MULTIPLAYER ROCK-PAPER-SCISSORS

CHARLOTTE ATEN

Abstract. We study a class of algebras we regard as generalized Rock-Paper-Scissors games. We determine when such algebras can exist, show that these algebras generate the varieties generated by (hyper)tournament algebras, count these algebras, study their automorphisms, and determine their congruence lattices. We produce a family of finite simple algebras.

Contents

1. Introduction

The game of Rock-Paper-Scissors (RPS) involves two players simultaneously choosing either rock (r), paper (p), or scissors (s). Informally, the rules of the game are that "rock beats scissors, paper beats rock, and scissors beats paper". That is, if one player selects rock and the other selects paper then the latter player wins, and so on. If two players choose the same item then the round is a tie.

A *magma* is an algebra $\mathbf{A} \coloneqq (A, f)$ consisting of a set A and a single binary operation $f\colon A^2 \to A$. We will view the game of RPS as a magma. We let $A \coloneqq \{r, p, s\}$ and define a binary operation $f\colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$. This operation is given by the table below and completely describes the rules of RPS. In order to play the first player selects a member of A, say x, at

Date: 2019.
email: caten2@u.rochester.edu
Thanks to Jonathan Pakianathan and Clifford Bergman for their helpful comments.
Thanks to Scott Kirila for pointing out the result we use in section 2.
This research was supported in part by the people of the Yosemite Valley.
A short version of this paper appeared in the proceedings of the 2018 Algebras and Lattices in Hawai'i conference[2].

the same time that the second player selects a member of A, say y. Each player who selected $f(x,y)$ is the winner. Note that it is possible for both players to win, in which case we have a tie.

	r	p	s
r	r	p	r
p	p	p	s
s	r	s	s

In general we have a class of *selection games*, which are games consisting of a collection of items A, from which a fixed number of players n each choose one, resulting in a tuple $a \in A^n$, following which the round's winners are those who chose $f(a)$ for some fixed rule $f\colon A^n \to A$. We refer to an algebra $\mathbf{A} \coloneqq (A, f)$ with a single basic n-ary operation $f\colon A^n \to A$ as an *n-ary magma* or an *n-magma*. We will sometimes abuse this terminology and refer to an n-ary magma $\mathbf{A}$ simply as a *magma*. Each such game can be viewed as an n-ary magma and each n-ary magma can be viewed as a game in the same manner, providing we allow for games where we keep track of who is "player 1", who is "player 2", etc. Again note that any subset of the collection of players might win a given round, so there can be multiple player ties.

The classic RPS game has several desirable properties. Namely, RPS is, in terms we proceed to define,

(1) conservative,
(2) essentially polyadic,
(3) strongly fair, and
(4) nondegenerate.

Let $\mathbf{A} \coloneqq (A, f)$ be an n-magma. We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \ldots, a_n \in A$ we have that $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$[3, p.94]. Similarly we call $\mathbf{A}$ *conservative* when f is conservative. We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$. Similarly we call $\mathbf{A}$ *essentially polyadic* when f is essentially polyadic. We say that f is *fair* when for all $a, b \in A$ we have $\left|f^{-1}(a)\right| = \left|f^{-1}(b)\right|$. Let A_k denote the members of A^n which have exactly k distinct components for some $k \in \mathbb{N}$. We say that f is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$. Similarly we call $\mathbf{A}$ *(strongly) fair* when f is (strongly) fair. Note that if f (respectively, $\mathbf{A}$) is strongly fair then f (respectively, $\mathbf{A}$) is fair, but the reverse implication does not hold. We say that f is *nondegenerate* when $|A| > n$. Similarly we call $\mathbf{A}$ *nondegenerate* when f is nondegenerate.

Thinking in terms of selection games we say that $\mathbf{A}$ is conservative when each round has at least one winning player. We say that $\mathbf{A}$ is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item. We say that $\mathbf{A}$ is fair when each item has the same probability of being the winning item (or tying). We say that $\mathbf{A}$ is strongly fair when each item has the same chance of being the winning item when exactly k distinct items are chosen for any $k \in \mathbb{N}$. Note that this is not the same as saying that each player has the same chance of choosing the winning item (respectively, when exactly k distinct items are chosen). When $\mathbf{A}$ is *degenerate* (i.e. not nondegenerate) we have that $|A| \leq n$. In this case we have that all members of $A_{|A|}$ have the same set of components.

If $\mathbf{A}$ is essentially polyadic with $|A| \leq n$ it is impossible for $\mathbf{A}$ to be strongly fair unless $|A| = 1$.

Extensions of RPS which allow players to choose from more than the three eponymous items are attested historically. The French variant of RPS gives a pair of players 4 items to choose among[11, p.140]. In addition to the usual rock, paper, and scissors there is also the well (w). The well beats rock and scissors but loses to paper. The corresponding Cayley table is given below. This game is not fair, as $\left|f^{-1}(r)\right| = 3$ yet $\left|f^{-1}(p)\right| = 5$. It is nondegenerate since there are 4 items for 2 players to chose among. It is also conservative and essentially polyadic.

	r	p	s	w
r	r	p	r	w
p	p	p	s	p
s	r	s	s	w
w	w	p	w	w

There has been some recent recreational interest in RPS variants with larger numbers of items from which two players may choose. For example, the game Rock-Paper-Scissors-Spock-Lizard[9] (RPSSL) is attested in the popular culture. The Cayley table for this game is given below, with v representing Spock and l representing lizard. This game is conservative, essentially polyadic, strongly fair, and nondegenerate.

	r	p	s	v	l
r	r	p	r	v	r
p	p	p	s	p	l
s	r	s	s	v	s
v	v	p	v	v	l
l	r	l	s	l	l

Variants of RPS with larger numbers of items appear in the literature as balanced tournaments[4]. Under this combinatorial definition it is well-established that only variants with an odd number of items may exist when the quantity of items to choose from is finite. We detail the connection between our generalization of RPS and tournaments in section 4. In our language we have an analogous odd-order result.

Theorem 1. *Let* $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A| \in \mathbb{N}$. *We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

Proof. Since $\mathbf{A}$ is nondegenerate and $n = 2$ we must have that $m > n = 2$ and hence $m \neq 1$. Games with $m = 1$ have only one item to choose from and all players always tie.

Since $\mathbf{A}$ is strongly fair we must have that $\left|f^{-1}(a) \cap A_2\right| = \left|f^{-1}(b) \cap A_2\right|$ for all $a, b \in A$. As the m distinct sets $f^{-1}(a) \cap A_2$ for $a \in A$ partition A_2 and are all the same size we require that $m \mid |A_2|$. Moreover, as we take $\mathbf{A}$ to be essentially polyadic we have that $f(a, b) = f(b, a)$ for all $a, b \in A$. This implies that each of the m items must be the winner among the same number of unordered pairs of distinct elements $\{a, b\}$. Let $\binom{A}{2}$ denote the collection of unordered pairs of distinct elements in A. That is, we write $\binom{A}{2}$ to indicate the collection of 2-*sets* of A. We have that $\left|\binom{A}{2}\right| = \binom{m}{2}$ so we require that $m \mid \left|\binom{A}{2}\right| = \binom{m}{2}$. This implies that $ms = \frac{m(m-1)}{2}$ for some $s \in \mathbb{N}$. Rearranging we find that $m = 2s + 1$ so m is odd.

It remains to show that such games $\mathbf{A}$ exist when $m \neq 1$ is odd. Let $\binom{A}{1}$ denote the collection of singletons of elements in A. That is, we write $\binom{A}{1}$ to indicate the collection of 1-*sets* of A. Since $\left|\binom{A}{1}\right| = |A| = m$ we can partition $\binom{A}{1}$ into m subcollections $C_1 := \{C_{1,r}\}_{r\in A}$ indexed on the m elements of A, each with $|C_{1,r}| = 1$. Since we assume that $m \neq 1$ is odd we have that $m = 2s + 1$ for some $s \in \mathbb{N}$. This implies that

$$\binom{m}{2} = \binom{2s+1}{2} = (2s + 1)s = ms$$

so we can partition $\binom{A}{2}$ into m subcollections $C_2 := \{C_{2,r}\}_{r\in A}$ indexed on the m elements of A, each with $|C_{2,r}| = s$. With respect to these partitions $C := \{C_1, C_2\}$ we define a binary operation $f\colon A^2 \to A$ by $f(a, b) := r$ when $\{a, b\} \in C_{k,r}$ for some $k \in \{1, 2\}$. This map is well-defined since each $\{a, b\}$ contains either 1 or 2 distinct elements and thus belongs to a unique member of one of the partitions C_k. In order to see that the resulting magma $\mathbf{A} := (A, f)$ is essentially polyadic choose some $a_0 \in A$ and let $g\colon \mathrm{Sb}(A) \to A$ be given by

$$g(U) := \begin{cases} r & \text{when } (\exists k \in \{1, 2\})(U \in C_{k,r}) \\ a_0 & \text{when } |U| > 2 \end{cases}.$$

By construction we have that $f(a_1, a_2) = g(\{a_1, a_2\})$ for all $a_1, a_2 \in A$. Note that the choice of a_0 was necessary to define a function but ultimately irrelevant. We now show that $\mathbf{A}$ is strongly fair. Given $r \in A$ we have that $f(a_1, a_2) = r$ with $\{a_1, a_2\} \in A_1$ when $\{a_1, a_2\} \in C_{1,r}$. There is only one way to obtain an ordered pair from $\{a_1, a_2\}$ with $a_1 = a_2$ so $\left|f^{-1}(r) \cap A_1\right| = 1\,|C_{1,r}| = 1$. Given $r \in A$ we have that $f(a_1, a_2) = r$ with $\{a_1, a_2\} \in A_2$ when $\{a_1, a_2\} \in C_{2,r}$. There are two ways to obtain an ordered pair from $\{a_1, a_2\}$ with $a_1 \neq a_2$ so $\left|f^{-1}(r) \cap A_2\right| = 2\,|C_{2,r}|$. As each of the $C_{2,r}$ have the same size we conclude that $\mathbf{A}$ is strongly fair. Since $m = 2s+1$ for some $s \in \mathbb{N}$ we have that $m \geq 3 > 2 = n$ so $\mathbf{A}$ is also nondegenerate. We see that an essentially polyadic, strongly fair, nondegenerate magma always exists when $m \neq 1$ is odd. □

Historically those games which have been played or described tend to be conservative but we did not need that assumption for our argument. We say a partition $P := \{P_i\}_{i\in I}$ of a set S is *regular* when $|P_i| = |P_j|$ for all $i, j \in I$. Note that we gave a description of all possible essentially polyadic, strongly fair, nondegenerate magmas $\mathbf{A}$ with $m \neq 1$ odd as any such magma will induce a regular partition C_1 of $\binom{A}{1}$ and a regular partition C_2 of $\binom{A}{2}$ and such a pair of partitions of $\binom{A}{1}$ and $\binom{A}{2}$ will yield a map $f\colon A^2 \to A$ with the desired properties. It is also the case that there is at least one conservative, essentially polyadic, strongly fair, and nondegenerate magma $\mathbf{A}$ for each odd $m \neq 1$. We produce examples of these games in section 3.

We explore selection games for more than 2 simultaneous players, in particular those which we see as generalized Rock-Paper-Scissors games. We give a numerical constraint on which n-magmas of order m can be essentially polyadic, strongly fair, and nondegenerate and show that this constraint is sharp. After giving examples of such magmas for all possible pairs (m, n) we go on to detail connections between RPS, tournament magmas, and hypertournaments. We proceed to count these magmas and study their automorphisms and congruences, concluding with the exhibition of an infinite family of finite simple magmas.

2. RPS Magmas

The magmas we are interested in are those corresponding to selection games which have the four desirable properties possessed by Rock-Paper-Scissors. As in the preceding section it will benefit us to first examine the larger class of magmas obtained by dropping the conservativity axiom.

Definition 1 (PRPS magma)**.** Let $\mathbf{A} \coloneqq (A, f)$ be an n-ary magma. When $\mathbf{A}$ is essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is a PRPS *magma* (read "*pseudo*-RPS *magma*"). When $\mathbf{A}$ is an n-magma of order $m \in \mathbb{N}$ with these properties we say that $\mathbf{A}$ is a PRPS(m, n) *magma*. We also use PRPS and PRPS(m, n) to indicate the classes of such magmas.

Our first theorem generalizes directly to selection games with more than 2 players.

Theorem 2. *Let $\mathbf{A} \in \mathrm{PRPS}(m, n)$ and let $\varpi(m)$ denote the least prime dividing m. We have that $n < \varpi(m)$. Conversely, for each pair (m, n) with $m \neq 1$ such that $n < \varpi(m)$ there exists such a magma.*

Proof. Since $\mathbf{A}$ is nondegenerate we must have that $m > n$.

Since $\mathbf{A}$ is strongly fair we must have that $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$ for all $k \in \mathbb{N}$. As the m distinct sets $f^{-1}(a) \cap A_k$ for $a \in A$ partition A_k and are all the same size we require that $m \mid |A_k|$. When $k > n$ we have that $A_k = \varnothing$ and obtain no constraint on m.

When $k \leq n$ we have that A_k is nonempty. As we take $\mathbf{A}$ to be essentially polyadic we have that $f(x) = f(y)$ for all $x, y \in A_k$ such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. Let $\binom{A}{k}$ denote the collection of unordered sets of k distinct elements of A. That is, $\binom{A}{k}$ consists of all k-*sets* in A. Note that the size of the collection of all members $x \in A_k$ such that $\{x_1, \ldots, x_n\} = \{z_1, \ldots, z_k\}$ for distinct $z_i \in A$ does not depend on the choice of distinct z_i. This implies that for a fixed $k \leq n$ each of the m items must be the winner among the same number of unordered sets of k distinct elements in A. We have that $\left|\binom{A}{k}\right| = \binom{m}{k}$ so we require that $m \mid \left|\binom{A}{k}\right| = \binom{m}{k}$ for all $k \leq n$.

