

TILINGS OF CONVEX POLYGONS BY EQUILATERAL TRIANGLES OF MANY DIFFERENT SIZES

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ABSTRACT. An equilateral triangle cannot be dissected into finitely many mutually incongruent equilateral triangles [17]. Therefore Tuza [18] asked for the largest number $s = s(n)$ such that there is a tiling of an equilateral triangle by n equilateral triangles of $s(n)$ different sizes. We solve that problem completely and consider the analogous questions for dissections of convex k -gons into equilateral triangles, $k = 4, 5, 6$. Moreover, we discuss all these questions for the subclass of tilings such that no two tiles are translates of each other.

1. OVERVIEW

1.1. Motivation. A (finite) *tiling* of a subset A of the Euclidean plane \mathbb{R}^2 is a family $\mathcal{T} = \{A_1, \dots, A_n\}$ of subsets of A , called *tiles*, such that $A = A_1 \cup \dots \cup A_n$ and the interiors of A_1, \dots, A_n are mutually disjoint. A tiling is called *perfect* if all tiles are images of each other under similarity transformations, but mutually incongruent under isometries. The concept of a perfect tiling has been introduced in [2] for the construction of perfect dissections of rectangles and even of squares into squares. Tutte showed that there are no perfect tilings of triangles by (at least two) equilateral triangles [17, Theorem 2·12] (see also [2, Section 10.3]). In fact, no convex polygon admits a perfect tiling by equilateral triangles, as has been shown by Buchman [3, Section 2] and Tuza [18, Theorem 1].

Therefore Tuza considered tilings of triangles by equilateral triangles that are nearly perfect in the sense that only few of the tiles are isometrically congruent: The *size* $\text{size}(T)$ of an equilateral triangle T is measured by the length of (one of) its sides. For a family $\mathcal{T} = \{T_1, \dots, T_n\}$ of equilateral triangles, we define $s(\mathcal{T}) = |\{\text{size}(T_1), \dots, \text{size}(T_n)\}|$, where $|\cdot|$ denotes the cardinality of a set. Tuza introduced the numbers

$$s_{\text{tri}}(n) = \max\{s(\mathcal{T}) : \mathcal{T} \text{ is a tiling of a triangle by } n \text{ equilateral triangles}\}.$$

He showed by an inflation argument that there exists a constant c , $\frac{5}{7} \leq c \leq 1$, such that $s_{\text{tri}}(n) = cn - o(n)$ as $n \rightarrow \infty$ [18, Theorem 2] and he asked for the exact values of $s_{\text{tri}}(n)$ [18, Problem 1] or at least for the exact value of c . We solve that problem completely in Theorem 1(a).

If a convex polygon P admits a tiling by equilateral triangles, its inner angles are of size $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ and P must be an equilateral triangle (all inner angles have size $\frac{\pi}{3}$), a trapezoid (sizes of angles are $\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$ in cyclic order), a parallelogram ($\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$), a pentagon ($\frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$) or a hexagon (six times $\frac{2\pi}{3}$). (*When speaking of trapezoids in the present paper we mean trapezoids that are no parallelograms.*) As we know that no convex polygon has a perfect tiling by equilateral

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triangles, we introduce relatives of Tuza's numbers, namely

$$s_{\text{trap}}(n) = \max\{s(\mathcal{T}) : \mathcal{T} \text{ is a tiling of some convex trapezoid by } n \text{ equilateral triangles}\}$$

and, similarly, $s_{\text{par}}(n)$, $s_{\text{pent}}(n)$ and $s_{\text{hex}}(n)$ with 'trapezoid' replaced by 'parallelogram', 'pentagon' and 'hexagon', respectively. Theorem 1 gives our results on these numbers including the determination of the *domains* of the functions $s_{\text{tri}}, \dots, s_{\text{hex}}$. These are the sets of all integers n such that there exists a tiling of a suitable convex polygon of the respective shape by n equilateral triangles.

Another possibility of weakening the property of perfectness is based on the fact that, if two tiles of a tiling of a convex set by equilateral triangles are congruent under isometries, then they are either congruent under some translation or under some rotation by an angle of π . Some authors call the tiling already perfect if no two tiles are congruent under translations [17, 5]. In order to avoid confusion with (isometric) perfectness, we shall speak of translational perfectness (or t-perfectness for short). That is, a tiling by equilateral triangles is called *t-perfect* if no two tiles are translates of each other. First examples of t-perfect tilings of parallelograms and triangles by equilateral triangles are given in [17]. Drápal and Hämäläinen [5] present a systematic computational approach to dissections of equilateral triangles into equilateral triangles with a particular emphasis on the t-perfect case. Illustrations of all t-perfect tilings of triangles by up to 19 equilateral triangles are given in [7].

We are interested in t-perfect tilings of convex polygons by equilateral triangles that are close to be (isometrically) perfect in so far as the number of tiles of equal size is as small as possible. More precisely, we study the numbers

$$s_{\text{tri}}^{\text{t-perf}}(n) = \max\{s(\mathcal{T}) : \mathcal{T} \text{ is a t-perfect tiling of a triangle by } n \text{ equilateral triangles}\}$$

as well as the analogous variants $s_{\text{trap}}^{\text{t-perf}}(n)$, $s_{\text{par}}^{\text{t-perf}}(n)$, $s_{\text{pent}}^{\text{t-perf}}(n)$ and $s_{\text{hex}}^{\text{t-perf}}(n)$ of $s_{\text{trap}}(n)$, $s_{\text{par}}(n)$, $s_{\text{pent}}(n)$ and $s_{\text{hex}}(n)$, respectively. Our corresponding results are summarized in Theorem 2.

Finally, let us point out that the study of dissections into incongruent equilateral triangles is a fruitful field of ongoing research, see e.g. [4, Section C11], [6, Exercise 2.4.10], [9, Problem 4] and [1, 8, 11, 12, 13, 14, 15].

1.2. Main results and open problems.

Theorem 1. (a) $\text{dom}(s_{\text{tri}}) = \{1, 4\} \cup \{6, 7, \dots\}$ and

$$s_{\text{tri}}(n) = \begin{cases} 1, & n = 1, 4, \\ 2, & n = 6, \\ n - 5, & n = 7, 8, \dots \end{cases}$$

(b) $\text{dom}(s_{\text{trap}}) = \{3\} \cup \{5, 6, \dots\}$ and

$$s_{\text{trap}}(n) = \begin{cases} 1, & n = 3, \\ 2, & n = 5, \\ n - 4, & n = 6, 7, \dots \end{cases}$$

(c) $\text{dom}(s_{\text{par}}) = \{2\} \cup \{4, 5, \dots\}$ and

$$s_{\text{par}}(n) = \begin{cases} 1, & n = 2, 4, \\ 2, & n = 5, \\ n - 4, & n = 6, 7, \dots \end{cases}$$

(d) $\text{dom}(s_{\text{pent}}) = \{4, 5, \dots\}$ and

$$s_{\text{pent}}(n) = \begin{cases} 2, & n = 4, \\ n - 3, & n = 5, 6, \dots \end{cases}$$

(e) $\text{dom}(s_{\text{hex}}) = \{6, 7, \dots\}$ and

$$s_{\text{hex}}(n) \begin{cases} = n - 5, & n = 6, 7, 8, \\ = n - 4, & n = 9, 10, \dots, 19; 21, 22; 24, 25, \\ \in \{n - 5, n - 4\}, & n = 20; 23; 26, 27, \dots \end{cases}$$

Here only the case of hexagons is not completely solved. All tilings we know to attain the upper bound $s_{\text{hex}}(n) \leq n - 4$ ($n \geq 9$) are displayed in the appendix (tilings (a)-(s)). Is this list complete?

The situation of t-perfect tilings appears more difficult. Only for pentagons we have a full solution. However, in the open cases the differences between upper and lower estimates are at most 2. In the case of hexagons it remains open if there are t-perfect tilings by 12 or 13 triangles. We conjecture that $12, 13 \notin \text{dom}(s_{\text{hex}}^{\text{t-perf}})$.

Theorem 2. (a) $\text{dom}(s_{\text{tri}}^{\text{t-perf}}) = \{1\} \cup \{15, 16, \dots\}$ and

$$s_{\text{tri}}^{\text{t-perf}}(n) \begin{cases} = 1, & n = 1, \\ = n - 5, & n = 15; 17, 18, \dots, 26; 28, \\ = n - 6, & n = 16, \\ \in \{n - 6, n - 5\}, & n = 27; 29, 30, \dots \end{cases}$$

(b) $\text{dom}(s_{\text{trap}}^{\text{t-perf}}) = \{13, 14, \dots\}$ and

$$s_{\text{trap}}^{\text{t-perf}}(n) \begin{cases} = n - 4, & n = 14; 16, 17, \dots, 25; 27, \\ \in \{n - 5, n - 4\}, & n = 13; 15; 26; 28, 29, \dots \end{cases}$$

(c) $\text{dom}(s_{\text{par}}^{\text{t-perf}}) = \{2\} \cup \{13, 14, \dots\}$ and

$$s_{\text{par}}^{\text{t-perf}}(n) \begin{cases} = 1, & n = 2, \\ = n - 4, & n = 15; 18, 19; 21, 22, 23; 26, \\ \in \{n - 5, n - 4\}, & n = 13, 14; 16, 17; 20; 24, 25; 27, 28, \dots \end{cases}$$

(d) $\text{dom}(s_{\text{pent}}^{\text{t-perf}}) = \{12, 13, \dots\}$ and

$$s_{\text{pent}}^{\text{t-perf}}(n) = n - 4 \text{ for all } n = 12, 13, \dots$$

