \atop m} \right\}. \tag{2.8}$$

We will also make use of the Von Staudt-Clausen theorem which states that the denominator of B_k in reduced form, is the product of all primes p such that $p - 1$ divides k. In particular, any prime may divide the denominator of a Bernoulli number once at most.

We shall need the Legendre formula which states that the highest power of a prime p which divides $j!$ is $\frac{j - s_p(j)}{p-1}$, where $s_p(j)$ be the sum of the standard base-p digits of j and the Kummer theorem which says that the highest power of p which divides $\binom{n}{k}$ is the number of carries when doing the addition $k + (n - k)$ in base p. In particular, we will make use of two consequences of Kummer theorem: (i) we have $\binom{2n}{2k} \equiv \binom{n}{k} \bmod 2$ and (ii) $\binom{n}{k}$ is even when n is even and k is odd. We recall the Wilson theorem which states that $(p-1)! \equiv -1 \bmod p$ for any prime p, and some other well-known congruences, valid for any prime p:

$$\binom{p}{j} \equiv \big[j = 0 \text{ or } j = p\big] \pmod{p}, \tag{2.9}$$

$$\left\{ {p \atop j} \right\} \equiv \big[j = 1 \text{ or } j = p\big] \pmod{p}, \tag{2.10}$$

$$\binom{p-1}{j} \equiv (-1)^j \big[0 \le j \le p-1\big] \pmod{p}, \tag{2.11}$$

$$\sum_{p-1 \ge j \ge 1} j^k \equiv (-1)\big[p-1 \text{ divides } k\big] \pmod{p}, \tag{2.12}$$

so that, from (2.6), when $n > 0$, we have

$$\left\{ {n \atop p-1} \right\} \equiv \big[p-1 \text{ divides } n\big] \pmod{p}. \tag{2.13}$$

3. Lemmas

Some lemmas are also going to be used. Since they may not be as well-known as the classic results of the previous section, they are given hereafter with proofs, for the sake of self-containment.

Lemma 3.1. *For the parity of the Sirling numbers of the first kind, we have*

$$\left[{n \atop k} \right] \equiv \binom{\lfloor \frac{n}{2} \rfloor}{k - \lceil \frac{n}{2} \rceil} \pmod{2}. \tag{3.1}$$

Proof. We reproduce the proof from [17]. Starting from (2.3), we have

$$\begin{aligned} \sum_{k=0}^{n} \left[{n \atop k} \right] x^k &= \prod_{k=0}^{n-1} (x+k) \\ &\equiv x(x+1)x(x+1)\cdots \pmod{2} \\ &\equiv x^{\lceil \frac{n}{2} \rceil}(x+1)^{\lfloor \frac{n}{2} \rfloor} \pmod{2}, \end{aligned}$$

hence

$$\begin{aligned} \left[{n \atop k} \right] &\equiv [[x^k]] \left(x^{\lceil \frac{n}{2} \rceil}(x+1)^{\lfloor \frac{n}{2} \rfloor} \right) \pmod{2} \\ &\equiv [[x^{k-\lceil \frac{n}{2} \rceil}]] \left((x+1)^{\lfloor \frac{n}{2} \rfloor} \right) \pmod{2} \\ &\equiv \binom{\lfloor \frac{n}{2} \rfloor}{k - \lceil \frac{n}{2} \rceil} \pmod{2}. \end{aligned}$$

□

Lemma 3.2. *Let $n, m, k \geq 0$ be integers, the following identity holds*

$$\left\{ n \atop m \right\} = \sum_{j=0}^{k} \sum_{i=0}^{k} (-1)^{i+j} m^{k-j} \binom{k}{j} \left\{ j \atop i \right\} \left\{ n-k \atop m-i \right\}. \tag{3.2}$$

Proof. This is Proposition 2.2 in [14]. Equation (3.2) can be rewritten as

$$\left\{ n+k \atop m \right\} = \sum_{i=0}^{k} (-1)^i P_{k,i}(m) \left\{ n \atop m-i \right\} \tag{3.3}$$

where $(P_{k,i};\ k, i \geq 0)$, is a family of polynomials of degree $\max(0, k-i)$, defined as

$$P_{k,i}(m) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left\{ j \atop i \right\} m^{k-j}. \tag{3.4}$$

We give a proof of Equation (3.3) by induction on k. It is clear that $P_{k,i} = 0$, when $i > k$. Equation (3.3) is trivially true for $k = 0$, for all $n, m \geq 0$, since $P_{0,0} = 1$. Our induction hypothesis is that Equation (3.3) holds for k and for all m, n. Then

$$\begin{aligned}
\left\{ n+k+1 \atop m \right\} &= m \left\{ n+k \atop m \right\} + \left\{ n+k \atop m-1 \right\} \\
&= \sum_{i=0}^{k} (-1)^i m P_{k,i}(m) \left\{ n \atop m-i \right\} + \sum_{i=0}^{k} (-1)^i P_{k,i}(m-1) \left\{ n \atop m-(i+1) \right\} \\
&= \sum_{i=0}^{k+1} (-1)^i \big(m P_{k,i}(m) - P_{k,i-1}(m-1) \big) \left\{ n \atop m-i \right\}.
\end{aligned}$$

Then, there remains to show that $mP_{k,i}(m) - P_{k,i-1}(m-1) = P_{k+1,i}(m)$. We have

$$\begin{aligned}
& m P_{k,i}(m) - P_{k,i-1}(m-1) \\
&= \sum_{j \geq 0} (-1)^j \binom{k}{j} \left\{ j \atop i \right\} m^{k-j+1} - \sum_{j \geq 0} (-1)^j \binom{k}{j} \left\{ j \atop i-1 \right\} (m-1)^{k-j} \\
&= \sum_{j \geq 0} (-1)^j \binom{k}{j} \left\{ j \atop i \right\} m^{k-j+1} \\
&- \sum_{j \geq 0} (-1)^j \binom{k}{j} \left\{ j \atop i-1 \right\} \sum_{h \geq 0} \binom{k-j}{h} m^h (-1)^{k-j-h} \\
&= \sum_{j \geq 0} (-1)^j \binom{k}{j} \left\{ j \atop i \right\} m^{k-j+1} \\
&- \sum_{h \geq 0} m^h (-1)^h \sum_{j \geq 0} (-1)^j \binom{k}{j} \left\{ j \atop i-1 \right\} \binom{k-j}{h} (-1)^{k-j}.
\end{aligned}$$

That is

$$
\begin{aligned}
&mP_{k,i}(m) - P_{k,i-1}(m-1)\\
&= \sum_{j\geq 0}(-1)^j \binom{k}{j}\left\{ {j \atop i} \right\} m^{k-j+1}\\
&- \sum_{h\geq 0} m^h(-1)^h \sum_{j\geq 0}(-1)^j \frac{k!}{j!}\left\{ {j \atop i-1} \right\}\frac{1}{h!(k-j-h)!}(-1)^{k-j}\\
&= \sum_{j\geq 0}(-1)^j \binom{k}{j}\left\{ {j \atop i} \right\} m^{k-j+1}\\
&- \sum_{h\geq 0}\binom{k}{h} m^h(-1)^h \sum_{j\geq 0}(-1)^j \left\{ {j \atop i-1} \right\}\binom{k-h}{j}(-1)^{k-j}\\
&= \sum_{j\geq 0}(-1)^j \binom{k}{j}\left\{ {j \atop i} \right\} m^{k-j+1} - \sum_{h\geq 0}\binom{k}{h} m^{k-h}(-1)^h \left\{ {h+1 \atop i} \right\} \text{ by (2.5)}\\
&= \sum_{j\geq 0}(-1)^j \binom{k}{j}\left\{ {j \atop i} \right\} m^{k-j+1} + \sum_{h\geq 1}\binom{k}{h-1} m^{k-h+1}(-1)^h \left\{ {h \atop i} \right\}\\
&= \left\{ {0 \atop i} \right\} m^{k+1} + \sum_{j\geq 1}(-1)^j \binom{k+1}{j}\left\{ {j \atop i} \right\} m^{k-j+1}\\
&= \sum_{j=0}^{k+1}(-1)^j \binom{k+1}{j}\left\{ {j \atop i} \right\} m^{k+1-j} = P_{k+1,i}(m).
\end{aligned}
$$

□

Lemma 3.3. *Let p be prime and $n \geq 0$, $q \geq 1$, we have*

$$
\left\{ {n \atop pq} \right\} \equiv \big[p-1 \textit{ divides } n-q\big]\binom{\frac{n-q}{p-1}-1}{q-1} \pmod{p}. \tag{3.5}
$$

