

EMERGENT GEOMETRY OF MATRIX MODELS WITH EVEN COUPLINGS

JIAN ZHOU

ABSTRACT. We show that to the modified GUE partition function with even coupling introduced by Dubrovin, Liu, Yang and Zhang, one can associate n -point correlation functions in arbitrary genera which satisfy Eynard-Orantin topological recursions. Furthermore, these n -point functions are related to intersection numbers on the Deligne-Mumford moduli spaces.

1. INTRODUCTION

In the early 1990s there appeared two different approaches to 2D topological gravity. The first one is via the double scaling limits of large N Hermitian one-matrix models. In this approach a connection to KdV hierarchy and Virasoro constraints [2] is established. Witten [14] introduced another approach via the intersection theory on the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves [14]. He conjectured a famous connection between intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and KdV hierarchy and Virasoro constraints based on the conjectural equivalence between these two approaches. Of course, at the same time, a connection between matrix models and intersection numbers of moduli spaces $\overline{\mathcal{M}}_{g,n}$ of Riemann surfaces was suggested. In the mathematical literature, a connection between matrix models and the orbifold Euler characteristics of $\mathcal{M}_{g,n}$ appeared earlier in the works of Harer-Zagier [7] and Penner [13].

Witten's conjecture was proved by Kontsevich [9] by introducing a different kind of matrix models, now called the Kontsevich model. His result establishes a connection between Kontsevich model with intersection numbers on $\overline{\mathcal{M}}_{g,n}$. We are interested in establishing a connection between the original Hermitian one-matrix models and intersection theory on $\overline{\mathcal{M}}_{g,n}$, without taking any double scaling limit.

A result of this type has already appeared recently in the work of Dubrovin, Liu, Yang, and Zhang [1], and so we will first examine their results in this paper to test our idea for the general case. They considered some modified partition function of Hermitian one-matrix model with only even couplings and identified with the generating series of some special Hodge integrals. Our goal is to consider the Hermitian one-matrix model with all possible couplings. We will achieve this goal in a subsequent paper which is modelled on this paper.

The method of [1] is as follows. Starting with the Virasoro constraints for GUE partition functions with all possible couplings, they derived the Virasoro constraints for modified partition function of Hermitian one-matrix model with only even couplings. On the other hand, they made a clever modifications of some Virasoro constraints for Hodge integrals derived by the author in an unpublished paper to get the same Virasoro constraints for some special Hodge integrals. The success of this method depending on knowing the intersection numbers (in this case, some

Hodge integrals) to be identified with. In this paper we will use an approach without this knowledge a priori. This will be particularly useful in our subsequent paper when we study Hermitian one-matrix model with all couplings, without knowing in advance what type of intersections numbers will be involved.

We will start with the Virasoro constraints derived in [1] for modified GUE partition function with even couplings. Our approach is based on the topological recursions developed by Eynard-Orantin [6], its connection with intersection numbers discovered by Eynard [4] inspired by results in the theory of matrix models, and the idea of emergent geometry from Virasoro constraints developed by the author [16, 17]. Our key result (Theorem 2) is that not only the spectral curve, but also the process of topological recursions, emerge from the Virasoro constraints. This is exactly in the same spirit as our treatment of the case of Witten-Konsevich tau-function in [15]. The spectral curve and the Bergman kernel emerge as a result of studying the genus zero one-point function and two-point functions respectively, and the Eynard-Orantin topological recursions emerge as one reformulate the Virasoro constraints in terms of residues on the spectral curve. As a result, we can use Eynard's result to relate the corresponding n -point functions to intersection numbers (cf. Theorem 3). We leave the problem of rederiving the result of [1] from our approach to future investigations.

As for the case of Hermitian one-matrix model with all couplings, the emergence of the spectral curve has been addressed in [17]. In [18] we will discuss the corresponding emergence of the topological recursions. In that case since the spectral curve has two branchpoints, we will use the generalization of [4] made by Eynard himself in [5] to relate to intersection numbers.

2. EYNARD-ORANTIN TOPOLOGICAL RECURSIONS AND INTERSECTIONS ON $\overline{\mathcal{M}}_{g,n}$

In this Section we recall the n -point multilinear differentials $\omega_{g,n}$ obtained from Eynard-Orantin topological recursions and their relationship with the intersections numbers on $\overline{\mathcal{M}}_{g,n}$.

