COUNTING 3-STACK-SORTABLE PERMUTATIONS

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Abstract. We prove a "Decomposition Lemma" that allows us to count preimages of certain sets of permutations under West's stack-sorting map s. As a first application, we give a new proof of Zeilberger's formula for the number $W_2(n)$ of 2-stack-sortable permutations in S_n. Our proof generalizes, allowing us to find an algebraic equation satisfied by the generating function that counts 2-stack-sortable permutations according to length, number of descents, and number of peaks. This is also the first proof of this formula that generalizes to the setting of 3-stack-sortable permutations. Indeed, the same method allows us to obtain a recurrence relation for $W_3(n)$, the number of 3-stack-sortable permutations in S_n. Hence, we obtain the first polynomial-time algorithm for computing these numbers. We compute $W_3(n)$ for $1 \leq n \leq 174$, vastly extending the 13 terms of this sequence that were known before. We also prove the first nontrivial lower bound for $\lim_{n\to\infty} W_3(n)^{1/n}$, showing that it is at least 8.659702. Invoking a result of Kremer, we also prove that $\lim_{n\to\infty} W_t(n)^{1/n} \geq (\sqrt{t}+1)^2$ for all $t \geq 1$, which we use to improve a result of Smith concerning a variant of the stack-sorting procedure. Our computations allow us to disprove a conjecture of Bóna, although we do not yet know for sure which one.

In fact, we can refine our methods to obtain a recurrence for $W_3(n,k,p)$, the number of 3-stack-sortable permutations in S_n with k descents and p peaks. This allows us to gain a large amount of evidence supporting a real-rootedness conjecture of Bóna. Using part of the theory of valid hook configurations, we give a new proof of a γ-nonnegativity result of Brändén, which in turn implies an older result of Bóna. We then answer a question of the current author by producing a set $A \subseteq S_{11}$ such that $\sum_{\sigma\in s^{-1}(A)} x^{\mathrm{des}(\sigma)}$ has nonreal roots. We interpret this as partial evidence against the same real-rootedness conjecture of Bóna that we found evidence supporting. Examining the parities of the numbers $W_3(n)$, we obtain strong evidence against yet another conjecture of Bóna. We end with some conjectures of our own.

1. Introduction

1.1. **The Stack-Sorting Map.** We use the word "permutation" to refer to a permutation of a set of positive integers written in one-line notation. Let S_n denote the set of permutations of the set $[n]$. If π is a permutation of length n, then the *normalization* of π is the permutation in S_n obtained by replacing the i^{th}-smallest entry in π with i for all $i \in [n]$. We say a permutation is *normalized* if it is equal to its normalization. A *descent* of a permutation $\pi = \pi_1 \cdots \pi_n$ is an index $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. A *peak* of π is an index $i \in \{2, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. Let $\mathrm{des}(\pi)$ and $\mathrm{peak}(\pi)$ denote the number of descents of π and the number of peaks of π, respectively.

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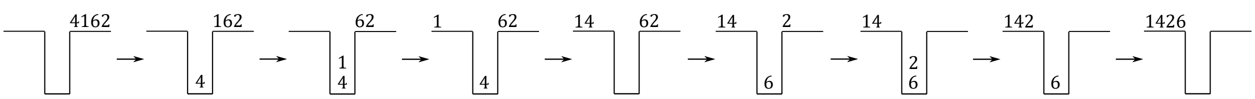
Definition 1.1. Given $\tau \in S_m$, we say a permutation $\sigma = \sigma_1 \cdots \sigma_n$ *contains the pattern* τ if there exist indices $i_1 < \cdots < i_m$ in $[n]$ such that the normalization of $\sigma_{i_1} \cdots \sigma_{i_m}$ is τ. We say σ *avoids* τ if it does not contain τ. Let $\text{Av}(\tau^{(1)}, \ldots, \tau^{(r)})$ denote the set of normalized permutations that avoid the patterns $\tau^{(1)}, \ldots, \tau^{(r)}$. Let $\text{Av}_n(\tau^{(1)}, \ldots, \tau^{(r)}) = \text{Av}(\tau^{(1)}, \ldots, \tau^{(r)}) \cap S_n$.

The study of permutation patterns is now a major area of research; it began with Knuth's analysis of a certain "stack-sorting algorithm" [35]. In his dissertation, West [41] defined a deterministic variant of Knuth's algorithm. This variant is a function, which we call the "stack-sorting map" and denote by s, that sends permutations to permutations. The stack-sorting map has now been studied extensively [2, 5–8, 10–14, 16–28, 30–32, 34, 39–41, 43]. The reader seeking further historical background and motivation should see one of the references [2, 7, 18–28].

To define the function s, let us begin with an input permutation $\pi = \pi_1 \cdots \pi_n$. At any point in time during this procedure, if the next entry in the input permutation is smaller than the entry at the top of the stack or if the stack is empty, the next entry in the input permutation is placed at the top of the stack. Otherwise, the entry at the top of the stack is annexed to the end of the growing output permutation. This process terminates when the output permutation has length n, and $s(\pi)$ is defined to be this output permutation. The following illustration shows that $s(4162) = 1426$.



Definition 1.2. We say a permutation π is *t-stack-sortable* if $s^t(\pi)$ is an increasing permutation, where s^t denotes the t-fold iterate of s. Let $\mathcal{W}_t(n)$ be the set of t-stack-sortable permutations in S_n, and let $\mathcal{W}_t(n,k) = \{\pi \in \mathcal{W}_t(n) : \text{des}(\pi) = k\}$ and $\mathcal{W}_t(n,k,p) = \{\pi \in \mathcal{W}_t(n,k) : \text{peak}(\pi) = p\}$. Let

$$W_t(n) = |\mathcal{W}_t(n)|, \quad W_t(n,k) = |\mathcal{W}_t(n,k)|, \quad \text{and} \quad W_t(n,k,p) = |\mathcal{W}_t(n,k,p)|.$$

Knuth simultaneously initiated the study of stack-sorting and the investigation of permutation patterns with the following theorem.

Theorem 1.1 ([35])**.** *A permutation is* 1*-stack-sortable if and only if it avoids the pattern* 231*. Furthermore,*

$$W_1(n) = |\text{Av}_n(231)| = C_n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the* n^{th} *Catalan number.*

In his dissertation, West conjectured a formula for $W_2(n)$, which Zeilberger later proved.

Theorem 1.2 ([43])**.** *We have*

$$W_2(n) = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

Combinatorial proofs of Zeilberger's theorem emerged later in [17, 30, 31, 34]. Some authors have investigated the enumeration of 2-stack-sortable permutations according to various statistics [5, 10, 12, 30]. The articles [29] and [33] give different proofs that new combinatorial objects called "fighting fish" are counted by the numbers $W_2(n)$. The authors of [1] studied what they called "n-point dominoes," and they have found that there are $W_2(n+1)$ such objects.

There is very little known about t-stack-sortable permutations when $t \geq 3$. Úlfarsson [40] characterized 3-stack-sortable permutations in terms of new "decorated patterns," but the characterization is too unwieldy to yield any additional information. The best known general upper bound for $W_t(n)$, which follows from a theorem of Stankova and West [7, Theorem 3.4], is the estimate

$$W_t(n) \leq (t+1)^{2n}. \tag{1}$$

The current author [25] showed that

$$\lim_{n\to\infty} W_3(n)^{1/n} < 12.53296 \quad \text{and} \quad \lim_{n\to\infty} W_4(n)^{1/n} < 21.97225. \tag{2}$$

The limits in (2) are known to exist (see Section 6). Recently, Bóna has obtained a new proof of the first inequality in (2) using "stack words." It also follows from Theorem 1.2 that

$$\lim_{n\to\infty} W_t(n)^{1/n} \geq 6.75 \quad \text{for all} \quad t \geq 2. \tag{3}$$

When $t \geq 3$, we refer to (3) as a "trivial" lower bound for the growth rate of $W_t(n)$, even though it relies on the highly nontrivial enumeration of 2-stack-sortable permutations. Remarkably, (3) was the best known lower bound for $\lim_{n\to\infty} W_t(n)^{1/n}$ for all $t \geq 2$ until now.

Bóna [8] proved that the polynomial $\sum_{k=0}^{n-1} W_t(n,k)x^k = \sum_{\sigma\in\mathcal{W}_t(n)} x^{\text{des}(\sigma)}$ is symmetric and unimodal (see Section 8 for the relevant definitions). In fact, his proof actually shows that $\sum_{\sigma\in s^{-1}(A)} x^{\text{des}(\sigma)}$ is symmetric and unimodal for every set $A \subseteq S_n$. Brändén strengthened this result with the following theorem.

Theorem 1.3 ([13])**.** *If $A \subseteq S_n$, then*

$$\sum_{\sigma\in s^{-1}(A)} x^{\text{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(A) : \text{peak}(\sigma) = m\}|}{2^{n-1-2m}} x^m(1+x)^{n-1-2m}.$$

In particular, $\sum_{\sigma\in s^{-1}(A)} x^{\text{des}(\sigma)}$ *is γ-nonnegative.*

In the present article, we concern ourselves with the following four conjectures of Bóna. Recall that a sequence $(a_n)_{n\geq1}$ of positive numbers is called *log-convex* if $(a_{n+1}/a_n)_{n\geq1}$ is nondecreasing.