Proof. This is also quite well-known. See for instance [3], [8] and [4] for the more general case where the modulus is a prime power. The proof generally involves the generating function of the Stirling partition numbers. Here, we give a different proof. We shall first recall and prove another known [14] congruence:

$$
\left\{ {n \atop m} \right\} \equiv \left\{ {n-p \atop m-p} \right\} + \left\{ {n-p+1 \atop m} \right\} \pmod{p}. \tag{3.6}
$$

Indeed, when letting $k = p$ into Equation (3.2) and acounting for (2.9) and (2.10), we have

$$\begin{aligned}
\left\{ {n \atop m} \right\} &= \sum_{j=0}^{p}\sum_{i=0}^{p}(-1)^{i+j}m^{p-j}\binom{p}{j}\left\{ {j \atop i} \right\}\left\{ {n-p \atop m-i} \right\} \\
&\equiv m^p\left\{ {n-p \atop m} \right\} + \sum_{i\geq 0}(-1)^{p+i}\left\{ {p \atop i} \right\}\left\{ {n-p \atop m-i} \right\} \pmod p \\
&\equiv m^p\left\{ {n-p \atop m} \right\} + (-1)^{p+1}\left\{ {n-p \atop m-1} \right\} + \left\{ {n-p \atop m-p} \right\} \pmod p \\
&\equiv m^p\left\{ {n-p \atop m} \right\} + \left\{ {n-p \atop m-1} \right\} + \left\{ {n-p \atop m-p} \right\} \pmod p \\
&\equiv m\left\{ {n-p \atop m} \right\} + \left\{ {n-p \atop m-1} \right\} + \left\{ {n-p \atop m-p} \right\} \pmod p \\
&\equiv \left\{ {n-p+1 \atop m} \right\} + \left\{ {n-p \atop m-p} \right\} \pmod p.
\end{aligned}$$

Now, for the proof of Lemma 3.3, we write (3.6) with $n-jp$ for n and $(q-j)p$ for m. That is

$$\left\{ {n-jp \atop (q-j)p} \right\} \equiv \left\{ {n-(j+1)p \atop \big(q-(j+1)\big)p} \right\} + \left\{ {n-(j+1)p+1 \atop (q-j)p} \right\} \pmod p,$$

then, summing from $j=0$ to $j=q-1$ and telescoping, we obtain

$$\left\{ {n \atop qp} \right\} \equiv \big[n=qp\big] + \sum_{j=1}^{q}\left\{ {n-qp-(p-1)+jp \atop jp} \right\} \pmod p. \tag{3.7}$$

We fix p and we are going to prove Lemma 3.3 by induction on n. It is easy to see directly that it is true for $n=0$, $n=1$ or $n=2$. We suppose (induction hypothesis) that for any $r\geq 1$ and all $N<n$, $\left\{ {N \atop rp} \right\} \equiv \big[p-1 \text{ divides } N-r\big]\binom{\frac{N-r}{p-1}-1}{r-1} \pmod p$. Actually, we don't need the induction hypothesis to see that (3.5) is true if $n\leq pq$. When $n<pq$, $\left\{ {n \atop pq} \right\}=0$ and $\big[p-1 \text{ divides } n-q\big]=0$ since $n-q<q(p-1)\leq p-1$. When $n=pq$, $\left\{ {n \atop pq} \right\}=1$ and $\big[p-1 \text{ divides } n-q\big]\binom{\frac{n-q}{p-1}-1}{q-1}=\binom{q-1}{q-1}=1$. So we only need to consider the case $n>pq$. Since $j\leq q$, we have $n-(q-j+1)p+1\leq n-p+1<n$ and then, by the induction hypothesis, we have

$$\begin{aligned}
\left\{ {n-qp-(p-1)+jp \atop jp} \right\} &\equiv \big[p-1 \text{ divides } n-q\big]\binom{\frac{n-qp-(p-1)+jp-j}{p-1}-1}{j-1} \pmod p \\
&\equiv \big[p-1 \text{ divides } n-q\big]\binom{\frac{n-q}{p-1}+j-2-q}{j-1} \pmod p \\
&\equiv \big[p-1 \text{ divides } n-q\big]\binom{\frac{n-q}{p-1}+j-2-q}{\frac{n-q}{p-1}-1-q} \pmod p.
\end{aligned}$$

Then, accounting for $n > pq$, Equation (3.7) becomes

$$\left\{ {n \atop pq} \right\} \equiv \left[p-1 \text{ divides } n-q \right] \sum_{j=1}^{q} \binom{\frac{n-q}{p-1}+j-2-q}{\frac{n-q}{p-1}-1-q} \pmod{p}.$$

If $pq < n < pq + p - 1$, $\binom{\frac{n-q}{p-1}+j-2-q}{\frac{n-q}{p-1}-1-q} = 0$ and $p-1$ does not divides $n-q$, then

$$\left\{ {n \atop pq} \right\} \equiv 0 = \left[p-1 \text{ divides } n-q \right] \binom{\frac{n-q}{p-1}-1}{q-1} \pmod{p}.$$

If $n \geq pq + p - 1$, by (2.1) we have $\sum_{j=1}^{q} \binom{\frac{n-q}{p-1}+j-2-q}{\frac{n-q}{p-1}-1-q} = \binom{\frac{n-q}{p-1}-1}{\frac{n-q}{p-1}-q} = \binom{\frac{n-q}{p-1}-1}{q-1}$, then

$$\left\{ {n \atop pq} \right\} \equiv \left[p-1 \text{ divides } n-q \right] \binom{\frac{n-q}{p-1}-1}{q-1} \pmod{p}.$$

□

Lemma 3.4. *Let p be prime, for any $n \geq 0$ and $g, k \geq 0$, we have*

$$\sum_{i \geq 1} \binom{i-1}{k} \left[{n \atop i(p-1)+k+1-g} \right] \equiv \left\{ {g \atop (k+1)p-n} \right\} \pmod{p}. \tag{3.8}$$

Proof. To the author's knowledge, this congruence involving both kinds of Stirling numbers does not seem to have been published already. Let

$$\begin{aligned} a_n(g,k) &:= \sum_{i \geq 1} \binom{i-1}{k} \left[{n \atop i(p-1)+k+1-g} \right] \\ &= \sum_{i=k+1}^{\frac{n+g-k-1}{p-1}} \binom{i-1}{k} \left[{n \atop i(p-1)+k+1-g} \right]. \end{aligned}$$

Lemma 3.4 is true when $n = 0$, since

$$\begin{aligned} a_0(g,k) &= \sum_{i \geq 1} \binom{i-1}{k} \left[{0 \atop i(p-1)+k+1-g} \right] \\ &= \left[p-1 \text{ divides } g-k-1 \right] \binom{\frac{g-k-1}{p-1}-1}{k} \\ &\equiv \left\{ {g \atop (k+1)p} \right\} \pmod{p} \quad \text{from Lemma 3.3.} \end{aligned}$$

Now, we suppose (induction hypothesis) that for a given n, for all $g, k \geq 0$, we have

$$a_n(g,k) \equiv \left\{ {g \atop (k+1)p-n} \right\} \pmod{p}.$$

Then

$$\begin{aligned}
a_{n+1}(g,k) &= \sum_{i=k+1}^{\frac{n+g-k}{p-1}} \binom{i-1}{k} \left[{n+1 \atop i(p-1)+k+1-g} \right] \\
&= n \sum_{i=k+1}^{\frac{n+g-k}{p-1}} \binom{i-1}{k} \left[{n \atop i(p-1)+k+1-g} \right] \\
&+ \sum_{i=k+1}^{\frac{n+g-k}{p-1}} \binom{i-1}{k} \left[{n \atop i(p-1)+k-g} \right] \\
&= n \sum_{i=k+1}^{\frac{n+g-k-1}{p-1}} \binom{i-1}{k} \left[{n \atop i(p-1)+k+1-g} \right] \\
&+ \sum_{i=k+1}^{\frac{n+g-k}{p-1}} \binom{i-1}{k} \left[{n \atop i(p-1)+k-g} \right].
\end{aligned}$$

Indeed, the first sum on the rigth-hand side may be limited to $\frac{n+g-k-1}{p-1}$ because even though $i = \frac{n+g-k}{p-1}$ may be integer, the corresponding summand is zero because $\left[{n \atop n+1} \right] = 0$. Then

$$\begin{aligned}
a_{n+1}(g,k) &= n \cdot a_n(g,k) + a_n(g+1,k) \\
&\equiv n \left\{ {g \atop (k+1)p-n} \right\} + \left\{ {g+1 \atop (k+1)p-n} \right\} \pmod{p} \\
&\equiv -\big((k+1)p-n\big) \left\{ {g \atop (k+1)p-n} \right\} + \left\{ {g+1 \atop (k+1)p-n} \right\} \pmod{p} \\
&\equiv -\big((k+1)p-n\big) \left\{ {g \atop (k+1)p-n} \right\} \\
&+ \big((k+1)p-n\big) \left\{ {g \atop (k+1)p-n} \right\} + \left\{ {g \atop (k+1)p-n-1} \right\} \pmod{p} \\
&\equiv \left\{ {g \atop (k+1)p-(n+1)} \right\} \pmod{p}.
\end{aligned}$$

