SIGNED PARTITIONS - A BALLS INTO URNS APPROACH

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Abstract. Using Reiner's definition of Stirling numbers of type B of the second kind, we provide a 'balls into urns' approach for proving a generalization of a well-known identity concerning the classical Stirling numbers of the second kind: $x^n = \sum_{k=0}^{n} S(n,k)[x]_k$.

1. Introduction

The partitions of the set $[n] = \{1, \dots, n\}$ in k blocks are enumerated by the *Stirling numbers of the second kind*, denoted by $S(n,k)$ (see [7, page 81]). These numbers arise in a variety of problems in enumerative combinatorics; they have many combinatorial interpretations, and have been generalized in various contexts and in many ways.

One of the celebrated results concerning Stirling numbers of the second kind is the following: Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have:

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k, \tag{1}$$

where $[x]_k := x(x-1)\cdots(x-k+1)$ is the *falling factorial of degree k* and $[x]_0 := 1$.

This identity arises when one expresses the standard basis of the polynomial ring $\mathbb{R}[x]$ as a linear combination of the basis consisting of the falling factorials (see e.g. the survey of Boyadzhiev [3]).

There are some known proofs for this identity. A combinatorial one, realizing x^n as the number of functions from the set $\{1, \dots, n\}$ to the set $\{1, \dots, x\}$ (for an integer x), is presented by Stanley [7, Eqn. (1.94d); its proof is in page 83], and we quote it here (where $\#N = n$ and $\#X = x$):

> "The left-hand side is the total number of functions $f : N \to X$. Each such function is surjective onto a unique subset $Y = f(N)$

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of X satisfying $\#Y \leq n$. If $\#Y = k$, then there are $k!S(n,k)$ such functions, and there are $\binom{x}{k}$ choices of subsets Y of X with $\#Y = k$. Hence:

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k} = \sum_{k=0}^{n} S(n,k)[x]_k.\text{”}$$

There is a nice generalization of Identity (1), which appears in Remmel and Wachs [5] and Bala [2]. In order to demonstrate this generalization combinatorially, we use the Stirling numbers of type B of the second kind, denoted by $S_B(n,k)$, which are related to the Coxeter group of type B. The exact definition will be given in Section 2. Their generalization is:

Theorem 1.1. *Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have:*

$$x^n = \sum_{k=0}^{n} S_B(n,k)[x]_k^B, \tag{2}$$

where $[x]_k^B := (x-1)(x-3)\cdots(x-2k+1)$ and $[x]_0^B := 1$.

Remmel and Wachs [5] proved this equality by using the combinatorial interpretation of $S_B(n,k)$ as counting configurations of k-non attacking rooks (specifically, this is $S_{n,k}^{0,2}(1,1)$ in their notation). Bala [2] proved this equality using a generating-functions technique ($S_{(2,0,1)}$ in his notation).

In [1], a geometric way to prove Equation (2), interpreting x^n as counting the number of points in a cubical lattice, is presented.

The purpose of this note (see Section 3) is a simple combinatorial proof, which interprets both sides of Equation (2), using a balls into urns approach. Note that our proof is actually a generalization for Coxeter groups of type B of the proof for Equation (1), that we have quoted above.

2. Signed partitions

We define the objects which the Stirling numbers of type B of the second kind count, introduced by Reiner [4]. Denote $[\pm n] := \{\pm 1, \ldots, \pm n\}$ and $-C := \{-i \mid i \in C\}$ for a set $C \subseteq [\pm n]$.

Definition 2.1. A *signed partition* is a set partition of $[\pm n]$ into blocks, which satisfies the following conditions:

- There exists at most one block satisfying $-C = C$, called the *zero-block*. It is a subset of $[\pm n]$ of the form $\{\pm i \mid i \in S\}$ for some $S \subseteq [n]$.

- If C appears as a block in the partition, then $-C$ also appears in that partition.