(e) $\{11\} \cup \{14, 15, \dots\} \subseteq \text{dom}(s_{\text{hex}}^{\text{t-perf}}) \subseteq \{11, 12, \dots\}$ and

$$s_{\text{hex}}^{\text{t-perf}}(n) \begin{cases} = n - 4, & n = 11; 14, 15; 17, 18, 19; 22, \\ \in \{n - 5, n - 4\}, & n = 16; 20, 21; 23, \\ \in \{n - 6, n - 5, n - 4\}, & n = 24, 25, \dots \end{cases}$$

2. PROOF OF THEOREM 1

2.1. Spiral pentagons and related tilings. The *Padovan spiral numbers* [16, 10] are defined recursively by

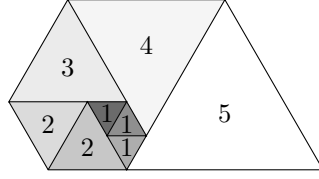
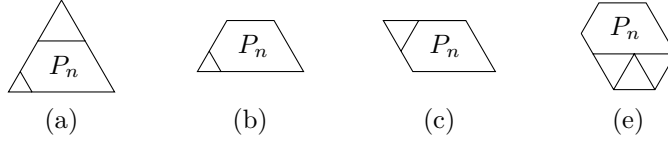
$$(1) \quad p(0) = p(1) = p(2) = 1 \quad \text{and} \quad p(n) = p(n-3) + p(n-2) \text{ for } n = 3, 4, \dots$$

Lemma 3. (a) *The Padovan spiral numbers satisfy*

- $p(0) = p(1) = p(2) = 1$, $p(3) = p(4) = 2$ and $p(n) > p(n-1)$ for $n \geq 5$,
- $p(n-3) < \frac{1}{2}p(n) < p(n-2)$ for $n \notin \{3, 4, 6\}$.

(b) *For every $n \in \{4, 5, \dots\}$, there is a convex pentagon P_n with sides of lengths $p(n-4)$, $p(n-3)$, $p(n-2)$, $p(n-1)$ and $p(n)$ that admits a tiling by n equilateral triangles T_i with $\text{size}(T_i) = p(i-1)$, $i = 1, \dots, n$ (see [16]).*

Proof. The first part of (a) follows from (1) by induction. It implies the second one by $p(n) = p(n-3) + p(n-2)$. Claim (b) can be found in [16], see Figure 1: the pentagon P_{n+1} is obtained from P_n by adding a triangle of size $p(n)$ at the longest side of P_n . \square

FIGURE 1. The spiral pentagon P_n (here with $n = 8$), cf. [16]FIGURE 2. Modifications of P_n by added equilateral triangles

Corollary 4. (a) $\{6, 7, \dots\} \subseteq \text{dom}(\text{s}_{\text{tri}})$ and

$$\text{s}_{\text{tri}}(n) \geq \begin{cases} 2, & n = 6, \\ n - 5, & n = 7, 8, \dots \end{cases}$$

(b) $\{5, 6, \dots\} \subseteq \text{dom}(\text{s}_{\text{trap}})$ and

$$\text{s}_{\text{trap}}(n) \geq \begin{cases} 2, & n = 5, \\ n - 4, & n = 6, 7, \dots \end{cases}$$

(c) $\{5, 6, \dots\} \subseteq \text{dom}(\text{s}_{\text{par}})$ and

$$\text{s}_{\text{par}}(n) \geq \begin{cases} 2, & n = 5, \\ n - 4, & n = 6, 7, \dots \end{cases}$$

(d) $\{4, 5, \dots\} \subseteq \text{dom}(\text{s}_{\text{pent}})$ and

$$\text{s}_{\text{pent}}(n) \geq \begin{cases} 2, & n = 4, \\ n - 3, & n = 5, 6, \dots \end{cases}$$

(e) $\{7, 8, \dots\} \subseteq \text{dom}(\text{s}_{\text{hex}})$ and $\text{s}_{\text{hex}}(n) \geq n - 5$ for all $n = 7, 8, \dots$

Proof. The pentagons P_n , $n = 4, 5, \dots$, from Lemma 3(b) prove (d). For (a), we add two triangles of sizes $p(n - 4)$ and $p(n - 2)$ at the respective sides of P_n , cf. Figure 2. For (b), we add only one of these two triangles. For (c), we add a triangle of size $p(n - 3)$. Every added triangle is of the same size as one of the original tiling of P_n from Lemma 3(b).

For (e), we add three triangles of size $\frac{1}{2}p(n)$ at the longest side of length $p(n)$ if $n \neq 6$, see Figure 2. By Lemma 3(a), the size of the new triangles differs from all sizes of the given tiling of P_n if $n = 5$ or $n \geq 7$, whereas that size appears already in the tiling of P_n if $n = 4$. For $n = 6$, we add three triangles of size $\frac{1}{2}p(5) = \frac{3}{2} \notin \{p(0), \dots, p(5)\}$ at the side of size $p(5) = 3$ of P_6 . \square

2.2. Domains and lower estimates. If a convex polygon P has a tiling \mathcal{T} by equilateral triangles, its inner angles are of sizes $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. A side between two vertices of sizes α and β is called an (α, β) -side of P . All vertices of triangles from \mathcal{T} are called *vertices of \mathcal{T}* . We speak of $\frac{\pi}{3}$ -vertices or $\frac{2\pi}{3}$ -vertices of \mathcal{T} , if they coincide with vertices of P of the respective sizes, of π -vertices of \mathcal{T} , if they are no vertices of P but on the boundary of P , and of 2π -vertices of \mathcal{T} if they are in the interior of P . A triangle $T \in \mathcal{T}$ is called *exposed* if it contains a $\frac{\pi}{3}$ -vertex of \mathcal{T} or, equivalently, if $P \setminus T$ is still convex.

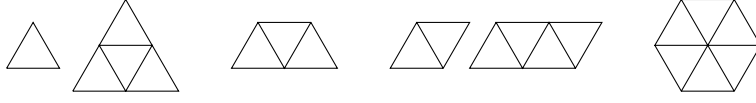


FIGURE 3. Tilings of small cardinality and with congruent tiles

Let us show that

- (2) $2, 3; 5 \notin \text{dom}(s_{\text{tri}}),$
(3) $1, 2; 4 \notin \text{dom}(s_{\text{trap}}),$
(4) $1; 3 \notin \text{dom}(s_{\text{par}}),$
(5) $1, 2, 3 \notin \text{dom}(s_{\text{pent}}),$
(6) $1, 2, 3, 4, 5 \notin \text{dom}(s_{\text{hex}}).$

To see (2), let n be the cardinality of a tiling \mathcal{T} of some triangle T and let v_π be the number of π -vertices of \mathcal{T} . If $v_\pi = 0$ then $n = 1$. If $1 \leq v_\pi \leq 3$ then $v_\pi = 3$ and \mathcal{T} splits into three congruent exposed triangles and a tiling \mathcal{T}' of the remaining triangle T' of T . The cardinality $|\mathcal{T}'| = n - 3$ cannot be two, whence $n = 4$ (if $|\mathcal{T}'| = 1$) or $n \geq 6$ (if $|\mathcal{T}'| \geq 3$). If $v_\pi \geq 4$ then one side of T contains two consecutive π -vertices v_1, v_2 . Hence $n \geq 6$, because \mathcal{T} contains three exposed triangles, one triangle covering the segment v_1v_2 and two triangles for covering the yet uncovered remainders of neighbourhoods of v_1 and v_2 .

For (3), note that $n \in \text{dom}(s_{\text{trap}})$ implies $n + 1 \in \text{dom}(s_{\text{tri}})$, since a trapezoid can be transformed into a triangle by adding one triangle at its $(\frac{2\pi}{3}, \frac{2\pi}{3})$ -side. Hence (2) implies (3). Similarly, $n \in \text{dom}(s_{\text{par}})$ implies $n + 1 \in \text{dom}(s_{\text{trap}})$ by adding a suitable triangle. Thus (4) is a consequence of (3).

A tiling of a convex pentagon contains at least four tiles, because it must contain one exposed triangle at the $\frac{\pi}{3}$ -vertex and at least one tile for each of the three $(\frac{2\pi}{3}, \frac{2\pi}{3})$ -sides. This gives (5). Similarly, we obtain (6), since a tiling of a convex hexagon contains at least one tile for each of the six sides of the hexagon.

Claims (2)-(6) together with Figure 3 and Corollary 4 show that the domains of s_{tri} , s_{trap} , s_{par} , s_{pent} and s_{hex} are as claimed in Theorem 1. Moreover, Corollary 4 gives the lower estimates of these functions from Theorem 1 apart from $s_{\text{hex}}(n) \geq n - 4$ for $n \in \{9, \dots, 19\} \cup \{21, 22, 24, 25\}$. These last estimates are justified by the tilings (a)-(s) presented in the appendix. We do not know other tilings of hexagons that realize $s_{\text{hex}}(n) \geq n - 4$. Table 2 gives the parameters of the illustrated tilings.

2.3. A necessary condition.

Lemma 5. *Let \mathcal{T} be a tiling of a convex m -gon P by at least two equilateral triangles, and let v_π be the number of π -vertices of \mathcal{T} . Then there are at least $m + v_\pi - 3$ pairs of distinct triangles from \mathcal{T} having a side in common.*

Proof. We can assume that

$$(7) \quad v_\pi \geq 9 - 2m.$$

Indeed, this is trivial if $m \geq 5$. If $m = 3$ then $v_\pi \geq 3$, since the trivial tiling $\mathcal{T} = \{P\}$ is excluded. If $m = 4$ and $v_\pi = 0$ then \mathcal{T} is a tiling of a rhombus P by two equilateral triangles, and the claim of the lemma is obvious.