□

Lemma 3.5. *Let n, k be non-negative integers such that $k \leq n$. There exists a polynomial $Q_k \in \mathbb{Q}[X]$ of degree $2k$, such that $Q_k(n)$ coincides with $\left[{n \atop n-k} \right]$. Moreover, $Q_k(-1) = 1$ and, if $k > 0$ then $0, 1, \cdots, k$ are roots of the polynomial Q_k.*

Proof. This can be found in [11]. Our proof is by induction on k. it is clearly true for $k = 0$, then suppose that Q_{k-1} is a polynomial function with rational coefficients of degree $2k-2$, then by the fundamental recurrence relation for Stirling cycle numbers, we have:

$$Q_k(n) - Q_k(n-1) = (n-1)Q_{k-1}(n-1)$$

Then, by telescoping

$$Q_k(n) - Q_k(1) = \sum_{j=1}^{n-1} jQ_{k-1}(j)$$

By expanding $Q_{k-1}(j)$ in powers of j, we see that the right-hand side in the above equation is a sum of sums of powers of j, the maximal exponent being $2k-2+1 = 2k-1$. But is is well-known that a sum of consecutive powers is a polynomial function of the last index, with rational coefficients, the degree of which is the exponent $+1$. Then the right-hand side is a polynomial function of $n-1$ with rational coefficient, the degree of which is $2k-1+1$ and this completes the proof that $Q_k(n)$ is indeed a polynomial fonction with rational coefficient of degree $2k$. It is clear that $Q_k(k) = \left[{k \atop 0}\right] = 0$ for $k > 0$. Since $Q_k(n+1) - Q_k(n) = nQ_{k-1}(n)$, letting $n = k-1$, we get $Q_k(k-1) = 0$, and then by iteration of the same process with $n = k-2, k-3, \cdots$ till $n = 0$, we see that $0, 1, \cdot\cdot, k$ are roots of Q_k. Now, since we have $Q_k(0) - Q_k(-1) = -1 \cdot Q_{k-1}(-1)$, we see that $Q_k(-1) = Q_{k-1}(-1)$ for any $k > 0$. Then $Q_k(-1) = 1$ for any $k \geq 0$ since $Q_0(-1) = 1$. □

4. Two identities for the Stirling cycle numbers

In this section, we will demonstrate two identities for the Stirling cycle numbers.

Theorem 4.1. *Let m, n be non-negative integers. We have*

$$\left[{n+1 \atop m+1}\right] = (-1)^{n-m} \sum_{h \geq 0} \binom{h+m}{m} \left[{n \atop h+m}\right] (-n)^h. \tag{4.1}$$

Moreover, if $n > 0$,

$$\left[{n \atop m}\right] = (-1)^{n-m} \sum_{h \geq 0} \binom{h+m-1}{m-1} \left[{n \atop h+m}\right] (-n)^h. \tag{4.2}$$

In more symetric formulations, these two identies also read

$$(-1)^{n-m} \left[{n+1 \atop m+1}\right] (-n)^m = \sum_h \binom{h}{m} \left[{n \atop h}\right] (-n)^h \tag{4.3}$$

and, if $n > 0$,

$$(-1)^{n-m}\begin{bmatrix} n \\ m \end{bmatrix}(-n)^m = \sum_h \binom{h-1}{m-1}\begin{bmatrix} n \\ h \end{bmatrix}(-n)^h. \tag{4.4}$$

Remark. In spite of their similarity to (2.4), these identities do not seem to be very well-known. They are not in [11] where quite many finite sums, recurrences and convolutions involving Stirling numbers are reported. Our equation (4.1) may be obtained as a particular case of Theorem 3 in [2]. An identity equivalent to our equation (4.2) is obtained incidentally in [1], where it is not even labelled. Another identity, equivalent to our equation (4.2) is the equation (18) in [15], where it is said to be new.

Proof of Theorem 4.1. Like in [1], our proof will highlight that (4.1) and (4.2) are actually closely related to the convolution identity (1.1) between the Harmonic and Stirling cycle numbers. Let $f_n(x) := \prod_{h=0}^{n-1}(x-h)$. We are going to show that, for $m \geq 1$,

$$m\frac{D^m f_n(x)}{m!} = -\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^h f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m-h}}. \tag{4.5}$$

It is true for $m = 1$, since $Df_n(x) = \sum_{j=0}^{n-1}\prod_{h\neq j}(x-h) = f_n(x)\sum_{j=0}^{n-1}\frac{1}{x-j}$. We suppose (induction hypothesis) that is true for some m, then

$$\begin{aligned}
(m+1)\frac{D^{m+1} f_n(x)}{(m+1)!} &= D\frac{D^m f_n(x)}{m!} = \frac{1}{m}D\left(m\frac{D^m f_n(x)}{m!}\right) \\
&= \frac{1}{m}\left(-\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^{h+1} f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m-h}}\right) \\
&+ \frac{1}{m}\left(\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^h f_n(x)}{h!}(m-h)\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right) \\
&= \frac{1}{m}\left(\sum_{h=1}^{m}(-1)^{m-h}\frac{D^h f_n(x)}{(h-1)!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right) \\
&+ \frac{1}{m}\left(m\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^h f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right) \\
&- \frac{1}{m}\left(\sum_{h=1}^{m-1}(-1)^{m-h}\frac{D^h f_n(x)}{(h-1)!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)
\end{aligned}$$

$$= \frac{1}{m}\left(\frac{D^m f_n(x)}{(m-1)!}\sum_{j=0}^{n-1}\frac{1}{x-j}\right)$$

$$+ \frac{1}{m}\left(m\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^h f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)$$

$$= \sum_{h=0}^{m}(-1)^{m-h}\frac{D^h f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\ .$$

This establishes the validity of (4.5). Now, when $x = n$, (4.5) reads

$$m\frac{D^m f_n(n)}{m!} = -\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^h f_n(n)}{h!}H_n^{(m-h)}. \tag{4.6}$$

This is the same recurrence as in (1.1), with the same inital value, since by definition $f_n(n) = n! = \left[{n+1 \atop 1}\right]$. Then

$$\frac{D^m f_n(n)}{m!} = \left[{n+1 \atop m+1}\right].$$

On the other hand, from (2.3), we have $f_n(x) = \sum_{h=0}^{n}\left[{n \atop h}\right](-1)^{n-h}x^h$, then

$$D^m f_n(x) = m!\sum_{h=0}^{n}\left[{n \atop h}\right]\binom{h}{m}(-1)^{n-h}x^{h-m},$$

$$\frac{D^m f_n(n)}{m!} = \sum_{h=0}^{n}\left[{n \atop h}\right]\binom{h}{m}(-1)^{n-h}n^{h-m}.$$

Hence

$$\left[{n+1 \atop m+1}\right] = \sum_{h=0}^{n}\left[{n \atop h}\right]\binom{h}{m}(-1)^{n-h}n^{h-m}$$

$$= \sum_{h=0}^{n}(-1)^{n-m}\left[{n \atop h+m}\right]\binom{h+m}{m}(-n)^h.$$

This completes the proof of (4.1). Now, for the proof of (4.2), we also use an induction argument, but on m and backward. Our induction hypothesis is

$$\left[{n \atop m+1}\right] = (-1)^{n-(m+1)}\sum_{h=0}^{n-(m+1)}\binom{h+m}{m}\left[{n \atop h+m+1}\right](-n)^h.$$

Then $$\left[{n \atop m+1}\right] = -(-1)^{n-m}\sum_{h=1}^{n-m}\binom{h+m-1}{m}\left[{n \atop h+m}\right](-n)^{h-1}.$$

Hence $$n\left[{n \atop m+1}\right] = (-1)^{n-m}\sum_{h=1}^{n-m}\binom{h+m-1}{m}\left[{n \atop h+m}\right](-n)^h.$$

We soustract the latter equation from (4.1), and we obtain

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} - n\begin{bmatrix} n \\ m+1 \end{bmatrix} = (-1)^{n-m}\sum_{h=1}^{n-m}\left(\binom{h+m}{m} - \binom{h+m-1}{m}\right)\begin{bmatrix} n \\ h+m \end{bmatrix}(-n)^h.$$