2.1. Eynard-Orantin topological recursions. Recall a *spectral curve* is a parameterized curve

$$(1) \quad x = x(z), \quad y = y(z)$$

together with a Bergman kernel $B(z_1, z_2)$ with the following property

$$(2) \quad B(z, z') \sim \left(\frac{1}{(z - z')^2} + O(1) \right) dz dz'.$$

For the purpose of this work, assume that x has only one nondegenerate critical point a :

$$(3) \quad x'(a) = 0, \quad x''(a) \neq 0.$$

Near a there is an involution σ such that $x(\sigma(z)) = x(z)$. A sequence $\omega_{g,n}(z_1, \dots, z_n)$ of multidifferential with $n \geq 1$ and $g \geq 0$ are defined by Eynard-Orantin [6] as follows:

$$\begin{aligned} \omega_{0,1}(z) &= y(z)dx(z), \\ \omega_{0,2}(z, z_0) &= B(z, z_0), \end{aligned}$$

and for $2g - 2 + n > 0$,

$$(4) \quad \begin{aligned} \omega_{g,n+1}(z_0, z_1, \dots, z_n) &= \text{Res}_{z \rightarrow a} K(z_0, z) \left[\omega_{g-1, n+2}(z, \sigma(z), z_{[n]}) \right. \\ &\left. + \sum_{h=0}^g \sum_{I \subset [n]} \omega_{h, |I|+1}(z, z_I) \omega_{g-h, n-|I|+1}(\sigma(z), z_{[n]-I}) \right], \end{aligned}$$

the recursion kernel K near a is:

$$(5) \quad K(z_0, z) = \frac{\int_{z'=\sigma(z)}^z B(z_0, z')}{2(y(z) - y(\sigma(z)))dx(z)}.$$

2.2. Expansions near the branch point. Near the branch point $z = a$, one can introduce a new local coordinate ζ :

$$(6) \quad \zeta(z) = \sqrt{x(z) - x(a)},$$

In other words,

$$(7) \quad x = x(a) + \zeta^2.$$

This local coordinate is called the *local Airy coordinate*.

With the introduction of the local Airy coordinates, one can express everything involved in the Eynard-Orantin recursion in terms of it. For z near a , let $\sigma(z)$ denote the unique point near a such that

$$(8) \quad \zeta(\sigma(z)) = -\zeta(z).$$

The function y can be expanded in Taylor series:

$$(9) \quad y(z) \sim \sum_{k=0} t_{k+2} \zeta^k.$$

Similarly, the Bergman kernel can be expanded:

$$(10) \quad B(z, z') = \left[\frac{1}{(\zeta - \zeta')^2} + \sum_{k,l} B_{k,l} \zeta(z_1)^k \zeta(z_2)^l \right] d\zeta(z_1) d\zeta(z_2).$$

The differential $d\zeta_k(z)$ is defined by:

$$(11) \quad d\zeta_k(z) = -\frac{(2k-1)!!}{2^k} \text{Res}_{z' \rightarrow a} B(z, z') \frac{1}{\zeta(z')^{2k+1}}.$$

From the expansion (10) one gets:

$$(12) \quad d\zeta_k(z) = -\frac{(2k+1)!! d\zeta(z)}{2^k \zeta(z)^{2k+2}} - \frac{(2k-1)!!}{2^k} \sum_l B_{2k,l} \zeta(z)^l d\zeta(z),$$

therefore, $d\zeta_k(z)$ is the differential of the function $\zeta_k(z)$ defined by:

$$(13) \quad \zeta_k(z) = \frac{(2k-1)!!}{2^k} \left(\frac{1}{\zeta(z)^{2k+1}} - \sum_l B_{2k,l} \frac{\zeta(z)^{l+1}}{l+1} \right).$$

2.3. Laplace transform and intersection numbers. To make connection with intersection numbers on $\overline{\mathcal{M}}_{g,n}$, one needs to perform the Laplace transforms of $\omega_{g,n}$.