The following graph-theoretic arguments generalize a similar approach from [17, 3]. We associate a bipartite planar graph Γ to \mathcal{T} : In the interior of every triangle from \mathcal{T} we place a white node of Γ . All vertices of \mathcal{T} , except for the $\frac{\pi}{3}$ -vertices, are the black nodes of Γ . A black and a white node of Γ are joined by an edge if the black one is a vertex of the triangle represented by the white one, see Figure 4. Let v , e and f be the numbers of nodes, edges and faces of Γ , respectively.

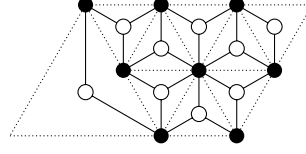


FIGURE 4. A tiling of a parallelogram (dotted) and the associated graph

All nodes of Γ have degree 2, 3 or 6. Denoting the respective numbers by v_2 , v_3 and v_6 , we have

$$(8) \quad v = v_2 + v_3 + v_6.$$

Since P is a convex m -gon whose inner angles have sizes of $\frac{\pi}{3}$ or $\frac{2\pi}{3}$, it has $6 - m$ inner angles of size $\frac{\pi}{3}$ and $2m - 6$ inner angles of size $\frac{2\pi}{3}$. A white node of Γ has degree 2 if and only if it represents a triangle that covers an angle of P of size $\frac{\pi}{3}$. A black node of Γ has degree 2 if and only if it corresponds to an angle of P of size $\frac{2\pi}{3}$. Thus

$$(9) \quad v_2 = (6 - m) + (2m - 6) = m.$$

Counting the edges of Γ in terms of the nodes we obtain

$$(10) \quad 2e = 2v_2 + 3v_3 + 6v_6.$$

Since Γ is bipartite, we have

$$(11) \quad f = \sum_{i=2}^{\infty} f_{2i}$$

where f_j is the number of faces with j edges. The boundary of P contains exactly $(2m - 6) + v_\pi$ black nodes of Γ : $2m - 6$ of them represent inner angles of size $\frac{2\pi}{3}$ and v_π of them are π -vertices. Hence the unbounded face of Γ has $2((2m - 6) + v_\pi)$ edges, and in turn

$$(12) \quad f_{2(2m-6+v_\pi)} \geq 1.$$

Counting the edges of Γ in terms of the faces we obtain

$$(13) \quad 2e = \sum_{i=2}^{\infty} 2i f_{2i}.$$

Euler's formula for Γ gives

$$\begin{aligned} 2 &= f - e + v \\ &= \left(f - \frac{1}{6} 2e \right) + \left(v - \frac{1}{3} 2e \right) \\ &\stackrel{(8,10,11,13)}{=} \left(\sum_{i=2}^{\infty} f_{2i} - \frac{1}{6} \sum_{i=2}^{\infty} 2i f_{2i} \right) + \left(v_2 + v_3 + v_6 - \frac{1}{3} (2v_2 + 3v_3 + 6v_6) \right) \\ &\stackrel{(9)}{=} \frac{1}{3} \left(f_4 - \sum_{i=3}^{\infty} (i-3) f_{2i} \right) + \frac{1}{3} (m - 3v_6). \end{aligned}$$

Thus

$$f_4 = 6 + \sum_{i=3}^{\infty} (i-3) f_{2i} - m + 3v_6.$$

Using $2m - 6 + v_\pi \geq 3$ (as a consequence of (7)) and $v_6 \geq 0$ we get

$$f_4 \geq 6 + ((2m - 6 + v_\pi) - 3) f_{2(2m-6+v_\pi)} - m$$

and, by (12),

$$f_4 \geq 6 + ((2m - 6 + v_\pi) - 3) - m = m + v_\pi - 3.$$

This estimate completes the proof, because every face of Γ with four edges represents two triangles of \mathcal{T} having a side in common. \square

2.4. Upper estimates. We shall use the following observation.

Lemma 6. *If a tiling \mathcal{T} of a trapezoid by equilateral triangles satisfies one of the conditions*

- (α) *all π -vertices of \mathcal{T} are contained in the $(\frac{\pi}{3}, \frac{\pi}{3})$ -side of the trapezoid or*
- (β) $|\mathcal{T}| = 3$,

then \mathcal{T} is a similar image of the respective tiling from Figure 3 and $s(\mathcal{T}) = 1$.

Proof. We can assume (α), since (β) implies (α). Let $T_1, T_2 \in \mathcal{T}$ be the tiles covering the two $(\frac{\pi}{3}, \frac{2\pi}{3})$ -sides of the trapezoid and let $T_3 \in \mathcal{T}$ be the triangle that covers the $(\frac{2\pi}{3}, \frac{2\pi}{3})$ -side. Since the two $(\frac{\pi}{3}, \frac{2\pi}{3})$ -sides have the same length, $\text{size}(T_1) = \text{size}(T_2)$. We can exclude the situations $\text{size}(T_3) < \text{size}(T_1)$, because then the interiors of T_1 and T_2 would overlap, and $\text{size}(T_3) > \text{size}(T_1)$, since then the third vertex of T_3 would be outside the trapezoid. Hence $\text{size}(T_1) = \text{size}(T_2) = \text{size}(T_3)$ and T_3 covers the remainder of the trapezoid between T_1 and T_2 . The claim follows. \square

For completing the proof of Theorem 1, it remains to show that

$$(14) \quad s_{\text{tri}}(n) \leq \begin{cases} 1, & n = 1, 4, \\ 2, & n = 6, \\ n - 5, & n = 7, 8, \dots, \end{cases}$$

$$(15) \quad s_{\text{trap}}(n) \leq \begin{cases} 1, & n = 3, \\ 2, & n = 5, \\ n - 4, & n = 6, 7, \dots, \end{cases}$$

$$(16) \quad s_{\text{par}}(n) \leq \begin{cases} 1, & n = 2, 4, \\ 2, & n = 5, \\ n - 4, & n = 6, 7, \dots, \end{cases}$$

$$(17) \quad s_{\text{pent}}(n) \leq \begin{cases} 2, & n = 4, \\ n - 3, & n = 5, 6, \dots, \end{cases}$$

$$(18) \quad s_{\text{hex}}(n) \leq \begin{cases} n - 5, & n = 6, 7, 8, \\ n - 4, & n = 9, 10, \dots \end{cases}$$

In all situations we consider a tiling \mathcal{T} of a respective polygon P . The cardinality and the number of π -vertices of \mathcal{T} are denoted by $n = |\mathcal{T}|$ and v_π , respectively. We have to show that $s(\mathcal{T})$ is bounded from above by the claimed upper estimate of $s_{\text{tri}}(n)$, $s_{\text{trap}}(n)$, $s_{\text{par}}(n)$, $s_{\text{pent}}(n)$ or $s_{\text{hex}}(n)$, respectively.

Proof of (14). We proceed by induction over n . The claim is trivial if $n = 1$. If $n = 4$ then \mathcal{T} contains three exposed triangles and the fourth tile covers the remainder of P . We obtain the respective tiling from Figure 3 by four congruent tiles. This yields $s(\mathcal{T}) = 1$. Now let $n \geq 6$.

Case 1: $v_\pi \leq 3$. Then $v_\pi = 3$ and \mathcal{T} splits into three exposed triangles of the same size and a tiling \mathcal{T}' of the remaining equilateral triangle P' of P , P' being of that size as well. Now $|\mathcal{T}'| = n - 3 \geq 3$, and the induction hypothesis gives $s(\mathcal{T}') \leq |\mathcal{T}'| - 3 = n - 6$. Since the tiles in P' are smaller than the exposed tiles of \mathcal{T} , we obtain $s(\mathcal{T}) = s(\mathcal{T}') + 1 \leq (n - 6) + 1 = n - 5$.

Case 2: $v_\pi = 4$ and $n \neq 6$. Now \mathcal{T} contains three exposed triangles T_1, T_2 and T_3 such that T_1 shares a vertex with each of T_2 and T_3 . Thus $\text{size}(T_2) = \text{size}(T_3)$, but T_2 and T_3 are disjoint. By Lemma 5, \mathcal{T} contains at least $3 + v_\pi - 3 = 4$ pairs of triangles that share a side. We introduce a graph Γ whose nodes are the triangles

of \mathcal{T} . Two triangles form an edge if these are T_2 and T_3 or if they have a side in common. This graph has at least $1 + 4 = 5$ edges and does not contain a cycle of size less than six. (Cycles not involving the edge $\{T_2, T_3\}$ have size at least six. If a cycle contains $\{T_2, T_3\}$ then all triangles of $\mathcal{T} \setminus \{T_1\}$ are of the same size. By $v_\pi = 4$, the side of P that meets both T_2 and T_3 contains exactly two π -vertices. Hence there are exactly five triangles in $\mathcal{T} \setminus \{T_1\}$ and in turn $n = |\mathcal{T}| = 6$, which is excluded.) We pick five edges of Γ . Since they do not contain a cycle and since each of them connects two congruent triangles, we have $s(\mathcal{T}) \leq |\mathcal{T}| - 5 = n - 5$.

Case 3: $v_\pi = 4$ and $n = 6$. Lemma 5 shows that \mathcal{T} contains at least $3 + v_\pi - 3 = 4$ pairs of triangles having a side in common. This yields $s(\mathcal{T}) \leq |\mathcal{T}| - 4 = 2$.