We denote by $S_B(n,k)$ the number of signed partitions of $[\pm n]$ having exactly k pairs of nonzero blocks. These numbers are called *Stirling numbers of type B of the second kind.* They form the sequence A039755 in the OEIS [6]. Table 1 records these numbers for small values of n and k.

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	4	1				
3	1	13	9	1			
4	1	40	58	16	1		
5	1	121	330	170	25	1	
6	1	364	1771	1520	395	36	1

TABLE 1. Stirling numbers of type B of the second kind $S_B(n,k)$.

Example 2.2. The following partitions

$$P_1 = \{\{3,-3\},\{-2,1\},\{2,-1\},\{-4,5\},\{4,-5\}\},$$
$$P_2 = \{\{3\},\{-3\},\{-2,1\},\{2,-1\},\{-4,5\},\{4,-5\}\},$$

are respectively a signed partition of $[\pm 5]$ with a zero block $\{3,-3\}$ and a signed partition of $[\pm 5]$ without a zero-block.

3. THE COMBINATORIAL PROOF

In this section, we supply a direct combinatorial proof for Theorem 1.1, where x^n is interpreted as the number of assignments of n balls numbered 1 to n into x distinguishable urns. As will be explained below, we can assume that x is an integer.

Direct combinatorial proof. Since Equation (2) is a polynomial identity, it is sufficient to prove it for odd natural numbers. Let $m \in \mathbb{N}$ be an odd number. We will show:

$$m^n = \sum_{k=0}^{n} S_B(n,k)[m]_k^B. \tag{3}$$

The left-hand side of Equation (3) is the number of assignments of n balls numbered 1 to n into m urns. In the right-hand side, we associate

$$[m]_k^B = (m-1)(m-3)\cdots(m-2k+1)$$

assignments to each one of the $S_B(n,k)$ signed partitions, and then we sum them up to get the total number of assignments, thus proving the identity.

Let $\mathcal{B} = \{B_0, B_1, -B_1, \dots, B_k, -B_k\}$ be a signed partition, where B_0 is the zero-block (which possibly does not exist). Note that by the definition of $[x]_k^B$ which appeared in Theorem 1.1, we have $[x]_k^B = 0$ for $m < 2k$, so we may assume that $k < \frac{m}{2}$.

For our convenience, we impose the following order on the blocks of the signed partition: the blocks B and $-B$ are adjacent and the pairs of blocks of $\mathcal{B}$ are ordered in such a way that pairs of blocks which have smaller minimal positive elements precede (except for the zero-block B_0 which is always located as the first block). For each pair of blocks B and $-B$, the internal order between B and $-B$ is chosen in such a way that the block which contains the minimal positive element of $B \cup -B$ is located first. For example,

$$\{\{5,-5\},\{1,-3\},\{-1,3\},\{2,4\},\{-2,-4\}\}$$

is properly ordered.

For convenience, we consider an assignment of n balls into m urns as a function $f : [n] \to [m]$, and associate with $\mathcal{B}$ the set of ball assignments according to the following procedure:

- For any *positive* $i \in B_0$, define: $f(i) = 1$.
- Choose a number p out of the $m-1$ remaining numbers $(2 \le p \le m)$, and send the positive elements of B_1 to p. The absolute values of the negative elements of B_1 (i.e. the positive elements of $-B_1$, if they exist) will be sent to the next number in cyclical order excluding the number 1 (which might have already been occupied by the positive elements of the zero-block). This can be done in $m-1$ different ways.
- We pass to the pair of blocks B_2 and $-B_2$. Similarly, choose a new number p' out of the $m-3$ remaining numbers (the number 1 is occupied by the positive elements of the zero-block, and two additional numbers are already occupied by the elements of the pair of blocks B_1 and $-B_1$), and send the positive elements of B_2 to p'. For each negative $i \in B_2$, the absolute value of i will be sent to the next unoccupied number in cyclical order. This may be done in $m-3$ different ways.
- Proceeding this way, we associate a set of $[m]_k^B$ functions from $[n]$ to $[m]$ to each signed partition having k pairs of nonzero blocks.