Conjecture 1.1 ([2, 7])**.** *For all $n, t \geq 1$, we have*

$$W_t(n) \leq \binom{(t+1)n}{n}.$$

Conjecture 1.2 ([4])**.** *For every $t \geq 1$, the sequence $(W_t(n))_{n\geq1}$ is log-convex.*

Conjecture 1.3 ([32])**.** *If t is even, then $W_t(n)$ is frequently odd. If t is odd, then $W_t(n)$ is rarely odd.*

Conjecture 1.4 ([8])**.** *For all $n, t \geq 1$, the polynomial* $\sum_{\sigma\in\mathcal{W}_t(n)} x^{\text{des}(\sigma)}$ *has only real roots.*

Remark 1.1. Bóna's motivation for formulating Conjecture 1.1 came from the idea of encoding elements of $\mathcal{W}_t(n)$ as n-uniform words over a $(t+1)$-element alphabet (see [6] and [7] for more details).

His motivation behind Conjecture 1.2 came from an observation that the sequences $(W_t(n))_{n\geq 1}$ appear to be similar to the sequences that enumerate principle permutation classes, which he has also conjectured are log-convex. For example, Bóna has observed that his methods in [3] can be used to show that for fixed $n, t \geq 1$, the number of t-stack-sortable permutations of length n with c components is monotonically decreasing as a function of c. Similarly, his methods allow one to prove that the generating functions $\sum_{n\geq 1} W_t(n)x^n$ are not rational. Bóna formulated Conjecture 1.4 after observing that it holds when $t = 1$ and when $t = n-1$ (it also holds when $t \geq n$ because this is equivalent to the $t = n-1$ case). Brändén [14] proved this conjecture in the cases $t = 2$ and $t = n-2$, but the remaining cases are still open.

Conjecture 1.3 requires some explanation. Using Bóna's result that $\sum_{\sigma\in\mathcal{W}_t(n)} x^{\text{des}(\sigma)}$ is symmetric, one can easily deduce that $W_t(n)$ is even whenever n is even. Therefore, it is natural to consider the parity of $W_t(n)$ when n is odd. Let $\mathfrak{g}_t(m)$ be the number of integers n with $1 \leq n \leq m$ such that $W_t(n)$ is odd. Let F_r denote the r^{th} Fibonacci number (with $F_1 = F_2 = 1$). Using Theorems 1.1 and 1.2, one can show that $\mathfrak{g}_1(2^r) = r$ and $\mathfrak{g}_2(2^r) = F_r$ for all positive integers r. Bóna [32] interpreted this as saying $W_1(n)$ is rarely odd while $W_2(n)$ is frequently odd, and this led him to formulate Conjecture 1.3. One could formalize this by saying that $W_t(n)$ is *rarely odd* if $\limsup_{m\to\infty} \frac{\log \mathfrak{g}_t(m)}{\log m} = 0$ and is *frequently odd* $\liminf_{m\to\infty} \frac{\log \mathfrak{g}_t(m)}{\log m} > 0$ (although Bóna did not use this formalism). Bóna's motivation behind Conjecture 1.3 also came from the idea of encoding t-stack-sortable permutations with words.

1.2. **Summary of Main Results.** In Section 2, we formulate a "Decomposition Lemma," which provides a new method for analyzing preimages of permutations under the stack-sorting map. We actually prove a stronger lemma, which we call the Refined Decomposition Lemma, that allows us to take the statistics des and peak into account. In Section 3, we briefly review some formulas arising from the theory of new combinatorial objects called "valid hook configurations." In Section 4, we use the Decomposition Lemma to give a new proof of Zeilberger's formula for $W_2(n)$. We also use the Refined Decomposition Lemma to find an algebraic equation satisfied by the generating function of the numbers $W_2(n, k, p)$. This equation is new.

Our new proof of Zeilberger's formula is the first one that generalizes to the setting of 3-stack-sortable permutations. In Section 5, we use the Refined Decomposition Lemma to prove a recurrence relation for the numbers $W_3(n, k, p)$. Specializing this theorem gives us a recurrence for $W_3(n, k)$, and specializing further gives a recurrence for $W_3(n)$. This yields the first polynomial-time algorithm for computing $W_3(n)$. According to Wilf [42], we have solved the problem of counting 3-stack-sortable permutations. More precisely, he would say that we have "p-solved" this problem.

Before now, the values of $W_3(n)$ were only known up to $n = 13$. Indeed, the only algorithm that was used to compute these numbers before now relied on a brute-force approach. Using our recurrence, we have generated the values of $W_3(n)$ for $1 \leq n \leq 174$. We have added these terms to sequence A134664 in the Online Encyclopedia of Integer Sequences [38]. There are two significant theoretical implications of these computations. First, we will see in Section 6 that Bóna's Conjectures 1.1 and 1.2 cannot both be true. Thus, we have disproven a conjecture of Bóna, although we do not yet know with absolute certainty which one. Let us remark, however, that the data suggests very strongly that Conjecture 1.2 is true while Conjecture 1.1 is false. Furthermore, it appears that our recurrence coupled with sufficient computing time (and clever computing!) should allow one to completely disprove Conjecture 1.1. Second, we will prove that $\lim_{n\to\infty} W_3(n)^{1/n} \geq 8.659702$; this is the first nontrivial lower bound for $\lim_{n\to\infty} W_3(n)^{1/n}$. In Section

7, we prove that $\lim_{n\to\infty} W_t(n)^{1/n} \geq (\sqrt{t}+1)^2$ for every $t \geq 1$, yielding the first nontrivial lower bounds for these growth rates for all $t \geq 4$. As a corollary, we improve a result of Smith concerning permutations that can be sorted by t stacks in series using the so-called "left-greedy algorithm" [39]. Although there are multiple ways one could rigorously interpret Bóna's Conjecture 1.3, we will see in Section 6 that every reasonable interpretation of the conjecture is likely to be false.

We have also computed the numbers $W_3(n,k)$ for $1 \leq n \leq 43$, allowing us to verify Conjecture 1.4 when $t=3$ and $1 \leq n \leq 43$ (see OEIS sequence A324916 [38]). In Section 8, we show that the formulas from Section 3 easily implies Brändén's Theorem 1.3. We also provide a two-element set $A \subseteq S_{11}$ such that $\sum_{\sigma\in s^{-1}(A)} x^{\text{des}(\sigma)}$ is not real-rooted. This provides a negative answer to the last part of Question 12.1 in [26], which we interpret as a small amount of evidence against Bóna's Conjecture 1.4. Section 9 concludes the paper with a new conjecture about $\lim_{n\to\infty} W_3(n)^{1/n}$, several conjectures about the numbers $\mathfrak{g}_3(m)$ (defined in Remark 1.1), and two new conjectures about unimodality and log-concavity.

Before we proceed, let us make one additional remark about the usefulness of the Decomposition Lemma that we prove in Section 2. In a subsequent paper [20], we apply this lemma in order to settle several conjectures of the current author from [26]. More precisely, we complete the project of determining $|s^{-1}(\text{Av}_n(\tau^{(1)},\ldots,\tau^{(r)}))|$ for every subset $\{\tau^{(1)},\ldots,\tau^{(r)}\} \subseteq S_3$ with the exception of the singleton set $\{321\}$. This allows us to enumerate a new permutation class, find a new example of an unbalanced Wilf equivalence, and prove a conjecture of Hossain concerning the so-called "Boolean-Catalan numbers." Hence, one can even view the Decomposition Lemma as a bridge that allows one to use the stack-sorting map s as a tool for proving results that were conjectured without any reference to stack-sorting.

2. The Decomposition Lemma

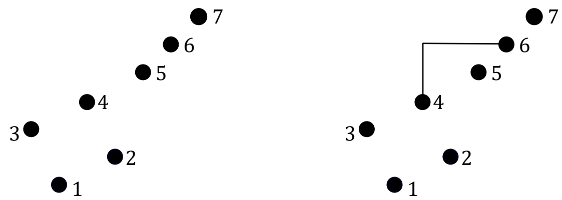
West [41] defined the *fertility* of a permutation π to be $|s^{-1}(\pi)|$, the number of preimages of π under s. He then went to great lengths to compute the fertilities of the permutations of the forms

$$23\cdots k1(k+1)\cdots n, \quad 12\cdots(k-2)k(k-1)(k+1)\cdots n, \quad \text{and} \quad k12\cdots(k-1)(k+1)\cdots n.$$

Bousquet-Mélou [11] found a method for determining whether or not a given permutation is *sorted*, meaning that its fertility is positive. She then asked for a general method for computing the fertility of any given permutation. The current author achieved this in even greater generality in [24–26,28] using new combinatorial objects called "valid hook configurations" ([28] is joint with Kravitz). In this section, we prove the Refined Decomposition Lemma and the Decomposition Lemma, which provide a new method for analyzing fertilities of permutations.

The *plot* of a permutation $\pi = \pi_1\cdots\pi_n$ is the figure showing the points (i,π_i) for all $i \in [n]$. For example, the image on the left in Figure 1 is the plot of 3142567. A *hook* of π is obtained by starting at a point (i,π_i) in the plot of π, drawing a vertical line segment moving upward, and then drawing a horizontal line segment to the right that connects with a point (j,π_j). In order for this to make sense, we must have $i<j$ and $\pi_i<\pi_j$. The point (i,π_i) is called the *southwest endpoint* of the hook, while (j,π_j) is called the *northeast endpoint*. Let $\mathsf{SW}_i(\pi)$ be the set of hooks of π with southwest endpoint (i,π_i). The right image in Figure 1 shows a hook of 3142567. This hook is in $\mathsf{SW}_3(3142567)$ because its southwest endpoint is $(3,4)$.