Case 4: $v_\pi \geq 5$. Now Lemma 5 shows that \mathcal{T} contains at least $3 + v_\pi - 3 \geq 5$ pairs of triangles having a side in common. This yields $s(\mathcal{T}) \leq |\mathcal{T}| - 5 = n - 5$. \square

Proof of (15). We add an equilateral triangle at the $(\frac{2\pi}{3}, \frac{2\pi}{3})$ -side of P and obtain a triangle P' and a corresponding tiling \mathcal{T}' of P' with $|\mathcal{T}'| = n + 1$. Then

$$s(\mathcal{T}) \leq s(\mathcal{T}') \leq s_{\text{tri}}(n+1) \stackrel{(14)}{\leq} \begin{cases} 1, & n = 3, \\ 2, & n = 5, \\ n - 4, & n = 6, 7, \dots \end{cases}$$

\square

Proof of (16). W.l.o.g. $n \geq 4$, since $n \in \text{dom}(s_{\text{par}}) = \{2\} \cup \{4, 5, \dots\}$ and the situation is clear for $n = 2$.

Case 1: One side length of P is larger than $\max\{\text{size}(T) : T \in \mathcal{T}\}$. We add two triangles at two sides of P that meet at a $\frac{\pi}{3}$ -vertex of P . This way we obtain a tiling \mathcal{T}' of a triangle P' with $|\mathcal{T}'| = n + 2$ and $s(\mathcal{T}') \geq s(\mathcal{T}) + 1$, since one of the new triangles has a new size. Consequently,

$$s(\mathcal{T}) \leq s(\mathcal{T}') - 1 \leq s_{\text{tri}}(n+2) - 1 \stackrel{(14)}{\leq} \begin{cases} 2 - 1 = 1, & n = 4, \\ 2 - 1 < 2, & n = 5, \\ ((n+2) - 5) - 1 = n - 4, & n = 6, 7, \dots \end{cases}$$

Case 2: No side length of P is larger than $\max\{\text{size}(T) : T \in \mathcal{T}\}$. Since no side of P can be shorter than $\max\{\text{size}(T) : T \in \mathcal{T}\}$, P is a rhombus and one tile $T_0 \in \mathcal{P}$ represents half of P . Then $\mathcal{T}' = \mathcal{T} \setminus \{T_0\}$ is a tiling of the other half P' of P . In particular, P' is a triangle, $s(\mathcal{T}') \geq s(\mathcal{T}) - 1$ and

$$n \in \{5\} \cup \{7, 8, \dots\},$$

because $n \in \text{dom}(s_{\text{par}}) \setminus \{2\} = \{4, 5, \dots\}$ and $n - 1 = |\mathcal{T}'| \in \text{dom}(s_{\text{tri}}) = \{1, 4\} \cup \{6, 7, \dots\}$. We obtain

$$s(\mathcal{T}) \leq s(\mathcal{T}') + 1 \leq s_{\text{tri}}(n-1) + 1 \stackrel{(14)}{\leq} \begin{cases} 1 + 1 = 2, & n = 5, \\ 2 + 1 = n - 4, & n = 7, \\ ((n-1) - 5) + 1 < n - 4, & n = 8, 9, \dots \end{cases}$$

\square

Proof of (17). *Case 1:* $v_\pi = 0$. After removing the exposed triangle from \mathcal{T} as well as from P , we obtain a tiling \mathcal{T}' of a trapezoid P' such that all π -vertices of \mathcal{T}' are on the $(\frac{\pi}{3}, \frac{\pi}{3})$ -side of P' . By Lemma 6, \mathcal{T}' is a similar image of the tiling of cardinality three from Figure 3. Hence \mathcal{T} consists of $n = 4$ tiles, three of them being of the same size. So we have $s(\mathcal{T}) \leq 2$, the required upper estimate for $n = 4$.

Case 2: $v_\pi \geq 1$. By Lemma 5, \mathcal{T} contains at least $5 + v_\pi - 3 \geq 3$ pairs of triangles that have a side in common. This yields $s(\mathcal{T}) \leq |\mathcal{T}| - 3 = n - 3$. \square

Proof of (18). *Case 1:* $v_\pi = 0$ or $n \leq 6$. We have $v_\pi = 0$. Indeed, if $n \leq 6$ then $n = 6$ by (6). Since no tile from \mathcal{T} covers segments of more than one side of the hexagon P , each side of P is a side of some triangle from \mathcal{T} and in turn $v_\pi = 0$.

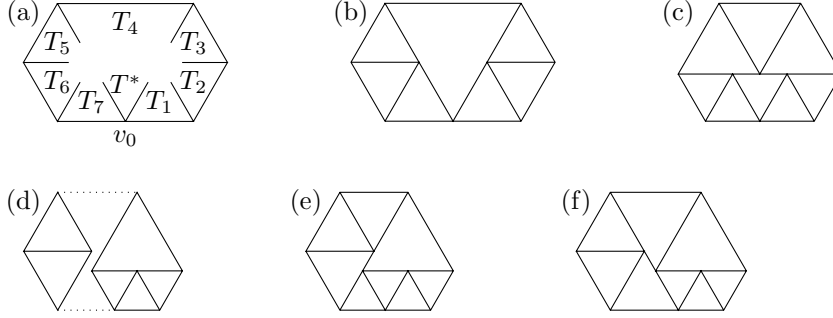


FIGURE 5. Proof of (18), Case 2

Let $T_1, \dots, T_6 \in \mathcal{T}$ be the tiles that cover the sides of P in successive order. It is enough to show that $\text{size}(T_1) = \dots = \text{size}(T_6)$, since then P is a regular hexagon of side length $\text{size}(T_1)$ that is completely tiled by T_1, \dots, T_6 , hence $n = 6$ and $s(\mathcal{T}) = 1$, as claimed in (18).

To obtain a contradiction, suppose that, say, $\text{size}(T_6) > \text{size}(T_1)$. Then

$$(19) \quad \text{size}(T_6) > \text{size}(T_1) \geq \text{size}(T_2) \geq \text{size}(T_3),$$

because tiles do not overlap. The side lengths of P satisfy

$$(20) \quad \text{size}(T_2) + \text{size}(T_3) = \text{size}(T_5) + \text{size}(T_6).$$

Then (19) shows that $\text{size}(T_6) > \text{size}(T_5)$, which implies

$$(21) \quad \text{size}(T_6) > \text{size}(T_5) \geq \text{size}(T_4) \geq \text{size}(T_3)$$

as above. Adding the inequalities $\text{size}(T_2) < \text{size}(T_6)$ from (19) and $\text{size}(T_3) \leq \text{size}(T_5)$ from (21) gives a contradiction to (20).

Case 2: $v_\pi = 1$ and $n \in \{7, 8\}$. Let v_0 be the π -vertex, let $T_1, \dots, T_7 \in \mathcal{T}$ be the tiles that form the boundary of P ordered consecutively such that v_0 is a common vertex of T_1 and T_7 , and let $T^* \in \mathcal{T}$ be the third tile having v_0 as a vertex, see Figure 5(a). If $n = 7$ then necessarily $T^* = T_4$, the hexagon P splits into T_4 and two trapezoids tiled by T_1, T_2, T_3 and T_5, T_6, T_7 , respectively, Lemma 6 shows that \mathcal{T} is similar to Figure 5(b), and we obtain our claim $s(\mathcal{T}) \leq 2$.

Now we consider $n = 8$. If v_0 was a vertex of T_4 then one of the two trapezoids besides T_4 would be tiled by four triangles, contrary to (3). Thus v_0 is a vertex of $T_8 \in \mathcal{T}$. Consequently, the union of the lower sides of T_3 and T_5 is the union of the upper sides of T_2, T_6 and T_8 .

If the lower sides of T_3 and T_5 together form a segment, then $\text{size}(T_3) = \text{size}(T_5)$ and $\text{size}(T_2) = \text{size}(T_6) = \text{size}(T_8)$, the tiling \mathcal{T} is similar to Figure 5(c), and we obtain $s(\mathcal{T}) = 2 < 3$.

If the lower sides of T_3 and T_5 do not form a segment, then w.l.o.g. the lower side of T_3 is the union of the upper sides of T_2 and T_8 and the lower side of T_5 coincides with the upper one of T_6 , see Figure 5(d). Then $\text{size}(T_1) = \text{size}(T_2) = \text{size}(T_8)$, $\text{size}(T_3) = 2 \text{size}(T_1)$ and $\text{size}(T_5) = \text{size}(T_6) = \frac{3}{2} \text{size}(T_1)$. Now it follows easily that either $\text{size}(T_4) = \frac{3}{2} \text{size}(T_1)$ (and in turn $\text{size}(T_7) = \text{size}(T_1)$, see Figure 5(e)) or $\text{size}(T_4) = 2 \text{size}(T_1)$ (and in turn $\text{size}(T_7) = \frac{3}{2} \text{size}(T_1)$, see Figure 5(f)). In both the situations we obtain our claim $s(\mathcal{T}) \leq 3$.

Case 3: $v_\pi = 1$ and $n \geq 9$. Lemma 5 gives at least $6 + v_\pi - 3 = 4$ pairs of triangles in \mathcal{T} that have a common side. This yields $s(\mathcal{T}) \leq |\mathcal{T}| - 4 = n - 4$.