Conversely, we now recover the signed partition from a given function $f : [n] \to [m]$. Define:

$$B_0 = \{\pm i \mid f(i) = 1\}.$$

Mark the number 1 as used. Let $k \in [n]$ be the minimal positive number such that $f(k) \neq 1$. Denote $a := f(k)$. Let $b \in [m] - \{1\}$ be the next unused number in cyclical order. Define:

$$B_1 = \{i \in [n] \mid f(i) = a\} \cup \{-i \mid f(i) = b\},$$

and add the blocks B_1 and $-B_1$ to the signed partition. Now mark the numbers a, b as used. Proceeding along these lines, we arrive at the signed partition which induces the function f. □

Example 3.1. Let $n = 6$ and $m = 7$. Consider the signed partition

$$\mathcal{B} = \{\{\pm 1\}, \{2, -3, 5\}, \{-2, 3, -5\}, \{4, -6\}, \{-4, 6\}\}$$

of $[\pm 6]$. Every function $f : [6] \to [7]$ which is induced by $\mathcal{B}$ sends 1 (which is the content of the zero block) to 1 (see Figure 1).

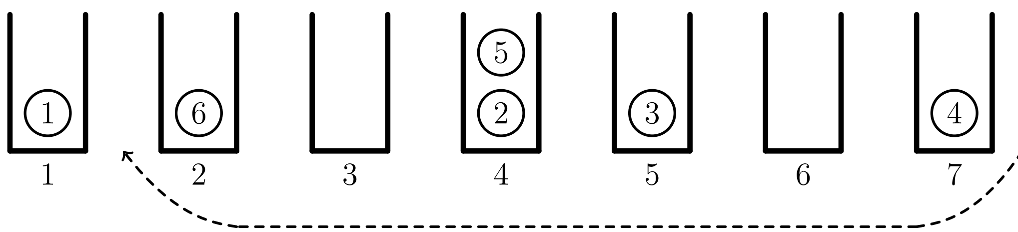

FIGURE 1. An assignment of 6 balls into 7 urns

Now we pass to the first block $\{2, -3, 5\}$ (together with its negative block $\{-2, 3, -5\}$): we have to choose a value for the images of 2 and 5 out of 6 possibilities. Take for example $f(2) = f(5) = 4$. Then we have to assign $f(3) = 5$, which is the next free value in cyclical order.

The next block is $\{4, -6\}$ (together with its negative block $\{-4, 6\}$). For this block we are left with 4 possibilities for assigning values. Choose for instance $f(4) = 7$ and so we must assign $f(6) = 2$, which is the next free value in cyclical order. The resulting balls into urns assignment is depicted in Figure 1.

Conversely, given the balls into urns assignment obtained above:

$$f(1) = 1,\ f(2) = 4,\ f(3) = 5,\ f(4) = 7,\ f(5) = 4, f(6) = 2.$$

In order to recover the signed partition $\mathcal{B}$ which induced this assignment, we act as follows:

- Only 1 is sent to 1, so we have the zero block $B_0 = \{\pm 1\}$.

- The current minimal unused element is 2 which is sent by f to 4, so the positive part of the next block will be $f^{-1}(\{4\}) = \{2, 5\}$ and the negative part will be $f^{-1}(\{5\}) = \{3\}$ (since 5 is the next value after 4 in cyclical order). Hence we get the pair of blocks: $\{2, 5, -3\}$ and $\{-2, -5, 3\}$.
- Now, the current minimal unused element is 4 which is sent by f to 7, so the positive part of the block will be $f^{-1}(\{7\}) = \{4\}$ and the negative part will be $f^{-1}(\{2\}) = \{6\}$ (since 2 is the next value after 7 in cyclical order). Hence we get the pair of blocks: $\{4, -6\}$ and $\{6, -4\}$.

So we get that the signed partition is:

$$\mathcal{B} = \{\{\pm 1\}, \{2, 5, -3\}, \{-2, -5, 3\}, \{4, -6\}, \{-4, 6\}\},$$

which indeed was our original signed partition.

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