Define the *tail length* of a permutation $\pi=\pi_1\ldots\pi_n \in S_n$, denoted $\text{tl}(\pi)$, to be the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n-\ell$. We make the convention that $\text{tl}(1\cdots n)=n$. The



(A) (B)

FIGURE 1. The left image is the plot of 3142567. The right image shows this plot along with a single hook.

tail of π is the sequence of points $(n - \text{tl}(\pi) + 1, n - \text{tl}(\pi) + 1), \ldots, (n, n)$ in the plot of π. For example, the tail length of the permutation 3142567 shown in Figure 1 is 3, and the tail of this permutation is $(5, 5), (6, 6), (7, 7)$. We say a descent d of π is *tail-bound* if every hook in $\mathsf{SW}_d(\pi)$ has its northeast endpoint in the tail of π. The only tail-bound descent of 3142567 is 3.

Suppose H is a hook of a permutation $\pi = \pi_1 \cdots \pi_n$ with southwest endpoint (i, π_i) and northeast endpoint (j, π_j). Let $\pi_U^H = \pi_1 \cdots \pi_i \pi_{j+1} \cdots \pi_n$ and $\pi_S^H = \pi_{i+1} \cdots \pi_{j-1}$. The permutations π_U^H and π_S^H are called the *H-unsheltered subpermutation of* π and the *H-sheltered subpermutation of* π, respectively. For example, if $\pi = 3142567$ and H is the hook shown on the right in Figure 1, then $\pi_U^H = 3147$ and $\pi_S^H = 25$. In all of the cases we consider in this paper, the plot of π_S^H lies completely below the hook H in the plot of π (it is "sheltered" by the hook H).

Lemma 2.1 (Refined Decomposition Lemma)**.** *If d is a tail-bound descent of a permutation $\pi \in S_n$, then*

$$\sum_{\sigma \in s^{-1}(\pi)} x^{\text{des}(\sigma)+1} y^{\text{peak}(\sigma)+1}$$

$$= \sum_{H \in \mathsf{SW}_d(\pi)} \left(\sum_{\mu \in s^{-1}(\pi_U^H)} x^{\text{des}(\mu)+1} y^{\text{peak}(\mu)+1} \right) \left(\sum_{\lambda \in s^{-1}(\pi_S^H)} x^{\text{des}(\lambda)+1} y^{\text{peak}(\lambda)+1} \right).$$

Proof. If the tail of π is empty, then both sides of the desired equation are 0 because $s^{-1}(\pi)$ and $\mathsf{SW}_d(\pi)$ are empty. Hence, we may assume $\text{tl}(\pi) \geq 1$. Let $a = \pi_d$. Given $\sigma \in s^{-1}(\pi)$, we let f_σ be the entry that forces a to leave the stack when we apply the stack-sorting procedure (described in the introduction) to σ. More precisely, f_σ is the leftmost entry that appears to the right of a in σ and is larger than a. Note that f_σ appears to the right of a in π. Because d is tail-bound, this means that the point (f_σ, f_σ) is in the tail of π. Given a point (j, j) in the tail of π, let E_j be the set of permutations $\sigma \in s^{-1}(\pi)$ such that $f_\sigma = j$.

Now fix a point (j, j) in the tail of π, and let H be the hook in $\mathsf{SW}_d(\pi)$ with northeast endpoint (j, j). We will show that

$$\sum_{\sigma \in E_j} x^{\text{des}(\sigma)+1} y^{\text{peak}(\sigma)+1} = \left(\sum_{\mu \in s^{-1}(\pi_U^H)} x^{\text{des}(\mu)+1} y^{\text{peak}(\mu)+1} \right) \left(\sum_{\lambda \in s^{-1}(\pi_S^H)} x^{\text{des}(\lambda)+1} y^{\text{peak}(\lambda)+1} \right),$$

from which the lemma will follow. We can write $\pi = L\, a\, \pi_S^H\, j\, R$, where $L = \pi_1 \cdots \pi_{d-1}$ and $R = (j+1) \cdots n$. Suppose $\sigma \in E_j$. Let us write $\sigma = \tau\, j\, \tau'$. Because $j = f_\sigma$, it follows from the stack-sorting procedure that every entry in τ that is smaller than a must appear to the left of a in $s(\sigma) = \pi$. This implies that every entry in π_S^H that is smaller than a is in τ'. In particular, π_{d+1}

is in τ' (we know that $a > \pi_{d+1}$ because d is a descent of π). Now suppose b is an entry in π_S^H that is larger than a. If b is in τ, then we can appeal to the stack-sorting procedure again to see that b must appear to the left of π_{d+1} in π. This is impossible, so every entry in π_S^H is in τ'. The stack-sorting procedure forces every entry in L to be in τ, so every entry in τ' that is not in π_S^H must be an entry in R. Furthermore, an entry in R that is also in τ' cannot appear to the left of one of the entries from π_S^H in τ' (otherwise, j would appear to the right of one of the entries from π_S^H in π). This proves that we can write $\tau' = \lambda\tau''$, where λ is a permutation of the entries in π_S^H. Moreover, every entry in τ'' is in R.

Now let $\mu = \tau\tau''$. One can verify that $s(\mu) = \pi_U^H$ and $s(\lambda) = \pi_S^H$. Let $\delta = 1$ if 1 is a descent of τ'', and let $\delta = 0$ otherwise. Because $j = f_\sigma$, the leftmost entry in τ'' is the leftmost entry in μ that appears to the right of a in μ and is larger than a (if no such entry exists, then τ'' is empty). Also, the rightmost entry in τ is less than j. Combining these observations, we find that $\operatorname{des}(\sigma) + 1 = \operatorname{des}(\tau) + 1 + \operatorname{des}(\lambda) + \operatorname{des}(\tau'') + 1 = \operatorname{des}(\mu) + 1 + \operatorname{des}(\lambda) + 1$ and $\operatorname{peak}(\sigma) + 1 = \operatorname{peak}(\tau) + 1 + \operatorname{peak}(\lambda) + \operatorname{peak}(\tau'') + \delta + 1 = \operatorname{peak}(\mu) + 1 + \operatorname{peak}(\lambda) + 1$.

We have shown how to take a permutation $\sigma \in E_j$ and decompose it into permutations $\mu \in s^{-1}(\pi_U^H)$ and $\lambda \in s^{-1}(\pi_S^H)$ with $\operatorname{des}(\sigma) + 1 = \operatorname{des}(\mu) + 1 + \operatorname{des}(\lambda) + 1$ and $\operatorname{peak}(\sigma) + 1 = \operatorname{peak}(\mu) + 1 + \operatorname{peak}(\lambda) + 1$. We can easily reverse this procedure. Namely, if we are given μ and λ, we can write $\mu = \tau\tau''$ so that the leftmost entry in τ'' is the leftmost entry in μ that appears to the right of a in μ and is larger than a. We then recover σ by letting $\sigma = \tau\, j\, \lambda\, \tau''$. ☐

Corollary 2.1 (Decomposition Lemma). *If d is a tail-bound descent of a permutation $\pi \in S_n$, then*

$$|s^{-1}(\pi)| = \sum_{H \in \mathsf{SW}_d(\pi)} |s^{-1}(\pi_U^H)| \cdot |s^{-1}(\pi_S^H)|.$$

Proof. Set $x = y = 1$ in Lemma 2.1. ☐

3. Fertility Formulas

The purpose of this brief section is to establish some terminology and state some formulas from [24] that we will use in Section 8. We will also use a very special consequence of Theorem 3.1 in Section 4 when we analyze the generating function of the numbers $W_2(n, k, p)$.

A *composition of b into a parts* is an a-tuple of positive integers that sum to b. For example, $(3, 4, 3, 1)$ is a composition of 11 into 4 parts. Let $\operatorname{Comp}_a(b)$ denote the set of compositions of b into a parts. Let $C_r = \frac{1}{r+1}\binom{2r}{r}$ denote the r^{th} Catalan number. Let

$$N(r,i) = \frac{1}{r}\binom{r}{i}\binom{r}{i-1} \quad \text{and} \quad V(r,j) = 2^{r-2j+1}\binom{r-1}{2j-2}C_{j-1}. \tag{4}$$

Let

$$N_r(x) = \sum_{i=1}^{r} N(r,i)x^i \quad \text{and} \quad V_r(y) = \sum_{j=1}^{r} V(r,j)y^j. \tag{5}$$

The numbers $N(r, i)$ are called *Narayana numbers.* They are given in the OEIS sequence A001263 and constitute the most common refinement of the Catalan numbers [38]. The polynomials $N_r(x)$ are called *Narayana polynomials.* Among many other things, the Narayana numbers $N(r, i)$ count binary plane trees with r vertices and $i - 1$ right edges. The numbers $V(r, j)$, which count binary plane trees with r vertices and j leaves, are given in the OEIS sequence A091894. Let $L(r, i, j)$

be the number of binary plane trees with r vertices, $i-1$ right edges, and j leaves. Letting $F(w,x,y)=\sum_{r,i,j\geq 0} L(r,i,j)w^rx^iy^j$, we have

$$F(w,x,y)=x+wxy+w(F(w,x,y)+1)(F(w,x,y)-x). \tag{6}$$

This yields

$$F(w,x,y)=\frac{1-w+wx-\sqrt{(1-w+wx)^2-4wx(1-w+wy)}}{2w},$$

from which one obtains

$$L(r,i,j)=\frac{1}{r+1-j}\binom{r-1}{r-j}\binom{r+1-j}{j}\binom{r+1-2j}{i-j}. \tag{7}$$