That is

$$\begin{bmatrix} n \\ m \end{bmatrix} = (-1)^{n-m}\sum_{h=1}^{n-m}\binom{h+m-1}{m-1}\begin{bmatrix} n \\ h+m \end{bmatrix}(-n)^h.$$

To finish the proof, we just need that (4.2) be true for $m = n$, which is obvious. □

5. Extended congruences for the Harmonic numbers $G_{p-1}^{(j+1)}$

Theorem 5.1. *Let $k \geq 0$ an integer and p a prime number, we have*

$$G_{p-1}^{(k)} = (-1)^k \sum_{j\geq 0}(-1)^j\binom{j+k}{j}G_{p-1}^{(k+j)}p^j. \tag{5.1}$$

In particular, when $k = 0$, we have

$$\sum_{j\geq 0}(-1)^j G_{p-1}^{(j+1)}p^j = 0. \tag{5.2}$$

Proof. Letting $n = p$ a prime number, and $m = k+1$ in (4.2), and dividing throughout by $(p-1)!$ provides the desired result. □

Recall [7] that when $k \geq 1$, the generalized Harmonic numbers $H_{p-1}^{(k)}$ admit the following p-adically converging expansion:

$$H_{p-1}^{(k)} = (-1)^k \sum_{j\geq 0}\binom{j+k-1}{j}H_{p-1}^{(k+j)}p^j. \tag{5.3}$$

It is interesting to point out the similarity of (5.3) and (5.1), but also some differences. Contrary to (5.3), the sum on the right-hand side of (5.1) is finite. It is actually limited to $j = p-1-k$; we also notice that the sign alternates in (5.1) and that there is a slight difference in the binomial coefficient.

It is also kwown [16] that, for odd prime p,

$$\sum_{j\geq 0}\binom{j+2k}{2k}B_j H_{p-1}^{(2k+j+1)}(-p)^j = 0, \tag{5.4}$$

the convergence of the series being understood p-adically. More precisely [7] when $p \geq 5$, the following congruence was shown:

$$\sum_{j=0}^{2n+1} \binom{j+2k}{2k} B_j H_{p-1}^{(2k+j+1)} (-p)^j \equiv 0 \pmod{p^{2n+3}}. \tag{5.5}$$

Now, we look for an equation similar to (5.4), but for the Stirling cycle numbers. In the case where $k = 0$, we have, for $p \geq 5$

$$\sum_{j=0}^{2n+1} B_j H_{p-1}^{(j+1)} (-p)^j \equiv 0 \pmod{p^{2n+3}}. \tag{5.6}$$

For the lowest values of n, $n = 0, 1, 2...$, these congruences read

$$H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} \equiv 0 \pmod{p^3} \ ,$$
$$H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} + \frac{p^2}{6} H_{p-1}^{(3)} \equiv 0 \pmod{p^5} \ ,$$
$$H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} + \frac{p^2}{6} H_{p-1}^{(3)} - \frac{p^4}{30} H_{p-1}^{(5)} \equiv 0 \pmod{p^7} \text{ ..., respectively.}$$

A clue for our search of the analogous to (5.4) is obtained by making use of (1.1) in order to recursively compute $H_{p-1}^{(j+1)}$ as function of the $G_{p-1}^{(i+1)}$, with $i \leq j$, then substituting $H_{p-1}^{(j+1)}$ in the above congruences and finally reducing modulo p^{2n+3} as much as possible, by accounting for any previous congruence involving the $G_{p-1}^{(i+1)}$. In doing so, it is found that, for $p \geq 5$

$$G_{p-1} - p G_{p-1}^{(2)} \equiv 0 \pmod{p^3} \ ,$$
$$G_{p-1} - p G_{p-1}^{(2)} + \frac{p^2}{2} G_{p-1}^{(3)} \equiv 0 \pmod{p^5} \ ,$$
$$G_{p-1} - p G_{p-1}^{(2)} + \frac{p^2}{2} G_{p-1}^{(3)} - \frac{p^4}{6} G_{p-1}^{(5)} \equiv 0 \pmod{p^7} \ ,$$
$$G_{p-1} - p G_{p-1}^{(2)} + \frac{p^2}{2} G_{p-1}^{(3)} - \frac{p^4}{6} G_{p-1}^{(5)} + \frac{p^6}{6} G_{p-1}^{(7)} \equiv 0 \pmod{p^9} \text{ ... etc.}$$

The calculations become increasingly laborious as n increases, but we are able to guess that

$$\sum_{j=0}^{2n+1} (j+1) B_j G_{p-1}^{(j+1)} p^j \equiv 0 \pmod{p^{2n+3}}. \tag{5.7}$$

In an even broader generalization, we anticipate the following theorem

Theorem 5.2. *Let $i \geq 1$ an integer and p a prime number, we have*

$$\sum_{j\geq 0} B_j \binom{j+2i-1}{j} G_{p-1}^{(j+2i-1)} p^j = 0 \tag{5.8}$$

or equivalently,

$$\sum_{j\geq 0} B_j \binom{j+2i-1}{j} \left[\begin{matrix} p \\ j+2i \end{matrix} \right] p^j = 0. \tag{5.9}$$

Proof. Note that the sums in (5.8) and (5.9) are actually finite. The proof will be given in the next section. □

6. The aerated triangular array $\mathcal{A}_{n,k}$

For performing numerical verifications of (5.9), we now introduce the number $\mathcal{A}_{n,k}$.

Definition. Let n, k be non-negative integers, we define the number $\mathcal{A}_{n,k}$ by

$$\mathcal{A}_{n,k} := \sum_{h\geq 0} B_h \binom{k+h-1}{h} \left[\begin{matrix} n \\ h+k \end{matrix} \right] n^h. \tag{6.1}$$

It is clear from this definition that $\mathcal{A}_{n,k}$ is zero when $k > n$ and that $\mathcal{A}_{n,n} = 1$. The first terms of the sequence $(\mathcal{A}_{n,k})$ are computed numerically and displayed in the following table:

n	$\mathcal{A}_{n,0}$	$\mathcal{A}_{n,1}$	$\mathcal{A}_{n,2}$	$\mathcal{A}_{n,3}$	$\mathcal{A}_{n,4}$	$\mathcal{A}_{n,5}$	$\mathcal{A}_{n,6}$	$\mathcal{A}_{n,7}$	$\mathcal{A}_{n,8}$	$\mathcal{A}_{n,9}$	$\mathcal{A}_{n,10}$
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0
3	0	−1	0	1	0	0	0	0	0	0	0
4	0	0	−5	0	1	0	0	0	0	0	0
5	0	24	0	−15	0	1	0	0	0	0	0
6	0	0	238	0	−35	0	1	0	0	0	0
7	0	−3396	0	1281	0	−70	0	1	0	0	0
8	0	0	−51508	0	4977	0	−126	0	1	0	0
9	0	1706112	0	−408700	0	15645	0	−210	0	1	0
10	0	0	35028576	0	-2267320	0	42273	0	−330	0	1

Table 1: The triangular array $\mathcal{A}_{n,k}$ for n, k in the range 0 to 10.

It is striking that these numbers seem to be zero when $n - k$ is odd, which, if true, would imply the validity of Theorem 5.2. This will be demonstrated in the next theorem. It is also striking that they seem to be all integers. This will be shown right after.

Theorem 6.1. *Let n, k be non-negative integers and*

$$\mathcal{A}_{n,k} := \sum_{h\geq 0} B_h \binom{k+h-1}{h} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h,$$

then $\mathcal{A}_{n,k} = (-1)^{n-k}\mathcal{A}_{n,k}$. Equivalently, $\mathcal{A}_{n,k} = 0$ when $n-k$ is odd.

Proof. When $n = 0$, this is obviously true. Suppose $n > 0$, we have

$$\begin{aligned}
\mathcal{A}_{n,k} &= \sum_{h} B_h \binom{k+h-1}{h} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h \\
&= (-1)^n \sum_{h} \frac{B_h}{n^k} \binom{k+h-1}{h} (-1)^{n-(h+k)} \begin{bmatrix} n \\ h+k \end{bmatrix} (-n)^{h+k} \\
&= (-1)^n \sum_{h} \frac{B_h}{n^k} \binom{k+h-1}{h} \sum_{g} \binom{g-1}{h+k-1} \begin{bmatrix} n \\ g \end{bmatrix} (-n)^g. \qquad \text{(by (4.4))}
\end{aligned}$$

But, it is easy to see that $\binom{k+h-1}{h}\binom{g-1}{h+k-1} = \binom{g-k}{h}\binom{g-1}{k-1}$, so that

$$\begin{aligned}
\mathcal{A}_{n,k} &= (-1)^{n-k} \sum_{g} \sum_{h} B_h \binom{g-k}{h} \binom{g-1}{k-1} \begin{bmatrix} n \\ g \end{bmatrix} (-n)^{g-k} \\
&= (-1)^{n-k} \sum_{g} (-1)^{g-k} B_{g-k} \binom{g-1}{k-1} \begin{bmatrix} n \\ g \end{bmatrix} (-n)^{g-k} \qquad \text{(by (2.7))} \\
&= (-1)^{n-k} \sum_{g} B_g \binom{k+g-1}{g} \begin{bmatrix} n \\ g+k \end{bmatrix} n^g = (-1)^{n-k}\mathcal{A}_{n,k}
\end{aligned}$$

□

Theorem 6.2. *Let n, k be non-negative integers, B_h a Bernoulli number, $\begin{bmatrix} n \\ k \end{bmatrix}$ an unsigned Stirling number of first kind, and*

$$\mathcal{A}_{n,k} := \sum_{h\geq 0} B_h \binom{k+h-1}{h} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h,$$

then $\mathcal{A}_{n,k}$ is a triangular array of integers.