Case 4: $v_\pi \geq 2$ and $n \geq 7$. Lemma 5 shows that there are at least $6 + v_\pi - 3 \geq 5$ pairs of tiles in \mathcal{T} having a side in common. This gives

$$s(\mathcal{T}) \leq |\mathcal{T}| - 5 = n - 5 \leq \begin{cases} n - 5, & n = 7, 8, \\ n - 4, & n = 9, 10, \dots \end{cases}$$

□

3. PROOF OF THEOREM 2

3.1. Negative results on domains. For

$$(22) \quad 2, 3, \dots, 14 \notin \text{dom} \left(s_{\text{tri}}^{\text{t-perf}} \right)$$

we refer to [5, Section 3.2]. It remains to show that

$$(23) \quad 1, 2, \dots, 12 \notin \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right),$$

$$(24) \quad 1; 3, 4, \dots, 12 \notin \text{dom} \left(s_{\text{par}}^{\text{t-perf}} \right),$$

$$(25) \quad 1, 2, \dots, 11 \notin \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right),$$

$$(26) \quad 1, 2, \dots, 10 \notin \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right).$$

These claims are prepared by a lemma.

Lemma 7. (a) *Let $*$ stand for tri, trap, par, pent or hex. If $n \in \text{dom} \left(s_*^{\text{t-perf}} \right)$*

then $2s_(n) \geq n$.*

(b) *If $n \in \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right)$ then $n - 2 \in \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right)$.*

(c) *If $n \in \text{dom} \left(s_{\text{par}}^{\text{t-perf}} \right)$ then*

$$n - 2 \in \{0\} \cup \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{par}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right).$$

(d) *If $n \in \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right)$ then $n - 1 \in \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right)$.*

Proof. (a) If $n \in \text{dom} \left(s_*^{\text{t-perf}} \right)$ then there exists a corresponding t-perfect tiling \mathcal{T} of cardinality n . By t-perfectness, there are no three triangles of the same size in \mathcal{T} . Consequently, $n \leq 2s_*^{\text{t-perf}}(n)$. Combining this with the trivial estimate $s_*^{\text{t-perf}}(n) \leq s_*(n)$ we obtain the claim $n \leq 2s_*(n)$.

(b) Let $n \in \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right)$ be represented by a t-perfect tiling of some trapezoid P . Cutting off the two exposed triangles of the tiling from P results in a smaller polygon P' with a t-perfect tiling of cardinality $n - 2$. The polygon P' is neither a parallelogram nor a triangle, since in the latter case the two exposed triangles would have been congruent under a translation. So P' is a trapezoid, a pentagon or a hexagon. Thus $n - 2 \in \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right) \cup \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right)$.

(c) Now let $n \in \text{dom} \left(s_{\text{par}}^{\text{t-perf}} \right)$ be represented by a t-perfect tiling of some parallelogram P . Cutting off the two exposed triangles from P , we get a polygon P' that may be empty (if $n = 2$) or a trapezoid, a parallelogram, a pentagon or a hexagon and has a t-perfect tiling of cardinality $n - 2$. This yields (c).

(d) Here we start with a t-perfect tiling of cardinality n of some pentagon P . We cut off the one exposed triangle and get a t-perfect tiling of cardinality $n - 1$ of some trapezoid, pentagon or hexagon P' . □

Corollary 8.

$$\begin{aligned} 1, 2, \dots, 10 &\notin \text{dom}\left(s_{\text{trap}}^{\text{t-perf}}\right), \\ 1; 3, 4, \dots, 10 &\notin \text{dom}\left(s_{\text{par}}^{\text{t-perf}}\right), \\ 1, 2, \dots, 9 &\notin \text{dom}\left(s_{\text{pent}}^{\text{t-perf}}\right), \\ 1, 2, \dots, 8 &\notin \text{dom}\left(s_{\text{hex}}^{\text{t-perf}}\right). \end{aligned}$$

Proof. The trivial inclusion $\text{dom}\left(s_*^{\text{t-perf}}\right) \subseteq \text{dom}(s_*)$, Theorem 1 and Lemma 7(a) give

$$\begin{aligned} 1, 2, 3, 4, 5, 6, 7 &\notin \text{dom}\left(s_{\text{trap}}^{\text{t-perf}}\right), \\ 1; 3, 4, 5, 6, 7 &\notin \text{dom}\left(s_{\text{par}}^{\text{t-perf}}\right), \\ 1, 2, 3; 5 &\notin \text{dom}\left(s_{\text{pent}}^{\text{t-perf}}\right), \\ 1, 2, 3, 4, 5, 6, 7, 8 &\notin \text{dom}\left(s_{\text{hex}}^{\text{t-perf}}\right). \end{aligned}$$

Now fourfold application of Lemma 7(d) yields

$$4; 6, 7, 8 \notin \text{dom}\left(s_{\text{pent}}^{\text{t-perf}}\right).$$

Then Lemma 7(b) and (c) show that

$$8, 9, 10 \notin \text{dom}\left(s_{\text{trap}}^{\text{t-perf}}\right) \quad \text{and} \quad 8, 9, 10 \notin \text{dom}\left(s_{\text{par}}^{\text{t-perf}}\right).$$

Finally, by Lemma 7(d),

$$9 \notin \text{dom}\left(s_{\text{pent}}^{\text{t-perf}}\right).$$

□

Lemma 9. $9, 10 \notin \text{dom}\left(s_{\text{hex}}^{\text{t-perf}}\right)$.

Proof. To obtain a contradiction, suppose that there is a t-perfect tiling $\mathcal{T} = \{T_1, \dots, T_n\}$ of some convex hexagon P with $n \in \{9, 10\}$. Let v_π be the number of π -vertices of \mathcal{T} .

Case 1: $v_\pi = 0$. As in Case 1 of the proof of (18) in Subsection 2.4, we see that $n = 6$, a contradiction.

Case 2: $v_\pi \geq 3$. Lemma 5 says that \mathcal{T} contains at least $6 - v_\pi + 3 \geq 6$ pairs of triangles that share a side. Since $n < 12$, there must be three triangles of the same size, which contradicts the t-perfectness of \mathcal{T} .

Case 3: $v_\pi = 2$. Now Lemma 5 gives $6 + v_\pi - 3 = 5$ pairs of triangles with a common side. Since \mathcal{T} is t-perfect, this shows that $n = 10$ and the five pairs constitute five rhombi R_1, \dots, R_5 of mutually different side lengths that tile P .

For a vector $w \in \mathbb{R}^2 \setminus \{0\}$, we define the functional $\varphi_w(Q)$ of a polygon Q by

$$\varphi_w(Q) = \sum_{\substack{s \text{ is a side of } Q \text{ with} \\ \text{outer normal vector } w}} \text{length}(s) - \sum_{\substack{s \text{ is a side of } Q \text{ with} \\ \text{outer normal vector } -w}} \text{length}(s).$$

Clearly $\varphi_w(R_i) = 0$ and, since φ_w is additive under tiling, $\varphi_w(P) = \varphi_w(R_1) + \dots + \varphi_w(R_5) = 0$ for every w . That is, opposite sides of P have the same length. By the assumption $v_\pi = 2$, there are two opposite sides s_1, s_2 of P that do not contain π -vertices of \mathcal{T} . So s_1 and s_2 are sides of two triangles $T_1, T_2 \in \mathcal{T}$ whose union is one of the rhombi, say R_1 , because $\text{length}(s_1) = \text{length}(s_2)$. But then R_1, \dots, R_5 cannot tile P , a contradiction.

Case 4: $v_\pi = 1$. Now we are in the situation of the left-hand side of Figure 6, where T_9 (and possibly T_{10}) are not displayed. Clearly $T_4 \neq T_8$, because otherwise

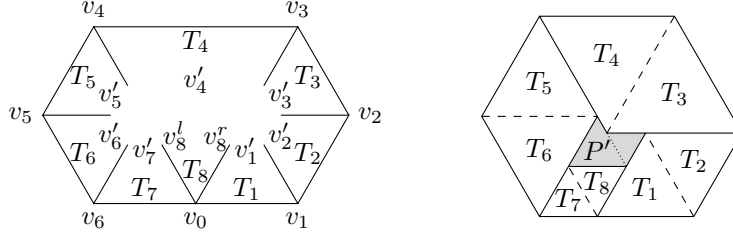


FIGURE 6. Proof of Lemma 9, Case 4

the trapezoids $v_0v_1v_2v_3$ and $v_0v_4v_5v_6$ would be tiled by a total number of eight or nine triangles, a contradiction to Corollary 8. The 2π -vertices of the triangles T_1, \dots, T_8 are denoted by $v'_1, \dots, v'_7, v_8^l, v_8^r$. We will discuss coincidences between these vertices.

First note that two of these nine points cannot agree if their lower indices differ by more than one modulo 8: Indeed, if a 2π -vertex of T_i coincides with a 2π -vertex of T_j such that T_i and T_j are no neighbours in the cyclic order of T_1, \dots, T_8 , then there is a polygonal arc a along sides of T_i and T_j that connects a vertex of T_i with a vertex of T_j and passes through their common 2π -vertex such that a dissects P into two polygons P' and P'' both being tiled by at least three (and in turn at most seven) triangles of \mathcal{T} and such that one of P' and P'' is convex. (E.g., if $v'_1 = v'_3$ we can pick $a = v_0v'_1v_3$, if $v'_2 = v_8^r$ we can pick $a = v_2v'_2v_8^lv_0$.) Thus some convex polygon admits a t-perfect tiling of a cardinality between 3 and 7. This is impossible by (22) and Corollary 8. Consequently, *the only possible coincidences among $v'_1, \dots, v'_7, v_8^l, v_8^r$ are*

$$v'_1 = v'_2, v'_2 = v'_3, v'_3 = v'_4, v'_4 = v'_5, v'_5 = v'_6, v'_6 = v'_7, v'_7 = v_8^l, v_8^r = v'_1.$$

When counted cyclically, no two consecutive of these identities are satisfied, because they would give rise to three tiles of the same size, which contradicts t-perfectness.