Let

$$d(m, n) \coloneqq \gcd\left(\left\{ \binom{m}{k} \;\middle|\; 1 \leq k \leq n \right\}\right).$$

Since $m \mid \binom{m}{k}$ for all $k \leq n$ we must have that $m \mid d(m, n)$. Joris, Oestreicher, and Steinig showed that when $m > n$ we have

$$d(m, n) = \frac{m}{\operatorname{lcm}(\{ k^{\varepsilon_k(m)} \mid 1 \leq k \leq n \})}$$

where $\varepsilon_k(m) = 1$ when $k \mid m$ and $\varepsilon_k(m) = 0$ otherwise[7, p.103]. Since we have that $m \mid d(m, n)$ and $d(m, n) \mid m$ it must be that $m = d(m, n)$ and hence

$$\operatorname{lcm}\left(\left\{ k^{\varepsilon_k(m)} \;\middle|\; 1 \leq k \leq n \right\}\right) = 1.$$

This implies that $\varepsilon_k(m) = 0$ for all $2 \leq k \leq n$. That is, no k between 2 and n inclusive divides m. This is equivalent to having that no prime $p \leq n$ divides m, which is in turn equivalent to having that $n < \varpi(m)$, as desired.

It remains to show that such games $\mathbf{A}$ exist when $m \neq 1$ and $n < \varpi(m)$. By this assumption we have that $k \nmid m$ whenever $2 \leq k \leq n$. Since

$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!} = m\frac{(m-1)\cdots(m-k+1)}{k(k-1)\cdots(2)}$$

and none of the nontrivial factors of $k!$ divide m it must be that $m \mid \binom{m}{k}$ for each $2 \leq k \leq n$. This implies that $m \mid \left|\binom{A}{k}\right|$ for each $k \leq n$ so for each $k \leq n$ we can partition $\binom{A}{k}$ into m subcollections $C_k \coloneqq \{C_{k,r}\}_{r\in A}$ indexed on the m elements of A, each with $|C_{k,r}| = \frac{1}{m}\binom{m}{k}$. With respect to this collection of partitions $C \coloneqq \{C_k\}_{1\leq k\leq n}$ we define an n-ary operation $f\colon A^n \to A$ by $f(a_1,\ldots,a_n) \coloneqq r$ when $\{a_1,\ldots,a_n\} \in C_{k,r}$ for some $k \in \{1,\ldots,n\}$. This map is well-defined since each $\{a_1,\ldots,a_n\}$ contains exactly k distinct elements for some $k \in \{1,\ldots,n\}$ and thus belongs to a unique member of one of the partitions C_k. In order to see that the resulting magma $\mathbf{A} \coloneqq (A,f)$ is essentially polyadic choose some $a_0 \in A$ and let $g\colon \mathrm{Sb}(A) \to A$ be given by

$$g(U) \coloneqq \begin{cases} r & \text{when } (\exists k \in \{1,\ldots,n\})(U \in C_{k,r}) \\ a_0 & \text{when } |U| > n \end{cases}.$$

By construction we have that $f(a_1,\ldots,a_n) = g(\{a_1,\ldots,a_n\})$ for all $a_1,\ldots,a_n \in A$. Again the choice of a_0 was immaterial. We now show that $\mathbf{A}$ is strongly fair. Given $r \in A$ we have that $f(a_1,\ldots,a_n) = r$ with $(a_1,\ldots,a_n) \in A_k$ when $\{a_1,\ldots,a_n\} \in C_{k,r}$. Again note that the number of members of A_k whose coordinates form the set $\{a_1,\ldots,a_n\}$ is the same as the number of members of A_k whose coordinates form the set $\{b_1,\ldots,b_n\}$ for some other $(b_1,\ldots,b_n) \in A_k$. This implies that each of the $\left|f^{-1}(r) \cap A_k\right|$ have the same size for a fixed k and hence $\mathbf{A}$ is strongly fair. To see that $\mathbf{A}$ is nondegenerate observe that if $1 \neq m \leq n$ then there is some prime p dividing m. Since $p \leq m \leq n$ is prime we require that $p \nmid m$, a contradiction. We see that a $\mathrm{PRPS}(m,n)$ magma always exists when $m \neq 1$ and $n < \varpi(m)$. $\square$

As in the case of $n = 2$ we did not use the conservativity axiom. We have given a description of all possible finite PRPS magmas. To see this, note that any PRPS magma of order m satisfying our numerical condition will induce a regular partition C_k of $\binom{A}{k}$ for each $1 \leq k \leq n$ and any such collection of partitions will yield a map $f\colon A^n \to A$ with the desired properties. As before it is always possible to find conservative $\mathrm{PRPS}(m,n)$ magmas when this numerical condition is met. These magmas are those which possess all the desirable properties of the game RPS.

Definition 2 (RPS magma)**.** Let $\mathbf{A} \coloneqq (A,f)$ be an n-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an n-magma of order $m \in \mathbb{N}$ with these properties we say that $\mathbf{A}$ is an $\mathrm{RPS}(m,n)$ *magma*. We also use RPS and $\mathrm{RPS}(m,n)$ to indicate the classes of such magmas.

More succinctly, RPS magmas are conservative PRPS magmas. We proceed to exhibit members of this class.

3. Examples of RPS Magmas

We give examples of $\mathrm{RPS}(m, n)$ magmas for all m and n satisfying the numerical constraint of Theorem 2. This shows that such pairs (m, n) are precisely those for which such magmas exist. Our construction makes use of group actions and we first give a lemma in that direction.

Definition 3 (k-extension of an action)**.** Given a group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ of $\mathbf{G}$ on a set A and some $1 \leq k \leq |A|$ define for each $s \in G$ a map $\alpha_k(s)\colon \binom{A}{k} \to \binom{A}{k}$ by

$$(\alpha_k(s))(U) \coloneqq \{\, (\alpha(s))(a) \mid a \in U \,\}.$$

The function $\alpha_k\colon G \to \binom{A}{k}^{\binom{A}{k}}$ is called the *k-extension* of α.

Lemma 1. *The k-extension of a group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ is a group action.*

Proof. We show that $\alpha_k(s) \in \mathrm{Perm}(\binom{A}{k})$. Let $U, V \in \binom{A}{k}$ such that $(\alpha_k(s))(U) = (\alpha_k(s))(V)$. This implies that

$$\{\, (\alpha(s))(a) \mid a \in U \,\} = \{\, (\alpha(s))(a) \mid a \in V \,\}.$$

Given $a \in U$ we have that

$$(\alpha(s))(a) \in (\alpha_k(s))(U) = (\alpha_k(s))(V)$$

so $a = (\alpha(s))^{-1}((\alpha(s))(a)) \in V$. We find that $U \subset V$ and by symmetry $V \subset U$ so $U = V$. We see that $\alpha_k(s)$ is injective.

Suppose that $U \in \binom{A}{k}$ and define $V \coloneqq \left\{\, (\alpha(s))^{-1}(a) \;\middle|\; a \in U \,\right\}$. Since $\alpha(s) \in \mathrm{Perm}(A)$ it must be that $|V| = |U|$ and hence $V \in \binom{A}{k}$. Observe that

$$(\alpha_k(s))(V) = \{\, (\alpha(s))(b) \mid b \in V \,\} = \left\{\, (\alpha(s))((\alpha(s))^{-1}(a)) \;\middle|\; a \in U \,\right\} = U.$$

Thus, $\alpha_k(s)$ is surjective and hence is a bijection.

We show that $\alpha_k\colon G \to \mathrm{Perm}(\binom{A}{k})$ is a group homomorphism. Let e denote the identity element of G. We have that $\alpha_k(e) = \mathrm{id}_{\binom{A}{k}}$ immediately. Similarly immediately we have that $\alpha_k(s)$ and $\alpha_k(s^{-1})$ are inverse functions of each other. Observe that for each $U \in \binom{A}{k}$ we have

$$\begin{aligned}(\alpha_k(s_1 s_2))(U) &= \{\, (\alpha(s_1 s_2))(a) \mid a \in U \,\} \\ &= \{\, (\alpha(s_1))((\alpha(s_2))(a)) \mid a \in U \,\} \\ &= (\alpha_k(s_1))((\alpha_k(s_2))(U)) \\ &= (\alpha_k(s_1) \circ \alpha_k(s_2))(U)\end{aligned}$$

so $\alpha_k\colon \mathbf{G} \to \mathbf{Perm}(\binom{A}{k})$ is indeed a group action. □

Certain group actions on a set A yield families of magmas with universe A. Recall that a group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(X)$ is called *free* when for any $s, t \in G$ and any $x \in X$ we have that $(\alpha(s))(x) = (\alpha(t))(x)$ implies $s = t$, *transitive* when for any $x, y \in X$ there exists some $s \in G$ such that $(\alpha(s))(x) = y$, and *regular* when α is both free and transitive.

Definition 4 (α-action magma)**.** Fix a group $\mathbf{G}$, a set A, and some $n < |A|$. Given a regular group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ such that each of the k-extensions of α is free for $1 \leq k \leq n$ let $\Psi_k \coloneqq \left\{\, \mathrm{Orb}(U) \;\middle|\; U \in \binom{A}{k} \,\right\}$ where $\mathrm{Orb}(U)$ is the orbit of U under α_k. Let $\beta \coloneqq \{\beta_k\}_{1 \leq k \leq n}$ be a sequence of choice functions $\beta_k\colon \Psi_k \to \binom{A}{k}$ such that $\beta_k(\psi) \in \psi$ for each $\psi \in \Psi_k$. Let $\gamma \coloneqq \{\gamma_k\}_{1 \leq k \leq n}$ be a sequence of functions $\gamma_k \colon \Psi_k \to A$ such that $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$. Choose some $a_0 \in A$ and let $g \colon \mathrm{Sb}(A) \to A$ be given by

$$g(U) \coloneqq \begin{cases} (\alpha(s))(\gamma_k(\psi)) & \text{when } U = (\alpha_k(s))(\beta_k(\psi)) \\ a_0 & \text{when } |U| > n \end{cases}.$$

Define $f \colon A^n \to A$ by $f(a_1, \ldots, a_n) \coloneqq g(\{a_1, \ldots, a_n\})$. The *$\alpha$-action magma* induced by (β, γ) is $\mathbf{A} \coloneqq (A, f)$.

Note that the choice of a_0 is again immaterial here. The function g, and hence f, is well-defined as we assume that each of the α_k is free and hence there is a unique $s \in G$ such that $U = (\alpha_k(s))(\beta_k(\psi))$ for each $U \in \binom{A}{k}$.

Theorem 3. *Let $\mathbf{A}$ be an α-action magma induced by (β, γ). We have that $\mathbf{A} \in$ RPS.*

Proof. We show that $\mathbf{A}$ is conservative. Let $a_1, \ldots, a_n \in A$ and define $U \coloneqq \{a_1, \ldots, a_n\} \in \binom{A}{k}$. Suppose that $U \in \psi \in \Psi_k$ with $U = (\alpha_k(s))(\beta_k(\psi))$. Observe that

$$f(a_1, \ldots, a_n) = g(U) = (\alpha(s))(\gamma_k(\psi)).$$

By assumption $\gamma_k(\psi) \in \beta_k(\psi)$ so

$$f(a_1, \ldots, a_n) = (\alpha(s))(\gamma_k(\psi)) \in (\alpha_k(s))(\beta_k(\psi)) = U,$$

as desired.

By definition of f via $g \colon \mathrm{Sb}(A) \to A$ we have that $\mathbf{A}$ is essentially polyadic.

In order to see that $\mathbf{A}$ is strongly fair note that $|\psi| = |G|$ for each $\psi \in \Psi_k$ as we assume the action of α_k on $\binom{A}{k}$ to be free. For each orbit ψ we have that g takes on each value in A exactly once as the action of α on A is assumed to be transitive. This shows that each orbit $\psi \in \Psi_k$ contributes the same number of elements to each of the sets $f^{-1}(a) \cap A_k$. It follows that $\mathbf{A}$ is strongly fair.

By definition of an α-action magma we must have that $n < |A|$ so $\mathbf{A}$ is nondegenerate. □

In order to have an α-action magma for $\alpha \colon \mathbf{G} \to \mathbf{Perm}(A)$ we must have that α is regular. Recall that every regular $\mathbf{G}$-action is isomorphic (in the category of $\mathbf{G}$-sets) to the left multiplication action $L \colon \mathbf{G} \to \mathbf{Perm}(G)$. Isomorphic actions determine equivalent orbits so without loss of generality we may only consider the left-multiplication actions of groups on themselves. Fortunately this class of actions is highly compatible with our construction.

Theorem 4. *Let $\mathbf{G}$ be a nontrivial finite group and let $L \colon \mathbf{G} \to \mathbf{Perm}(G)$ be the left-multiplication action. For each $1 \leq k < \varpi(|G|)$ we have that the k-extension L_k of L is free.*

Proof. Suppose that $s, t \in G$ and $U \in \binom{A}{k}$ such that $(L_k(s))(U) = (L_k(t))(U)$. It follows that $(L_k(t^{-1}s))(U) = U$ so the restriction $h \coloneqq L(t^{-1}s)|_U$ belongs to $\mathrm{Perm}(U)$. The order of h in $\mathbf{Perm}(U)$ must divide the order of $\mathbf{Perm}(U)$, which is $k!$. It follows that the order of h is coprime with $|G|$. The order of h must divide the order of $L(t^{-1}s)$ but the order of $L(t^{-1}s)$ is the order of $t^{-1}s$ in $\mathbf{G}$, which divides $|G|$. Thus, the order of h is both coprime with $|G|$ and divides $|G|$ so the order of h is 1 and hence $h = \mathrm{id}_U$. This implies that $s = t$ so L_k is a free action. □

Since we know that our numerical condition for the existence of an RPS(m,n) magma must hold we have characterized all $\mathbf{G}$-actions which give rise to a finite RPS magma through this construction.