Proof. In the definition of $\mathcal{A}_{n,k}$, we replace the Bernoulli numbers by the expression (2.8) in terms of Stirling numbers of second kind, so that

$$\begin{aligned}
\mathcal{A}_{n,k} &= \sum_{h\geq 0} B_h \binom{k+h-1}{h} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h \\
&= \sum_{h\geq 0} \sum_{m\geq 0} (-1)^m \frac{m!}{m+1} \begin{Bmatrix} h \\ m \end{Bmatrix} \binom{k+h-1}{k-1} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h \\
&= \sum_{m\geq 0} (-1)^m \frac{m!}{m+1} \sum_{h\geq 0} \begin{Bmatrix} h \\ m \end{Bmatrix} \binom{k+h-1}{k-1} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h.
\end{aligned}$$

We split the sum in four parts: $m+1=1$, $m+1=4$, $m+1>4$ composite, and $m+1$ prime, so that

$$\begin{aligned}\mathcal{A}_{n,k} &= \begin{bmatrix} n \\ k \end{bmatrix} - \frac{3}{2}\sum_{h\geq 3}\left\{ h \atop 3 \right\}\binom{k+h-1}{k-1}\begin{bmatrix} n \\ h+k \end{bmatrix} n^h \\ &\quad + \sum_{\substack{m+1>4 \\ m+1 \text{ composite}}} (-1)^m \frac{m!}{m+1}\sum_{h\geq 0}\left\{ h \atop m \right\}\binom{k+h-1}{k-1}\begin{bmatrix} n \\ h+k \end{bmatrix} n^h \\ &\quad + \sum_{p \text{ prime}} (-1)^{p-1}\frac{(p-1)!}{p}\sum_{h\geq p-1}\left\{ h \atop p-1 \right\}\binom{k+h-1}{k-1}\begin{bmatrix} n \\ h+k \end{bmatrix} n^h.\end{aligned} \tag{6.2}$$

The first term on the right-hand side of (6.2) is clearly an integer. We now show that the second term on the right-hand side of (6.2) is also integer. It is obvious that $\left\{ h \atop 3 \right\}\binom{k+h-1}{k-1}\left[n \atop h+k \right] n^h$ is even when n is even, since $h \geq 3$. It is also even when h is even ($h = 2g$, $g \geq 1$). Indeed, by the explict expression (2.6), we have $\left\{ h \atop 3 \right\} = \frac{3^{h-1}+1-2^h}{2} = 2\left(\frac{3^{2g-1}+1}{4} - 4^{g-1}\right)$ which is even, since $g \geq 1$. Now we suppose that n is odd and h is odd ($n = 2m+1$ and $h = 2g+1$). By Kummer theorem $\binom{k+h-1}{k-1}$ is even when $k > 0$ is even and h is odd, since there is at least one carry when doing the addition $h + (k-1)$ in base 2. So it suffices to consider the case where k is odd ($k = 2q+1$) and we now need to prove that $Q_{m,q}$ is even, with

$$Q_{m,q} := \sum_{g\geq 1}\left\{ 2g+1 \atop 3 \right\}\binom{2q+2g+1}{2q}\begin{bmatrix} 2m+1 \\ 2g+2q+2 \end{bmatrix}(2m+1)^{2g+1}.$$

Since $(2m+1)^{2g+1} \equiv 1 \bmod 2$ and $\left\{ 2g+1 \atop 3 \right\} = \frac{3^{2g}+1}{2} - 2^{2g} \equiv 1 \bmod 2$, we have

$$\begin{aligned} Q_{m,q} &\equiv \sum_{g\geq 1}\binom{2q+2g+1}{2q}\begin{bmatrix} 2m+1 \\ 2g+2q+2 \end{bmatrix} \pmod 2 \\ &\equiv \sum_{g\geq 1}\binom{2q+2g+1}{2q}\begin{bmatrix} 2m \\ 2g+2q+1 \end{bmatrix} \pmod 2 \\ &\equiv \sum_{g\geq 1}\left(\binom{2q+2g}{2q} + \binom{2q+2g}{2q-1}\right)\begin{bmatrix} 2m \\ 2g+2q+1 \end{bmatrix} \pmod 2 \\ &\equiv \sum_{g\geq 1}\binom{2q+2g}{2q}\begin{bmatrix} 2m \\ 2g+2q+1 \end{bmatrix} \pmod 2 \quad \text{by Kummer theorem} \\ &\equiv \sum_{g\geq 1}\binom{2q+2g}{2q}\binom{m}{2m-2q-2g-1} \pmod 2 \qquad \text{by Lemma 3.1} \\ &\equiv \sum_{g=q+1}^{m-1}\binom{2g}{2q}\binom{m}{2m-2g-1} \pmod 2. \end{aligned}$$

If m is even, $\binom{m}{2m-2g-1} \equiv 0 \bmod 2$ by Kummer theorem, then $Q_{m,q} \equiv 0 \bmod 2$. So it suffices to prove that $Q_{2\ell+1,q}$ is even ($\ell \geq 0$). We have

$$\begin{aligned}
Q_{2\ell+1,q} &\equiv \sum_{g=q+1}^{2\ell} \binom{2g}{2q}\binom{2\ell+1}{4\ell-2g+1} \pmod 2 \\
&\equiv \sum_{g=q+1}^{2\ell} \binom{2g}{2q}\left(\binom{2\ell}{4\ell-2g+1} + \binom{2\ell}{4\ell-2g}\right) \pmod 2 \\
&\equiv \sum_{g=q+1}^{2\ell} \binom{2g}{2q}\binom{2\ell}{4\ell-2g} \pmod 2 \quad \text{by Kummer theorem} \\
&\equiv \sum_{g=q+1}^{2\ell} \binom{g}{q}\binom{\ell}{2\ell-g} \pmod 2 \quad \text{again by Kummer theorem} \\
&\equiv \sum_{g=q}^{2\ell} \binom{g}{q}\binom{\ell}{2\ell-g} - \binom{\ell}{2\ell-q} \pmod 2 \\
&\equiv \sum_{g=0}^{2\ell-q} \binom{g+q}{q}\binom{\ell}{2\ell-g-q} - \binom{\ell}{2\ell-q} \pmod 2.
\end{aligned}$$

Now we have $\sum_{g\geq 0} \binom{g+q}{q} x^g = (1-x)^{-q-1}$ and $\sum_{g\geq 0} \binom{\ell}{g} x^g = (1+x)^\ell$, so that

$$\frac{(1+x)^\ell}{(1-x)^{q+1}} - (1+x)^\ell = \sum_{h\geq 0}\sum_{g=0}^{h} \binom{g+q}{q}\binom{\ell}{h-g} x^h - \sum_{h\geq 0}\binom{\ell}{h} x^h,$$

hence

$$\begin{aligned}
Q_{2\ell+1,q} &\equiv [[x^{2\ell-q}]]\left(\frac{(1+x)^\ell}{(1-x)^{q+1}} - (1+x)^\ell\right) \pmod 2 \\
&\equiv [[x^{2\ell-q}]]\left((1-x)^{\ell-q-1} - (1+x)^\ell\right) \pmod 2.
\end{aligned}$$

If $q+1 \leq \ell$ then $2\ell - q \geq \ell + 1$, hence $2\ell - q > \ell$ and $2\ell - q > \ell - q - 1 \geq 0$ hence

$$\begin{aligned}
Q_{2\ell+1,q} &\equiv (-1)^q \binom{\ell-q-1}{2\ell-q} - \binom{\ell}{2\ell-q} \pmod 2 \\
&\equiv 0 - 0 = 0 \pmod 2.
\end{aligned}$$

And if $q+1 > \ell$,

$$\begin{aligned}
Q_{2\ell+1,q} &\equiv \binom{q-\ell+2\ell-q}{2\ell-q} - \binom{\ell}{2\ell-q} \pmod 2 \\
&\equiv \binom{\ell}{2\ell-q} - \binom{\ell}{2\ell-q} = 0 \pmod 2.
\end{aligned}$$

The third term on the right-hand side of (6.2) is also integer. Indeed, if $m+1$ is composite and $m+1>4$ then $m+1$ divides $m!$. For suppose $m+1>4$ is composite and not the square of a prime p, then there exists integers m_1, m_2 such that $2 \le m_1 < m_2 \le \frac{m+1}{2} < m$, and $m+1 = m_1 m_2$. Then $m+1$ obviously divides $m!$. And if $m+1>4$ is a squared prime, $m+1=p^2$, with $p \ge 3$, then $m=(p-1)p+(p-1)$ and the sum of the base-p digits of m is $2p-2$ so that by Legendre formula $p^{\frac{p^2-1-2p+2}{p-1}} = p^{p-1}$ divides $m!$, then p^2 divides $m!$, since $p \ge 3$.