Let

$$L_r(x,y)=\sum_{i=1}^{r}\sum_{j=1}^{r}L(r,i,j)x^iy^j \tag{8}$$

so that

$$L_r(x,1)=N_r(x) \quad \text{and} \quad L_r(1,y)=V_r(y).$$

Theorem 3.1 ([24][1]). *If $n\geq 1$ and $\pi=\pi_1\cdots\pi_n$ has exactly k descents, then there exists a set $\mathcal{V}(\pi)\subseteq \mathrm{Comp}_{k+1}(n-k)$ such that*

$$\sum_{\sigma\in s^{-1}(\pi)} x^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}=\sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)}\prod_{t=0}^{k}L_{q_t}(x,y). \tag{9}$$

In particular,

$$\sum_{\sigma\in s^{-1}(\pi)} x^{\mathrm{des}(\sigma)+1}=\sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)}\prod_{t=0}^{k}N_{q_t}(x) \tag{10}$$

and

$$\sum_{\sigma\in s^{-1}(\pi)} y^{\mathrm{peak}(\sigma)+1}=\sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)}\prod_{t=0}^{k}V_{q_t}(y). \tag{11}$$

Thus,

$$|s^{-1}(\pi)|=\sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)}\prod_{t=0}^{k}C_{q_t}. \tag{12}$$

Remark 3.1. If $\pi=123\cdots n$, then the above theorem, along with Theorem 1.1, tells us that

$$\sum_{\sigma\in s^{-1}(123\cdots n)} x^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}=\sum_{\sigma\in \mathrm{Av}_n(231)} x^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}=L_n(x,y).$$

[1] Strictly speaking, the first statement in Theorem 3.1 has not been stated explicitly before. However, the proofs of Corollary 5.1 and Theorem 5.2 in [24] immediately generalize to yield that statement.

4. A New Proof of the Formula for $W_2(n)$

Recall from Section 2 the definition of the tail length $\mathrm{tl}(\pi)$ of a permutation π. Let $B_\ell(n)$ (respectively, $B_{\geq \ell}(n)$) be the number of 2-stack-sortable permutations $\sigma \in \mathcal{W}_2(n+\ell)$ such that $\mathrm{tl}(s(\sigma)) = \ell$ (respectively, $\mathrm{tl}(s(\sigma)) \geq \ell$). Let

$$\mathcal{D}_\ell(n) = \{\pi \in \mathrm{Av}_{n+\ell}(231) : \mathrm{tl}(\pi) = \ell\} \quad \text{and} \quad \mathcal{D}_{\geq \ell}(n) = \{\pi \in \mathrm{Av}_{n+\ell}(231) : \mathrm{tl}(\pi) \geq \ell\}.$$

Because $\mathcal{W}_2(n) = s^{-1}(\mathcal{W}_1(n)) = s^{-1}(\mathrm{Av}_n(231))$ by Theorem 1.1, we can write

$$B_\ell(n) = |s^{-1}(\mathcal{D}_\ell(n))| \quad \text{and} \quad B_{\geq \ell}(n) = |s^{-1}(\mathcal{D}_{\geq \ell}(n))|.$$

Suppose $\pi \in \mathcal{D}_\ell(n+1)$ is such that $\pi_{n+1-i} = n+1$ (where $n \geq 0$). Then $n+1-i$ is a tail-bound descent of π. The Decomposition Lemma (Corollary 2.1) tells us that $|s^{-1}(\pi)|$ is equal to the number of triples (H, μ, λ), where $H \in \mathsf{SW}_{n+1-i}(\pi)$, $\mu \in s^{-1}(\pi_U^H)$, and $\lambda \in s^{-1}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \ldots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. The permutation π and the choice of H determine the permutations π_U^H and π_S^H. On the other hand, the choices of H and the permutations π_U^H and π_S^H uniquely determine π. It follows that $B_\ell(n+1)$, which is the number of ways to choose an element of $s^{-1}(\mathcal{D}_\ell(n+1))$, is also the number of ways to choose j, the permutations π_U^H and π_S^H, and the permutations μ and λ. Let us fix a choice of j.

Because π avoids 231, π_U^H must be a permutation of the set $\{1, \ldots, n-i\} \cup \{n+1\} \cup \{n+2+j, \ldots, n+\ell+1\}$, while π_S^H must be a permutation of $\{n-i+1, \ldots, n+j\} \setminus \{n+1\}$. Therefore, choosing π_U^H and π_S^H is equivalent to choosing their normalizations. The normalization of π_U^H is in $\mathcal{D}_{\geq \ell-j+1}(n-i)$, while the normalization of π_S^H is in $\mathcal{D}_{\geq j-1}(i)$ (see Figure 2). Any element of $\mathcal{D}_{\geq \ell-j+1}(n-i)$ can be chosen as the normalization of π_U^H, and any element of $\mathcal{D}_{\geq j-1}(i)$ can be chosen as the normalization of π_S^H. Also, π_U^H and π_S^H have the same fertilities as their normalizations. Combining these facts, we find that the number of choices for π_U^H and μ is $|s^{-1}(\mathcal{D}_{\geq \ell-j+1}(n-i))| = B_{\geq \ell-j+1}(n-i)$. Similarly, the number of choices for π_S^H and λ is $B_{\geq j-1}(i)$. Hence,

$$B_\ell(n+1) = \sum_{i=1}^{n} \sum_{j=1}^{\ell} B_{\geq \ell-j+1}(n-i) B_{\geq j-1}(i). \tag{13}$$

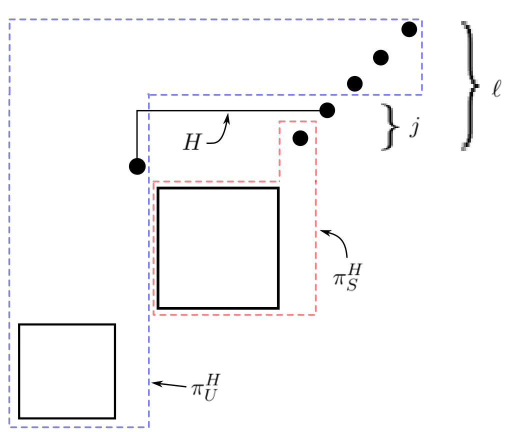


Figure 2. The decomposition of π into π_U^H and π_S^H.

Let

$$G_\ell(w) = \sum_{n\geq 0} B_{\geq \ell}(n) w^n \quad \text{and} \quad I(w,z) = \sum_{\ell \geq 0} G_\ell(w) z^\ell.$$

Note that

$$G_\ell(0) = B_{\geq \ell}(0) = |s^{-1}(\mathcal{D}_{\geq \ell}(0))| = |s^{-1}(123\cdots\ell)| = C_\ell$$

by Theorem 1.1. Let $C(z) = \sum_{n\geq 0} C_n z^n = \dfrac{1-\sqrt{1-4z}}{2z}$ be the generating function of the Catalan numbers. Because $B_{\geq 0}(n) = W_2(n)$ is the total number of 2-stack-sortable permutations in S_n, our goal is to understand the generating function

$$I(w,0) = G_0(w) = \sum_{n\geq 0} B_{\geq 0}(n) w^n = \sum_{n\geq 0} W_2(n) w^n.$$

By (13), we have

$$\sum_{\ell\geq 0}\sum_{n\geq 0} B_\ell(n+1) w^n z^\ell = \sum_{\ell\geq 0}\sum_{j=1}^{\ell}\sum_{n\geq 0}\sum_{i=1}^{n} B_{\geq \ell-j+1}(n-i) B_{\geq j-1}(i) w^n z^\ell$$

$$= \sum_{\ell\geq 0}\sum_{j=1}^{\ell} G_{\ell-j+1}(w)(G_{j-1}(w) - G_{j-1}(0)) z^\ell = \sum_{\ell\geq 0}\sum_{j=1}^{\ell} G_{\ell-j+1}(w)(G_{j-1}(w) - C_{j-1}) z^\ell$$

$$= \left(\sum_{r\geq 0} G_{r+1}(w) z^r\right)\left(\sum_{j\geq 1} (G_{j-1}(w) - C_{j-1}) z^j\right) = (I(w,z) - I(w,0))(I(w,z) - C(z)). \tag{14}$$

On the other hand,

$$B_\ell(n+1) = B_{\geq \ell}(n+1) - B_{\geq \ell+1}(n),$$

so

$$\sum_{\ell\geq 0}\sum_{n\geq 0} B_\ell(n+1) w^n z^\ell = \sum_{\ell\geq 0}\sum_{n\geq 0} B_{\geq \ell}(n+1) w^n z^\ell - \sum_{\ell\geq 0}\sum_{n\geq 0} B_{\geq \ell+1}(n) w^n z^\ell$$

$$= \frac{1}{w}\sum_{\ell\geq 0}(G_\ell(w) - C_\ell) z^\ell - \frac{1}{z}\sum_{\ell\geq 0} G_{\ell+1}(w) z^{\ell+1} = \frac{I(w,z) - C(z)}{w} - \frac{I(w,z) - I(w,0)}{z}. \tag{15}$$

Combining (14) and (15) yields the equation

$$(I(w,z) - I(w,0))(I(w,z) - C(z)) = \frac{I(w,z) - C(z)}{w} - \frac{I(w,z) - I(w,0)}{z}. \tag{16}$$

At this point, we employ the techniques described in [9]; the reader seeking additional details can consult that article. There is a unique fractional power series (Puiseux series) $Z = Z(w)$ such that $Z(w) = w + O(w^2)$ and

$$(I(w,Z) - I(w,0))(I(w,Z) - C(Z)) = \frac{I(w,Z) - C(Z)}{w} - \frac{I(w,Z) - I(w,0)}{Z}. \tag{17}$$

Indeed, we can compute the coefficients of $Z(w)$ one at a time from the equation (17) after we have initially computed the first few terms of $I(w,z)$ via its combinatorial definition. We can now solve (17) for $C(Z)$, use the standard Catalan functional equation $ZC(Z)^2 + 1 - C(Z) = 0$, and clear denominators to obtain a polynomial

$$\begin{aligned} Q(u,v,w,z) = & -vw + z + 2vwz + v^2w^2z + (w - z - 2wz - 2vw^2z + v^2w^2z)u \\ & + (w^2z - 2vw^2z + z^2 + 2vwz^2 + v^2w^2z^2)u^2 + (w^2z - 2wz^2 - 2vw^2z^2)u^3 + w^2z^2u^4 \end{aligned}$$

such that $Q(I(w,Z),I(w,0),w,Z)=0$.