Definition 5 (Regular RPS magma)**.** Let $\mathbf{G}$ be a nontrivial finite group and fix $n < \varpi(|G|)$. We denote by $\mathbf{G}_n(\beta,\gamma)$ the L-action n-magma induced by (β,γ), which we refer to as a *regular* RPS *magma*.

The games Rock-Paper-Scissors and Rock-Paper-Scissors-Spock-Lizard are isomorphic to regular RPS magmas. To obtain the classic RPS take $\mathbf{G} = \mathbb{Z}_3$ and $n = 2$. We have $\binom{A}{1} = \{\{0\}, \{1\}, \{2\}\}$ and $\binom{A}{2} = \{\{0,1\}, \{0,2\}, \{1,2\}\}$. Under the action of $\mathbb{Z}_3$ we have

$$\Psi_1 = \{\{\{0\}, \{1\}, \{2\}\}\}$$

and

$$\Psi_2 = \{\{\{0,1\}, \{0,2\}, \{1,2\}\}\}.$$

There is only one orbit $\psi_{1,1} \in \Psi_1$, for which we choose $\beta_1(\psi_{1,1}) := \{0\}$ as a representative. There is also only one orbit $\psi_{2,1} \in \Psi_2$, for which we choose $\beta_2(\psi_{2,1}) := \{0,1\}$ as a representative. We choose $\gamma_1(\psi_{1,1}) := 0$ and $\gamma_2(\psi_{2,1}) := 1$. We have that $\{0\} = 0 + \{0\}$, $\{1\} = 1 + \{0\}$, $\{2\} = 2 + \{0\}$, $\{0,1\} = 0 + \{0,1\}$, $\{1,2\} = 1 + \{0,1\}$, and $\{0,2\} = 2 + \{0,1\}$. The resulting values of g are $g(\{0\}) = 0$, $g(\{1\}) = 1$, $g(\{2\}) = 2$, $g(\{0,1\}) = 1$, $g(\{1,2\}) = 2$, and $g(\{0,2\}) = 0$. The Cayley table for the operation f obtained from g is given below. Observe that under the identification $0 \mapsto r$, $1 \mapsto p$, and $2 \mapsto s$ the magma $(\mathbb{Z}_3)_2(\beta,\gamma)$ is the original game of RPS.

	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2

To obtain the game Rock-Paper-Scissors-Spock-Lizard take $\mathbf{G} = \mathbb{Z}_5$ and $n = 2$. In order to simplify notation we will write sets as strings. For example 012 denotes $\{0,1,2\}$. The sets $\binom{A}{k}$ are then $\binom{A}{1} = \{0,1,2,3,4\}$ (where a represents the singleton set $\{a\}$) and $\binom{A}{2} = \{01, 02, 03, 04, 12, 13, 14, 23, 24, 34\}$. Under the action of $\mathbb{Z}_5$ we have

$$\Psi_1 = \{\{0,1,2,3,4\}\}$$

and

$$\Psi_2 = \{\{01, 12, 23, 34, 40\}, \{02, 13, 24, 30, 41\}\}.$$

There is only one orbit $\psi_{1,1} \in \Psi_1$, for which we choose $\beta_1(\psi_{1,1}) := 0$ as a representative. There are two orbits in Ψ_2, $\psi_{2,1} := \{01, 12, 23, 34, 40\}$ and $\psi_{2,2} := \{02, 13, 24, 30, 41\}$. We choose $\beta_2(\psi_{2,1}) := 01$ as a representative of $\psi_{2,1}$ and $\beta_2(\psi_{2,2}) := 02$ as a representative of $\psi_{2,2}$. We choose $\gamma_1(\psi_{1,1}) := 0$, $\gamma_2(\psi_{2,1}) := 1$, and $\gamma_2(\psi_{2,2}) := 2$. The Cayley table resulting from these choices is given below. Observe that under the identification $0 \mapsto r$, $1 \mapsto p$, $2 \mapsto s$, $3 \mapsto v$, and $4 \mapsto l$ the magma $(\mathbb{Z}_5)_2(\beta,\gamma)$ is the game of RPSSL.

	0	1	2	3	4
0	0	1	0	3	0
1	1	1	2	1	4
2	0	2	2	3	2
3	3	1	3	3	4
4	0	4	2	4	4

We now give an example of one of the games with 3 players. We know that such a game must have at least $m = 5$ items to choose from so we take $\mathbf{G} = \mathbb{Z}_5$. We make the same choices for the orbits in Ψ_1 and Ψ_2 as in RPSSL but we must now examine

$$\Psi_3 = \{\{012, 123, 234, 340, 401\}, \{013, 124, 230, 341, 402\}\}.$$

Choose $\beta_3(\psi_{3,1}) := 012$ as a representative of $\psi_{3,1} := \{012, 123, 234, 340, 401\}$ and $\beta_3(\psi_{3,2}) := 013$ as a representative of $\psi_{3,2} := \{013, 124, 230, 341, 402\}$. We choose $\gamma_3(\psi_{3,1}) := 0$ and $\gamma_3(\psi_{3,2}) := 0$. We give the Cayley table for the resulting ternary operation as five binary tables, one for $(x, y) \mapsto f(0, x, y)$, one for $(x, y) \mapsto f(1, x, y)$, one for $(x, y) \mapsto f(2, x, y)$, one for $(x, y) \mapsto f(3, x, y)$, and one for $(x, y) \mapsto f(4, x, y)$.

0	0	1	2	3	4
0	0	1	0	3	0
1	1	1	0	0	4
2	0	0	0	2	4
3	3	0	2	3	3
4	0	4	4	3	0

1	0	1	2	3	4
0	1	1	0	0	4
1	1	1	2	1	4
2	0	2	2	1	1
3	0	1	1	1	3
4	4	4	1	3	4

2	0	1	2	3	4
0	0	0	0	2	4
1	0	2	2	1	1
2	0	2	2	3	2
3	2	1	3	3	2
4	4	1	2	2	2

3	0	1	2	3	4
0	3	0	2	3	3
1	0	1	1	1	3
2	2	1	3	3	2
3	3	1	3	3	4
4	3	3	2	4	4

4	0	1	2	3	4
0	0	4	4	3	0
1	4	4	1	3	4
2	4	1	2	2	2
3	3	3	2	4	4
4	0	4	2	4	4

In practice it is easier to actually play by using a map $g\colon \mathrm{Sb}(A) \to A$ which shows that $\mathbf{A}$ is essentially polyadic. We give the relevant parts of such maps for RPS, RPSSL, and our example of an RPS$(5, 3)$ magma. The orbits are separated by vertical dividers.

U	0	1	2	01	12	20
$g(U)$	0	1	2	0	1	2

RPS

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

RPSSL

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

U	012	123	234	340	401	013	124	230	341	402
$g(U)$	0	1	2	3	4	0	1	2	3	4

RPS$(5, 3)$ example

Not all RPS magmas can be obtained by the preceding construction. Consider the RPS$(7, 2)$ magma $\mathbf{A}$ given by the table below.

	0	1	2	3	4	5	6
0	0	1	0	3	4	0	0
1	1	1	2	1	1	5	6
2	0	2	2	3	2	5	2
3	3	1	3	3	4	3	6
4	4	1	2	4	4	5	4
5	0	5	5	3	5	5	6
6	0	6	2	6	4	6	6

This magma corresponds to the graph Chamberland and Herman call HexagonalPyramid, which they show has automorphism group isomorphic to $\mathbb{Z}_3$[4, p.7]. In section 6 we show that the automorphism group of $\mathbf{G}_2(\beta,\gamma)$ must contain a copy of $\mathbf{G}$. It follows that a regular RPS magma of order 7 cannot have an automorphism group with less than 7 elements so $\mathbf{A}$ cannot be isomorphic to a regular RPS magma.

4. Tournaments and RPS Magmas

Now that we have specialized from PRPS magmas to RPS magmas to regular RPS magmas and obtained some basic information we give more general context to our discussion. We detail the relationship between RPS magmas, tournament algebras, and the hypertournaments considered in hypergraph theory.

4.1. Tournament Magmas.

The following is a classic object considered in graph theory.

Definition 6 (Tournament)**.** A *tournament* is a directed graph without multiple edges $\mathbf{T} := (T,\tau)$ where for each pair of vertices $u, v \in T$ we have that τ contains exactly one of the directed edges (u, v) or (v, u).

When $(u, v) \in \tau$ we say that u *dominates* v and write $u \to v$ to indicate this. Note that when $u = v$ we have $(u, v) = (v, u)$ and thus every vertex of a tournament carries a single loop. Tournaments are so named because we can imagine each vertex of a tournament $\mathbf{T}$ as being a team or player taking part in a tournament in the sense that each pair of teams plays a game and a winner is decided. Ties are forbidden, except in the degenerate case of a team "playing itself", and $u \to v$ is taken to indicate that team u beat team v when the two played each other.

If we make the stipulation that our directed graphs do not have multiple edges but are allowed loops we may more succinctly define a tournament as a complete directed graph. Similarly we can think of a tournament as an orientation of a complete undirected graph without multiple edges where each vertex carries a single loop.

This idea has an algebraic formulation, as well, which was introduced by Hedrlín and Chvátal in 1965.

Definition 7 (Tournament magma)**.** Given a tournament $\mathbf{T} := (T,\tau)$ the *tournament magma* obtained from $\mathbf{T}$ is the (binary) magma $\mathbf{A} := (T, f)$ where for $u, v \in T$ we define

$$f(u, v) := \begin{cases} u & \text{when } u \to v \\ v & \text{when } v \to u \end{cases}.$$

Note that the tournament magmas are precisely those magmas which are conservative and essentially polyadic. Thus we have the equivalent definition which follows.

Definition 8 (Tournament magma)**.** A *tournament magma* is a (binary) magma which is conservative and essentially polyadic.

Tournaments, both as graphs and magmas, have been studied extensively[5]. Earlier treatments are more likely to characterize tournament magmas as those magmas which are commutative and conservative (and idempotent, although this is redundant). Since we will soon consider an n-ary analogue of this situation we note that in the case of a binary operation we might as well take essential polyadicity as our axiom.

Let PRPS_2 denote the class of all binary PRPS magmas, let RPS_2 denote the class of all binary RPS magmas, and let Tour_2 denote the class of all (binary) tournament magmas. We discuss the relationship between the three classes PRPS_2, RPS_2, and Tour_2. By definition we have that $\mathrm{RPS}_2 \subset \mathrm{PRPS}_2$. In our earlier analysis we never addressed whether the conservativity axiom actually added anything. That is, we only provided examples of RPS_2 magmas and did not consider whether there exist any PRPS_2 magmas which are not RPS_2 magmas. It is easy to see that there exist PRPS_2 magmas which are not conservative. Let $\mathbf{A} \coloneqq (A, f)$ be the canonical $\mathrm{RPS}(3,2)$ magma corresponding to the original RPS game. Let $\sigma \in \mathrm{Perm}(A)$ be given by $\sigma(r) \coloneqq p$, $\sigma(p) \coloneqq s$, and $\sigma(s) \coloneqq r$. Define a new magma $\mathbf{A}_\sigma \coloneqq (A, f_\sigma)$ where for $x, y \in A$ we set $f_\sigma(x, y) \coloneqq \sigma(f(x, y))$. The Cayley table for $\mathbf{A}_\sigma$ is given below. Note that $\mathbf{A}_\sigma$ is essentially polyadic, strongly fair, and nondegenerate but unlike $\mathbf{A}$ we have that $\mathbf{A}_\sigma$ is not conservative. This shows that the class PRPS_2 properly contains RPS_2.

	r	p	s
r	p	s	p
p	s	s	r
s	p	r	r

Since in general PRPS_2 magmas are not conservative we also have that Tour_2 is not contained in PRPS_2. In the reverse direction we have that there exist tournament magmas which are not strongly fair, such as the minimal example given by the Cayley table below. It follows that the classes PRPS_2 and Tour_2 are incomparable.

	a	b
a	a	a
b	a	b

Again by definition we have that RPS_2 is contained in Tour_2 so it remains to determine whether this containment is proper. The previous example of a tournament which is not strongly fair shows that this containment is indeed proper. We summarize these considerations in the following theorem.

Theorem 5. *We have that* $\mathrm{RPS}_2 \subsetneq \mathrm{PRPS}_2$*,* $\mathrm{RPS}_2 \subsetneq \mathrm{Tour}_2$*, and neither of* PRPS_2 *and* Tour_2 *contains the other. Moreover,* $\mathrm{RPS}_2 = \mathrm{PRPS}_2 \cap \mathrm{Tour}_2$*.*

Proof. See above for the arguments that $\mathrm{RPS}_2 \subsetneq \mathrm{PRPS}_2$, $\mathrm{RPS}_2 \subsetneq \mathrm{Tour}_2$, and PRPS_2 and Tour_2 are incomparable. It remains to show that RPS_2 is precisely the intersection of PRPS_2 and Tour_2. This is also an immediate consequence of the definitions. □

None of the classes PRPS_2, RPS_2, and Tour_2 are varieties. That Tour_2 is not closed under taking products has already been demonstrated[5, p.98]. We included a nondegeneracy condition in our definitions of PRPS and RPS in order to avoid repeatedly disavowing certain trivial situations. Since PRPS_2 and RPS_2 do not contain any trivial members we have immediately that PRPS_2 and RPS_2 are not varieties. Even if we expand these classes to include trivial algebras they still do not form varieties, as we now show. Let RPS_n denote the class of n-ary RPS magmas. As it happens, the classes RPS_n for $n > 1$ are as far from being closed under products as possible.