The total size of all inner angles of $T_1, \dots, T_9, (T_{10})$ is 9π or 10π . The size of those who are placed on the boundary of P is 5π . Thus the sizes of inner angles in 2π -vertices of \mathcal{T} sum up to 4π or 5π . Since every 2π -vertex requires inner angles of total size π or 2π , \mathcal{T} has at most five 2π -vertices. All of $v'_1, \dots, v'_7, v_8^l, v_8^r$ are 2π -vertices. So there are at least four identities between them. Taking the above observation into account, we obtain

$$(27) \quad v'_1 = v'_2, \quad v'_3 = v'_4, \quad v'_5 = v'_6, \quad v'_7 = v_8^l$$

(the alternative situation $v'_2 = v'_3, v'_4 = v'_5, v'_6 = v'_7, v_8^r = v'_1$ would be equivalent).

So $T_1 \cup T_2, T_3 \cup T_4, T_5 \cup T_6$ and $T_7 \cup T_8$ are rhombi and the sizes of T_1, T_3, T_5, T_7 are mutually different. Since $\text{size}(T_2) + \text{size}(T_3) = \text{size}(T_5) + \text{size}(T_6) = 2 \text{size}(T_5)$, we have either $\text{size}(T_2) < \text{size}(T_5) < \text{size}(T_3)$ or $\text{size}(T_2) > \text{size}(T_5) > \text{size}(T_3)$. As the latter situation would cause an overlap of T_2 and T_5 , the former inequality applies and we are in the situation of the right-hand side of Figure 6. The remainder P' of P after cutting off T_1, \dots, T_8 must be a pentagon with inner angles of sizes $\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{3}$ and $\frac{\pi}{3}$ in consecutive order. This excludes $n = 9$, because P' cannot be tiled by one single triangle T_9 . If $n = 10$, the only possibility of tiling P' by T_9 and T_{10} is dotted in the figure. But then the tiling of P is not t-perfect. This contradiction completes the proof. \square

Proof of (23), (24), (25) and (26). By Corollary 8 and Lemma 9, it remains to prove that

$$11, 12 \notin \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right), \quad 11, 12 \notin \text{dom} \left(s_{\text{par}}^{\text{t-perf}} \right), \quad 10, 11 \notin \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right).$$

The third claim follows from Corollary 8 and Lemma 9 by twofold application of Lemma 7(d). Then the remainder is a consequence of Lemma 7(b) and (c). \square

3.2. New spiral pentagons and corresponding t-perfect tilings. We define a sequence of integers $q(n)$, $n = 8, 9, \dots$, by

$$(28) \quad q(8) = 8, \quad q(9) = 11, \quad q(10) = 9 \quad \text{and} \quad q(n) = q(n-3) + q(n-2) \quad \text{for } n \geq 11.$$

Lemma 10. (a) *The above numbers satisfy*

- $q(n) > q(n-1)$ and $q(n) > 12$ for $n \geq 11$,
- $\frac{1}{3}(q(n-1) - q(n-2)) \notin \{2, 3, 5, 7, 8, 9, 11, 12\} \cup \{q(m) : m \geq 8\}$ for $n = 12, 13, 14, 15; 17, 18, \dots$

(b) *For every $n \in \{12, 13, \dots\}$, there is a convex pentagon Q_n with sides of lengths $q(n-4)$, $q(n-3)$, $q(n-2)$, $q(n-1)$ and $q(n)$ (in consecutive order) with an inner angle of size $\frac{\pi}{3}$ between the sides of lengths $q(n-1)$ and $q(n)$. That pentagon admits a t-perfect tiling \mathcal{T}_n by n equilateral triangles T_i , $i = 1, \dots, n$, such that*

- $\text{size}(T_1) = \text{size}(T_2) = 2$, $\text{size}(T_3) = 3$, $\text{size}(T_4) = \text{size}(T_5) = 5$, $\text{size}(T_6) = \text{size}(T_7) = 7$, $\text{size}(T_8) = \text{size}(T_9) = 8$, $\text{size}(T_{10}) = 9$, $\text{size}(T_{11}) = 11$, $\text{size}(T_{12}) = 12$ and $\text{size}(T_i) = q(i-1)$ for $i = 13, \dots, n$,
- *if one side length of Q_n is a side length of some triangle $T_i \in \mathcal{T}_n$ and different from 8 (the latter is always the case if $n \geq 13$), then that side of Q_n is a side of T_i and there is only one triangle of that size in \mathcal{T}_n .*

Proof. The first part of (a) follows from (28) by induction. The second part of (a) is shown for $n \leq 24$ by explicit computation. For $n \geq 25$, it is proved by inductive verification of

$$q(n-10) < \frac{1}{3}(q(n-1) - q(n-2)) = \frac{q(n-6)}{3} < q(n-9)$$

as follows: First note that

$$q(n-1) - q(n-2) = q(n-4) + q(n-3) - q(n-5) - q(n-4) = q(n-3) - q(n-5) = q(n-6).$$

The base cases for $n = 25, 26, 27$, obtained by computation, are $q(15) = 48 < \frac{q(19)}{3} = \frac{154}{3} < q(16) = 67 < \frac{q(20)}{3} = \frac{202}{3} < q(17) = 87 < \frac{q(21)}{3} = \frac{269}{3} < q(18) = 115$. Finally, the step case for $n \geq 28$ is

$$\frac{q(n-6)}{3} = \frac{q(n-9)}{3} + \frac{q(n-8)}{3} \begin{cases} > q(n-13) + q(n-12) = q(n-10), \\ < q(n-12) + q(n-11) = q(n-9). \end{cases}$$

Also claim (b) is shown by induction: The pentagon Q_{12} and the tiling \mathcal{T}_{12} is illustrated in Figure 7 (see also tiling (c) in the appendix). Then Q_{n+1} and \mathcal{T}_{n+1} are obtained from Q_n and \mathcal{T}_n by adding a triangle T_{n+1} of size $q(n)$ at the side of size $q(n)$ of Q_n , cf. Figure 7. \square

3.3. Positive results on domains and lower estimates. We start with a direct consequence of Lemma 10.

Corollary 11. (a) $\{15, 16, \dots\} \subseteq \text{dom} \left(s_{\text{tri}}^{\text{t-perf}} \right)$ and $s_{\text{tri}}^{\text{t-perf}}(n) \geq n - 6$ for all $n = 15, 16, \dots$

(b) $\{13, 14, \dots\} \subseteq \text{dom} \left(s_{\text{trap}}^{\text{t-perf}} \right)$ and $s_{\text{trap}}^{\text{t-perf}}(n) \geq n - 5$ for all $n = 13, 14, \dots$

(c) $\{13, 14, \dots\} \subseteq \text{dom} \left(s_{\text{par}}^{\text{t-perf}} \right)$ and $s_{\text{par}}^{\text{t-perf}}(n) \geq n - 5$ for all $n = 13, 14, \dots$

(d) $\{12, 13, \dots\} \subseteq \text{dom} \left(s_{\text{pent}}^{\text{t-perf}} \right)$ and $s_{\text{pent}}^{\text{t-perf}}(n) \geq n - 4$ for all $n = 12, 13, \dots$

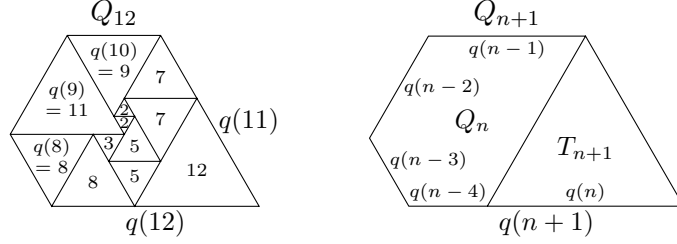
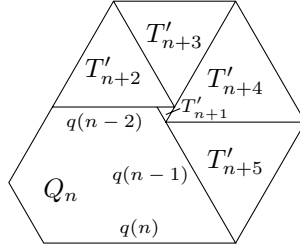
FIGURE 7. Pentagons Q_{12} and Q_{n+1} 

FIGURE 8. Proof of Corollary 11(e)

- (e) $\{17, 18, \dots\} \setminus \{21\} \subseteq \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right)$ and $s_{\text{hex}}^{\text{t-perf}}(n) \geq n - 6$ for all $n = 17, 18, 19, 20; 22, 23, \dots$

Proof. The pentagons Q_n and tilings \mathcal{T}_n from Lemma 10(b) prove (d).

For (a), we add two triangles of sizes $q(n-4)$ and $q(n-2)$ at the respective sides of Q_n , cf. Figure 2(a). This gives a t-perfect tiling of a triangle by $n+2$ tiles of at least $n-4 = (n+2) - 6$ different sizes if $n \geq 13$. (For $n = 12$, t-perfectness is violated, since the size $q(12-4) = 8$ is not allowed, see the last claim of Lemma 10(b).)

For (b) and (c), we add a triangle of size $q(n-2)$ or $q(n-3)$, respectively, to Q_n and \mathcal{T}_n for $n \geq 12$, cf. Figure 2.