The Laplace transform of the 1-form $\omega_{0,1} = ydx$ gives the times \tilde{t}_k :

$$(14) \quad e^{-\sum_k \tilde{t}_k u^{-k}} = \frac{2u^{3/2}e^{ux(a)}}{\sqrt{\pi}} \int_{\gamma} e^{-ux} y dx,$$

where γ is a steepest descent path from the branchpoint to $x = +\infty$, i.e. $x(\gamma) - x(a) = \mathbb{R}_+$. More precisely

$$\begin{aligned} e^{-\sum_k \tilde{t}_k u^{-k}} &= \frac{2u^{3/2}e^{ux(a)}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u(x(a)+\zeta^2)} \sum_{k \geq 0} t_{k+2} \zeta^k d\zeta^2 \\ &= \sum_k (2k+1)!! t_{2k+3} u^{-k}. \end{aligned}$$

The double Laplace transform of the Bergman kernel gives the kernel

$$(15) \quad \hat{B}(u, v) = \sum_{k,l} \hat{B}_{k,l} u^k v^l$$

by

$$(16) \quad \begin{aligned} &\sum_{k,l} \hat{B}_{k,l} u^k v^l \\ &= \frac{(uv')^{1/2} e^{(u+v)x(a)}}{2\pi} \int_{z \in \gamma} \int_{z' \in \gamma} e^{-ux(z)} e^{-u'x(z')} \left(B(z, z') - B_0(z, z') \right), \end{aligned}$$

where the integral is regularized by subtracting the singular part with double pole

$$B^0(z_1, z_2) = \frac{dx(z_1)dx(z_2)}{4\sqrt{x(z_1)-x(a)}\sqrt{x(z_2)-x(a)}} \frac{1}{(\sqrt{x(z_1)-x(a)} - \sqrt{x(z_2)-x(a)})^2}.$$

More concretely,

$$(17) \quad \hat{B}_{k,l} = (2k-1)!!(2l-1)!! 2^{-k-l-1} B_{2k,2l}.$$

The main result of Eynard [4] is that the Laplace transforms of $\omega_{g,n}$ when $2g-2+n > 0$ are given by intersection numbers on $\overline{\mathcal{M}}_{g,n}$:

$$(18) \quad \begin{aligned} &\prod_{i=1}^n \sqrt{\frac{\mu_i}{\pi}} e^{\mu_i x(a)} \int_{z_1 \in \gamma} \dots \int_{z_2 \in \gamma} \prod_{i=1}^n e^{-\mu_i x(z_i)} \omega_{g,n}(z_1, \dots, z_n) \\ &= 2^{3g-3+2n} \left\langle \prod_{i=1}^n \hat{B}(\mu_i, 1/\psi_i) e^{1/2 \sum_{\delta} \sum_{k,l} \hat{B}_{k,l} \mu_{\delta} \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \end{aligned}$$

Equivalently,

$$(19) \quad \begin{aligned} &\omega_{g,n}(z_1, \dots, z_n) \\ &= 2^{3g-3+n} \sum_{d_1 + \dots + d_n \leq 3g-3+n} \prod_i d\zeta_{d_i}(z_i) \left\langle e^{\frac{1}{2} \sum_{\delta} \iota_{\delta} \hat{B}(\psi, \psi')} e^{\sum_k \tilde{t}_k \kappa_k} \prod_i \psi_i^{d_i} \right\rangle_{g,n} \end{aligned}$$

When $n = 0$:

$$(20) \quad F_g = 2^{3g-3} \left\langle e^{\frac{1}{2} \sum_{\delta} \iota_{\delta} \hat{B}(\psi, \psi')} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,0}$$

The following are some examples from [4]:

$$(21) \quad \omega_{0,3}(z_1, z_2, z_3) = e^{\tilde{t}_0} d\zeta_0(z_1)d\zeta_0(z_2)d\zeta_0(z_3) = \frac{1}{2t_3} d\zeta_0(z_1)d\zeta_0(z_2)d\zeta_0(z_3).$$

$$(22) \quad \omega_{1,1}(z) = \frac{1}{24t_3} d\zeta_1(z) + \left(\frac{B_{0,0}}{4t_3} - \frac{t_5}{16t_3} \right) d\zeta_0(z),$$

$$(23) \quad \begin{aligned} \omega_{0,4}(z_1, z_2, z_3, z_4) &= \frac{1}{2t_3^2} (d\zeta_1(z_1)d\zeta_0(z_2)d\zeta_0(z_3)d\zeta_0(z_4) + \text{perm.}) \\ &+ \frac{3(B_{0,0} - t_5/t_3)}{4t_3^2} d\zeta_0(z_1)d\zeta_0(z_2)d\zeta_0(z_3)d\zeta_0(z_4). \end{aligned}$$

The following are some well-known special cases from [4].