Theorem 6. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* RPS *n-magmas with $n > 1$. The magma* $\mathbf{A} \times \mathbf{B}$ *is not an* RPS *magma.*

Proof. We show that $\mathbf{A} \times \mathbf{B}$ cannot be conservative. Let $\mathbf{A} := (A, f)$ and let $\mathbf{B} := (B, g)$. Let $x_1, \ldots, x_n \in A$ be distinct and let $y_1, \ldots, y_n \in B$ be distinct. Since f and g are conservative we have that $f(x) = x_i$ and $g(y) = y_j$ for some i and j. It follows that $(f \times g)((x_1, y_1), \ldots, (x_n, y_n)) = (x_i, y_j)$. Either $i \neq j$, in which case (x_i, y_j) is not one of the (x_i, y_i) and $\mathbf{A} \times \mathbf{B}$ is not conservative, or $i = j$. In this latter case we have that f is essentially polyadic so $f(\sigma(x)) = f(x)$ for any permutation σ of the x_i. This implies that

$$f(x_2, \ldots, x_n, x_1) = f(x_1, \ldots, x_n) = x_i$$

so

$$(f \times g)((x_2, y_1), \ldots, (x_n, y_{n-1}), (x_1, y_n)) = (x_i, y_i).$$

Now (x_i, y_i) does not appear among the arguments of $f \times g$ so again we see that $\mathbf{A} \times \mathbf{B}$ cannot be conservative and hence cannot be an RPS magma. □

As we do not require that magmas in PRPS be conservative we must give a different nontrivial reason that PRPS_2 is not a variety. Let PRPS_n denote the class of n-ary PRPS magmas. These classes do not support taking subalgebras when $n > 1$.

Theorem 7. *Let $n > 1$. There exists a magma* $\mathbf{A}$ *belonging to* PRPS_n *and a subalgebra* $\mathbf{B} \leq \mathbf{A}$ *such that* $\mathbf{B}$ *is not a* PRPS *magma.*

Proof. Since PRPS magmas are conservative we have that any subset of a PRPS magma's universe is a subuniverse. Take $\mathbf{A} := (\mathbb{Z}_p)_n(\beta, \gamma)$ for a prime $p > n$ and any valid choice of (β, γ). Take $\mathbf{B}$ to be any subalgebra of $\mathbf{A}$ of order $p-1$. Observe that $\mathbf{A}$ is a member of PRPS_n. However, the order of $\mathbf{B}$ is even and $n > 1$ so it cannot be that $n < \varpi(p-1)$. This shows that $\mathbf{B}$ is not a PRPS magma. □

We have actually been overzealous in that the previous argument also shows that RPS_n is not in general closed under taking subalgebras. If we restrict our attention to the binary case the examples we have provided are already sufficient to see that PRPS_2 and RPS_2 are not closed under taking subalgebras. Note that both the original RPS magma and the magma for the French variant are contained in the magma for RPSSL so subalgebras of RPS_2 magmas may or may not be RPS_2 magmas. Note that $\{r, s, l\}$ is a subuniverse for RPSSL and the corresponding subalgebra satisfies the numerical condition necessary for PRPS_2 magmas (being binary and of order 3), yet the corresponding subalgebra fails to be strongly fair and as such does not belong to PRPS_2.

In any case we certainly have that PRPS_2, RPS_2, and Tour_2 are not varieties. Let $\mathcal{T}_2 \coloneqq \mathbf{V}(\mathrm{Tour}_2)$ be the variety generated by all tournaments, let $\mathcal{P}_2 \coloneqq \mathbf{V}(\mathrm{PRPS}_2)$ be the variety generated by all binary PRPS magmas, and let $\mathcal{R}_2 \coloneqq \mathbf{V}(\mathrm{RPS}_2)$ be the variety generated by all binary RPS magmas. We study the relationship between these varieties.

Theorem 8. *We have that $\mathcal{T}_2 = \mathcal{R}_2$. Moreover, $\mathcal{T}_2$ is generated by the finite members of* RPS_2.

Our proof concerns the graph-theoretic analogue of RPS magmas.

Definition 9 (Balanced tournament)**.** A *balanced tournament* is a tournament $\mathbf{T} \coloneqq (T, \tau)$ such that for all $u, v \in T$ we have

$$|\{ w \in T \mid (u, w) \in \tau \}| = |\{ w \in T \mid (v, w) \in \tau \}| .$$

This is to say that each vertex in a balanced tournament has the same out-degree. Equivalently, each vertex in a balanced tournament dominates the same number of vertices, where "number" is meant in the sense of cardinality for infinite collections of vertices T. Just as tournament magmas are in natural correspondence with tournaments we have that binary RPS magmas are in natural correspondence with balanced tournaments. We exploit this connection.

Lemma 2. *Every finite tournament is a subgraph of a finite balanced tournament.*

Proof. Let $\mathbf{T} \coloneqq (T, \tau)$ be a finite tournament with r vertices. Define

$$\Phi \coloneqq \{ u_1 \mid u \in T \} \cup \{ u_2 \mid u \in T \} \cup \{\eta\}$$

where η is a new vertex distinct from the u_i. Define a relation $\phi \subset \Phi^2$ as follows. Given $u_i \in \Phi$ we set $(u_i, v_i) \in \phi$ when $(u, v) \in \tau$. We set $(u_i, v_j) \in \phi$ for $u \neq v$ and $i \neq j$ when $(v, u) \in \tau$. We set $(u_1, u_2) \in \phi$, $(u_2, \eta) \in \phi$, $(\eta, u_1) \in \phi$, and $(\eta, \eta) \in \phi$. We claim that $\mathbf{\Phi} \coloneqq (\Phi, \phi)$ is a balanced tournament.

It is immediate that $\mathbf{\Phi}$ is a tournament. To see that $\mathbf{\Phi}$ is balanced observe that $\mathbf{\Phi}$ consists of two copies of $\mathbf{T}$ along with the new vertex η. The vertex u_i dominates some k vertices other than u_i in the i copy of $\mathbf{T}$ and dominates the $n-k-1$ vertices in the $j \neq i$ copy of $\mathbf{T}$ corresponding to the $n-k-1$ vertices in $\mathbf{T}$ other than u which dominate u, giving $n-1$ vertices dominated by u_i. We have that u_1 dominates u_2 and is in turn dominated by η. We also have that u_1 dominates itself. It follows that each u_1 has out-degree $n+1$. Similarly, each u_2 dominates itself and η and is in turn dominated by u_1 so each u_2 has out-degree $n+1$. The new vertex η dominates the n vertices u_1 as well as itself and is dominated by the vertices u_2 so η also has out-degree $n+1$. We find that $\mathbf{\Phi}$ is a balanced tournament. The vertices u_i for a fixed i induce a subgraph of $\mathbf{\Phi}$ isomorphic to $\mathbf{T}$. $\square$

We give the proof of Theorem 8.

Proof. It follows from the previous lemma that every finite tournament magma is a subalgebra of a binary RPS magma of finite order. Since the finite tournament magmas generate $\mathcal{T}_2$[5, p.99] we have that the finite binary RPS magmas generate a variety containing $\mathcal{T}_2$ and hence $\mathcal{T}_2 \leq \mathcal{R}_2$. Since $\mathrm{RPS}_2 \subset \mathrm{Tour}_2$ we have that $\mathcal{R}_2 \leq \mathcal{T}_2$ so $\mathcal{T}_2 = \mathcal{R}_2$. $\square$

By the proof that the finite tournament magmas generate $\mathcal{T}_2$ (loc. cit.) we have that $\mathcal{T}_2 \models \alpha(x_1, \dots, x_k) \approx \beta(x_1, \dots, x_k)$ if and only if $\alpha \approx \beta$ is satisfied by each binary RPS magma of order $2k+1$.

With a little more work we can give a stronger result in this direction.

Theorem 9. *The variety $\mathcal{T}_2$ is generated by the class of finite regular* RPS_2 *magmas.*

In order to make this argument we refer to the graph-theoretic analogue of regular RPS magmas.

Definition 10 (Regular balanced tournament)**.** A *regular balanced tournament* is a tournament $\mathbf{T} := (T, \tau)$ such that there exists a regular RPS magma $\mathbf{G}_2(\beta, \gamma)$ where $G = T$ and $u \to v$ in $\mathbf{T}$ when $uv = u$ in $\mathbf{G}_2(\beta, \gamma)$.

Note that by definition of an RPS magma every regular balanced tournament is balanced. We will need an alternative characterization of regular RPS magmas.

Definition 11 (β-chirality)**.** Given a nontrivial group $\mathbf{G}$ of odd order let Ψ_k denote the collection of orbits of $\binom{G}{k}$ under the k-extension of the left-multiplication action of $\mathbf{G}$ on itself. Given a sequence of choice functions $\beta = \{\beta_1, \beta_2\}$ where $\beta_k \colon \Psi_k \to \binom{G}{k}$ we say that a sequence of maps $\gamma = \{\gamma_1, \gamma_2\}$ such that $\gamma_k \colon \Psi_k \to G$ with $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$ is a *β-chirality* of $\mathbf{G}$. We denote by $\mathrm{Chir}(\mathbf{G}, \beta)$ the collection of all β-chiralities of $\mathbf{G}$.

For each fixed β the members γ of $\mathrm{Chir}(\mathbf{G}, \beta)$ are precisely the data for each possible regular RPS magma obtained from $\mathbf{G}$.

Definition 12 (Sign function)**.** Given a nontrivial group $\mathbf{G}$ of odd order let $\mathrm{Sgn}(\mathbf{G})$ denote the set of all choice functions on $\left\{ \{x, x^{-1}\} \,\middle|\, x \in G \setminus e \right\}$ where e denotes the identity element of $\mathbf{G}$. We refer to a member $\lambda \in \mathrm{Sgn}(\mathbf{G})$ as a *sign function* on $\mathbf{G}$.

We can turn a β-chirality into a sign function.

Definition 13 (β-signor)**.** Define the *β-signor* $\zeta_\beta \colon \mathrm{Chir}(\mathbf{G}, \beta) \to \mathrm{Sgn}(\mathbf{G})$ as follows. Let f_γ denote the basic operation of $\mathbf{G}_2(\beta, \gamma)$. Given $\gamma \in \mathrm{Chir}(\mathbf{G}, \beta)$ define

$$(\zeta_\beta(\gamma))(\{x, x^{-1}\}) := \begin{cases} x & \text{when } f_\gamma(x, e) = e \\ x^{-1} & \text{otherwise} \end{cases}.$$

For a particular choice of β the sign functions on a group $\mathbf{G}$ are equivalent to the β-chiralities.

Lemma 3. *The map $\zeta_\beta \colon \mathrm{Chir}(\mathbf{G}, \beta) \to \mathrm{Sgn}(\mathbf{G})$ is a bijection.*

Proof. Let $\zeta := \zeta_\beta$. Suppose that $\gamma, \gamma' \in \mathrm{Chir}(\mathbf{G}, \beta)$ with $\zeta(\gamma) = \zeta(\gamma')$. It follows that for each $x \in G \setminus \{e\}$ we have $f_\gamma(x, e) = f_{\gamma'}(x, e)$. Given any $a, b \in G$ we have that $f(a, b) = c$ for some $c \in G$ if and only if $f(b^{-1}a, e) = b^{-1}c$[1]. It follows that a regular RPS magma operation is determined by the products $f(x, e)$ for $x \in G \setminus \{e\}$. This implies that $f_\gamma = f_{\gamma'}$. If γ and γ' were to differ then there would be some orbit $\psi \in \Psi_2$ for which $\gamma_2(\psi) \neq \gamma_2'(\psi)$. This would imply that $f_\gamma \neq f_{\gamma'}$, so it must be that $\zeta(\gamma) = \zeta(\gamma')$ implies that $\gamma = \gamma'$. That is, ζ is injective.

[1] We argue this in the proof of Theorem 17.

The map ζ is also surjective. Given a sign function $\lambda \in \operatorname{Sgn}(\mathbf{G})$, $\psi \in \Psi_2$, and $\beta_2(\psi) = \{a, b\}$ define $\gamma_2 \colon \Psi_2 \to G$ by

$$\gamma_2(\psi) \coloneqq \begin{cases} a & \text{when } \lambda(\{b^{-1}a, a^{-1}b\}) = a^{-1}b \\ b & \text{when } \lambda(\{b^{-1}a, a^{-1}b\}) = b^{-1}a \end{cases}.$$

We claim that for this choice of γ_2 we have $\zeta(\gamma) = \lambda$. To see this, observe that given $\{x, x^{-1}\}$ with $\psi = \operatorname{Orb}(\{x, e\})$ and $\beta_2(\psi) = \{a, b\}$ we have that $\{x, e\} = c\{a, b\}$ for some $c \in G$. Suppose that $x = ca$ and $e = cb$. We have that $c = b^{-1}$ and $x = b^{-1}a$. It follows that $(\zeta(\gamma))(\{x, x^{-1}\}) = x$ if and only if $f_\gamma(x, e) = e$, which occurs if and only if $f_\gamma(b^{-1}a, e) = e$. This is equivalent to having that $f_\gamma(a, b) = b$, which occurs when $\lambda(\{b^{-1}a, a^{-1}b\}) = b^{-1}a$. That is, $(\zeta(\gamma))(\{x, x^{-1}\}) = x$ when $\lambda(\{x, x^{-1}\}) = x$.

If instead we take $x = cb$ and $e = ca$ we find that $c = a^{-1}$ so $x = a^{-1}b$ and our analysis is identical to that given previously. We find that $\zeta(\gamma) = \lambda$, so ζ is indeed surjective and hence a bijection. □

This lemma says that in the case of binary regular RPS magmas we can always work with choice functions from $\operatorname{Sgn}(\mathbf{G})$ rather than with the more unwieldy collection of maps $\operatorname{Chir}(\mathbf{G}, \beta)$. Fixing a choice of β we write $\mathbf{G}_2(\lambda)$ to indicate $\mathbf{G}_2(\beta, \zeta_\beta^{-1}(\lambda))$. Note that the choice of β is actually irrelevant as $\mathbf{G}_2(\beta, \zeta_\beta^{-1}(\lambda)) = \mathbf{G}_2(\beta', \zeta_{\beta'}^{-1}(\lambda))$ for any β and β'. That is, $\mathbf{G}_2(\lambda)$ is well-defined without reference to a particular choice of orbit representatives β.

We are now ready to prove Theorem 9.