Let $\Delta_u Q(v,w,z)$ be the discriminant of $Q(u,v,w,z)$ with respect to the variable u. A computer can explicitly compute this discriminant as $\Delta_u Q(v,w,z)=w^6(1-4z)^2z^3\widehat{Q}(v,w,z)$, where

$$\widehat{Q}(v,w,z)=z^3+2wz^2(-3+2vz)+w^4z(1+v+v^2z)^2+w^2z(9+(2-10v)z+6v^2z^2)\\ +2w^3(-2+(5-3v)z-(-2+v)vz^2+2v^3z^3).$$

It follows from the method described in [9] that $z=Z(w)$ is a repeated root of $\Delta_u Q(I(w,0),w,z)$. Since $Z(w)=w+O(w^2)$, we know that $w^6(1-4Z)^2Z^3\neq 0$. Therefore, $z=Z(w)$ is a repeated root of $\widehat{Q}(I(w,0),w,z)$. The discriminant of a polynomial with a repeated root must be 0. This means that $\Delta_z\widehat{Q}(I(w,0),w)=0$, where $\Delta_z\widehat{Q}(v,w)$ is the discriminant of $\widehat{Q}(v,w,z)$ with respect to z. Computing $\Delta_z\widehat{Q}(v,w)$ explicitly and ignoring extraneous factors, we find that $R(I(w,0),w)=0$, where

$$R(v,w)=-1+11w+w^2+v^3w^2+v^2w(2+3w)+v(1-14w+3w^2).$$

To complete our new proof of Theorem 1.2, we follow the proof of Proposition 5.2 in [11]. Namely, we consider the power series $U(w)$ defined by $U(w)=w(1+U(w))^3$. We then verify that $R(1+U(w)-U(w)^2,w)=0$ and deduce that $I(w,0)=1+U(w)-U(w)^2$. Lagrange inversion then completes the proof that

$$I(w,0)=\sum_{n\geq 0}\frac{2}{(n+1)(2n+1)}\binom{3n}{n}w^n.$$

The above argument generalizes as follows. Let

$$I_{x,y}(w,z)=\sum_{\ell\geq 0}\sum_{n\geq 0}\sum_{\sigma\in s^{-1}(\mathcal{D}_{\geq\ell}(n))}x^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}w^nz^\ell.$$

Note that $I_{x,y}(w,0)=\sum_{n\geq 0}\sum_{\sigma\in\mathcal{W}_2(n)}w^nx^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}$. We make the convention that the empty permutation has 0 descents and -1 peaks so that $I_{x,y}(0,0)=x$. Let F be the generating function in (6). If we replace $B_\ell(n)$ with $\sum_{\sigma\in s^{-1}(\mathcal{D}_\ell(n))}x^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}$, replace $B_{\geq\ell}(n)$ with $\sum_{\sigma\in s^{-1}(\mathcal{D}_{\geq\ell}(n))}x^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}$, use the Refined Decomposition Lemma instead of the Decomposition Lemma, and use Remark 3.1 instead of Theorem 1.1, then the above argument produces the equation

$$(I_{x,y}(w,z)-I_{x,y}(w,0))(I_{x,y}(w,z)-F(z,x,y))=\frac{I_{x,y}(w,z)-F(z,x,y)}{w}-\frac{I_{x,y}(w,z)-I_{x,y}(w,0)}{z}$$

in place of (16). We then continue the argument, using the functional equation (6) instead of the Catalan functional equation for $C(Z)$, in order to arrive at the following theorem concerning the generating function of the numbers $W_2(n,k,p)$.

Theorem 4.1. *Let*

$$R(v,w,x,y)=-x+(4x+8x^2-xy)w+(-6x-16x^2-16x^3+3xy+36x^2y)w^2+(4x+8x^2-3xy-36x^2y\\ +27x^2y^2)w^3+(-x+xy)w^4+(1+(-4-12x)w+(6+20x+32x^2-33xy)w^2+(-4-4x+16x^2+30xy\\ -36x^2y)w^3+(1-4x+3xy)w^4)v+(4w+(-4-22x)w^2+(-4-20x+8x^2+33xy)w^3+(4-6x+3xy)w^4)v^2\\ +(6w^2+(4-12x)w^3+(6-4x+xy)w^4)v^3+(4w^3+(4-x)w^4)v^4+w^4v^5.$$

We have $R(I_{x,y}(w,0),w,x,y)=0$, *where*

$$I_{x,y}(w,0)=\sum_{n\geq 0}\sum_{\sigma\in\mathcal{W}_2(n)}w^nx^{\mathrm{des}(\sigma)+1}y^{\mathrm{peak}(\sigma)+1}.$$

5. 3-Stack-Sortable Permutations

In the previous section, we counted 2-stack-sortable permutations by viewing them as preimages of 231-avoiding permutations under the stack-sorting map. In doing so, we had to keep track of the tail lengths of the 231-avoiding permutations under consideration. In this section, we count 3-stack-sortable permutations by viewing them as preimages of 2-stack-sortable permutations. We will again keep track of tail lengths, but we will also need an additional new statistic.

Definition 5.1. Given $\pi = \pi_1 \cdots \pi_n \in S_n$ and $a \in \{0, \ldots, n\}$, we say the open interval $(a, a+1)$ is a *legal space for* π if there do not exist indices $i_1 < i_2 < i_3$ such that $\pi_{i_3} \leq a < \pi_{i_1} < \pi_{i_2}$. Let $\text{leg}(\pi)$ be the number of legal spaces of π.

For example, if $\pi \in S_n$, then $\text{leg}(\pi) = n+1$ if and only if π avoids 231. The legal spaces of 145326 are $(0,1), (1,2), (4,5), (5,6), (6,7)$, so $\text{leg}(145326) = 5$. Imagine taking the plot of a permutation π and adding a new point to the left of all other points. One can think of the legal spaces of π as the vertical positions where the new point can be inserted so as to not form a new 2341 pattern. This is relevant for us because of the following characterization of 2-stack-sortable permutations due to West.

Theorem 5.1 ([41])**.** *A permutation is* 2*-stack-sortable if and only if it avoids the pattern* 2341 *and also avoids any* 3241 *pattern that is not part of a* 35241 *pattern.*

We are now in a position to state and prove the main theorems of this article. In what follows, let $B_{\geq \ell}^{(g)}(n)$ be the number of 3-stack-sortable permutations $\sigma \in \mathcal{W}_3(n+\ell)$ such that $\text{tl}(s(\sigma)) \geq \ell$ and $\text{leg}(s(\sigma)) = \ell + g$. Also, recall the definitions from Section 2.

Theorem 5.2. *If* $n \geq 1$*, then*

$$W_3(n) = \sum_{g=1}^{n+1} B_{\geq 0}^{(g)}(n).$$

We have $B_{\geq \ell}^{(0)}(n) = 0$ *and*

$$B_{\geq \ell}^{(g)}(1) = \begin{cases} 0, & \text{if } g \neq 2; \\ C_{\ell+1}, & \text{if } g = 2. \end{cases}$$

If $n, g \geq 1$ *and* $\ell \geq 0$*, then*

$$B_{\geq \ell}^{(g)}(n+1) = \sum_{j=1}^{\ell} \left(\sum_{a=2}^{n} \sum_{b=\max\{2,g-a\}}^{g-1} \sum_{i=a-1}^{n-b+1} B_{\geq j-1}^{(a)}(i) B_{\geq \ell-j+1}^{(b)}(n-i) + B_{\geq j-1}^{(g-1)}(n) C_{\ell-j+1} \right) + B_{\geq \ell+1}^{(g-1)}(n).$$

Proof. The first statement and the fact that $B_{\geq \ell}^{(0)}(n) = 0$ are clear from the definitions we have given. The permutations σ counted by $B_{\geq \ell}^{(g)}(1)$ are in $S_{\ell+1}$ and satisfy $\text{tl}(s(\sigma)) \geq \ell$, so they must actually satisfy $s(\sigma) = 123 \cdots (\ell+1)$. Since $\text{leg}(123 \cdots (\ell+1)) = \ell + 2$, the formula for $B_{\geq \ell}^{(g)}(1)$ follows from Theorem 1.1.