For (e), we add five triangles $T'_{n+1}, \dots, T'_{n+5}$ of sizes

$$\text{size}(T'_k) = \begin{cases} \frac{1}{3}(q(n-1) - q(n-2)), & k = n+1, \\ \frac{2}{3}q(n-2) + \frac{1}{3}q(n-1), & k = n+2, n+3, \\ \frac{1}{3}q(n-2) + \frac{2}{3}q(n-1), & k = n+4, n+5, \end{cases}$$

over the sides of lengths $q(n-2)$ and $q(n-1)$ of Q_n , see Figure 8. By Lemma 10(a), $\text{size}(T'_{n+1}) \neq \text{size}(T_i)$ for $i = 1, \dots, n$ if $n \in \{12, 13, \dots\} \setminus \{16\}$. Moreover, the monotonicity from Lemma 10(a) gives

$$q(n-2) < \text{size}(T'_{n+2}) = \text{size}(T'_{n+3}) < \text{size}(T'_{n+4}) = \text{size}(T'_{n+5}) < q(n-1),$$

so that the sizes of the new triangles are different from those of \mathcal{T}_n . Thus \mathcal{T}_n together with $T'_{n+1}, \dots, T'_{n+5}$ constitutes a t-perfect tiling by $n+5$ triangles of $(n-4) + 3 = (n+5) - 6$ different sizes. This proves (e). \square

TABLE 1. Tilings proving claims (30), (31) and (32): the table gives the respective initial tilings from the appendix and the sizes of new triangles in the order of their addition.

Claim (30)		Claim (31)	
n		n	
15	(c); 12, 19, 20, 11	14	(c); 12, 19, 20
17	(c); 12, 19, 28, 39, 20, 28	16	(c); 12, 19, 28, 39, 20
18	(j); 27, 44, 47, 24	17	(j); 27, 44, 47
19	(k); 20, 32, 33, 17	18	(k); 20, 32, 33
20	(j); 27, 44, 67, 91, 47, 67	19	(j); 27, 44, 67, 91, 47
21	(m); 64, 106, 111, 59	20	(m); 64, 106, 111
22	(n); 102, 157, 162, 84	21	(n); 102, 157, 162
23	(o); 138, 213, 220, 114	22	(o); 138, 213, 220
24	(n); 102, 157, 235, 319, 162, 235	23	(n); 102, 157, 235, 319, 162
25	(o); 138, 213, 319, 433, 220, 319	24	(o); 138, 213, 319, 433, 220
26	(q); 325, 533, 534, 283	25	(q); 325, 533, 534
28	(q); 325, 533, 784, 1067, 534, 784	27	(q); 325, 533, 784, 1067, 534
Claim (32)			
n		n	
15	(c); 12, 19, 28, 20	22	(n); 102, 157, 235, 162
18	(j); 27, 44, 67, 47	23	(o); 138, 213, 319, 220
19	(k); 20, 32, 48, 33	26	(q); 325, 533, 784, 534
21	(m); 64, 106, 158, 111		

Now the remaining (non-trivial) claims from Theorem 2 on domains and lower estimates are

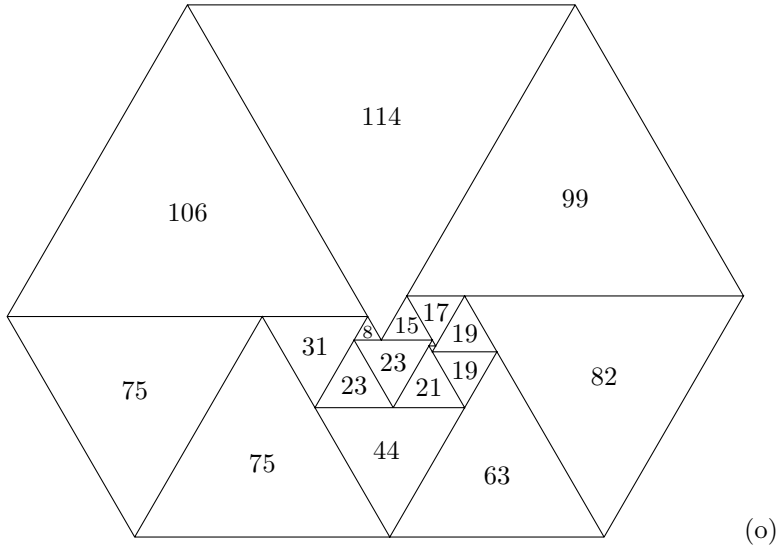
$$\begin{aligned}
(29) \quad & 11; 14, 15, 16; 21 \in \text{dom} \left(s_{\text{hex}}^{\text{t-perf}} \right), \\
(30) \quad & s_{\text{tri}}^{\text{t-perf}}(n) \geq n - 5, \quad n = 15; 17, 18, \dots, 26; 28, \\
(31) \quad & s_{\text{trap}}^{\text{t-perf}}(n) \geq n - 4, \quad n = 14; 16, 17, \dots, 25; 27, \\
(32) \quad & s_{\text{par}}^{\text{t-perf}}(n) \geq n - 4, \quad n = 15; 18, 19; 21, 22, 23; 26, \\
(33) \quad & s_{\text{hex}}^{\text{t-perf}}(n) \geq \begin{cases} n - 4, & n = 11; 14, 15; 17, 18, 19; 22, \\ n - 5, & n = 16; 20, 21; 23. \end{cases}
\end{aligned}$$

Proof of (29) and (33). It is enough to show that the lower bounds from (33) are attained by t-perfect tilings of the respective cardinalities. We refer to the tilings (c), (j), (k), (m), (n), (o), (q) and (t), (u), (v), (w) from the appendix and Table 2. \square

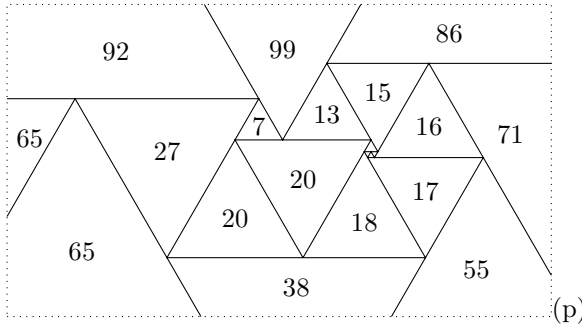
Proof of (30), (31) and (32). These estimates are established by tilings constructed as follows: We start with a particular tiling from the appendix. Then we add successively triangles of prescribed sizes such that in each step the new triangle is placed over the side of the respective length of the previously tiled polygon. Table 1 summarizes all parameters.

Several of the above tilings showing that $s_{\text{tri}}^{\text{t-perf}}(n) \geq n - 5$ can be found in the literature: for $n = 15$ see [17, Figure 1 and the text thereafter], [18, Figure 4], [5, Figure 8, second tiling] and [7, perfect_dissection_size15_595_r5_c3], for $n = 17$ see [5, Figure 10, first tiling] and [7, perfect_dissection_size17_3091_r3_c0], for $n = 18$ see [7, perfect_dissection_size18_30413_r6_c3], for $n = 19$ see [7, perfect_dissection_size19_130975_r6_c4]. \square

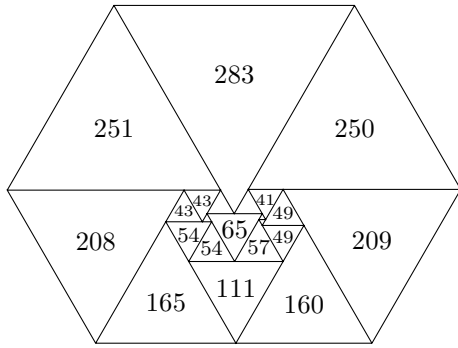
3.4. Upper estimates. The property of t-perfectness gives a new upper estimate for pentagons.



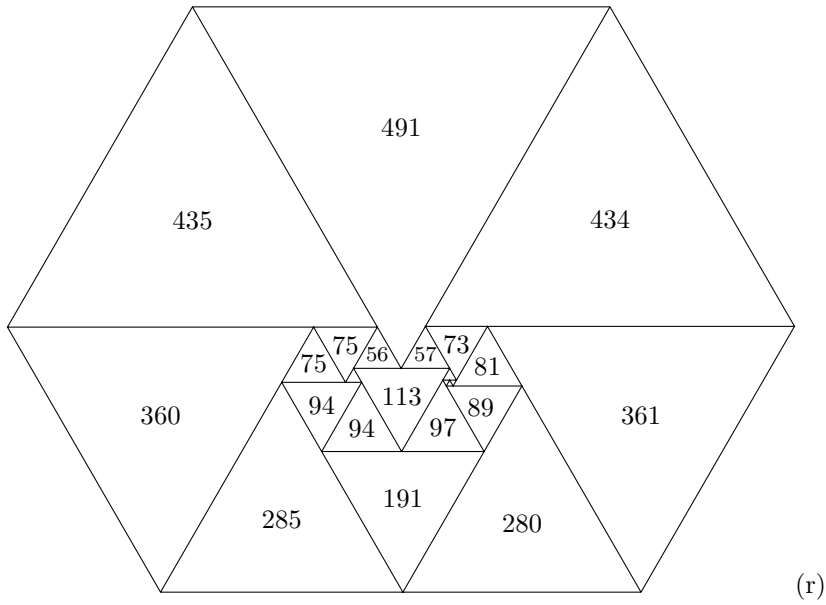
(o)