Example 2.1. When the spectral curve is the Airy curve given by the parameterization

$$x = \frac{1}{2}z^2, \quad y = z,$$

with the Bergman kernel

$$B(x_1, x_2) = \frac{dx_1 dx_2}{(x_1 - x_2)^2},$$

one has

$$(24) \quad \omega_{g,n}(z_1, \dots, z_n) = (-1)^n \sum_{d_1 + \dots + d_n = 3g - 3 + n} \prod_i \frac{(2d_i + 1)!! d\zeta_i}{\zeta_i^{2d_i + 2}} \cdot \left\langle \prod_i \psi_i^{d_i} \right\rangle_{g,n}.$$

The Eynard-Orantin topological recursions in this case are equivalent to the DVV Virasoro constraints for Witten-Kontsevich tau-function [15].

Example 2.2. When the spectral curve is the Airy curve given by the parameterization

$$x = \frac{1}{2}z^2, \quad y = z + \sum_{k=2}^{\infty} u_k z^k,$$

with the Bergman kernel

$$B(x_1, x_2) = \frac{dx_1 dx_2}{(x_1 - x_2)^2},$$

one has

$$(25) \quad \begin{aligned} &\omega_{g,n}(z_1, \dots, z_n) \\ &= (-1)^n \sum_{d_1 + \dots + d_n = 3g - 3 + n} \prod_i \frac{(2d_i + 1)!! d\zeta_i}{\zeta_i^{2d_i + 2}} \cdot \left\langle \exp \sum_{n=1}^{\infty} s_n \kappa_n \cdot \prod_i \psi_i^{d_i} \right\rangle_{g,n}, \end{aligned}$$

where the parameters $\{s_n\}_{n \geq 1}$ are related to $\{u_k\}_{k \geq 2}$ by

$$(26) \quad \exp\left(-\sum_{n=1}^{\infty} s_n z^n\right) = \sum_{m \geq 1} (2m + 1)!! u_{2m+1} z^{2m}.$$

The Eynard-Orantin topological recursions in this case are related to higher Weil-Petersson volumes in [3]. They are shown to be equivalent to the Virasoro constraints for higher Weil-Petersson volumes proved by Mulase-Safnuk [11] and Liu-Xu [10] in [15].

3. COMPUTING CORRELATION FUNCTIONS OF MODIFIED HERMITIAN ONE-MATRIX MODEL WITH EVEN COUPLING CONSTANTS BY VIRASORO CONSTRAINTS

In this Section we define the n -point functions of the modified partition function of Hermitian one-matrix model and derive an algorithm to compute them by Virasoro constraints.

3.1. Modified GUE partition function and modified Virasoro constraints.

Let Z_{even} denote the GUE partition function with even couplings. Define \tilde{Z} by

$$(27) \quad \log Z_{even} = (\Lambda^{1/2} + \Lambda^{-1/2}) \log \tilde{Z}.$$

Then \tilde{Z} satisfies the followings system of equations which are called the Virasoro constraints:

$$(28) \quad \frac{1}{2} \frac{\partial F_g}{\partial s_2} = \sum_{k \geq 1} k s_{2k} \frac{\partial F_g}{\partial s_{2k}} + \frac{t^2 \delta_{g,0}}{4} - \frac{\delta_{g,1}}{16},$$

$$(29) \quad \frac{1}{2} \frac{\partial F_g}{\partial s_{2n+2}} = \sum_{k=1}^{n-1} \frac{\partial F_g}{\partial s_{2k}} \frac{\partial F_g}{\partial s_{2n-2k}} + \sum_{k=1}^{n-1} \frac{\partial^2 F_g}{\partial s_{2k} \partial s_{2n-2k}} + t \frac{\partial F_g}{\partial s_{2n}} + \sum_{k \geq 1} k s_{2k} \frac{\partial F_g}{\partial s_{2k+2n}},$$

$n \geq 1$, where

$$(30) \quad \log \tilde{Z} = \sum_{g \geq 0} \epsilon^{2g-2} F_g.$$

In the above $t = N\epsilon$, and $\Lambda = e^{\epsilon \partial_t}$. See [1] for notations and the proof of the above results.

