Double-Recurrence Fibonacci Numbers and Generalizations

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Abstract

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the recurrence $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. There are several generalizations of this sequence and also several interesting identities. In this paper, we investigate a homogeneous recurrence relation that, in a way, extends the linear recurrence of the Fibonacci sequence for two variables, called *double-recurrence Fibonacci numbers*, given by $F(m, n) = F(m-1, n-1) + F(m-2, n-2)$, for $n, m \geq 2$, where $F(m, 0) = F_m$, $F(m, 1) = F_{m+1}$, $F(0, n) = F_n$ and $F(1, n) = F_{n+1}$. We exhibit a formula to calculate the values of this double recurrence, only in terms of Fibonacci numbers, such as certain identities for their sums are outlined. Finally, a general case is studied.

1 Introduction

Fibonacci numbers are known for their amazing properties, association with geometric figures, among others [7, 4]. Using the usual notation for such numbers, $(F_n)_{n\geq 0}$, they are given by the following linear recurrence of order two: $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by studying high order recurrences with similar initial conditions [6, 3].

Our interest relies in a generalization that uses a recurrence for two indices (called a *double-recurrence*), such as the one studied by Hosoya [2], who defined a set of integers $\{f_{m,n}\}$ satisfying:

$$f_{m,n} = f_{m-1,n} + f_{m-2,n},$$

$$f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2},$$

for all $m \geq 2$, $m \geq n \geq 0$, where

$$f_{0,0} = f_{1,0} = f_{1,1} = f_{2,1} = 1 \ .$$

Those numbers, when arranged triangularly, provide the famous *Fibonacci Triangle* (also known as *Hosoya's Triangle*). One of our goals is to construct an analogue of the Fibonacci Triangle, studying a similar double-recurrence. The set of numbers $\{F(m,n)\}$, will be required to satisfy the following,

$$F(m,n) = F(m-1,n-1) + F(m-2,n-2), \text{ for } \ m,n \geq 2, \tag{1}$$

with initial values

$$\begin{array}{ll} F(m,0) = F_m, & F(1,n) = F_{n+1}, \\ F(m,1) = F_{m+1}, & F(0,n) = F_n \ . \end{array}$$

The initial conditions above, along with (1), are sufficient to calculate the value of $F(m,n)$ at each $(m,n) \in \mathbb{N}^2$. We call the values of the set $\{F(m,n)\}$, *double-recurrence Fibonacci numbers*. Note that $F(m,n)$ is a symmetric function, since the initial conditions above and below the main diagonal are the same, and that $F(k,i) = F(k,k-i)$ for all $0 \leq i \leq \lfloor k/2 \rfloor$. Figure 1, displays a few values for $F(m,n)$, considering the bottom left corner as the origin $(0,0)$, and the (m,n) coordinate having the value for $F(m,n)$.

Consider the value of the coordinate $(7,4)$, given by $F(7,4) = 19$, and then draw a parallel to the antidiagonal from this point towards the axis, where the interactions begin with initial values $F(3,0) = F_3$ and $F(4,1) = F_5$. This means that, in order to determine $F(7,4)$, we only needed the pair F_3 and F_5, in other words, only Fibonacci numbers. The following proposition, asserts that this property is true for all $F(m,n)$, meaning that these values can be obtained using only Fibonacci numbers.

13	21	18	19	19	18	21	13
8	13	11	12	11	13	8	21
5	8	7	7	8	5	13	18
3	5	4	5	3	8	11	19
2	3	3	2	5	7	12	19
1	2	1	3	4	7	11	18
1	1	2	3	5	8	13	21
0	1	1	2	3	5	8	13