Finally, in order to demonstrate that $\mathcal{A}_{n,k}$ is integer, it suffices to show that for any integers $n, k \ge 0$ and any prime p

$$\sum_{h=p-1}^{n-k} \left\{ {h \atop p-1} \right\} \binom{k+h-1}{k-1} \left[{n \atop h+k} \right] n^h \equiv 0 \pmod{p}.$$

This is obvious when p divides n, and it suffices to show that this is true when p and n are coprime. By Fermat little theorem, and by (2.13), we then see that we just need to show that for any $k \ge 1$,

$$(p\ ,\ n) \text{ coprime} \implies \sum_{p-1 \text{ divides } h} \binom{k+h-1}{k-1} \left[{n \atop h+k} \right] \equiv 0 \pmod{p}. \qquad (6.3)$$

In Equation (3.8) from Lemma 3.4, we let $g=0$ and $i(p-1)=h$ and we introduce $k-1$ instead of k . Then, we have

$$\sum_{\frac{h}{p-1} \ge 1} \binom{\frac{h}{p-1}-1}{k-1} \left[{n \atop h+k} \right] \equiv \left\{ {0 \atop kp-n} \right\} \pmod{p}.$$

Now

$$\begin{aligned} \binom{\frac{h}{p-1}-1}{k-1} &= \frac{\left(\frac{h}{p-1}-1\right)\left(\frac{h}{p-1}-2\right)\cdot\cdot\left(\frac{h}{p-1}-k+1\right)}{(k-1)!} \\ &\equiv \frac{(-h-1)(-h-2)\cdot\cdot(-h-(k-1))}{(k-1)!} \pmod{p} \\ &\equiv (-1)^{k-1}\binom{k+h-1}{k-1} \pmod{p}. \end{aligned}$$

Then

$$(-1)^{k-1} \sum_{p-1 \text{ divides } h} \binom{k+h-1}{k-1} \left[{n \atop h+k} \right] \equiv [n = kp] \pmod{p},$$

which imply the validity of (6.3). ☐

7. The dual triangle $\mathcal{B}_{n,k}$

In this section, we will point out some similarities with the usual Stirling numbers, and show that there exists a dual triangle $(\mathcal{B}_{n,k})$ which is to $(\mathcal{A}_{n,k})$ what the Stirling numbers of the second kind are to the Stirling numbers of the first kind.

Theorem 7.1. *Let n, k be non-negative integers such that $0 \leq k \leq n$. There exists a polynomial $P_k \in \mathbb{Q}[X]$, of degree $2k$, such that $P_k(n)$ coincides with $\mathcal{A}_{n,n-k}$. Moreover, when $k > 0$, $-1, 0, \cdots, k$ are $k+2$ roots of $P_k(x)$.*

Proof. We have

$$\mathcal{A}_{n,n-k} = \sum_{h=0}^{k} B_h \binom{n-1-(k-h)}{h} \begin{bmatrix} n \\ n-(k-h) \end{bmatrix} n^h,$$

where the binomial coefficient is a polynomial in n from $\mathbb{Q}[X]$, of degree h, and $\left[{n \atop n-(k-h)}\right]$ is known ([11] or Lemma 3.5) to also be a polynomial in n from $\mathbb{Q}[X]$, of degree $2(k-h)$. Therefore $\mathcal{A}_{n,n-k}$ is also a polynomial in n from $\mathbb{Q}[X]$, of degree $2k$. Let $k > 0$, and Q_j the polynomial such that $Q_j(n) = \left[{n \atop n-j}\right]$. We have

$$\begin{aligned} P_k(u) &= \sum_{h=0}^{k} B_h \binom{u+h-1-k}{h} Q_{k-h}(u) u^h \\ &= \sum_{h=0}^{k} B_h \frac{(u+h-1-k)\cdot\cdot(u-k)}{h!} Q_{k-h}(u) u^h \end{aligned}$$

From Lemma 3.5 we know that if $k > h$ then $0, 1, \cdots, k-h$ are roots of the polynomial function $Q_{k-h}(x)$. Moreover when $k \geq h$, $Q_{k-h}(-1) = 1$.
Then, when $u = 0$, we have $P_k(0) = B_0 \binom{-1-k}{0} Q_k(0) 0^0 = 0$, since $k > 0$.
When $u > 0$, we have

$$\begin{aligned} P_k(u) &= \sum_{h=k-u+1}^{k} B_h \frac{(u-k)\cdot\cdot(u-k+h-1)}{h!} Q_{k-h}(u) u^h \\ &= \sum_{h=k-u+1}^{k} (-1)^h B_h \frac{(k-u)\cdot\cdot(k-u-h+1)}{h!} Q_{k-h}(u) u^h \end{aligned}$$

If $0 < u \leq k$, for any h in the set $\{k-u+1, \cdot\cdot, k\}$ the product $(k-u)\cdot\cdot(k-u-h+1)$ must vanish because we see that it has one factor which is zero, and then we also have $P_k(u) = 0$.

Finally, if $u = -1$,

$$\begin{aligned}P_k(-1) &= \sum_{h=0}^{k} B_h \frac{(h-2-k)\cdot\cdot(-1-k)}{h!}(-1)^h Q_{k-h}(-1)\\ &= \sum_{h=0}^{k} B_h \frac{(k+2-h)\cdot\cdot(k+1)}{h!}\\ &= \sum_{h=0}^{k} \binom{k+1}{h} B_h = 0 \quad \text{(by 2.7).}\end{aligned}$$

□

Example. We have $\mathcal{A}_{n,n-2} = \frac{-n-1}{4}\binom{n}{3} = -\binom{n+1}{4}$, whereas $\left[{n \atop n-2}\right] = \frac{3n-1}{4}\binom{n}{3}$. We then see that $\mathcal{A}_{n,n-2} - \left[{n \atop n-2}\right] = -n\binom{n}{3}$. More generally, we will have the following theorem:

Theorem 7.2. *Let n, k be positive integers, B_h a Bernoulli number and $\left[{n \atop k}\right]$ an unsigned Stirling number of first kind, then*

$$\mathcal{A}_{n,k} \equiv \frac{1+(-1)^{n-k}}{2}\begin{bmatrix} n \\ k \end{bmatrix} \pmod{n}.$$

Proof. From (4.2), we have if $n > 0$,

$$\frac{\left[{n \atop k}\right] - (-1)^{n-k}\left[{n \atop k}\right]}{2} + \frac{(-1)^{n-k}kn}{2}\begin{bmatrix} n \\ k+1 \end{bmatrix} = \frac{(-1)^{n-k}}{2}\sum_{h\geq 2}\binom{h+k-1}{h}\begin{bmatrix} n \\ h+k \end{bmatrix}(-n)^h.$$

Hence

$$\frac{-(-1)^{n-k}\left[{n \atop k}\right] + \left[{n \atop k}\right]}{2} \equiv \frac{kn}{2}\begin{bmatrix} n \\ k+1 \end{bmatrix} \pmod{n}.$$

On the other hand, from the definition of $\mathcal{A}_{n,k}$, we have

$$\mathcal{A}_{n,k} - \begin{bmatrix} n \\ k \end{bmatrix} + \frac{kn}{2}\begin{bmatrix} n \\ k+1 \end{bmatrix} = \sum_{h\geq 2} B_h \binom{k+h-1}{h}\begin{bmatrix} n \\ h+k \end{bmatrix} n^h.$$

Then

$$\mathcal{A}_{n,k} - \frac{1+(-1)^{n-k}}{2}\begin{bmatrix} n \\ k \end{bmatrix} \equiv n\sum_{h\geq 2} nB_h \binom{k+h-1}{h}\begin{bmatrix} n \\ h+k \end{bmatrix} n^{h-2} \pmod{n}.$$

But each summand in the sum on the right-hand side is p-integral for all prime p that divides n. To see this, we make use of the Von Staudt-Clausen theorem whereby p may divide the denominator of B_h once at most. Then nB_h is p-integral and then the right-hand side is 0 modulo n. □

As a corollary to Theorem 7.2, we have the following Wilson-like theorem for $\mathcal{A}_{n,1}$, illustrating the similarity between the first column in Table 1 and the factorial $(n-1)!$.