3.2. Virasoro constraints in terms of correlators. Define the n -point correlators by

$$(31) \quad \langle p_{2a_1} \cdots p_{2a_n} \rangle_g^c := \frac{\partial^n}{\partial s_{2a_1} \cdots \partial s_{2a_n}} F_g \Big|_{s_{2k}=0, k \geq 1}.$$

Then the equation (28) can be written in terms of the correlators as follows:

$$(32) \quad \frac{1}{2} \langle p_2 \cdot p_{2a_1} \cdots p_{2a_n} \rangle_g^c = \sum_{j=1}^n a_j \cdot \langle p_{2a_1} \cdots p_{2a_n} \rangle_g,$$

together with initial value:

$$(33) \quad \langle p_2 \rangle_0 = \frac{t^2}{2}, \quad \langle p_2 \rangle_1 = -\frac{1}{8}.$$

And for $m \geq 1$,

$$(34) \quad \begin{aligned} \frac{1}{2} \langle p_{2m+2} \cdot p_{2a_1} \cdots p_{2a_n} \rangle_g^c &= \sum_{j=1}^n a_j \cdot \langle p_{2a_1} \cdots p_{2a_j+2m} \cdots p_{2a_n} \rangle_g^c \\ &+ t \langle p_{2m} \cdot p_{2a_1} \cdots p_{2a_n} \rangle_g^c + \sum_{k=1}^{m-1} \langle p_{2k} p_{2m-2k} \cdot p_{2a_1} \cdots p_{2a_n} \rangle_{g-1}^c \\ &+ \sum_{k=1}^{m-1} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2=[n]}} \langle p_{2k} \cdot \prod_{i \in I_1} p_{2a_i} \rangle_{g_1}^c \cdot \langle p_{2m-2k} \cdot \prod_{i \in I_2} p_{2a_i} \rangle_{g_2}^c, \end{aligned}$$

where $[n] = \{1, \dots, n\}$.

For example,

$$\begin{aligned}\langle p_2^2 \rangle_0^c &= 2\langle p_2 \rangle_0^c = t^2, \\ \langle p_2^3 \rangle_0^c &= 2 \cdot 2\langle p_2 p_2 \rangle_0^c = 4t^2, \\ \langle p_4 \rangle_0^c &= 2t\langle p_2 \rangle_0^c = t^3.\end{aligned}$$

Now we define the genus g , n -point functions by:

$$(35) \quad G_{g,n}(z_1, \dots, z_n) := \sum_{a_1, \dots, a_n \geq 0} \langle p_{2a_1} \cdots p_{2a_n} \rangle_{g,n}^c \frac{1}{z_1^{a_1+1}} \cdots \frac{1}{z_n^{a_n+1}}.$$

Our goal is to compute these functions and interpret the result geometrically.

3.3. Computation of genus zero one-point function by Virasoro constraints.

From

$$\begin{aligned}\langle p_2 \rangle_0^c &= t^2/2, \\ \langle p_{2m} \rangle_0^c &= 2 \sum_{k=1}^{m-2} \langle p_{2k} \rangle_0^c \cdot \langle p_{2m-2-2k} \rangle_0^c + 2t \cdot \langle p_{2m-2} \rangle_0^c, \quad m \geq 2.\end{aligned}$$

we get:

$$\begin{aligned}G_{0,1}(x) &= \frac{t^2}{2x^2} + \sum_{m \geq 2} \frac{1}{x^{m+1}} \left(2 \sum_{k=1}^{m-2} \langle p_{2k} \rangle_0^c \cdot \langle p_{2m-2-2k} \rangle_0^c + 2t \cdot \langle p_{2m-2} \rangle_0^c \right) \\ &= \frac{t^2}{2x^2} + \frac{2t}{x} G_{0,1}(x) + 2G_{0,1}(x)^2.\end{aligned}$$

After solving for $G_{0,1}(x)$ we then get:

$$(36) \quad G_{0,1}(x) = \frac{1}{4} \left(1 - \frac{2t}{x} - \sqrt{\left(1 - \frac{2t}{x} \right)^2 - \frac{4t^2}{x^2}} \right) = \frac{1}{4} \left(1 - \frac{2t}{x} - \sqrt{1 - \frac{4t}{x}} \right).$$