Figure 1: Double-Fibonacci Numbers

Proposition 1. *Let* $m, n \in \mathbb{N}$, *and* $F(m,n)$ *be a double-recurrence Fibonacci number, with* $k := \min\{m,n\}$. *Then,*

$$F(m,n) = F_k F_{|m-n|+2} + F_{k-1} F_{|m-n|}. \tag{2}$$

Proof. We proceed by the induction principle for two variables. It Is straightforward that $F(0,0) = F_0 = F_0 F_2 + F_{-1} F_0$. So, supposing that (2) holds for all $i \le m$ and $j \le n$, we have

$$\begin{aligned} F(m+1,n) &= F(m,n-1) + F(m-1,n-2) \\ &= F_{k'} F_{|m-n+1|+2} + F_{k'-1} F_{|m-n+1|} + F_{k'-1} F_{|m-n+1|+2} + F_{k'-2} F_{|m-n+1|}, \end{aligned}$$

where $k' = \min\{m, n-1\} \Rightarrow k' - 1 = \min\{m-1, n-2\}$. Therefore,

$$F(m+1,n) = F_{k'+1} F_{|(m+1)-n|+2} + F_{k'} F_{|(m+1)-n|},$$

and since $k' + 1 = \min\{m+1, n\}$, the identity holds in this case. Analogously, following the same steps, the identity also holds for $F(m, n+1)$, which completes the proof. □

In the homogeneous double-recurrence (1), one could replace the initial conditions by a general linear recurrence sequence of order two, or even arithmetic functions. In other words, we have the following:

Definition 2. Let $m, n \in \mathbb{N}$. The function $H(m,n)$ satisfying

$$H(m,n) = H(m-1,n-1) + H(m-2,n-2) \tag{3}$$

for all $m, n \ge 2$, where the following initial conditions are given

$$H(m,0)=H_1(m), \qquad H(0,n)=H_2(n),$$
$$H(m,1)=H_1^2(m), \qquad H(1,n)=H_2^1(n),$$

with H_1, H_2, H_1^2 and H_2^1 arithmetic functions, is called a double-recurrence function. If H_1, H_2, H_1^2 and H_2^1 are linear recurrence sequences of order two, the function satisfying (3) is called a spin Function.

In this way, double-recurrence Fibonacci numbers are values of a spin Function, such as every Fibonacci and Lucas numbers. Now, let $H(m,n)$ be a spin Function, where

$$\begin{aligned} H(m,0)&=H_1(m) \quad \text{with} \quad H_1(0)=a \ \text{ and } \ H_1(1)=b,\\ H(m,1)&=H_1^2(m) \quad \text{with} \quad H_1^2(0)=d \ \text{ and } \ H_1^2(1)=c,\\ H(0,n)&=H_2(n) \quad \text{with} \quad H_2(0)=a \ \text{ and } \ H_2(1)=b,\\ H(1,n)&=H_2^1(n) \quad \text{with} \quad H_2^1(0)=d \ \text{ and } \ H_2^1(1)=c, \end{aligned} \tag{4}$$

and if $m=n$, we have a linear recurrence sequence of order two, given by:

$$H(m,m)=H_1^1(m), \text{ with } H_1^1(0)=a \ \text{ and } \ H_1^1(1)=c. \tag{5}$$

The motivation for the term *spin function*, relies on the way that we can reach, from the initial terms, all pairs of $(m,n)\in\mathbb{N}^2$, where the function is evaluated, using every secondary diagonal on it, that we refer as *strings*. A graphical representation of it, can be seen next.

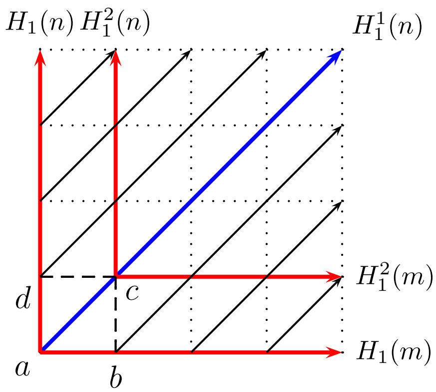

Figure 2: A spin Function and its strings

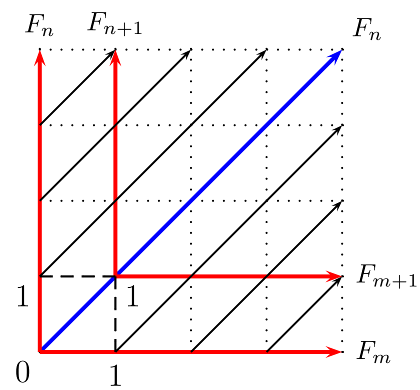

Figure 3: Double-recurrence Fibonacci function

2 Properties and Identities

Among several generalizations for Fibonacci numbers, we now consider the ones that satisfies the Fibonacci recurrence relation, but with arbitrary initial conditions.