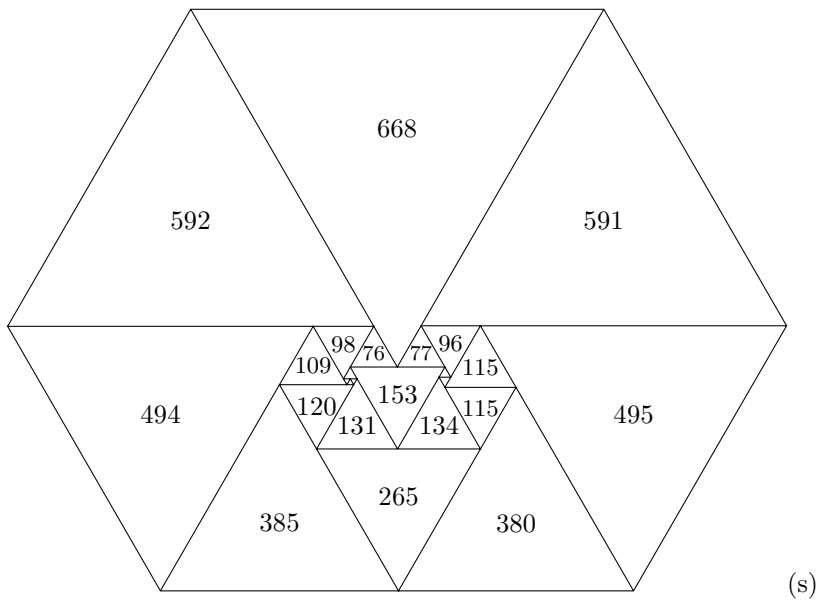
(p)



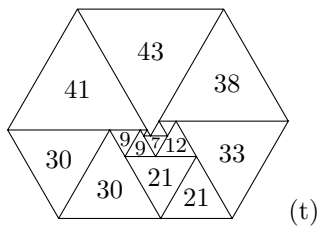
(q)



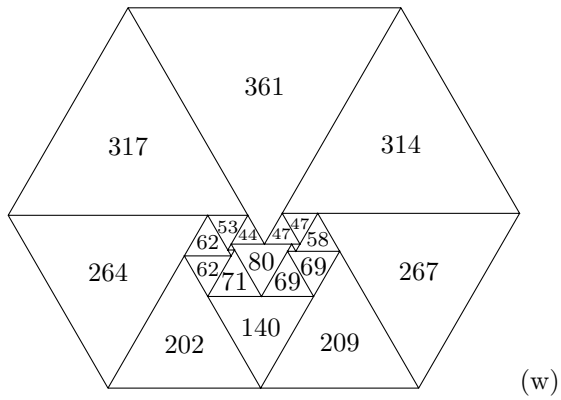
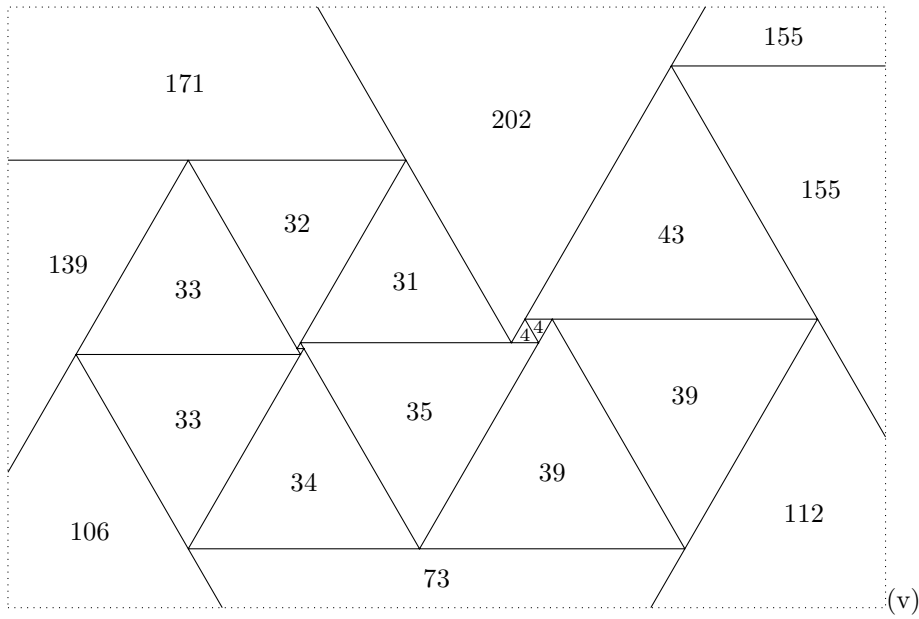
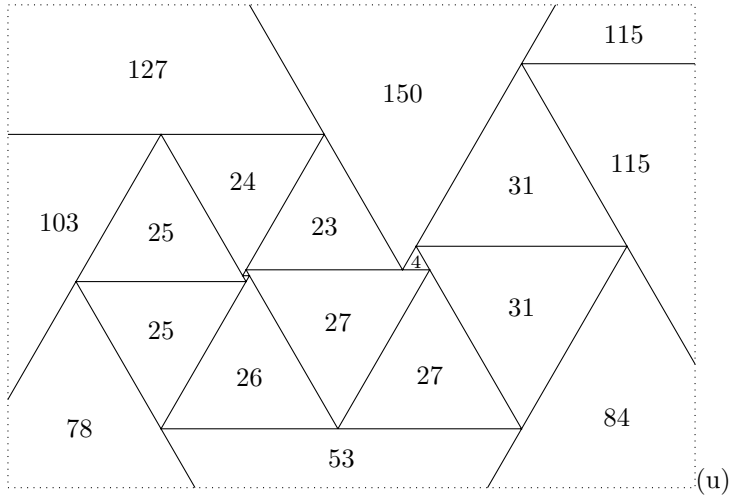
(r)



(s)



(t)



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TABLE 2. Parameters of the tilings from the appendix

Tilings showing that $s_{\text{hex}}(n) \geq n - 4$ or even $s_{\text{hex}}^{\text{t-perf}}(n) \geq n - 4$			
	n	sizes of tiles (references of related tilings)	t-perfect
(a)	9	1, 3, 3, 3, 4, 4, 5, 5, 6	no
(b)	10	1, 1, 4, 4, 4, 5, 6, 6, 7, 8 (cf. [5, p. 1488, third tiling])	no
(c)	11	2, 2, 3, 5, 5, 7, 7, 8, 8, 9, 11 (cf. [17, Figure 1], [18, Figure 4], [5, Figure 8], [7, perfect_dissection_size15_595_r5_c3])	yes
(d)	11	1, 3, 4, 4, 4, 7, 7, 8, 9, 9, 10	no
(e)	12	1, 3, 3, 4, 7, 7, 7, 10, 11, 12, 12, 13	no
(f)	12	1, 1, 1, 2, 7, 7, 7, 9, 10, 11, 12, 14	no
(g)	13	2, 2, 2, 4, 5, 9, 9, 11, 13, 14, 14, 15, 19	no
(h)	14	3, 3, 3, 5, 5, 6, 11, 14, 16, 16, 17, 20, 21, 26	no
(i)	14	2, 3, 3, 3, 6, 8, 11, 11, 14, 17, 19, 21, 21, 23	no
(j)	14	1, 4, 5, 5, 6, 6, 11, 11, 16, 17, 20, 20, 23, 24 (cf. [5, p. 1488, last tiling], [7, perfect_dissection_size18_30413_r6_c3])	yes
(k)	15	1, 2, 2, 3, 3, 4, 5, 8, 8, 12, 12, 13, 15, 16, 17 (cf. [7, perfect_dissection_size18_30411_r3_c6])	yes
(l)	16	3, 5, 8, 8, 9, 11, 11, 11, 14, 22, 25, 30, 34, 34, 38, 43	no
(m)	17	3, 3, 7, 10, 12, 13, 13, 16, 16, 19, 29, 35, 42, 47, 47, 52, 59	yes
(n)	18	2, 6, 11, 13, 13, 15, 15, 17, 17, 23, 32, 47, 55, 55, 60, 73, 78, 84	yes
(o)	19	2, 2, 8, 15, 17, 19, 19, 21, 23, 23, 31, 44, 63, 75, 75, 82, 99, 106, 114	yes
(p)	21	1, 1, 1, 2, 7, 13, 15, 16, 17, 18, 20, 20, 27, 38, 55, 65, 65, 71, 86, 92, 99	no
(q)	22	8, 8, 11, 32, 33, 41, 43, 43, 49, 49, 54, 54, 57, 65, 111, 160, 165, 208, 209, 250, 251, 283	yes
(r)	24	8, 8, 8, 16, 19, 56, 57, 73, 75, 75, 81, 89, 94, 94, 97, 113, 191, 280, 285, 360, 361, 434, 435, 491	no
(s)	25	11, 11, 11, 19, 19, 22, 76, 77, 96, 98, 109, 115, 115, 120, 131, 134, 153, 265, 380, 385, 494, 495, 591, 592, 668	no
Tilings showing that $s_{\text{hex}}^{\text{t-perf}}(n) \geq n - 5$			
	n	sizes of tiles (references of related tilings)	
(t)	16	2, 2, 5, 5, 7, 9, 9, 12, 21, 21, 30, 30, 33, 38, 41, 43 (cf. [7, perfect_dissection_size19_30749_r0_c2])	
(u)	20	1, 1, 4, 23, 24, 25, 25, 26, 27, 27, 31, 31, 53, 78, 84, 103, 115, 115, 127, 150	
(v)	21	1, 1, 4, 4, 31, 32, 33, 33, 34, 35, 39, 39, 43, 73, 106, 112, 139, 155, 155, 171, 202	
(w)	23	9, 9, 11, 11, 44, 47, 47, 53, 58, 62, 62, 69, 69, 71, 80, 140, 202, 209, 264, 267, 314, 317, 361	

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