Proof. We show that every finite tournament embeds into a finite regular balanced tournament. Since we know that the finite tournament magmas generate $\mathcal{T}_2$ this will show that the finite regular RPS magmas generate $\mathcal{T}_2$.

Let $\mathbf{T} \coloneqq (T, \tau)$ be a finite tournament. Consider the group $\mathbf{G} \coloneqq \bigoplus_{u \in T} \mathbb{Z}_{\alpha_u}$ where each α_u is an odd natural other than 1. Identify each $u \in T$ with a generator of $\mathbb{Z}_{\alpha_u}$. Define a choice function $\lambda \in \operatorname{Sgn}(\mathbf{G})$ as follows. Given distinct vertices $u, v \in T$ with $(u, v) \in \tau$ we define $\lambda(\{v - u, u - v\}) \coloneqq u - v$. The function λ may take on any values for other pairs $\{x, -x\}$. By the previous lemma λ yields a regular RPS magma $\mathbf{G}_2(\lambda)$. It follows that $\mathbf{T}$ embeds into the corresponding regular balanced tournament. □

This result has a few notable specializations.

Corollary 1. *Every finite tournament of order m is a subgraph of a finite regular balanced tournament of order 3^m.*

Proof. Take each $\alpha_u \coloneqq 3$ in the previous argument. □

This graph-theoretic statement has an algebraic analogue.

Corollary 2. *The finite regular* RPS *magmas of the form $(\mathbb{Z}_3^m)_2(\lambda)$ generate $\mathcal{T}_2$.*

Proof. Again, take each $\alpha_u \coloneqq 3$ in the previous argument. □

Instead of working with regular RPS magmas obtained from the groups $\mathbb{Z}_3^m$ we may instead work with only the regular RPS magmas obtained from certain cyclic groups.

Corollary 3. *Let p_k indicate the k^{th} prime and let $\alpha(m) := \prod_{k=2}^{m+1} p_k$. The finite regular* RPS *magmas of the form $(\mathbb{Z}_{\alpha(m)})_2(\lambda)$ generate $\mathcal{T}_2$.*

Proof. Order the $u \in T$ as $\{u_i\}_{i=1}^m$ and set $\alpha_{u_i} := p_{i+1}$. We have that $\mathbf{G} = \bigoplus_{i=1}^m \mathbb{Z}_{p_{k+1}} \cong \mathbb{Z}_{\alpha(m)}$. □

Since we know the relationship between $\mathcal{T}_2$ and $\mathcal{R}_2$ we turn to the variety $\mathcal{P}_2$. We know that $\mathcal{P}_2 \models xy \approx yx$ so $\mathcal{P}_2$ has a nontrivial equational theory. It turns out that $\mathcal{T}_2$ is properly contained in $\mathcal{P}_2$.

Theorem 10. *We have $\mathcal{T}_2 < \mathcal{P}_2$.*

Proof. Since $\mathrm{RPS}_2 \subset \mathrm{PRPS}_2$ and $\mathcal{T}_2 = \mathcal{R}_2$ we have that $\mathcal{T}_2 \leq \mathcal{P}_2$. As $\mathcal{T}_2 \models (x(yz))z \approx x((xy)z)$[5, p.115] it suffices to exhibit a member of $\mathcal{P}_2$, such as a member of PRPS_2, which does not model this identity. Consider the magma $\mathbf{A}_\sigma$ described previously. Observe that

$$(r(ps))s = (rr)s = ps = r \neq p = rr = r(ss) = r((rp)s)$$

so indeed $\mathbf{A}_\sigma \not\models (x(yz))z \approx x((xy)z)$. □

Analysis of $\mathcal{P}_2$ would be aided by a combinatorial object playing the role of a balanced tournament. Although we will not pursue the structure of $\mathcal{P}_2$ any further in this paper, the next subsection deals with a relevant area of combinatorics.

4.2. **Hypertournaments.** In the previous subsection we dealt primarily with binary RPS magmas, which we discovered generate the variety $\mathcal{T}_2$ generated by tournament magmas. There is a natural n-ary generalization of these concepts. To wit, the following object may be considered in hypergraph theory.

Definition 14 (Pointed hypergraph)**.** A *pointed hypergraph* $\mathbf{S} := (S, \sigma, g)$ consists of a hypergraph (S, σ) and a map $g : \sigma \to S$ such that for each edge $e \in \sigma$ we have that $g(e) \in e$. The map g is called a *pointing* of (S, σ).

We make use of the following family of hypergraphs.

Definition 15 (n-complete hypergraph)**.** Given a set S we denote by $\mathbf{S}_n$ the *n-complete hypergraph* whose vertex set is S and whose edge set is $\bigcup_{k=1}^n \binom{S}{k}$.

Alternatively in the nondegenerate case that $|S| \geq n$ the n-complete hypergraph on S is the $(n-1)$-skeleton of the simplex with vertex set S viewed as an abstract simplicial complex.

Definition 16 (Hypertournament)**.** An *n-hypertournament* is a pointed hypergraph $\mathbf{T} := (T, \tau, g)$ where $(T, \tau) = \mathbf{S}_n$ for some set S.

Strictly speaking our definition of n-hypertournament does not generalize the usual definition of a tournament, but the $n = 2$ case corresponds to a tournament in an obvious way. We can think of the vertices of an n-hypertournament as players where for each collection of at most n players those players participate in a game for which a single winner is decided. Given an n-hypertournament $\mathbf{T} := (T, \tau, g)$, $V = \{v_1, \ldots, v_k\} \in \binom{T}{k}$ for $1 \leq k \leq n-1$, and $u \in T$ such that $g(\{u, v_1, \ldots, v_k\}) = u$ we say that u *dominates* V and write $u \to V$ to indicate this.

Our definition of hypertournament differs from that of Surmacs[10], for example. The benefit of our definition is that there is a natural algebraic formulation of this concept.

Definition 17 (Hypertournament magma)**.** Given an n-hypertournament $\mathbf{T} \coloneqq (T, \tau, g)$ the *hypertournament magma* obtained from $\mathbf{T}$ is the n-magma $\mathbf{A} \coloneqq (T, f)$ where for $u_1, \ldots, u_n \in T$ we define

$$f(u_1, \ldots, u_n) \coloneqq g(\{u_1, \ldots, u_n\}).$$

Note that hypertournament magmas are precisely those magmas which are conservative and essentially polyadic, allowing us to give the following equivalent definition.

Definition 18 (Hypertournament magma)**.** A *hypertournament magma* is an n-magma which is conservative and essentially polyadic.

Recall that we defined RPS_n to be the class of n-ary RPS magmas and PRPS_n to be the class of n-ary PRPS magmas. Let Tour_n denote the class of all n-ary hypertournament magmas. These classes satisfy the same containment relationships we had for RPS_2, PRPS_2, and Tour_2.

Theorem 11. *Let $n > 1$. We have that $\mathrm{RPS}_n \subsetneq \mathrm{PRPS}_n$, $\mathrm{RPS}_2 \subsetneq \mathrm{Tour}_n$, and neither of PRPS_n and Tour_n contains the other. Moreover, $\mathrm{RPS}_n = \mathrm{PRPS}_n \cap \mathrm{Tour}_n$.*

Proof. Given an RPS n-magma $\mathbf{A} \coloneqq (A, f)$ and some $\sigma \in \mathrm{Perm}(A)$ we can define a PRPS n-magma $\mathbf{A}_\sigma \coloneqq (A, f_\sigma)$ where for $x_1, \ldots, x_n \in A$ we set $f_\sigma(x_1, \ldots, x_n) \coloneqq \sigma(f(x, y))$. As before the resulting magma $\mathbf{A}_\sigma$ will always belong to PRPS_n but will not in general belong to RPS_n. As by definition we have $\mathrm{RPS}_n \subset \mathrm{PRPS}_n$ this shows that $\mathrm{RPS}_n \subsetneq \mathrm{PRPS}_n$.

We also have by definition that $\mathrm{RPS}_n \subset \mathrm{Tour}_n$. Consider the magma $\mathbf{A} \coloneqq (\{a, b\}, f)$ where given $x \in A^n$ we define $f(x) \coloneqq a$ when at least one component of x is a and $f(x) \coloneqq b$ otherwise. We have that $\mathbf{A}$ belongs to Tour_n but does not belong to RPS_n so $\mathrm{RPS}_n \subsetneq \mathrm{Tour}_n$.

We have immediately from the definitions that $\mathrm{RPS}_n = \mathrm{PRPS}_n \cap \mathrm{Tour}_n$. □

Given $n > 1$ we have that PRPS_n, RPS_n, and Tour_n are not varieties. We have seen previously that PRPS_n is not closed under taking subalgebras and RPS_n is as far from belong closed under products as possible. In order to see that Tour_n is not a variety recall that Tour_2 is not closed under taking products. Given a pair of magmas $\mathbf{A} \coloneqq (A, f)$ and $\mathbf{B} \coloneqq (B, g)$ in Tour_2 such that $\mathbf{A} \times \mathbf{B} \notin \mathrm{Tour}_2$ construct magmas $\mathbf{A}' \coloneqq (A, f')$ and $\mathbf{B}' \coloneqq (B, g')$ in Tour_n such that $f'(x_1, \ldots, x_n) = f(u, v)$ when $\{x_1, \ldots, x_n\} = \{u, v\}$ and similarly for g'. Since $\mathbf{A} \times \mathbf{B}$ is not in Tour_2 it cannot be the case that $\mathbf{A}' \times \mathbf{B}'$ belongs to Tour_n.

Let $\mathcal{T}_n \coloneqq \mathbf{V}(\mathrm{Tour}_n)$ be the variety generated by all n-hypertournaments, let $\mathcal{P}_n \coloneqq \mathbf{V}(\mathrm{PRPS}_n)$ be the variety generated by all n-ary PRPS magmas, and let $\mathcal{R}_2 \coloneqq \mathbf{V}(\mathrm{RPS}_n)$ be the variety generated by all n-ary RPS magmas. We generalize Theorem 9 to the n-ary case.

Theorem 12. *Let $n > 1$. We have that $\mathcal{T}_n = \mathcal{R}_n$. Moreover $\mathcal{T}_n$ is generated by the class of finite regular RPS_n magmas.*

In order to make this argument we refer to the hypergraph-theoretic analogue of regular RPS magmas.

Definition 19 (Regular balanced hypertournament)**.** A *regular balanced hypertournament* is a hypertournament $\mathbf{T} \coloneqq (T, \tau, g)$ such that there exists a regular RPS magma $\mathbf{G}_n(\beta, \gamma)$ where $G = T$ and $g(u_1, \ldots, u_n) = u_i$ when $f(u_1, \ldots, u_n) = u_i$ in $\mathbf{G}_n(\beta, \gamma)$.

We generalize the alternative characterization of regular RPS magmas given previously.

Definition 20 ((β, n)-chirality)**.** Given $n > 1$ and a nontrivial finite group $\mathbf{G}$ with $n < \varpi(|G|)$ let Ψ_k denote the collection of orbits of $\binom{G}{k}$ under the k-extension of the left-multiplication action of $\mathbf{G}$ on itself. Given a sequence of choice functions $\beta = \{\beta_1, \dots, \beta_n\}$ where $\beta_k \colon \Psi_k \to \binom{G}{k}$ we say that a sequence of maps $\gamma = \{\gamma_1, \dots, \gamma_n\}$ such that $\gamma_k \colon \Psi_k \to G$ with $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$ is a (β, n)-*chirality* of $\mathbf{G}$. We denote by $\mathrm{Chir}_n(\mathbf{G}, \beta)$ the collection of all (β, n)-chiralities of $\mathbf{G}$.

For each fixed (β, n) the members γ of $\mathrm{Chir}_n(\mathbf{G}, \beta)$ are precisely the data for each possible regular n-ary RPS magma obtained from $\mathbf{G}$.

Definition 21 (Obverse k-set)**.** Given $n > 1$, a nontrivial finite group $\mathbf{G}$ with $n < \varpi(|G|)$, $1 \le k \le n-1$, and $U, V \in \binom{G \setminus \{e\}}{k}$ we say that V is an *obverse* of U when $U = \{a_1, \dots, a_k\}$ and there exists some $a_i \in U$ such that $V = \left\{ a_i^{-1} \right\} \cup \left\{ a_i^{-1} a_j \,\middle|\, i \neq j \right\}$. We denote by $\mathrm{Obv}(U)$ the set consisting of all obverses V of U, as well as U itself.

Note that any $U \in \binom{G \setminus \{e\}}{k}$ has exactly k distinct obverses since given any obverse V of U we have that $(V \cup \{e\}) = a_i^{-1}(U \cup \{e\})$ and the left-multiplication action of $\mathbf{G}$ on itself extends to a free action on $\binom{G}{k+1}$ for $1 \le k \le n-1$. That is, the obverses of U are the nonidentity elements in the members of $\mathrm{Orb}(U \cup \{e\}) \setminus (U \cup \{e\})$ which contain e.

Definition 22 (n-sign function)**.** Given $n > 1$ and a nontrivial group $\mathbf{G}$ with $n < \varpi(|G|)$ let $\mathrm{Sgn}_n(\mathbf{G})$ denote the set of all choice functions on

$$\left\{ \mathrm{Obv}(U) \,\middle|\, (\exists k \in \{1, \dots, n-1\}) \left(U \in \binom{G \setminus \{e\}}{k} \right) \right\}.$$

We refer to a member $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ as an n-*sign function* on $\mathbf{G}$.

We can turn a (β, n)-chirality into an n-sign function.

Definition 23 ((β, n)-signor)**.** Define the (β, n)-*signor* $\zeta_{\beta,n} \colon \mathrm{Chir}_n(\mathbf{G}, \beta) \to \mathrm{Sgn}_n(\mathbf{G})$ as follows. Let f_γ denote the basic operation of $\mathbf{G}_n(\beta, \gamma)$. Given $\gamma \in \mathrm{Chir}_n(\mathbf{G}, \beta)$ and $U = \{a_1, \dots, a_k\} \in \binom{G \setminus \{e\}}{k}$ for some $1 \le k \le n-1$ define

$$(\zeta_{\beta,n}(\gamma))(\mathrm{Obv}(U)) \coloneqq U$$

when

$$f_\gamma(e, \dots, e, a_1, \dots, a_k) = e.$$

Just as in the binary case we choose from among U and its obverses the subset V of $G \setminus \{e\}$ such that e dominates V.