Definition 3. Let $(G_n)_n$ a linear recurrence sequence of order two, where $G_1 = a$, $G_2 = b$ and $G_{n+2} = G_{n+1} + G_n$, $n \geq 1$. The ensuing sequence is called a generalized Fibonacci sequence (GFS).

The following, is a classical result, that can be easily proved by induction, which states that every term on a GFS, can be written only in terms of Fibonacci numbers and their initial conditions.

Theorem 4. *Let G_n denote the* n*th term of the GFS. Then $G_{n+2} = bF_{n+1} + aF_n$, $n \geq 1$.*

Proof. See [5, Th. 7.1]. □

Note that, Proposition 1 can be seen as a generalization of Theorem 4 for double-recurrence Fibonacci numbers. Our immediate purpose is to show that an analogous result also holds for spin functions. In order to do so, we introduce a double-recurrence function that will play the same role as Fibonacci numbers on Theorem 4. Let $m, n, a, b \in \mathbb{N}$. Then, define

$$F_a^b(m,n) := bF_n F_{|m-n|+2} + aF_{n-1}F_{|m-n|} \ . \tag{6}$$

It is easy to see that $F_a^b(m,n)$ is a double-recurrence function, but not necessarily a spin Function, i.e.,

$$F_a^b(m+2,n+2) = F_a^b(m+1,n+1) + F_a^b(m,n),$$

but the functions on the initial conditions are not necessarily linear recurrence sequences of order two. For that, we have the following result.

Proposition 5. *Let $m, n \in \mathbb{N}$ and the spin function $H(m,n)$, such as on Definition 2. Then,*

i. If $n \leq m-1$, then $H(m,n) = F_{a+b}^c(m-1,n) + F_b^d(m-2,n)$.

ii. If $m-1 < n$, then $H(m,n) = F_{a+d}^c(n-1,m) + F_d^b(n-2,m)$.

Proof. Let $H(m,n)$ be a spin function for $n \leq m-1$, with functions H_1^2 and H_1 given by the initial conditions described previously. Similarly to the Proposition 1, we have

$$H(m,n) = F_n H_1^2(m-n+1) + F_{n-1}H_1(m-n),$$

and since H_1^2 and H_1 are linear recurrence sequences, using Theorem 4, we get

$$\begin{aligned} H(m,n) &= F_n(cF_{m-n+1} + dF_{m-n}) + F_{n-1}(bF_{m-n} + aF_{m-n-1}) \\ &= cF_nF_{m-n+1} + aF_{n-1}F_{m-n-1} + dF_nF_{m-n} + bF_{n-1}F_{m-n}. \end{aligned}$$

Using that $bF_{n-1}F_{m-n} = b \cdot (F_{n-1}F_{m-n-2} + F_{n-1}F_{m-n-1})$, we obtain

$$\begin{aligned} H(m,n) &= cF_nF_{m-n+1} + (a+b) \cdot F_{n-1}F_{m-n-1} + dF_nF_{m-n} + bF_{n-1}F_{m-n-2} \\ &= F_{a+b}^c(m-1,n) + F_b^d(m-2,n) \ . \end{aligned}$$