Now, let $B_{\ell}^{(g)}(n)$ be the number of 3-stack-sortable permutations $\sigma \in \mathcal{W}_3(n+\ell)$ such that $\text{tl}(s(\sigma)) = \ell$ and $\text{leg}(s(\sigma)) = \ell + g$. Let

$$\mathcal{D}_{\ell}^{(g)}(n) = \{\pi \in \mathcal{W}_2(n+\ell) : \text{tl}(\pi) = \ell, \ \text{leg}(\pi) = \ell + g\} \tag{18}$$

and

$$\mathcal{D}^{(g)}_{\geq \ell}(n) = \{\pi \in \mathcal{W}_2(n+\ell) : \mathrm{tl}(\pi) \geq \ell,\ \mathrm{leg}(\pi) = \ell + g\} \tag{19}$$

so that

$$B^{(g)}_{\ell}(n) = |s^{-1}(\mathcal{D}^{(g)}_{\ell}(n))| \quad \text{and} \quad B^{(g)}_{\geq \ell}(n) = |s^{-1}(\mathcal{D}^{(g)}_{\geq \ell}(n))|.$$

We have $B^{(g)}_{\geq \ell}(n+1) = B^{(g)}_{\ell}(n+1) + B^{(g-1)}_{\geq \ell+1}(n)$, so we need to show that

$$B^{(g)}_{\ell}(n+1) = \sum_{a=2}^{n} \sum_{b=\max\{2,g-a\}}^{g-1} \sum_{i=a-1}^{n-b+1} \sum_{j=1}^{\ell} B^{(a)}_{\geq j-1}(i) B^{(b)}_{\geq \ell-j+1}(n-i) + \sum_{j=1}^{\ell} B^{(g-1)}_{\geq j-1}(n) C_{\ell-j+1}. \tag{20}$$

Suppose $\pi \in \mathcal{D}^{(g)}_{\ell}(n+1)$ is such that $\pi_{n+1-i} = n+1$ (where $n \geq 0$). The Decomposition Lemma (Corollary 2.1) tells us that $|s^{-1}(\pi)|$ is equal to the number of triples (H, μ, λ), where $H \in \mathsf{SW}_{n+1-i}(\pi)$, $\mu \in s^{-1}(\pi^H_U)$, and $\lambda \in s^{-1}(\pi^H_S)$. Choosing H amounts to choosing the number $j \in \{1, \ldots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. The permutation π and the choice of H determine the permutations π^H_U and π^H_S. On the other hand, the choices of H and the permutations π^H_U and π^H_S uniquely determine π. It follows that $B^{(g)}_{\ell}(n+1)$, which is the number of ways to choose an element of $s^{-1}(\mathcal{D}^{(g)}_{\ell}(n+1))$, is also the number of ways to choose j, the permutations π^H_U and π^H_S, and the permutations μ and λ. Let us fix a choice of j.

Assume for the moment that $i \leq n-1$, and let r be the largest entry appearing to the left of $n+1$ in π. Because π is 2-stack-sortable, we can use Theorem 5.1 to see that π^H_U is a permutation of the set $\{1, \ldots, n-i-1\} \cup \{r, n+1\} \cup \{n+2+j, \ldots, n+\ell+1\}$ and that π^H_S is a permutation of $\{n-i, \ldots, n+j\} \setminus \{r, n+1\}$. Therefore, choosing π^H_U and π^H_S is equivalent to choosing their normalizations and the value of r. The normalization of π^H_S is in $\mathcal{D}^{(a)}_{\geq j-1}(i)$ for some $a \in \{2, \ldots, i+1\}$, while the normalization of π^H_U is in $\mathcal{D}^{(b)}_{\geq \ell-j+1}(n-i)$ for some $b \in \{2, \ldots, n-i+1\}$. Once we have chosen a and b, the number of choices for $\pi^H_U, \mu, \pi^H_S, \lambda$ is $B^{(a)}_{\geq j-1}(i) B^{(b)}_{\geq \ell-j+1}(n-i)$.

Suppose we have already chosen the value of a. The fact that π avoids 2341 and the definition of a legal space tell us that there are a possible values of r, say $\kappa_1 < \cdots < \kappa_a$ (see Example 5.1 for an illustration of this part of the proof). If we choose $r = \kappa_m$, then π has $a+b-m+1+\ell$ legal spaces. We are assuming that $\mathrm{leg}(\pi) = \ell+g$, so $g = a+b-m+1$. It follows that $2 \leq a \leq n$ and $\max\{2, g-a\} \leq b \leq g-1$. Since $a \in \{2, \ldots, i+1\}$ and $b \in \{2, \ldots, n-i+1\}$, we also have the constraint $a-1 \leq i \leq n-b+1$. This explains the expression $\sum_{a=2}^{n} \sum_{b=\max\{2,g-a\}}^{g-1} \sum_{i=a-1}^{n-b+1} \sum_{j=1}^{\ell} B^{(a)}_{\geq j-1}(i) B^{(b)}_{\geq \ell-j+1}(n-i)$ in (20).

The expression $\sum_{j=1}^{\ell} B^{(g-1)}_{\geq j-1}(n) C_{\ell-j+1}$ in (20) comes from the case in which $i = n$. In this case, π^H_S is in $\mathcal{D}^{(g-1)}_{\geq j-1}(n)$, and $\pi^H_U = (n+1)(n+2+j)(n+3+j)\cdots(n+\ell+1)$ is an increasing permutation of length $\ell-j+1$. The number of choices for π^H_S and λ is $B^{(g-1)}_{\geq j-1}(n)$. The number of choices for μ is $|s^{-1}(\pi^H_U)| = C_{\ell-j+1}$. $\square$

Example 5.1. Consider the part of the proof of Theorem 5.2 in which we have already chosen n, g, ℓ, j, i and have assumed $i \leq n-1$. Suppose $n = 8$, $\ell = 5$, $j = 2$, and $i = 5$. If we choose the normalization of π^H_U to be 24315678 and choose the normalization of π^H_S to be 315246, then $a = \mathrm{leg}(315246) - (j-1) = 5$ and $b = \mathrm{leg}(24315678) - (\ell-j+1) = 4$. The green dots in Figure 3 represent the possible choices for r, which are $\kappa_1 = 4$, $\kappa_2 = 5$, $\kappa_3 = 7$, $\kappa_4 = 8$, and $\kappa_5 = 9$. If $r = \kappa_m$, then we can refer to this figure to see that $\mathrm{leg}(\pi) = 15 - m = \ell + a + b - m + 1$. Hence, the choice of r is determined by the value of g.

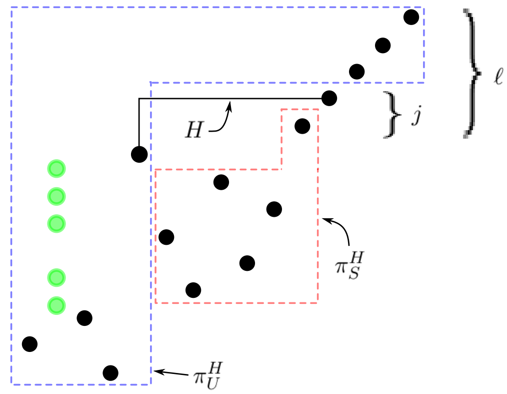


FIGURE 3. The decomposition of π into π_U^H and π_S^H along with the possible choices for r.

The proof of Theorem 5.2 generalizes, allowing us to obtain a recurrence for $W_3(n,k,p)$, the number of 3-stack-sortable permutations in S_n with k descents and p peaks. We actually state the following theorem in terms of polynomials, but one can easily obtain the desired recurrence by comparing coefficients. In what follows, let

$$E_{\geq \ell}^{(g)}(n) = \sum_{\sigma \in s^{-1}(\mathcal{D}_{\geq \ell}^{(g)}(n))} x^{\text{des}(\sigma)+1} y^{\text{peak}(\sigma)+1},$$

where $\mathcal{D}_{\geq \ell}^{(g)}(n)$ is as in (19). We have suppressed the dependence on x and y in our notation for readability. Let $L_r(x,y)$ be as in (8).

Theorem 5.3. *If $n \geq 1$, then*

$$\sum_{\sigma \in \mathcal{W}_3(n)} x^{\text{des}(\sigma)+1} y^{\text{peak}(\sigma)+1} = \sum_{g=1}^{n+1} E_{\geq 0}^{(g)}(n).$$

We have $E_{\geq \ell}^{(0)}(n) = 0$ and

$$E_{\geq \ell}^{(g)}(1) = \begin{cases} 0, & \text{if } g \neq 2; \\ L_{\ell+1}(x,y), & \text{if } g = 2. \end{cases}$$

If $n, g \geq 1$ and $\ell \geq 0$, then

$$E_{\geq \ell}^{(g)}(n+1) = \sum_{j=1}^{\ell} \left(\sum_{a=2}^{n} \sum_{b=\max\{2,g-a\}}^{g-1} \sum_{i=a-1}^{n-b+1} E_{\geq j-1}^{(a)}(i) E_{\geq \ell-j+1}^{(b)}(n-i) + E_{\geq j-1}^{(g-1)}(n) L_{\ell-j+1}(x,y) \right)$$
$$+ E_{\geq \ell+1}^{(g-1)}(n).$$

Proof. To derive the formula for $E_{\geq \ell}^{(g)}(1)$, we follow the same argument used to find the formula for $B_{\geq \ell}^{(g)}(1)$ in the proof of Theorem 5.2, except we use Remark 3.1 instead of Theorem 1.1. To derive the last statement in this theorem, we follow the rest of the proof of Theorem 5.2, except we invoke the Refined Decomposition Lemma instead of the Decomposition Lemma and again use Remark 3.1 instead of Theorem 1.1. ☐

6. Data Analysis

The *sum* of two permutations μ and λ, denoted $\mu \oplus \lambda$, is the permutation whose plot is obtained by placing the plot of λ above and to the right of the plot of μ. It is easy to check that the sum of two t-stack-sortable permutations is t-stack-sortable. It follows that $W_t(m+n) \geq W_t(m)W_t(n)$ for all $m, n \geq 1$. We express this by saying the sequence $(W_t(n))_{n\geq 1}$ is *supermultiplicative.* It follows from Fekete's lemma that

$$\lim_{n\to\infty} \frac{W_t(n+1)}{W_t(n)} = \lim_{n\to\infty} W_t(n)^{1/n} = \sup_{n\geq 1} W_t(n)^{1/n}. \tag{21}$$

We have used Theorem 5.2 to compute the numbers $W_3(n)$ for $1 \leq n \leq 174$. We have added these terms to the OEIS sequence A134664. This allows us to prove the first nontrivial lower bound for $\lim_{n\to\infty} W_3(n)^{1/n}$. Note that this is better than the lower bound of $(\sqrt{3}+1)^2$ obtained in Section 7.