For a particular choice of β the n-sign functions on a group $\mathbf{G}$ are equivalent to the (β, n)-chiralities.

Lemma 4. *The map* $\zeta_{\beta,n} \colon \mathrm{Chir}_n(\mathbf{G}, \beta) \to \mathrm{Sgn}_n(\mathbf{G})$ *is a bijection.*

Proof. Let $\zeta \coloneqq \zeta_{\beta,n}$. Suppose that $\gamma, \gamma' \in \mathrm{Chir}_n(\mathbf{G}, \beta)$ with $\zeta(\gamma) = \zeta(\gamma')$. A regular RPS magma operation is determined by choosing for each $1 \le k \le n$ and each orbit of $\binom{G}{k}$ the k-set U for which $g(U) = e$. Having that $\zeta(\gamma) = \zeta(\gamma')$ says that these choices are the same for both $\mathbf{G}_n(\beta, \gamma)$ and $\mathbf{G}_n(\beta, \gamma')$. Taking f_γ and $f_{\gamma'}$ to be the basic operations of $\mathbf{G}_n(\beta, \gamma)$ and $\mathbf{G}_n(\beta, \gamma')$, respectively, it follows that $f_\gamma = f_{\gamma'}$. If γ and γ' were to differ then there would be some orbit $\psi \in \Psi_k$ for some $1 \leq k \leq n$ for which $\gamma_k(\psi) \neq \gamma'_k(\psi)$. This would imply that $f_\gamma \neq f_{\gamma'}$, so it must be that $\zeta(\gamma) = \zeta(\gamma')$ implies that $\gamma = \gamma'$. That is, ζ is injective.

The map ζ is also surjective. Given an n-sign function $\lambda \in \mathrm{Sgn}(\mathbf{G})$, $\psi \in \Psi_k$, and $\beta_k(\psi) = \{a_1, \ldots, a_k\}$ set $U \coloneqq \left\{ a_1^{-1} a_i \;\middle|\; i \neq 1 \right\}$ and define $\gamma_k \colon \Psi_k \to G$ by $\gamma_k(\psi) \coloneqq a_1$ when $\lambda(\mathrm{Obv}(U)) = U$. We claim that for this choice of γ_k for each k we have $\zeta(\gamma) = \lambda$. To see this, observe that given $V = \{b_1, \ldots, b_{k-1}\} \in \binom{G \setminus \{e\}}{k-1}$ for $1 \leq k \leq n$ with $\psi = \mathrm{Orb}(\{e, b_1, \ldots, b_{k-1}\})$ and $\beta_k(\psi) = \{a_1, \ldots, a_k\}$ we have that $\{e, b_1, \ldots, b_{k-1}\} = c\{a_1, \ldots, a_k\}$ for some $c \in G$. Suppose that $e = ca_1$ and $b_i = ca_{i+1}$ for $1 \leq i \leq k-1$. We have that $c = a_1^{-1}$ and $b_i = a_1^{-1} a_{i+1}$. Taking $U = \left\{ a_1^{-1} a_i \;\middle|\; i \neq 1 \right\}$ it follows that $(\zeta(\gamma))(\mathrm{Obv}(U)) = U$ if and only if

$$f_\gamma(e, \ldots, e, a_1^{-1} a_2, \ldots, a_1^{-1} a_k) = e,$$

which occurs if and only if

$$f_\gamma(e, \ldots, e, a_1, \ldots, a_k) = a_1.$$

This is equivalent to having that $\lambda(\mathrm{Obv}(U)) = U$ by definition of γ. We find that $\zeta(\gamma) = \lambda$, so ζ is indeed surjective and hence a bijection. $\square$

This lemma says that for n-ary regular RPS magmas we can always work with choice functions from $\mathrm{Sgn}_n(\mathbf{G})$ rather than $\mathrm{Chir}_n(\mathbf{G}, \beta)$. For any β and β' we have that $\mathbf{G}_n(\beta, \zeta_{\beta,n}^{-1}(\lambda)) = \mathbf{G}_n(\beta', \zeta_{\beta',n}^{-1}(\lambda))$ so as in the binary case $\mathbf{G}_n(\lambda)$ is well-defined without reference to a particular choice of orbit representatives β.

We give the proof of Theorem 12.

Proof. Since n-hypertournament magmas are conservative we have that $\mathrm{Tour}_n \models \epsilon$ for some identity ϵ in m variables if and only if each n-hypertournament magma of order m models ϵ. Since $\mathcal{T}_n = \mathbf{V}(\mathrm{Tour}_n) = \mathrm{Mod}(\mathrm{Id}(\mathrm{Tour}_n))$ we have that $\mathcal{T}_n$ consists of all models of those identities which hold in the finite n-hypertournament magmas. We find that $\mathcal{T}_n$ is generated by the finite members of Tour_n. This is the same argument as the classical one given in the binary case.

We show that every finite n-hypertournament embeds into a finite regular balanced hypertournament. Since we know that the finite hypertournaments generate $\mathcal{T}_n$ this will establish that the finite regular balanced hypertournaments alone generate $\mathcal{T}_n$.

Let $\mathbf{T} \coloneqq (T, \tau, g)$ be a finite hypertournament. Consider the group $\mathbf{G} \coloneqq \bigoplus_{u \in T} \mathbb{Z}_{\alpha_u}$ where for each u we have that α_u is a natural other than 1 and $n < \varpi(\alpha_u)$. Identify $u \in T$ with a generator of $\mathbb{Z}_{\alpha_u}$. Define an n-sign function $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ as follows. Given distinct vertices $u_1, \ldots, u_k \in T$ for some $1 \leq k \leq n$ where $g(\{u_1, \ldots, u_k\}) = u_1$ we define

$$\lambda(\mathrm{Obv}(\{\, u_i - u_1 \mid i \neq 1 \,\})) \coloneqq \{\, u_i - u_1 \mid i \neq 1 \,\}.$$

The function λ may take on any values for other arguments. By the previous lemma λ yields a regular RPS magma $\mathbf{G}_n(\lambda)$. It follows that $\mathbf{T}$ embeds into the corresponding regular balanced hypertournament.

Since the variety $\mathcal{T}_n$ is generated by the finite regular members of RPS_n we have that $\mathcal{T}_n \leq \mathcal{R}_n$. Conversely we have that $\mathrm{RPS}_n \subset \mathrm{Tour}_n$ so $\mathcal{R}_n \leq \mathcal{T}_n$. We find that $\mathcal{T}_n = \mathcal{R}_n$. $\square$

Our corollaries also have higher-arity analogues.

Corollary 4. *Every finite n-hypertournament of order m is contained in a finite regular balanced hypertournament of order $\kappa(n)^m$ where $\kappa(n)$ is the least prime strictly greater than n.*

Proof. Take each $\alpha_u \coloneqq \kappa(n)$ in the previous argument. □

Corollary 5. *The finite regular* RPS *magmas of the form $(\mathbb{Z}^m_{\kappa(n)})(\lambda)$ generate $\mathcal{T}_n$.*

Proof. Again, take each $\alpha_u \coloneqq \kappa(n)$ in the previous argument. □

Corollary 6. *Let p_k indicate the k^{th} prime and suppose that $\kappa(n) = p_\ell$. Define $\alpha(m) \coloneqq \prod_{k=\ell}^{m+\ell-1} p_k$. The finite regular* RPS *magmas of the form $(\mathbb{Z}_{\alpha(m)})_n(\lambda)$ generate $\mathcal{T}_n$.*

Proof. Order the $u \in T$ as $\{u_i\}_{i=1}^m$ and let $\alpha_{u_i} \coloneqq p_{i+\ell-1}$. We have that $\mathbf{G} = \bigoplus_{i=1}^m \mathbb{Z}_{p_{i+\ell-1}} \cong \mathbb{Z}_{\alpha(m)}$. □

The final result of the previous section is easier to generalize.

Theorem 13. *We have $\mathcal{T}_n < \mathcal{P}_n$ for $n > 1$.*

Proof. Since $\mathrm{RPS}_n \subset \mathrm{PRPS}_n$ and $\mathcal{T}_n = \mathcal{R}_n$ we have that $\mathcal{T}_n \leq \mathcal{P}_n$. Given $\mathbf{A} \in \mathrm{RPS}_n$ with $\mathbf{A} \coloneqq (A, f)$ we have that $\mathbf{A}' \in \mathrm{RPS}_2$ where $\mathbf{A}' \coloneqq (A, f')$ is given by $f'(x,y) \coloneqq f(x,y,\ldots,y)$. Let $\alpha(x,y)$ denote the term $f(x,y,\ldots,y)$. It follows that since $\mathcal{T}_2 \models (x(yz))z \approx x((xy)z)$ we have that

$$\mathcal{T}_n \models \alpha(\alpha(x,\alpha(y,z)),z) \approx \alpha(x,\alpha(\alpha(x,y),z)).$$

Let $\mathbf{A} \coloneqq (\mathbb{Z}_{\kappa(n)})_n(\lambda)$ for some n-sign function λ. Note that $\mathbf{A}'$ must contain a copy of the canonical $\mathrm{RPS}(3,2)$ magma, for given two distinct elements u and v with $u \to v$ we have that u is dominated by $\frac{\kappa(n)-1}{2}$ elements other than u and v while v dominates $\frac{\kappa(n)-1}{2}$ elements other than u and v. There then exists some w with $u \to v \to w \to u$. Permute the outputs of the basic operation of $\mathbf{A}$ to obtain an algebra $\mathbf{A}_\sigma$ as in the argument for the binary case. The resulting n-magma is a PRPS_n magma which contains a copy of the three element PRPS_2 magma which failed to satisfy the identity in question. □

5. Counting RPS and PRPS Magmas

Given an order m and an arity n we determine the number of $\mathrm{PRPS}(m,n)$ magmas of a given order on a fixed set of size m. We denote the number of $\mathrm{PRPS}(m,n)$ magmas on a fixed set of size m by $|\mathrm{PRPS}(m,n)|$. We subsequently count how many regular RPS magmas $\mathbf{G}_n(\lambda)$ exist given a finite group $\mathbf{G}$. We denote the number of such magmas by $|\mathrm{RPS}(\mathbf{G},n)|$. Finally, we count the total number of RPS n-magmas on a fixed set of size m, which we denote by $|\mathrm{RPS}(m,n)|$.

In order to perform these counts we must determine the number of regular partitions of $\binom{A}{k}$ into m subsets of size $\frac{1}{m}|A|$. That is, we must give the amount $\mathcal{B}(m,k)$ of regular partitions of a set of size $\left|\binom{A}{k}\right| = \binom{m}{k}$ into m subsets of size $\frac{1}{m}\binom{m}{k}$. In order to do this we give the amount $\mathscr{P}(m,s)$ of regular partitions of a set of size s into m subsets of size $\frac{s}{m}$.

Lemma 5. *Let $m, s \in \mathbb{N}$ with $m \mid s$. We have that*

$$\mathscr{P}(m,s) = \frac{1}{m!}\prod_{\ell=0}^{m-1}\binom{s-\ell\frac{s}{m}}{\frac{s}{m}}.$$

Proof. Let X be a set of size s. By assumption $m \mid s$ so we can produce a regular partition of X into m subsets. We do this in the following manner. First choose $\frac{s}{m}$ elements from the s elements in X to form the first class of the partition, then choose $\frac{s}{m}$ elements from the $s - \frac{s}{m}$ elements in X not already chosen to form the second class of the partition, and so on. There are

$$\prod_{\ell=0}^{m-1} \binom{s - \ell\frac{s}{m}}{\frac{s}{m}}$$

ways to specify a partition in this way. Since partitions are unordered we have counted each regular partition of X exactly $m!$ times, once for each order we can place on the members of a partition. The result follows. □

The following special case will be relevant to us.

Corollary 7. *Let $m, k \in \mathbb{N}$ and suppose that $m \mid \binom{m}{k}$. We have that*

$$\mathcal{B}(m,k) = \frac{1}{m!} \prod_{\ell=0}^{m-1} \binom{\binom{m}{k} - \ell\frac{1}{m}\binom{m}{k}}{\frac{1}{m}\binom{m}{k}}.$$

Proof. Take $s = \binom{m}{k}$ in the previous lemma. □

We are now ready to compute $|\mathrm{PRPS}(m,n)|$.

Theorem 14. *Let $m, n \in \mathbb{N}$ with $m \neq 1$ and $n < \varpi(m)$. We have that*

$$|\mathrm{PRPS}(m,n)| = \prod_{k=1}^{n} \prod_{\ell=0}^{m-1} \binom{\binom{m}{k} - \ell\frac{1}{m}\binom{m}{k}}{\frac{1}{m}\binom{m}{k}}.$$

Proof. Let A be a fixed set with m elements. The members of $\mathrm{PRPS}(m,n)$ with A as their universe are in bijection with regular ordered partitions of $\binom{A}{k}$ into m subsets for $1 \leq k \leq n$. That is, in order to obtain a member of $\mathrm{PRPS}(m,n)$ on A first choose for each $1 \leq k \leq n$ a partition of $\binom{A}{k}$ into m subsets. There are $\mathcal{B}(m,k)$ ways to do this for any given k. Choose an order on the selected partitions. Since our partitions are partitions of $\binom{A}{k}$ into m subsets there are $m!$ ways to order each partition. This ordering corresponds to assigning to each equivalence class an element of A. It follows that there are a total of

$$\begin{aligned}\prod_{k=1}^{n} m!\,\mathcal{B}(m,k) &= \prod_{k=1}^{n} m!\left(\frac{1}{m!}\prod_{\ell=0}^{m-1}\binom{\binom{m}{k} - \ell\frac{1}{m}\binom{m}{k}}{\frac{1}{m}\binom{m}{k}}\right)\\ &= \prod_{k=1}^{n}\prod_{\ell=0}^{m-1}\binom{\binom{m}{k} - \ell\frac{1}{m}\binom{m}{k}}{\frac{1}{m}\binom{m}{k}}\end{aligned}$$

ways to make these choices. Each such choice corresponds to the basic operation of a $\mathrm{PRPS}(m,n)$ magma and each $\mathrm{PRPS}(m,n)$ with universe A is uniquely described by such a choice. □

The value of $|\mathrm{RPS}(\mathbf{G},n)|$ is an even more direct computation.