It is then easy to see that:

$$(37) \quad G_{0,1}(x) = \frac{1}{4} \sum_{n \geq 2} \frac{(2n-3)!!}{n!} \cdot (2t)^n x^{-n} = \frac{1}{2} \sum_{n \geq 2} \frac{(2n-2)!}{(n-1)! n!} \frac{t^n}{x^n}.$$

The following are the first few terms of $G_{0,1}(x)$:

$$G_{0,1}(x) = \frac{1}{2} \left(1 \cdot \frac{t^2}{x^2} + \frac{2t^3}{x^3} + \frac{5t^4}{x^4} + \frac{14t^5}{x^5} + \frac{42t^6}{x^6} + \frac{132t^7}{x^7} + \frac{429t^8}{x^8} + \cdots \right),$$

where the sequence 1, 2, 5, 24, 42, 132, 429 are the Catalan numbers. In terms of the correlators, we have shown that:

$$(38) \quad \langle p_{2n} \rangle_0^c = \frac{1}{2} \frac{(2n)!}{n!(n+1)!} t^{n+1}.$$

3.4. Computation of genus zero two-point function by Virasoro constraints. By equation (28),

$$\langle p_2 \cdot p_{2n} \rangle_0^c = 2n \cdot \langle p_{2n} \rangle_g^c,$$

and so

$$\begin{aligned} & \sum_{n=1}^{\infty} \langle p_2 \cdot p_{2n} \rangle_0^c \frac{1}{x_2^{-n-1}} = \sum_{n \geq 2} 2n \cdot \langle p_{2n} \rangle_0^c \frac{1}{x_2^{n+1}} \\ &= -\frac{d}{dx_2} \left(2x_2 \sum_{n \geq 2} \langle p_{2n} \rangle_0^c \frac{1}{x_2^{n+1}} \right) \\ &= -\frac{d}{dx_2} \left(\frac{x_2}{2} \left(1 - \frac{2t}{x_2} - \sqrt{1 - \frac{4t}{x_2}} \right) \right) \\ &= \frac{1 - \frac{2t}{x_2}}{2\sqrt{1 - \frac{4t}{x_2}}} - \frac{1}{2}. \end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \langle p_2 \cdot p_{2n} \rangle_0^c \frac{1}{x_2^{-n-1}} = \frac{t^2}{x_2^2} + \frac{4t^3}{x_2^3} + \frac{15t^4}{x_2^4} + \frac{56t^5}{x_2^5} + \frac{210t^6}{x_2^6} + \frac{792t^7}{x_2^7} + \dots,$$

where the sequences of the numbers 1, 4, 15, 56, 210, 792 are the sequence A001791 on [12], they are given by

$$(39) \quad a_n = \binom{2n}{n-1} = \frac{(2n)!}{(n-1)!n!}.$$

The second equation in the sequence can be written as

$$(40) \quad \langle p_4 \cdot p_{2n} \rangle_0^c = 2n \cdot \langle p_{2n+2} \rangle_0^c + 2t \cdot \langle p_2 \cdot p_{2n} \rangle_0^c.$$

By taking generating series, one gets:

$$\begin{aligned} & \sum_{n \geq 1} \langle p_4 \cdot p_{2n} \rangle_0^c \frac{1}{x_2^{n+1}} \\ &= \sum_{n \geq 1} \frac{1}{x_2^{n+1}} \left(2n \cdot \langle p_{2n+2} \rangle_0^c + 2t \cdot \langle p_2 \cdot p_{2n} \rangle_0^c \right) \\ &= 2 \sum_{n \geq 2} \frac{n-1}{x_2^n} \langle p_{2n} \rangle_0^c + \sum_{n \geq 1} \frac{2t}{x_2^{n+1}} \cdot \langle p_2 \cdot p_{2n} \rangle_0^c \\ &= -2 \frac{d}{dx_2} \left(x_2^2 \sum_{n \geq 2} \frac{1}{x_2^{2n+2}} \langle p_{2n} \rangle_0^c \right) + 2t \sum_{n \geq 1} \langle p_2 \cdot p_{2n} \rangle_0^c \frac{1}{x_2^{n+1}} \\ &= -2 \frac{d}{dx_2} \left(x_2^2 \left(\frac{1}{4} \left(1 - \frac{2t}{x_2} - \sqrt{1 - \frac{4t}{x_2}} \right) - \frac{t^2}{2x_2^2} \right) \right) + 2t \left(\frac{1 - \frac{2t}{x_2}}{2\sqrt{1 - \frac{4t}{x_2}}} - \frac{1}{2} \right) \\ &= x_2 \left(\frac{1 - \frac{2t}{x_2} - \frac{2t^2}{x_2^2}}{\sqrt{1 - \frac{4t}{x_2}}} - 1 \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{n \geq 1} \langle p_4 \cdot p_{2n} \rangle_0^c \frac{1}{z_2^{n+1}} \\ &= 2 \left(\frac{2t^3}{x^2} + \frac{9t^4}{x^3} + \frac{36t^5}{x^4} + \frac{140t^6}{2^5 x^5} + \frac{540t^7}{x^6} + \frac{2079t^8}{x^7} + \frac{8008t^9}{x^8} + \frac{30888t^{10}}{x^9} + \dots \right), \end{aligned}$$