Theorem 7.3. *Let n be a positive integer, we have*

$$\mathcal{A}_{n,1} + \big[n \text{ is an odd prime}\big] \equiv 0 \pmod{n}.$$

Proof. From Theorem 7.2, we have $\mathcal{A}_{n,1} \equiv \frac{1+(-1)^{n-1}}{2} \left[n \atop 1 \right] \bmod n$. But $\left[n \atop 1 \right] = (n-1)!$ then $\mathcal{A}_{n,1} \equiv \frac{1+(-1)^{n-1}}{2}(n-1)! \bmod n$. If n is an odd prime, by the Wilson theorem we have $\mathcal{A}_{n,1} \equiv -1 \bmod n$; otherwise, if n is even, clearly $\mathcal{A}_{n,1} \equiv 0 \bmod n$, and if n is an odd composite, we have already seen that n divides $(n-1)!$, so that we also have $\mathcal{A}_{n,1} \equiv 0 \bmod n$. ☐

Coming back to the polynomial P_k, we may extend the definition of $\mathcal{A}_{n,k}$ to non-positive indices since for non-negative n, k it is natural to define $\mathcal{A}_{-n,-n-k} := P_k(-n)$. Then, we have

$$\begin{aligned} \mathcal{A}_{-n,-n-k} &= \sum_{h=0}^{k} B_h \binom{-n-1-(k-h)}{h} Q_{k-h}(-n)(-n)^h \\ &= \sum_{h=0}^{k} B_h \binom{n+k}{h} Q_{k-h}(-n) n^h. \end{aligned}$$

That is

$$\mathcal{A}_{-n,-k} = \sum_{h=0}^{k-n} B_h \binom{k}{h} Q_{k-n-h}(-n) n^h \tag{7.1}$$

But from [11], we know that the definition of the Stirling numbers of both kinds may also be extended to negative indices, so that for any integers (positive or negative) n, k, we have

$$\left[-n \atop -k \right] = \left\{ k \atop n \right\} \tag{7.2}$$

then

$$Q_n(-x) = \left[-x \atop -x-n \right] = \left\{ x+n \atop x \right\} \tag{7.3}$$

and Equation (7.1) becomes

$$\mathcal{A}_{-n,-k} = \sum_{h=0}^{k-n} B_h \binom{k}{h} \left\{ k-h \atop n \right\} n^h.$$

Definition. Let n, k be positive integers, we define the number $\mathcal{B}_{n,k}$ by

$$\mathcal{B}_{n,k} := \sum_{h \geq 0} B_h \binom{n}{h} \left\{ n-h \atop k \right\} k^h. \tag{7.4}$$

It then clear that for all integers n, k, positive or negative, we have the duality:

$$\mathcal{A}_{-n,-k} = \mathcal{B}_{k,n} \tag{7.5}$$

which is similar to Equation (7.2) for the usual Stirling numbers.

It is clear from this definition that $\mathcal{B}_{n,k}$ is zero when $k > n$ and that $\mathcal{B}_{n,n} = 1$. Also note that we have $\mathcal{B}_{x+n,x} = P_n(-x)$.

The first $\mathcal{B}_{n,k}$ are computed numerically and displayed in the following table:

n	$\mathcal{B}_{n,1}$	$\mathcal{B}_{n,2}$	$\mathcal{B}_{n,3}$	$\mathcal{B}_{n,4}$	$\mathcal{B}_{n,5}$	$\mathcal{B}_{n,6}$	$\mathcal{B}_{n,7}$	$\mathcal{B}_{n,8}$	$\mathcal{B}_{n,9}$	$\mathcal{B}_{n,10}$	$\mathcal{B}_{n,11}$	$\mathcal{B}_{n,12}$
1	1	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0
4	0	−1	0	1	0	0	0	0	0	0	0	0
5	0	0	−5	0	1	0	0	0	0	0	0	0
6	0	3	0	−15	0	1	0	0	0	0	0	0
7	0	0	49	0	−35	0	1	0	0	0	0	0
8	0	−17	0	357	0	−70	0	1	0	0	0	0
9	0	0	−809	0	1701	0	−126	0	1	0	0	0
10	0	155	0	−13175	0	6195	0	−210	0	1	0	0
11	0	0	20317	0	-120395	0	18711	0	-330	0	1	0
12	0	-2073	0	706893	0	-760100	0	49203	0	-495	0	1

Table 2: The triangular array $\mathcal{B}_{n,k}$ for n, k in the range 1 to 12.

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-8	1															
-7		1														
-6	-70		1													
-5		-35		1												
-4	357		-15		1											
-3		49		-5		1										
-2	-17		3		-1		1									
-1		0		0		0		1								
0	0		0		0		0		1							
1		0		0		0		0		1						
2	0		0		0		0		0		1					
3		0		0		0		0		-1		1				
4	0		0		0		0		0		-5		1			
5		0		0		0		0		24		-15		1		
6	0		0		0		0		0		238		-35		1	
7		0		0		0		0		-3396		1281		-70		1

Table 3: $\mathcal{A}_{k,n}$ and $\mathcal{B}_{k,n}$ in tandem.

Again, we see on Table 2 that the $\mathcal{B}_{n,k}$ seem to be all integers and to vanish when $n-k$ is odd: this will be the next theorem. We can also, as was done in [11] for the usual Stirling numbers, display $(\mathcal{A}_{n,k})$ and $(\mathcal{B}_{k,n})$ in tandem. This is the purpose of Table 3, where we have left void the zero entries for $k > n$ and for odd $n-k$. The numbers which appear now in the diagonal lines are the values of the polynomial function $P_{n-k}(x)$ for integer arguments.

Theorem 7.4. *Let n, k be non-negative integers, B_h a Bernoulli number, $\left\{ n \atop k \right\}$ a Stirling number of second kind, and*

$$\mathcal{B}_{n,k} := \sum_{h \geq 0} B_h \binom{n}{h} \left\{ n-h \atop k \right\} k^h,$$

then $\mathcal{B}_{n,k}$ is a triangular array of integers such that $\mathcal{B}_{n,k} = 0$ when $n-k$ is odd. Moreover we have the inter-relations

$$\mathcal{B}_{x,x-n} = \sum_u \binom{n+x}{n-u} \binom{n-x}{n+u} \mathcal{A}_{n+u,u} \tag{7.6}$$

$$\mathcal{A}_{x,x-n} = \sum_u \binom{n+x}{n-u} \binom{n-x}{n+u} \mathcal{B}_{n+u,u}. \tag{7.7}$$

Remark *The equations* (7.6) *and* (7.7) *are formally the same as*

$$\left\{ x \atop x-n \right\} = \sum_u \binom{n+x}{n-u} \binom{n-x}{n+u} \left[u+n \atop u \right]$$

$$\left[x \atop x-n \right] = \sum_u \binom{n+x}{n-u} \binom{n-x}{n+u} \left\{ u+n \atop u \right\}$$

which hold [11] for the usual Stirling numbers.