Theorem 6.1. *We have*

$$\lim_{n\to\infty} W_3(n)^{1/n} \geq 8.659702.$$

Proof. The value of $W_3(174)$ is

1335109055832443343636882328903941541553316885478273864987091560565206631540380152
7870514001230180265889501841168312512206012823853129556966628901079194868270269904,

and the 174^{th} root of this number is slightly more than 8.659702. The proof follows from (21). $\square$

We can also show that Bóna's Conjectures 1.1 and 1.2 contradict each other.

Theorem 6.2. *If $(W_3(n))_{n\geq 1}$ is log-convex, then $W_3(n) > \binom{4n}{n}$ for all sufficiently large n.*

Proof. It follows from Stirling's formula that $\lim_{n\to\infty} \binom{4n}{n}^{1/n} = 256/27 \approx 9.4815$. Also, $\frac{W_3(174)}{W_3(173)} \approx 9.4907$. If $(W_3(n))_{n\geq 1}$ is log-convex, then $\lim_{n\to\infty} W_3(n)^{1/n} = \lim_{n\to\infty} \frac{W_3(n+1)}{W_3(n)} \geq 9.4907 > 9.4815$. $\square$

We now turn our attention to the parity of $W_3(n)$ and Bóna's Conjecture 1.3. Let $\varepsilon_t(n)$ be the number in $\{0,1\}$ with the same parity as $W_t(n)$. As mentioned in the introduction, $\varepsilon_t(n) = 0$ whenever n is even. The values of $\varepsilon_3(2n+1)$ for $0 \leq n \leq 86$ are

(22) $1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0,$
$0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0.$

Letting $\mathfrak{g}_t(m) = \sum_{n=1}^{m} \varepsilon_t(n)$, we can use this data to find that $\mathfrak{g}_1(m) < \mathfrak{g}_2(m) \leq \mathfrak{g}_3(m)$ whenever $13 \leq m \leq 660$. Therefore, it appears that $W_3(n)$ is odd more frequently than $W_2(n)$! If this is true, then Bóna's Conjecture 1.3 is certainly false. We state some new conjectures and open problems concerning the parities of the numbers $W_3(n)$ in Section 9.

Let us end this section by recording one final proposition, which verifies Bóna's Conjecture 1.4 in several new cases. We have obtained this proposition by computing several values of $W_3(n,k)$ via Theorem 5.3 (setting $y = 1$ in that theorem).

Proposition 6.1. *If $1 \leq n \leq 43$, then the polynomial $\sum_{\sigma\in\mathcal{W}_3(n)} x^{\text{des}(\sigma)}$ has only real roots.*

7. Lower Bounds for t-Stack-Sortable Permutations

Let Γ_t be the set of all $\kappa = \kappa_1 \cdots \kappa_{t+2} \in S_{t+2}$ such that $\kappa_{t+1} = t+2$ and $\kappa_{t+2} = 1$. Let $\mathrm{Av}_n(\Gamma_t)$ be the set of permutations in S_n that avoid all of the patterns in Γ_t. After applying a dihedral symmetry to the permutations in Γ_t, we can use a result of Kremer [36, 37] to see that

$$\sum_{n\geq t} |\operatorname{Av}_n(\Gamma_t)|x^n = (t-1)!x^{t-2}\frac{1+(t-1)x-\sqrt{1-2(t+1)x+(t-1)^2x^2}}{2}. \tag{23}$$

Some basic singularity analysis now shows that $\lim_{n\to\infty} |\operatorname{Av}_n(\Gamma_t)|^{1/n} = (\sqrt{t}+1)^2$.

We will prove by induction that $\mathrm{Av}_n(\Gamma_t) \subseteq \mathcal{W}_t(n)$. Since $\Gamma_1 = \{231\}$, this is certainly true for $t = 1$ (by Theorem 1.1). Now suppose that $t \geq 2$ and that $\mathrm{Av}_n(\Gamma_{t-1}) \subseteq \mathcal{W}_{t-1}(n)$. Choose a permutation $\pi \in S_n \setminus \mathcal{W}_t(n)$. This means that $s(\pi) \notin \mathcal{W}_{t-1}(n)$, so $s(\pi)$ contains a permutation in Γ_{t-1}. In other words, there exist entries $b_1, \ldots, b_{t-1}, c, a$ that appear in this order in $s(\pi)$ and satisfy $a < b_j < c$ for all $j \in \{1, \ldots, t-1\}$. Because c appears to the left of a in $s(\pi)$, there must be an entry $d > c$ that appears to the right of c and to the left of a in π. The entries $b_1, \ldots, b_{t-1}$ must appear to the left of d in π since they would appear to the right of c in $s(\pi)$ otherwise. The subpermutation of π formed by the entries $a, b_1, \ldots, b_{t-1}, c, d$ has a normalization that is in Γ_t, so $\pi \notin \mathrm{Av}_n(\Gamma_t)$. This completes the induction and proves the following theorem.

Theorem 7.1. *For every $t \geq 1$, we have*

$$\lim_{n\to\infty} W_t(n)^{1/n} \geq (\sqrt{t}+1)^2.$$

In [39], Smith investigated a variant of the stack-sorting map known as the "left-greedy algorithm." Let $\widehat{\mathcal{W}}_t(n)$ be the set of permutations in S_n that can be sorted by t stacks in series using the left-greedy algorithm (see her paper for definitions). Smith proved that $\mathcal{W}_t(n) \subseteq \widehat{\mathcal{W}}_t(n)$ and that $|\widehat{\mathcal{W}}_t(n)| \geq \dfrac{t!}{(t+1)^t}(t+1)^n$ whenever $n \geq t \geq 1$. In terms of exponential growth rates, this shows that $\lim_{n\to\infty} |\widehat{\mathcal{W}}_t(n)|^{1/n} \geq t+1$ (using Fekete's lemma, one can show that this limit exists). The following corollary of Theorem 7.1 improves this estimate.

Corollary 7.1. *For every $t \geq 1$, we have $\lim_{n\to\infty} |\widehat{\mathcal{W}}_t(n)|^{1/n} \geq (\sqrt{t}+1)^2$.*

8. Symmetry, Unimodality, γ-Nonnegativity, Log-Concavity, and Real-Rootedness

We devote this brief section to showing how Brändén's theorem concerning γ-nonnegativity (Theorem 1.3) follows easily from Theorem 3.1. We also show that the analogue of that theorem with "γ-nonnegative" replaced by "real-rooted" is false. Let us begin by recalling some definitions.

A polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{R}_{\geq 0}[x]$ is called

- *symmetric* if $a_i = a_{n-i}$ for all $i \in \{0, \ldots, n\}$; in this case, $n/2$ is called the *center of symmetry* of $p(x)$;
- *unimodal* if there exists $j \in \{0, \ldots, n\}$ such that $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$;
- *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i \in \{1, \ldots, n-1\}$;
- *real-rooted* if all of the complex roots of $p(x)$ are real.

If $p(x)$ is a symmetric polynomial with center of symmetry $n/2$, then it can be written in the form $p(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \gamma_m x^m (1+x)^{n-2m}$ for some real numbers γ_m. We then say $p(x)$ is *γ-nonnegative* if the numbers γ_m are all nonnegative. We have the following implications among these properties for polynomials in $\mathbb{R}_{\geq 0}[x]$ [15]:

$$\text{real-rooted} \implies \text{log-concave} \implies \text{unimodal};$$

$$\text{symmetric and real-rooted} \implies \gamma\text{-nonnegative} \implies \text{symmetric and unimodal}.$$

New Proof of Theorem 1.3.[2] Note that it suffices to prove Theorem 1.3 in the specific case in which $A = \{\pi\}$ is a singleton set. Indeed, the result for a general set $A \subseteq S_n$ then follows by summing over all $\pi \in A$. Thus, let us fix a permutation $\pi \in S_n$ with exactly k descents.