Theorem 15. *Let $m, n \in \mathbb{N}$ with $m \neq 1$ and $n < \varpi(m)$. Given a group $\mathbf{G}$ of order m we have that*

$$|\mathrm{RPS}(\mathbf{G},n)| = \prod_{k=1}^{n} k^{\frac{1}{m}\binom{m}{k}}.$$

Proof. Observe that for each $1 \leq k \leq n$ we have that $\binom{G}{k}$ breaks up into $\frac{1}{m}\binom{m}{k}$ orbits under the action of $\mathbf{G}$. Fix a choice of orbit representatives β. For each orbit ψ of $\binom{G}{k}$ we must choose one of the k elements of $\beta_k(\psi)$ to be $\gamma_k(\psi)$. For a given k there are thus $k^{\frac{1}{m}\binom{m}{k}}$ ways to make this choice, yielding a total of

$$\prod_{k=1}^{n} k^{\frac{1}{m}\binom{m}{k}}$$

choices for $\mathbf{G}_n(\beta, \gamma)$, as claimed. □

It seems to be more challenging to give an elementary formula for $|\mathrm{RPS}(m, n)|$. Given a polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$ and a monomial $x \coloneqq x_1^{s_1} \cdots x_n^{s_n}$ we write $\mathcal{C}(x, p)$ to indicate the coefficient of x in p. Given a finite set $S = \{z_1, \ldots, z_s\}$ of variables denote by $\sum S$ the polynomial $z_1 + \cdots + z_s$.

Theorem 16. *Let $m, n \in \mathbb{N}$ with $m \neq 1$ and $n < \varpi(m)$. Let $A \coloneqq \{x_1, \ldots, x_m\}$, let $y_k \coloneqq (x_1 \cdots x_m)^{\frac{1}{m}\binom{m}{k}}$, and define $p_k \coloneqq \prod_{S \in \binom{A}{k}} \sum S$. We have that*

$$|\mathrm{RPS}(m, n)| = \prod_{k=1}^{n} \mathcal{C}(y_k, p_k).$$

Proof. Each $\mathrm{RPS}(m, n)$ magma with universe A corresponds to a pointing g of $\mathbf{A}_n$, the n-complete hypergraph with vertex set A, such that $\left|g^{-1}(x_i) \cap \binom{A}{k}\right| = \left|g^{-1}(x_j) \cap \binom{A}{k}\right|$ for each i and j. Fixing some k, the number of such ways to choose an element from each k-set in A is $\mathcal{C}(y_k, p_k)$ as there is one factor of p_k for each k-set in A and we require that each x_i is chosen exactly $\frac{1}{m}\binom{m}{k}$ times from among these factors by taking the y_k coefficient. Taking the product we obtain the total number of ways to choose a pointing g. □

This result is the n-ary generalization of a formula given without a literature citation on the On-Line Encyclopedia of Integer Sequences for the number of labeled balanced tournaments[1], which is also the quantity $|\mathrm{RPS}(m, 2)|$. An asymptotic formula has been obtained via analytic methods[8].

6. Automorphisms of Regular RPS Magmas

We describe the automorphism groups of regular RPS magmas.

Theorem 17. *Let $\mathbf{A} \coloneqq \mathbf{G}_n(\beta, \gamma)$ be a regular* RPS *magma. There is a canonical embedding $h \colon \mathbf{G} \hookrightarrow \mathbf{Aut}(\mathbf{A})$.*

Proof. Let $h \colon \mathbf{G} \to \mathbf{Aut}(\mathbf{A})$ be given by $h(s) \coloneqq L_s$ where $L_s \colon G \to G$ is the left-multiplication map. We claim that h is an embedding.

We first show that $h(s) \in \mathrm{Aut}(\mathbf{A})$. Let $a_1, \ldots, a_n \in A$ with $U \coloneqq \{a_1, \ldots, a_n\} \in \psi \in \Psi_k$ where $U = t\beta_k(\psi)$. We have that $f(a_1, \ldots, a_n) = t\gamma_k(\psi)$. It follows that $\{h(s)(a_1), \ldots, h(s)(a_n)\} = \{sa_1, \ldots, sa_n\} = sU \in \psi \in \Psi_k$ with $sU = st\beta_k(\psi)$. This implies that

$$f(h(s)(a_1), \ldots, h(s)(a_n)) = st\gamma_k(\psi) = sf(a_1, \ldots, a_n) = h(s)(f(a_1, \ldots, a_n)).$$

Since $L_s \colon G \to G$ is always a bijection we have that $h(s)$ is an automorphism of $\mathbf{A}$.

In order to see that h is a group homomorphism note that h is obtained by restricting the codomain of the Cayley embedding of $\mathbf{G}$ into $\mathbf{Perm}(G)$ via $s \mapsto L_s$. Thus, we can always view $\mathbf{G}$ as a subgroup of $\mathbf{Aut}(\mathbf{A})$. □

We would like to know whether there are any other automorphisms of regular RPS magmas. By counting we know that there must be regular RPS magmas with automorphism groups larger than $\mathbf{G}$.

Theorem 18. *For each arity $n \in \mathbb{N}$ with $n \neq 1$ and each order $m \in \mathbb{N}$ with $n < \varpi(m)$ there exists a regular $\mathrm{RPS}(m,n)$ magma $\mathbf{A} \coloneqq \mathbf{G}_n(\lambda)$ such that $|\mathbf{Aut}(\mathbf{A})| > |\mathbf{G}|$.*

Proof. Suppose towards a contradiction that for each magma $\mathbf{A} \coloneqq \mathbf{G}_n(\lambda)$ we have that $\mathbf{Aut}(\mathbf{A}) \cong \mathbf{G}$. Each isomorphism class of magmas of the form $\mathbf{G}_n(\beta,\gamma)$ must contribute

$$\frac{|\mathrm{Perm}(G)|}{|G|} = \frac{m!}{m} = (m-1)!$$

to the value of $|\mathrm{RPS}(\mathbf{G},n)|$ so $(m-1)!$ divides $|\mathrm{RPS}(\mathbf{G},n)|$. Since $\varpi(m) \mid (m-1)!$ we have that $\varpi(m) \mid |\mathrm{RPS}(\mathbf{G},n)|$. As $|\mathrm{RPS}(\mathbf{G},n)| = \prod_{k=1}^{n} k^{\frac{1}{m}\binom{m}{k}}$ and each of the factors $k^{\frac{1}{m}\binom{m}{k}}$ is a natural number which is a power of $k \leq n < \varpi(m)$ we have that $\varpi(m) \nmid |\mathrm{RPS}(\mathbf{G},n)|$. This is a contradiction. □

In the case of a group $\mathbf{G}$ with nontrivial inner automorphism group and a particular choice of λ we can explicitly give other automorphisms of $\mathbf{G}_n(\lambda)$.

Theorem 19. *We denote by $c_b\colon G \to G$ the conjugation map given by $c_b(a) \coloneqq bab^{-1}$. Let $\mathbf{A} \coloneqq \mathbf{G}_n(\lambda)$ be a regular RPS magma such that given $U \in \binom{G\setminus\{e\}}{k}$ for $1 \leq k \leq n-1$ we have $\lambda(\mathrm{Obv}(U)) = U$ implies that $\lambda(\mathrm{Obv}(c_b(U))) = c_b(U)$. We have that $c_b \in \mathrm{Aut}(\mathbf{A})$.*

Proof. Let $a_1, \ldots, a_n \in A$ with $\{a_1, \ldots, a_n\} \in \psi \in \Psi_k$. Suppose that $f(a_1, \ldots, a_n) = a_1$. This occurs if and only if $\lambda(\mathrm{Obv}(U)) = U$ where $U \coloneqq \left\{ a_1^{-1}a_i \;\middle|\; i \neq 1 \right\}$. We have that $\lambda(\mathrm{Obv}(U)) = U$ if and only if $\lambda(\mathrm{Obv}(c_b(U))) = c_b(U)$. This in turn is equivalent to having that $f(c_b(a_1), \ldots, c_b(a_n)) = c_b(a_1)$. It follows that $c_b(f(a_1, \ldots, a_n)) = f(c_b(a_1), \ldots, c_b(a_n))$. Thus, c_b is an automorphism of $\mathbf{A}$. □

Those n-sign functions for which all conjugations are automorphisms will be of interest to us.

Definition 24 (Correlated n-sign function)**.** We say that an n-sign function $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ is *correlated* when given any $b \in G$ and any $U \in \binom{G\setminus\{e\}}{k}$ for $1 \leq k \leq n-1$ we have that $\lambda(\mathrm{Obv}(U)) = U$ implies that $\lambda(\mathrm{Obv}(c_b(U))) = c_b(U)$.

Corollary 8. *Let $\mathbf{A} \coloneqq \mathbf{G}_n(\lambda)$ for a correlated $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$. We have that $\mathbf{Inn}(\mathbf{G}) \hookrightarrow \mathbf{Aut}(\mathbf{A})$ where $\mathbf{Inn}(\mathbf{G})$ is the inner automorphism group of $\mathbf{G}$.*

Proof. Since the multiplication in both $\mathbf{Inn}(\mathbf{G})$ and $\mathbf{Aut}(\mathbf{A})$ is function composition we have that the inclusion map in question is an embedding of groups. □

7. A Group Action Constraint

Our construction of regular RPS magmas and our result constraining the pairs (m,n) for which $\mathrm{RPS}(m,n)$ magmas may exist can be combined to give a constraint on when the extensions of a regular group action are free.

Theorem 20. *Suppose that $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ is a regular action of a group $\mathbf{G}$ of order $m \neq 1$. The k-extensions α_k of α are all free for $1 \leq k \leq n$ if and only if $n < \varpi(m)$.*

Proof. We already argued that if $n < \varpi(m)$ then each of the α_k are free for $1 \leq k \leq n$ in the proof of Theorem 4 for the case that $A = G$. The argument in the general case is identical.

Conversely, suppose that each of the α_k are free for $1 \leq k \leq n$. By Theorem 3 we have that the corresponding α-action magma is an $\mathrm{RPS}(m, n)$ magma. By Theorem 2 we have that $n < \varpi(m)$. □

This implies that a particular extension is not free.

Corollary 9. *Suppose that $\alpha\colon \mathbf{G} \to \mathrm{Perm}(A)$ is a regular action of a group $\mathbf{G}$ of odd order $m \neq 1$. We have that the $\varpi(m)$-extension $\alpha_{\varpi(m)}$ of α is not free.*

Proof. Since the extensions α_k are free for $1 \leq k \leq \varpi(m) - 1$ and not all of the extensions α_k are free for $1 \leq k \leq \varpi(m)$ it must be that $\alpha_{\varpi(m)}$ is not a free action. □

We now have an alternative proof of a special case of Cauchy's theorem that if $\mathbf{G}$ is a finite group and p is a prime dividing the order of $\mathbf{G}$ then $\mathbf{G}$ has an element of order p.

Corollary 10. *Given a finite group $\mathbf{G}$ of order $m \neq 1$ we have that $\mathbf{G}$ contains an element of order $\varpi(m)$.*

Proof. Take α to be the left-multiplication action of $\mathbf{G}$ on itself. Since $\alpha_{\varpi(m)}$ is not free there exists some $U \subset G$ with $|U| = \varpi(m)$ and elements $s, t \in G$ such that $s \neq t$ yet $((\alpha_{\varpi(m)})(s))(U) = ((\alpha_{\varpi(m)})(t))(U)$. It follows that there exists some $\sigma \in \mathrm{Perm}(U)$ such that $su = t\sigma(u)$ for each $u \in U$. Note that $\sigma \neq \mathrm{id}_U$ since this would imply that $s = t$. We find that $t^{-1}su = \sigma(u)$ for each $u \in U$ so the order of $t^{-1}s$ in $\mathbf{G}$ must divide the order of $\mathbf{Perm}(U)$, which is $(\varpi(m))!$. Since the order $t^{-1}s$ must also divide the order of $\mathbf{G}$ and $s \neq t$ the only possible situation is that the order of $t^{-1}s$ is $\varpi(m)$. □

8. Congruences of Finite Regular RPS Magmas

We study the structure of the congruence lattices of finite regular RPS magmas.

Theorem 21. *Let $\theta \in \mathrm{Con}(\mathbf{A})$ for a regular $\mathrm{RPS}(m, n)$ magma $\mathbf{A} := \mathbf{G}_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $\mathbf{H} \leq \mathbf{G}$.*

Proof. Consider the principal congruence $\theta := \mathrm{Cg}(\{(e, a)\})$ generated by (e, a). Since $\mathbf{A}$ is conservative θ has at most one nontrivial equivalence class, which is the equivalence class containing e and a. We show that this equivalence class must contain $\mathrm{Sg}^{\mathbf{G}}(\{a\})$, the cyclic subgroup of $\mathbf{G}$ generated by a.

In order to do this we note that the map $\eta\colon G \to G$ given by $\eta(x) := x^2$ is a bijection. To see this observe that if $\eta(x) = \eta(y)$ then $x^2 = y^2$. Since $\mathbf{G}$ has odd order, say $2k + 1$, we have that

$$x = x^{2k+2} = (x^2)^{k+1} = (y^2)^{k+1} = y^{2k+2} = y.$$

Since η is an injection from a finite set to itself we have that η is a bijection. It follows that each element $a \in G$ has a unique *square root* $\sqrt{a} := \eta^{-1}(a)$ for which $(\sqrt{a})^2 = a$. Define $a^{\frac{1}{2^r}}$ inductively by $a^{\frac{1}{2}} := \sqrt{a}$ and $a^{\frac{1}{2^r}} := \sqrt{a^{\frac{1}{2^{r-1}}}}$ for $r > 1$.