Analogously, for $m-1 < n$, considering H_2^1 and H_2, we get

$$H(m,n) = F_{a+d}^c(n-1,m) + F_b^d(n-2,m),$$

which completes the proof. □

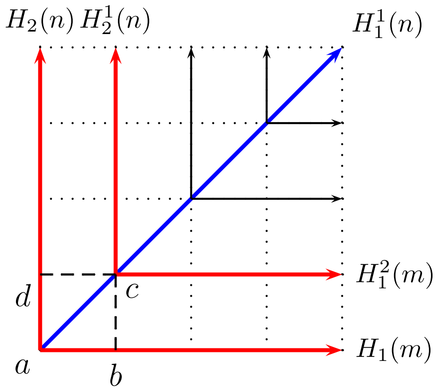

Figure 4: Graphical representation of Proposition 5

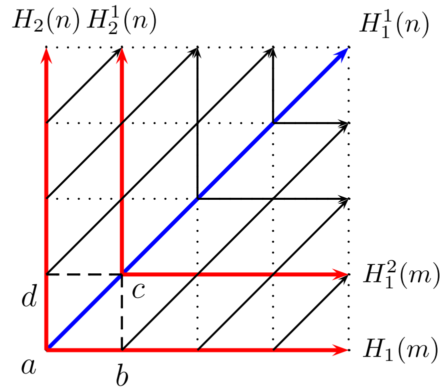

Figure 5: Combination of Proposition 5 and Definition 2

Now, we return our attention to sums of double-recurrence Fibonacci numbers. But first, we recall an interesting identity for Generalized Fibonacci Numbers [9], giving an alternative proof for it.

Proposition 6. *Let $(G_n)_n$ be a GFS, where $G_n = G_{n-1} + G_{n-2}$ with initial conditions $G_0 = g_0$ and $G_1 = g_1$. Then*

$$\sum_{i=1}^{n} iG_i = nG_{n+2} - G_{n+3} + G_3 \tag{7}$$

Proof. Straightforward from Theorem 4, we have $G_n = g_0F_{n-1} + g_1F_n$. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} iG_i &= g_0\sum_{i=1}^{n} iF_{i-1} + g_1\sum_{i=1}^{n} iF_i \\
&= g_0\sum_{i=0}^{n-1}(i+1)F_i + g_1\sum_{i=1}^{n} iF_i &(8)\\
&= g_0((n-1)F_{n+1} - F_{n+2} + 2 + F_{n+1} - 1) + g_1(nF_{n+2} - F_{n+3} + 2) &(9)\\
&= n(g_0F_{n+1} + g_1F_{n+2}) - (g_0F_{n+2} + g_1F_{n+3}) + 2g_0 + g_1 \\
&= nG_{n+2} - G_{n+3} + G_3
\end{aligned}$$

Where, from (8) to (9), the identity $\sum_{i=1}^{n} iF_i = nF_{n+2} - F_{n+3} + 2$, [8, p.16, Ex.10], is used. □

The following proposition, consists of a closed form to calculate the sums of Double-Fibonacci numbers, where the indices are in $\{1, \ldots, m\}^2$.

Proposition 7. *Let $F(i, j)$ be Double-Fibonacci Numbers, where $i, j \in \{0, 1, \ldots, m\}$. Then,* *i. The sum of all Double-Fibonacci Numbers with indices below the main diagonal, including it, is given by*

$$\sum_{\substack{i,j=0\\ j\le i}}^{m} F(i,j) = \frac{2}{5}\left(mL_{m+3} - L_{m+4} + 2F_{m+2}\right) + 2. \tag{10}$$

ii. The sum of all Double-Fibonacci Numbers, with indices on the square $m \times m$, is

$$\sum_{i,j=0}^{m} F(i,j) = \frac{4}{5}\left(mL_{m+3} - L_{m+4} + 2F_{m+2}\right) - F_{m+2} + 5.$$

Proof. First, we proceed to prove (i), and use it to prove (ii). Rewriting (10), and using the closed form on Proposition 1, we have

$$\begin{aligned}\sum_{\substack{i,j=0\\ i\ge j}}^{m} F(i,j) &= \sum_{i=0}^{m}\sum_{j=0}^{i} F_jF_{i-j+2} + F_{j-1}F_{i-j}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{i} F_jF_{i-j+1} + F_{j-1}F_{i-j} + F_jF_{i-j},\end{aligned}$$

and since $F_i = F_jF_{i-j+1} + F_{j-1}F_{i-j}$, it follows,

$$\begin{aligned}&= \sum_{i=0}^{m}\sum_{j=0}^{i} F_i + F_jF_{i-j}\\ &= \sum_{i=0}^{m}\left((i+1)F_i + \sum_{j=0}^{i} F_{i-j}F_j\right).\end{aligned}$$