where the numbers 2, 9, 36, 140, 540, 2079, 8008, 30888 are the sequence A007946 on [12], and they are given by

$$(41) \quad a_n = \frac{3(2n)!}{(n-1)!n!(n+2)}.$$

Proposition 3.1. *The following formula holds:*

$$(42) \quad G_{0,2}(x_1, x_2) = \frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2}}{2(x_1 - x_2)^2 \sqrt{(1 - \frac{4t}{x_1})(1 - \frac{4t}{x_2})}} - \frac{1}{2(x_1 - x_2)^2}.$$

Proof. For $m \geq 4$,

$$(43) \quad \langle p_{2m} \cdot p_{2n} \rangle_0^c = 2n \cdot \langle p_{2n+2m-2} \rangle_0^c + 4 \sum_{k=0}^{m-1} \langle p_{2k} \cdot p_{2n} \rangle_0^c \cdot \langle p_{2m-2-2k} \rangle_0^c,$$

where we use the following convention:

$$(44) \quad \langle p_0 \rangle_0^c = \frac{t}{2},$$

Therefore,

$$\begin{aligned} G_{0,2}(x_1, x_2) &= \sum_{m, n \geq 1} \langle p_{2m} \cdot p_{2n} \rangle_0^c \frac{1}{x_1^{m+1}} \frac{1}{x_2^{n+1}} \\ &= \sum_{m+n \geq 2} 2n \cdot \langle p_{2n+2m-2} \rangle_0^c \frac{1}{x_1^{m+1}} \frac{1}{x_2^{n+1}} \\ &+ 4 \sum_{m \geq 2, n \geq 1} \frac{1}{x_1^{m+1}} \frac{1}{x_2^{n+1}} \sum_{k=0}^{m-1} \langle p_{2k} \cdot p_{2n} \rangle_0^c \cdot \langle p_{2m-2-2k} \rangle_0^c \\ &= 2 \sum_{m+n \geq 2} n \cdot \langle p_{2n+2m-2} \rangle_0^c \frac{1}{x_1^{m+1}} \frac{1}{x_2^{n+1}} \\ &+ 4 \sum_{n, k \geq 1} \frac{1}{x_1^{k+1}} \frac{1}{x_2^{n+1}} \langle p_{2k} \cdot p_{2n} \rangle_0^c \cdot \sum_{j \geq 0} \langle p_{2j} \rangle_0^c \frac{1}{x_1^{j+1}} \\ &= 2 \sum_{m+n \geq 2} n \cdot \langle p_{2n+2m-2} \rangle_0^c \frac{1}{x_1^{m+1}} \frac{1}{x_2^{n+1}} \\ &+ 4W_{0,1}(z_1) \cdot W_{0,2}(z_1, z_2), \end{aligned}$$

where $W_{0,1}(z_1)$ is defined by:

$$(45) \quad W_{0,1}(x_1) = \frac{t}{2x_1} + G_{0,1}(z_1) = \sum_{j \geq 0} \langle p_{2j} \rangle_0^c \frac{1}{x_1^{j+1}} = \frac{1}{4} \left(1 - \sqrt{1 - \frac{4t}{x_1}} \right).$$