Previously we wrote $u \to V$ to indicate that u dominates V in a hypertournament. We expand on this notation slightly by writing $u \to v$ rather than $u \to \{v\}$ in the case that $V = \{v\}$ is a singleton. Either $e \to \sqrt{a}$, in which case applying the left-multiplication automorphism $L_{\sqrt{a}}$ for $\sqrt{a}$ yields that $\sqrt{a} \to a$ and we conclude that $(e, \sqrt{a}) \in \theta$ or $\sqrt{a} \to e$, in which case applying $L_{\sqrt{a}}$ yields that $a \to \sqrt{a}$ and we conclude that $(\sqrt{a}, a) \in \theta$. In either case if e/θ contains a then e/θ contains $\sqrt{a}$. It follows that e/θ is closed under taking square roots. Since G is finite we have that e/θ is also closed under squaring. We have that $e/\theta \supset \left\{e, a, a^2\right\}$.

We write $a\theta \coloneqq L_a(\theta)$ for $a \in G$ and $\theta \in \operatorname{Con}(\mathbf{A})$. Suppose inductively that $e/\theta \supset \left\{a^{s-2}, a^{s-1}, a^s\right\}$. We show that $a^{s+1} \in e/\theta$. Since e/θ is an equivalence class of the principal congruence generated by (e, a) applying L_a shows that $a/(a\theta) \supset \left\{a^{s-1}, a^s, a^{s+1}\right\}$ is an equivalence class of the principal congruence generated by (a, a^2). We know that $(a, a^2) \in \theta$ so $a\theta \leq \theta$. Since $a^{s-1}, a^s \in e/\theta$ it follows that $e/\theta \supset \left\{a^{s-1}, a^s, a^{s+1}\right\}$ so $a^{s+1} \in e/\theta$, as desired. We find that e/θ contains $\operatorname{Cg}^{\mathbf{G}}(\{a\})$.

We now know that for any congruence $\theta \in \operatorname{Con}(\mathbf{A})$ we have that e/θ is a union of cyclic subgroups of $\mathbf{G}$. Suppose towards a contradiction that $a, b \in e/\theta$ and $ab \notin e/\theta$. Note that $\theta \geq \operatorname{Cg}(\left\{(e, a), (e, b^{-1})\right\})$. Observe that

$$\begin{aligned}
\operatorname{Cg}(\left\{(e, a), (e, b^{-1})\right\}) &= b^{-1} \operatorname{Cg}(\{(b, ba), (b, e)\}) \\
&\geq b^{-1} \operatorname{Cg}(\{(e, ba)\}) \\
&\geq b^{-1} \operatorname{Cg}(\{(e, baba)\}) \\
&\geq \operatorname{Cg}(\{(e, aba)\})
\end{aligned}$$

so we have that e/θ contains aba.

In order for $ab \notin e/\theta$ we must have that ab either dominates everything in e/θ or ab is dominated by everything in e/θ. In the former case we have that $ab \to aba$. Applying $L_{(ab)^{-1}}$ we find that $e \to a$. Since ab dominates e/θ we have that $(ab)^{-1}$ is dominated by e/θ. It follows that $b^{-1} \to b^{-1}a^{-1}$ so $e \to a^{-1}$. By definition of a regular RPS magma it is impossible to have both $e \to a$ and $e \to a^{-1}$ so we have arrived at a contradiction. The same argument with dominance relations reversed yields a contradiction in the case that ab is dominated by everything in e/θ. This establishes that e/θ is a subset of G containing e which is closed under taking inverses and products. That is, e/θ is a subgroup of $\mathbf{G}$.

Given any $\theta \in \operatorname{Con}(\mathbf{A})$ and some $a \in G$ we have that

$$a/\theta = a(e/(a^{-1}\theta)) = aH$$

for some $\mathbf{H} \leq \mathbf{G}$. □

We characterize those subgroups $\mathbf{H} \leq \mathbf{G}$ for which $a/\theta = aH$ for some $a \in G$.

Definition 25 (λ-convex subgroup)**.** Given a group $\mathbf{G}$, an n-sign function $\lambda \in \operatorname{Sgn}_n(\mathbf{G})$, and a subgroup $\mathbf{H} \leq \mathbf{G}$ we say that $\mathbf{H}$ is λ-*convex* when there exists some $a \in G$ such that $a/\theta = aH$ for some $\theta \in \operatorname{Con}(\mathbf{G}_n(\lambda))$.

Trivially we have that the whole group $\mathbf{G}$ and the trivial subgroup with universe $\{e\}$ are both λ-convex for every λ.

Theorem 22. *Let $\mathbf{G}$ be a finite group of order m and let $n < \varpi(m)$. Take $\lambda \in \operatorname{Sgn}_n(\mathbf{G})$ and $\mathbf{H} \leq \mathbf{G}$. The following are equivalent:*

(1) The subgroup $\mathbf{H}$ is λ-convex.

(2) *There exists a congruence* $\psi \in \mathrm{Con}(\mathbf{G}_n(\lambda))$ *such that* $e/\psi = H$.
(3) *Given* $1 \leq k \leq n-1$ *and* $b_1, \ldots, b_k \notin H$ *either* $e \to \{b_1h_1, \ldots, b_kh_k\}$ *for every choice of* $h_1, \ldots, h_k \in H$ *or* $\{b_1h_1, \ldots, b_kh_k\} \to e$ *for every choice of* $h_1, \ldots, h_k \in H$.

Proof. To see that (1) implies (2) suppose that $\mathbf{H}$ is λ-convex. We have that there exists some $a \in G$ such that $a/\theta = aH$ for some $\theta \in \mathrm{Con}(\mathbf{G}_n(\lambda))$. Applying $L_{a^{-1}}$ we see that $e/(a^{-1}\theta) = H$ so we can take $\psi = a^{-1}\theta$.

To see that (2) implies (3) note that if $e/\psi = H$ then we can define a congruence θ_e where the only nontrivial equivalence class is H. It follows that $\theta \coloneqq \bigcup_{a\in G} a\theta_e$ is a congruence of $\mathbf{G}_n(\lambda)$. Condition (3) is implied by θ having the substitution property.

To see that (3) implies (1) note that if (3) holds then the equivalence relation θ defined previously is a congruence and hence $\mathbf{H}$ is λ-convex. □

The congruence lattice of $\mathbf{G}_n(\lambda)$ is thus determined by those subgroups of $\mathbf{G}$ which are λ-convex. For a fixed λ such subgroups must form a chain.

Theorem 23. *Suppose that* $\mathbf{H}, \mathbf{K} \leq \mathbf{G}$ *are both* λ*-convex. We have that* $\mathbf{H} \leq \mathbf{K}$ *or* $\mathbf{K} \leq \mathbf{H}$.

Proof. Suppose that $\mathbf{H}$ and $\mathbf{K}$ are incomparable. That is, let $h \in H \setminus K$ and let $k \in K \setminus H$. Without loss of generality take $h \to e$ and $k \to e$. We have that $hk^{-1} \to e$ since everything in the coset hK must dominate everything in K in order for K to be λ-convex. It follows that $e \to kh^{-1}$ so everything in H dominates everything in kH. This implies that $e \to k$, a contradiction. □

Note that when λ is correlated we have that every conjugation map on $\mathbf{G}$ is an automorphism of $\mathbf{G}_n(\lambda)$. The preceding result implies that only normal subgroups of $\mathbf{G}$ can be λ-convex in this case, for if $\mathbf{H} \leq \mathbf{G}$ is not normal then $\mathbf{H}$ has at least one conjugate other than itself, say $a\mathbf{H}a^{-1}$. Since $\mathbf{H}$ is λ-convex if and only if $a\mathbf{H}a^{-1}$ is λ-convex for correlated λ and both $\mathbf{H}$ and $a\mathbf{H}a^{-1}$ have the same number of elements neither can be contained in the other.

We can now give the structure of the congruence lattice of $\mathbf{G}_n(\lambda)$.

Definition 26 (λ-coset poset)**.** Given $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ set

$$P_\lambda \coloneqq \{\, aH \mid a \in G \text{ and } \mathbf{H} \text{ is } \lambda\text{-convex} \,\}$$

and define the λ-*coset poset* to be $\mathbf{P}_\lambda \coloneqq (P_\lambda, \subset)$.

Dilworth showed that the maximal antichains of a finite poset form a distributive lattice. We follow Freese's treatment of this[6]. Given a finite poset $\mathbf{P} \coloneqq (P, \leq)$ let $\mathbf{L}(\mathbf{P})$ be the lattice whose elements are maximal antichains in $\mathbf{P}$ where if $U, V \in L(\mathbf{P})$ then we say that $U \leq V$ in $\mathbf{L}(\mathbf{P})$ when for every $u \in U$ there exists some $v \in V$ such that $u \leq v$ in $\mathbf{P}$.

Theorem 24. *We have that* $\mathbf{Con}(\mathbf{G}_n(\lambda)) \cong \mathbf{L}(\mathbf{P}_\lambda)$.

Proof. Define $h\colon \mathrm{Con}(\mathbf{G}_n(\lambda)) \to L(\mathbf{P}_\lambda)$ by $h(\theta) \coloneqq \{\, a/\theta \mid a \in G \,\}$. By our previous work we have that $h(\theta)$ is a subset of P_λ. Note that since θ is an equivalence relation we have that $h(\theta)$ is an antichain and since every member of G must belong to some equivalence class under θ we have that $h(\theta)$ is a maximal antichain.

We have that h is a well-defined map. It remains to show that h is an isotone bijection with isotone inverse. Certainly h is injective, for if $h(\theta) = h(\psi)$ then θ

and ψ determine the same partition of G, which implies that $\theta = \psi$. To see that h is surjective consider some maximal antichain U in $\mathbf{P}_\lambda$. For each element $aH \in U$ define θ_{aH} to be the equivalence relation on G whose only nontrivial equivalence class is aH. We have that θ_{aH} is a congruence and thus so is $\theta \coloneqq \bigcup_{aH \in U} \theta_{aH}$. For this choice of θ we have that $h(\theta) = U$ so h is surjective.

To see that h and h^{-1} are isotone note that one equivalence relation contains another precisely when the corresponding partition of one contains the partition of the other. This is equivalent to the given order on the antichains of $\mathbf{P}_\lambda$. □

Since every lattice of maximal antichains is distributive we have that the finite regular RPS magmas are all congruence-distributive.

Our analysis also yields a family of simple magmas.

Theorem 25. *Suppose that $\mathbf{G} = \mathbb{Z}_{p^k}$ for a prime p and $n < p$. There exists a $\lambda \in \operatorname{Sgn}_n(\mathbf{G})$ for which $\mathbf{G}_n(\lambda)$ is simple.*

Proof. Order the nontrivial subgroups of $\mathbf{G}$ as $\mathbf{H}_1 \leq \cdots \leq \mathbf{H}_s = \mathbf{G}$. For each $1 \leq i \leq s-1$ choose a coset $a + H_i$ of H_i other than H_i itself which lies in H_{i+1}. Choose another element $b \in a + H_i$ with $b \neq a$. Set $\lambda(\{a, -a\}) \coloneqq a$ and $\lambda(\{b, -b\}) \coloneqq -b$. We have that $\mathbf{H}_i$ is not λ-convex for $1 \leq i \leq s-1$. It follows that $\mathbf{G}_n(\lambda)$ has no nontrivial proper λ-convex subgroups for this choice of λ so $\mathbf{G}_n(\lambda)$ is simple. □

References

[1] *A007079*. Jan. 26, 2019. URL: https://oeis.org/A007079 (cit. on p. 23).

[2] Charlotte Aten. “Multiplayer Rock-Paper-Scissors”. In: *Algebras and Lattices in Hawai'i*. Ed. by Kira Adaricheva, William DeMeo, and Jennifer Hyndman. Apr. 2018, pp. 12–19 (cit. on p. 1).

[3] Clifford Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman and Hall/CRC, 2011. ISBN: 978-1-4398-5129-6 (cit. on p. 2).

[4] Marc Chamberland and Eugene A. Herman. “Rock-Paper-Scissors Meets Borromean Rings”. In: *The Mathematical Intelligencer* 37.2 (June 2015), pp. 20–25 (cit. on pp. 3, 11).

[5] Siniša Crvenković, Igor Dolinka, and Petar Marković. “A Survey of Algebra of Tournaments”. In: *Novi Sad Journal of Mathematics* 29.2 (1999), pp. 95–130 (cit. on pp. 12–14, 17).

[6] Ralph Freese. “An Application of Dilworth's Lattice of Maximal Antichains”. In: *Discrete Mathematics* 7 (1974), pp. 107–109 (cit. on p. 27).

[7] H. Joris, C. Oestreicher, and J. Steinig. “The Greatest Common Divisor of Certain Sets of Binomial Coefficients”. In: *Journal of Number Theory* 21.1 (1985), pp. 101–119 (cit. on p. 5).

[8] Brendan D. McKay. “The Asymptotic Numbers of Regular Tournaments, Eulerian Digraphs and Eulerian Oriented Graphs”. In: *Combinatorica* 10.4 (1990), pp. 367–377 (cit. on p. 23).

[9] *Rock Paper Scissors Spock Lizard*. Jan. 16, 2018. URL: http://www.samkass.com/theories/RPSSL.html (cit. on p. 3).

[10] Michel Surmacs. “Regular Hypertournaments and Arc-Pancyclicity”. In: *Journal of Graph Theory* 84.2 (2016), pp. 176–190 (cit. on p. 17).

[11] Gisèle Umbhauer. *Game Theory and Exercises*. 1st ed. Routledge Advanced Texts in Economics and Finance. Routledge, 2016. ISBN: 978-0-415-60421-5 (cit. on p. 3).