Now, we observe that the sum $\sum_{j=0}^{i} F_{i-j}F_j$, is referenced as sequence A001629 on [1], where is established that it is equal to $((i-1)F_i + 2iF_{i-1})/5 = (iL_i - F_i)/5$, $(L_n)_{n\ge0}$ being the Lucas Sequence, and the last equality follows from [5, Eq. 32.13, p. 375]. Thus,

$$\sum_{\substack{i,j=0\\ i\ge j}}^{m} F(i,j) = \sum_{i=0}^{m}(i+1)F_i + \sum_{i=0}^{m}\frac{iL_i - F_i}{5}.$$

From Proposition 6 and $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, we have,

$$= m\left(\frac{L_{m+2}}{5} + F_{m+2}\right) - \left(\frac{L_{m+3}}{5} + F_{m+3}\right) + \frac{4}{5}F_{m+2} + 2,$$

then, finally by $L_{n-1} + L_{n+1} = 5F_n$ (see [5, Cor. 5.5, p. 80]), it follows that

$$= \frac{m\left(L_{m+2} + L_{m+1} + L_{m+3}\right)}{5} - \frac{\left(L_{m+4} + L_{m+2} + L_{m+3}\right)}{5} + \frac{4}{5}F_{m+2} + 2$$

$$\therefore \sum_{\substack{i,j=0 \\ i \geq j}}^{m} F(i,j) = \frac{2}{5}\left(mL_{m+3} - L_{m+4} + 2F_{m+2}\right) + 2,$$

completing the proof for (i). For (ii), we use the symmetry satisfied by double-recurrence Fibonacci Numbers, $F(m,n) = F(n,m)$, giving us that the sum on (ii) is two times the sum on (i), minus the sum for indices on the main diagonal:

$$\begin{aligned} \sum_{i,j=0}^{m} F(i,j) &= 2\sum_{\substack{i,j=0 \\ i \leq j}}^{m} F(i,j) - \sum_{i=0}^{m} F(i,i) \\ &= \frac{4}{5}\left(mL_{m+3} - L_{m+4} + 2F_{m+2}\right) + 4 - \sum_{i=0}^{m} F_i \\ &= \frac{4}{5}\left(mL_{m+3} - L_{m+4} + 2F_{m+2}\right) - F_{m+2} + 5. \end{aligned}$$

□

Out of curiosity, equation (10) happens to be the same formula for the path length of the Fibonacci tree of order n. (A178523 of [1])

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(Concerned with sequences A001629, A002940, A006478, A010049, A014286, A122491, A178523, A190062,)

Appendix

The following table explicit some interesting sequences founded on [1], that can be obtained from the sum of the terms of $H(i, j)$, with initial conditions a, b, c and d, considering $0 \le j < i \le n$, $0 \le i \le j \le n$, and all $i, j \in \{0, 1, \ldots, n\}^2$.

Initial Condition $[a,b,c,d]$	$\sum_{i=1}^{n}\sum_{j=0}^{i-1} H(i,j)$	$\sum_{j=0}^{n}\sum_{i=0}^{j} H(i,j)$	$\sum_{i,j=0}^{m} H(i,j)$
$[0,0,0,1]$	A006478(n)	A001629$(n+1)$	A006478$(n+1)$
$[0,0,1,0]$	A002940$(n-2)$	A006478$(n+1)$	-
$[0,0,1,1]$	-	A122491$(n+2)$	-
$[0,1,0,0]$	A001629$(n+1)$	A006478(n)	A006478$(n+1)$
$[0,1,0,1]$	A006478$(n+1)$	A006478$(n+1)$	A178523$(n+1)$
$[0,1,1,0]$	A014286(n)	A002940$(n-1)$	-
$[0,1,1,1]$	-	A178523$(n+1)$	-
$[1,0,0,0]$	A001629(n)	A010049$(n+1)$	-
$[1,0,0,1]$	A122491$(n+1)$	A001629$(n+2)$	-
$[1,0,1,0]$	A178523(n)	-	-
$[1,0,1,1]$	-	A006478$(n+2)$	-
$[1,1,0,0]$	A006478$(n+1)$	A190062$(n+1)$	-
$[1,1,1,0]$	A002940$(n-1)$	-	-
$[1,1,1,1]$	-	A014286$(n+11)$	-

Table 1: Related sequences.