Proof. Let p_n be a polynomial of degree n from $\mathbb{Q}[X]$. The binomial coefficents $\left\{ \binom{x}{k}; 0 \leq k \leq n \right\}$ form a basis for the vector space of all the polynomials from $\mathbb{Q}[X]$ of degree less than $k+1$, therefore there exists $a_{n,k}$ such that $p_n(x) = \sum_{k=0}^n a_{n,k} \binom{x}{k}$. By the inversion formula (2.2) it is easy to verify that $a_{n,k} = \sum_u (-1)^{k-u} \binom{k}{u} p_n(u)$. We have seen that $\mathcal{A}_{x,x-n} = P_n(x)$ where P_n is a polynomial of degree $2n$ from $\mathbb{Q}[X]$, so we can apply the above general inversion scheme to $P_n(-x) = \mathcal{B}_{x+n,x}$ and we obtain

$$\begin{aligned} \mathcal{B}_{x+n,x} &= \sum_{k=0}^{2n} \sum_{u=0}^{k} (-1)^{k-u} \binom{k}{u} \mathcal{A}_{u,u-n} \binom{-x}{k} \\ &= \sum_{k=n}^{2n} \sum_{u=n}^{k} (-1)^{u} \binom{k}{u} \binom{x+k-1}{k} \mathcal{A}_{u,u-n} \end{aligned}$$

which, given Theorems 6.1 and 6.2, clearly shows that $B_{n,k}$ is a triangular array of integers such that $\mathcal{B}_{n,k} = 0$ when $n-k$ is odd.
Now, since $\binom{k}{u}\binom{x+k-1}{k} = \binom{x+k-1}{k-u}\binom{x+u-1}{u}$, we have

$$\begin{aligned}
\mathcal{B}_{x+n,x} &= \sum_{k=n}^{2n}\sum_{u=n}^{k}(-1)^u\binom{x+k-1}{k-u}\binom{x+u-1}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=n}^{2n}(-1)^u\sum_{k=u}^{2n}\binom{x+k-1}{x+u-1}\binom{x+u-1}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=n}^{2n}(-1)^u\binom{2n+x}{2n-u}\binom{x+u-1}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=n}^{2n}\binom{2n+x}{2n-u}\binom{-x}{u}\mathcal{A}_{u,u-n} \\
\mathcal{B}_{x,x-n} &= \sum_{u=n}^{2n}\binom{n+x}{2n-u}\binom{n-x}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=0}^{n}\binom{n+x}{n-u}\binom{n-x}{n+u}\mathcal{A}_{u+n,u}
\end{aligned}$$

Similarly, let $R_n(x) = \mathcal{B}_{x,x-n}$, this is a polynomial of degree $2n$ from $\mathbb{Q}[X]$. We apply the inversion to $R_n(-x) = \mathcal{A}_{x+n,x}$ which gives the similar identity where $\mathcal{A}$ and $\mathcal{B}$ are exchanged and this completes the proof of the theorem. □

Also, like for the usual Stirling numbers for which we have the *Stirling convolution polynomials* $\sigma_n(x)$ [11], polynomials from $\mathbb{Q}[X]$ of degree $n-1$ such that

$$Q_n(x) = \begin{bmatrix} x \\ x-n \end{bmatrix} = x(x-1)\cdots(x-n)\sigma_n(x) \tag{7.8}$$

or equivalently

$$Q_n(-x) = \begin{Bmatrix} x+n \\ n \end{Bmatrix} = (-1)^{n+1}x(x+1)\cdots(x+n)\sigma_n(-x), \tag{7.9}$$

we can, accounting for Theorem 7.1, define for $\mathcal{A}_n(x)$ and $\mathcal{B}_n(x)$ the *convolution polynomial* $\mathcal{S}_n(x)$ of degree $n-2$, such that

$$P_n(x) = \mathcal{A}_{x,x-n} = (x+1)x(x-1)\cdots(x-n)\mathcal{S}_n(x) \tag{7.10}$$

or equivalently

$$P_n(-x) = \mathcal{B}_{x+n,x} = (-1)^{n+2}(x-1)x(x+1)\cdots(x+n)\mathcal{S}_n(-x). \tag{7.11}$$

In the following table, we give the first instances of $\mathcal{S}_n(x)$, together with the Stirling convolution polynomial $\sigma_n(x)$ from [11].

n	1	2	3	4
$\sigma_n(x)$	$\frac{1}{2}$	$\frac{1}{24}(3x-1)$	$\frac{1}{48}(x^2-x)$	$\frac{1}{5760}(15x^3-30x^2+5x+2)$
$\mathcal{S}_n(x)$	0	$-\frac{1}{24}$	0	$\frac{1}{5760}(7x^2+3x+2)$

Table 4: The convolution polynomials $\sigma_n(x)$ and $\mathcal{S}_n(x)$ for n in the range 1 to 4.

The second and third columns of Table 2 are known to the OEIS [13]. Up to the sign and discarding the zeros, we find in these columns the even index *Genocchi numbers* G_{2n} and *Glaisher's G numbers*, A001469 and A002111 at the OEIS, respectively, for which exponential generating functions are known. Actually, we have the following general exponential generating fonction for $\mathcal{B}(n,k)$:

Theorem 7.5. *Let n, k be non-negative integers, we have*

$$\sum_n \mathcal{B}_{n,k}\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}\frac{kx}{e^{kx}-1}. \tag{7.12}$$

Remark *For comparison, the usual Stirling numbers have the following exponential generating functions [11]*

$$\sum_n \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$$

$$\sum_n \left[{n \atop k} \right] \frac{x^n}{n!} = \frac{(-1)^k\big(\ln(1-x)\big)^k}{k!}.$$

Proof. The proof is straightforward: we use the rule of multiplication of exponential generating functions [17] and we have

$$\begin{aligned} \frac{(e^x-1)^k}{k!}\frac{kx}{e^{kx}-1} &= \left(\sum_j \left\{ {j \atop k} \right\} \frac{x^j}{j!}\right)\left(\sum_j B_j \frac{(kx)^j}{j!}\right) \\ &= \sum_u \left(\sum_{j+h=u} B_j k^j \binom{u}{h} \left\{ {h \atop k} \right\}\right)\frac{x^u}{u!} \\ &= \sum_u \left(\sum_j B_j \binom{u}{j} \left\{ {u-j \atop k} \right\} k^j\right)\frac{x^u}{u!}. \end{aligned}$$

□

Since the second column of the triangle $\mathcal{B}(n,k)$ corresponds to the Genocchi numbers, we might consider the other columns as some sort of generalized Genocchi numbers. However, these numbers are not the same as the already known generalized Genocchi numbers from [5], nor as those from [12].

Unfortunately, the derivation of a generating function for $\mathcal{A}(n,k)$ seems much more complicated.

We finish this section by pointing out a notable difference with the usual Stirling numbers. Whereas it is well-known that the Stirling matrices of both kinds are inverse of eachother, we don't see any evident relationship between the inverse of the matrix $\mathcal{A}(n,k)$ and the matrix $\mathcal{B}(n,k)$. The first entries of the inverse of $\mathcal{A}^{-1}(n,k)$ are displayed in the following table:

n	$\mathcal{A}'_{n,1}$	$\mathcal{A}'_{n,2}$	$\mathcal{A}'_{n,3}$	$\mathcal{A}'_{n,4}$	$\mathcal{A}'_{n,5}$	$\mathcal{A}'_{n,6}$	$\mathcal{A}'_{n,7}$	$\mathcal{A}'_{n,8}$	$\mathcal{A}'_{n,9}$	$\mathcal{A}'_{n,10}$
1	1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0	0
4	0	5	0	1	0	0	0	0	0	0
5	−9	0	15	0	1	0	0	0	0	0
6	0	−63	0	35	0	1	0	0	0	0
7	1485	0	−231	0	70	0	1	0	0	0
8	0	18685	0	−567	0	126	0	1	0	0
9	−844757	0	125515	0	−945	0	210	0	1	0
10	0	−14862727	0	600655	0	−693	0	330	0	1

Table 5: The triangular array $\mathcal{A}^{-1}_{n,k}$, up to $n = 10$.

8. Discussion and Questions

Much is known on the classical Genocchi numbers, most notably a combinatorial interpretation [6], and a recursion from which the following relation can be derived and used to compute $\mathcal{B}_{n,2}$ recursively for $n \geq 2$

$$\mathcal{B}_{n,2} = \frac{n}{2} - \frac{1}{2}\sum_{j=0}^{n-1}\binom{n}{j}\mathcal{B}_{j,2}. \tag{8.1}$$

To the author's knowledge, a combinatorial interpretation for Glaisher's G numbers ($\mathcal{B}_{n,3}$) has not been given yet. Here we raise the more general questions: for a given $k > 2$, find a recursion for $\mathcal{B}_{n,k}$ and find combinatorial objects that they enumerate.

As for the $\mathcal{A}_{n,k}$, we have even more questions. Apart from their appearance in the above investigation of congruences modulo prime powers for the Stirling cycle numbers, we don't know their mathematical interest. It is quite a pity that our demonstration of Theorem 6.2 has to be that long and technical, because technicalities may easily conceal much mathematical signification. Any recurrence that would allow to compute an entry in this triangular array from entries of previous lines would be more insightful, and would probably lead to a more direct proof for Theorem 6.2. Moreover $\mathcal{A}_{n,k}$ cries for a generating function, exponential or of any other sort, or at least a functional equation involving such a function. Eventually, there remains the problem of the combinatorial interpretation of $\mathcal{A}_{n,k}$.

By comparison to these quite complicated combinatorics questions, the study of the arthmetic properties of $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ would seem more easy, because it could be made use of their explicit expression in terms of Bernoulli and Stirling numbers and take advantage of the existing knowledge on the arithmetics of the latter.

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