Recall the notation from (4) and (5). One can show that

$$N_q(x) = \sum_{m=0}^{q} \frac{V(q, m+1)}{2^{q-1-2m}} x^{m+1} (1+x)^{q-1-2m}$$

for all $q \geq 1$. Therefore, for $(q_0, \ldots, q_k) \in \text{Comp}_{k+1}(n-k)$, we have

$$\prod_{t=0}^{k} N_{q_t}(x) = \prod_{t=0}^{k} \sum_{m_t=0}^{q_t} \frac{V(q_t, m_t+1)}{2^{q_t-1-2m_t}} x^{m_t+1} (1+x)^{q_t-1-2m_t}$$

$$= \sum_{m=0}^{n} \sum_{\substack{m_0+\cdots+m_k=m-k \\ m_0,\ldots,m_k \geq 0}} \frac{1}{2^{n-1-2m}} \left(\prod_{t=0}^{k} V(q_t, m_t+1) \right) x^{m+1} (1+x)^{n-1-2m}.$$

Let $\mathcal{V}(\pi) \subseteq \text{Comp}_{k+1}(n-k)$ be the set of compositions from Theorem 3.1. Invoking equation (10) from that theorem, we obtain

$$\sum_{\sigma \in s^{-1}(\pi)} x^{\text{des}(\sigma)+1} = \sum_{(q_0,\ldots,q_k) \in \mathcal{V}(\pi)} \prod_{t=0}^{k} N_{q_t}(x)$$

$$= \sum_{(q_0,\ldots,q_k) \in \mathcal{V}(\pi)} \sum_{m=0}^{n} \sum_{\substack{m_0+\cdots+m_k=m-k \\ m_0,\ldots,m_k \geq 0}} \frac{1}{2^{n-1-2m}} \left(\prod_{t=0}^{k} V(q_t, m_t+1) \right) x^{m+1} (1+x)^{n-1-2m}$$

$$= \sum_{m=0}^{n} \frac{1}{2^{n-1-2m}} x^{m+1} (1+x)^{n-1-2m} \sum_{(q_0,\ldots,q_k) \in \mathcal{V}(\pi)} \sum_{(m_0',\ldots,m_k') \in \text{Comp}_{k+1}(m+1)} \prod_{t=0}^{k} V(q_t, m_t'),$$

where we have made the substitution $m_i' = m_i + 1$. It turns out that

$$\sum_{(q_0,\ldots,q_k) \in \mathcal{V}(\pi)} \sum_{(m_0',\ldots,m_k') \in \text{Comp}_{k+1}(m+1)} \prod_{t=0}^{k} V(q_t, m_t')$$

is the coefficient of y^{m+1} in the polynomial on the right-hand side of (11), so it is equal to

$$|\{\sigma \in s^{-1}(\pi) : \text{peak}(\sigma) = m\}|.$$

[2] To deduce Bóna's symmetry and unimodality result from Theorem 3.1, one simply needs to observe that this theorem tells us that $\sum_{\sigma \in \mathcal{W}_t(n)} x^{\text{des}(\sigma)+1}$ is a sum of products of Narayana polynomials with the same center of symmetry and then use the well-known fact that Narayana polynomials are real-rooted.

Note that this is 0 if $m > \frac{n-1}{2}$. Hence,

$$\sum_{\sigma\in s^{-1}(\pi)} x^{\mathrm{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(\pi) : \mathrm{peak}(\sigma) = m\}|}{2^{n-1-2m}} x^m (1+x)^{n-1-2m}. \qquad \square$$

We now give an example to show that Theorem 1.3 is false if the term "γ-nonnegative" is replaced by "real-rooted." Let

$$\mu = 6\ 7\ 8\ 4\ 5\ 9\ 10\ 1\ 2\ 3\ 11 \quad \text{and} \quad \mu' = 6\ 7\ 8\ 9\ 2\ 3\ 4\ 5\ 10\ 1\ 11.$$

We claim that $\sum_{\sigma\in s^{-1}(\{\mu,\mu'\})} x^{\mathrm{des}(\sigma)}$ is not real rooted. To see this, we use the fact[3] that $\mathcal{V}(\mu) = \{(4,2,3),(3,3,3)\}$ and $\mathcal{V}(\mu') = \{(4,4,1)\}$. Using (10), we find that

$$\sum_{\sigma\in s^{-1}\{\mu,\mu'\}} x^{\mathrm{des}(\sigma)} = \frac{1}{x}\sum_{\sigma\in s^{-1}(\mu)} x^{\mathrm{des}(\sigma)+1} + \frac{1}{x}\sum_{\sigma\in s^{-1}(\mu')} x^{\mathrm{des}(\sigma)+1}$$

$$= \frac{1}{x}\left(N_4(x)N_2(x)N_3(x) + N_3(x)N_3(x)N_3(x)\right) + \frac{1}{x}N_4(x)N_4(x)N_1(x)$$

$$= 3x^2 + 31x^3 + 112x^4 + 169x^5 + 112x^6 + 31x^7 + 3x^8,$$

and this polynomial is not real-rooted. This example yields a negative answer to the last part of Question 12.1 in [26].

Remark 8.1. Theorem 1.3 diverges from Bóna's point of view in Conjecture 1.4 by replacing the sum over $s^{-1}(\mathcal{W}_{t-1}(n))$ with a sum over $s^{-1}(A)$ for an arbitrary set $A \subseteq S_n$. This different viewpoint suggests that the sets of the form $\mathcal{W}_t(n) = s^{-1}(\mathcal{W}_{t-1}(n))$ might not be too special when compared with arbitrary sets of the form $s^{-1}(A)$ for $A \subseteq S_n$. If one believes Conjecture 1.4, then the preceding example lends credence to the hypothesis that the sets $\mathcal{W}_t(n)$ are special. On the other hand, if one does not believe there is anything special about the sets $\mathcal{W}_t(n)$, then this example hints that Conjecture 1.4 might be false.

9. Conjectures and Open Problems

We saw in Theorem 6.2 that Conjectures 1.1 and 1.2 cannot both be true. Our data suggests that Conjecture 1.2 is true. Moreover, by plotting the points $(1/n, W_3(n))$ for $1 \le n \le 174$, we have arrived at the following new conjecture.

Conjecture 9.1. *We have*

$$9.702 < \lim_{n\to\infty} W_3(n)^{1/n} < 9.704.$$

We also believe that the Decomposition Lemma could be used (possibly along with a significant amount of work) to find a lower bound for $\lim\limits_{n\to\infty} W_4(n)^{1/n}$ that exceeds 9.704.

Turning back to the parities of the numbers $W_3(n)$, we have the following problem.

Problem 9.1. *Characterize those positive integers n such that $W_3(n)$ is odd.*

[3]The reader interested in seeing why this is the case can refer to [26] for the full definition of $\mathcal{V}(\pi)$ and a description of how to compute it. However, the reader wishing to avoid this definition can still compute $\sum_{\sigma\in s^{-1}(\{\mu,\mu'\})} x^{\mathrm{des}(\sigma)}$ using a brute-force computer program that simply finds all of the permutations in $s^{-1}(\{\mu,\mu'\})$. *A priori*, a brute-force computer program would not easily *find* this example since it would have to search over *subsets* of S_{11}.

Now that we have obtained a recurrence for the numbers $W_3(n)$ in Theorem 5.2, Problem 9.1 seems within reach. Indeed, it appears as though there could be some patterns in the sequence whose initial terms are listed in (22). Solving this problem could require going through the proof of Theorem 5.2 and seeing which terms in the various sums simplify when we reduce modulo 2.

Recall the definition of $\mathfrak{g}_t(m)$ from Section 6. We have the following conjectures. Conjectures 9.3, 9.4, and 9.5 each contradict Bóna's Conjecture 1.3.

Conjecture 9.2. *The limit* $\displaystyle\lim_{n\to\infty}\frac{\log\mathfrak{g}_3(m)}{\log m}$ *exists.*

Conjecture 9.3. *We have* $\displaystyle\liminf_{n\to\infty}\frac{\log\mathfrak{g}_3(m)}{\log m}>0.$

Conjecture 9.4. *For every integer* $m\geq 13$*, we have* $\mathfrak{g}_2(m)\leq\mathfrak{g}_3(m)$*.*

Conjecture 9.5. *We have* $\displaystyle\lim_{m\to\infty}(\mathfrak{g}_3(m)-\mathfrak{g}_2(m))=\infty.$

In [26], the current author conjectured that for every $\pi\in S_n$, the polynomial $\sum_{\sigma\in s^{-1}(\pi)}x^{\mathrm{des}(\sigma)}$ is real-rooted. The example given in the previous section shows that this conjecture is false if we replace the singleton set $\{\pi\}$ by an arbitrary subset of S_n. Nevertheless, we can make the following new conjecture.

Conjecture 9.6. *If* $A\subseteq S_n$*, then the polynomial* $\displaystyle\sum_{\sigma\in s^{-1}(A)}x^{\mathrm{des}(\sigma)}$ *is log-concave.*

In [26], the current author showed that $|s^{-1}(\mathrm{Av}_n(123\cdots m))|=0$ when $n\geq 2^{m-1}$ and conjectured that the sequence $(|s^{-1}(\mathrm{Av}_n(123\cdots m))|)_{n=1}^{2^{m-1}-1}$ is unimodal for every $m\geq 2$. We wish to generalize this conjecture as follows. One can define $\mathrm{Av}(\tau^{(1)},\tau^{(2)},\ldots)$ for an infinite list of permutation patterns $\tau^{(1)},\tau^{(2)},\ldots$ in the obvious way. There is also a natural definition of unimodality for infinite sequences of nonnegative numbers. We make the convention that monotonic sequences are unimodal.

Conjecture 9.7. *If* $\tau^{(1)},\tau^{(2)},\ldots$ *is a (finite or infinite) list of permutation patterns, then the sequence* $(|s^{-1}(\mathrm{Av}_n(\tau^{(1)},\tau^{(2)},\ldots))|)_{n\geq 1}$ *is unimodal.*

10. Acknowledgments

I would like to express my deepest gratitude to Niven Achenjang, Amanda Burcroff, and Eric Winsor for writing computer programs that calculated the numbers $W_3(n)$ much faster than the author's original program. I would also like to thank Jay Pantone for running one of these programs on his computer for several days and for analyzing the resulting data. The contributions that these people made were paramount to the analysis discussed in Section 6. I thank Miklós Bóna and Doron Zeilberger for helpful conversations. I also thank Caleb Ji, who wrote a poem that inexplicably predicted I would make progress in the study of 3-stack-sortable permutations.

The author was supported by a Fannie and John Hertz Foundation Fellowship and an NSF Graduate Research Fellowship.

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