Brazilian Primes Which Are Also Sophie Germain Primes

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Abstract

We disprove a conjecture of Schott that no Brazilian prime is a Sophie Germain prime. We compute all counterexamples up to 10^{46}.

1 Introduction

The term "Brazilian numbers" originated at the 1994 Iberoamerican Mathematical Olympiad [6] in Fonseca, Brazil, in a problem proposed by the Mexican math team. They became a topic of lively discussion on the forum mathematiques.net. Bernard Schott [4] summarized the results in the standard reference on Brazilian numbers.

Definition 1. A Brazilian number n is an integer whose base–b representation has all the same digits for some $1 < b < n-1$.

In other words, n is Brazilian if and only if $n = m\left(\frac{b^q-1}{b-1}\right) = mb^{q-1} + \cdots + mb + m$, with $q \geq 2$. These numbers are A125134 in the OEIS; the entry links to Schott's article [4].

A Brazilian prime (or "prime repunit") is a Brazilian number that is prime; by necessity, $m = 1$ and $q \geq 3$. See A085104 in the OEIS for the sequence of Brazilian primes. In 2010, Schott [4] conjectured that no Brazilian prime is also a Sophie Germain prime.

2 Enumerating Counterexamples

Recall that a Sophie Germain prime is a prime p such that $2p + 1$ is also prime. If p is a Sophie Germain prime, then we say that $2p + 1$ is a safe prime.

To aid our search, we use a few lemmas.

Lemma 2. *If $p = \frac{b^q-1}{b-1}$ is a Brazilian prime, then q is an odd prime.*

Proof. Recall $x^q - 1$ is divisible by the mth cyclotomic polynomial $\Phi_m(x)$ for $m|q$; therefore p can only be prime if q is also prime. Note that $q > 2$ because $b < p - 1$, so q is an odd prime. □

The preceding lemma is also Corollary 4.1 of Schott [4].

Lemma 3. *If p is a Brazilian prime and a Sophie Germain prime, then $p \equiv q \equiv 2 \pmod 3$ and $b \equiv 1 \pmod 3$.*

Proof. If p is a Sophie Germain prime, then 3 cannot divide the safe prime $2p+1$, so p cannot be congruent to 1 (mod 3). The number 3 is not Brazilian, so $p \neq 3$ and thus $p \equiv 2$ (mod 3).

If $3|b$, then

$$p = b^{q-1} + b^{q-2} + \cdots + b + 1 \equiv 1 \pmod 3,$$

which is a contradiction. Lemma 2 states that q is an odd prime, so if $b \equiv 2$ (mod 3), then $p \equiv 1$ (mod 3), a contradiction. We conclude that $b \equiv 1$ (mod 3), so that $q \equiv p$ (mod 3), and therefore $q \equiv 2$ (mod 3). □

For $q = 5$, we use a modification of the technique described in [5] to compute a table of length–5 Brazilian primes up to 10^{46}. We will describe this computation in full in a forthcoming paper [2]. Of these, 104890280 are Sophie Germain primes. The smallest is $28792661 = 73^4 + 73^3 + 73^2 + 73 + 1$. We very easily prove the primality of Sophie German primes with the Pocklington–Lehmer test.

For $q \geq 11$, we very quickly enumerate all possibilities for $b \leq 10^{46/(q-1)}$. We find 22 Brazilian Sophie Germain primes for $q = 11$, and none for larger q. (We have $q < \log_2 10^{46} + 1 < 154$.) The smallest is

$$14781835607449391161742645225951 = 1309^{10} + 1309^9 + \cdots + 1309 + 1.$$

While we disprove Schott's conjecture, we do have the related

Proposition 4. *The only Brazilian prime which is a safe prime is* 7.

Proof. Recall that a safe prime is a prime p where $\frac{p-1}{2}$ is also prime. (Equivalently, $\frac{p-1}{2}$ is a Sophie Germain prime.) If $p = b^{q-1} + \cdots + b + 1$ is a safe prime, then $\frac{p-1}{2} = \frac{1}{2}(b^{q-1} + \cdots + b)$ must also be prime. This expression, however is divisible by $\frac{b(b+1)}{2}$, which is only prime when $b = 2$ and $p = 7$. □

The list of Brazilian Sophie Germain primes is A306845 in the OEIS. The first few counterexamples were also discovered by Giovanni Resta and Michel Marcus; see the comments for A085104.

3 Conditional Results

The infinitude of these type of primes, as well as the asymptotic number of them, is the consequence of well–known conjectures.

Proposition 5. *Assuming Schnizel's Hypothesis H, there are infinitely many Brazilian Sophie Germain primes.*

Proof. Recall that Hypothesis H [3] says that any set of polynomials, whose product is not identically zero modulo any prime, is simultaneously prime infinitely often. Take our two polynomials to be $f_0(x) = x^4+x^3+x^2+x+1$ and $f_1(x) = 2x^4+2x^3+2x^2+2x+3$. Then $f_0(b)$ is Brazilian and $f_1(b) = 2f_0(b)+1$. Rather than checking congruences, it suffices to note the existence of the above primes of this form to see that the conditions of Hypothesis H are satisfied. □

The Bateman–Horn Conjecture [1] implies a more precise statement about the number of Brazilian Sophie Germain primes. The following proposition follows from it immediately.

Proposition 6. *For an odd prime q, let $\phi_q(x)$ be the qth cyclotomic polynomial. Assuming the Bateman–Horn Conjecture, the number of values of $b < x$ such that $\phi_q(b)$ and $2\phi_q(b) + 1$ are simultaneously prime is* 0 *or* $\sim C_q \frac{x}{\log x^2}$, *for some constant C_q, depending on whether $\phi_q(b)(2\phi_q(b) + 1)$ is identically zero for some prime p.*

If we instead look at the number of primes $< x$ that are of this form, we get $\sim C'_q \frac{x^{1/q-1}}{\log x^2}$. We sum over all $q \equiv 2 \pmod 3$ and notice that the $q = 5$ term dominates. This gives us the following corollary.

Corollary 7. *The number of Brazilian Sophie Germain primes is* $\sim C\frac{x^{1/4}}{\log x^2}$, *for some* C.

References

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