So we have:

$$\begin{aligned} G_{0,2}(z_1, z_2) &= \frac{2}{1-4W_{0,1}(x_1)} \cdot \sum_{l \geq 1} \langle p_{2l} \rangle_0^c \sum_{n=1}^l \frac{n}{x_1^{l+2-n}} \frac{1}{x_2^{n+1}} \\ &= \frac{2}{\sqrt{1-\frac{4t}{x_1}}} \sum_{l \geq 1} \langle p_{2l} \rangle_0^c \sum_{n=1}^l \frac{n}{x_1^{l+2-n}} \frac{1}{x_2^{n+1}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} & \sum_{l \geq 1} \langle p_{2l} \rangle_0^c \sum_{n=1}^l \frac{n}{x_1^{l+2-n}} \frac{1}{x_2^{n+1}} \\ &= \sum_{l \geq 1} \langle p_{2l} \rangle_0^c \frac{1}{x_1^{l+1} x_2} \sum_{n=1}^l \frac{n}{x_1^{1-n}} \frac{1}{x_2^{n-1}} \\ &= \sum_{l \geq 1} \langle p_{2l} \rangle_0^c \frac{1}{x_1^{l+1} x_2} \left(\frac{1 - (\frac{x_1}{x_2})^l}{(1 - \frac{x_1}{x_2})^2} - \frac{l(\frac{x_1}{x_2})^l}{1 - \frac{x_1}{x_2}} \right) \\ &= \frac{1}{x_1(x_1 - x_2)^2} \sum_{l \geq 1} \langle p_{2l} \rangle_0^c \left(\frac{1}{x_1^l} - \frac{1}{x_2^l} \right) + \frac{1}{x_1(x_1 - x_2)} \sum_{l=1}^{\infty} \langle p_{2l} \rangle_0^c \cdot \frac{l}{x_2^{l+1}} \\ &= \frac{1}{x_1(x_1 - x_2)^2} (x_1 G_{0,1}(x_1) - x_2 G_{0,1}(x_2)) - \frac{1}{x_1(x_1 - x_2)} \frac{d}{dx_2} (x_2 G_{0,1}(x_2)) \\ &= \frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2} - \sqrt{(1 - \frac{4t}{x_1})(1 - \frac{4t}{x_2})}}{4(x_1 - x_2)^2 \sqrt{(1 - \frac{4t}{x_2})}}, \end{aligned}$$

where in the last equality we have used (36). Therefore, we have proved (42). \square

Corollary 3.1. *The two-point function in genus zero has the following expansion:*

$$(46) \quad G_{0,2}(x_1, x_2) = 2 \sum_{l=2}^{\infty} \frac{t^l}{l} \sum_{m+n=l-2} \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \frac{1}{x_1^{m+2} x_2^{n+2}},$$

In terms of correlators we have

$$(47) \quad \langle p_{2m} p_{2n} \rangle_0^c = \frac{1}{2} \cdot \frac{(2m)!}{(m-1)!m!} \cdot \frac{(2n)!}{(n-1)!n!} \cdot \frac{t^{m+n}}{m+n}.$$

Proof. By (42) we have:

$$\begin{aligned} \frac{\partial}{\partial t} G_{0,2}(x_1, x_2) &= \frac{\partial}{\partial t} \left(\frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2}}{2(x_1 - x_2)^2 \sqrt{(1 - \frac{4t}{x_1})(1 - \frac{4t}{x_2})}} - \frac{1}{2(x_1 - x_2)^2} \right) \\ &= \frac{2t}{x_1^2 x_2^2} \frac{1}{(1 - \frac{4t}{x_1})^{3/2} (1 - \frac{4t}{x_2})^{3/2}} \\ &= \frac{2t}{x_1^2 x_2^2} \cdot \sum_{m=0}^{\infty} \frac{(2m+1)!}{(m!)^2} \frac{t^m}{x_1^m} \cdot \sum_{n=0}^{\infty} \frac{(2n+1)!}{(n!)^2} \frac{t^n}{x_2^n} \\ &= \frac{2}{x_1^2 x_2^2} \cdot \sum_{l=2}^{\infty} t^{l-1} \sum_{m+n=l-2} \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \frac{1}{x_1^m x_2^n}. \end{aligned}$$

Then (46) is proved by integration. \square

3.5. Computation of n -point functions in arbitrary genera by Virasoro constraints. By (34) we have:

$$\begin{aligned}
& G_{g,n}(x_0, x_1, \dots, x_n) \\
&= \sum_{m, a_1, \dots, a_n \geq 1} \langle p_{2m} \cdot p_{2a_1} \cdots p_{2a_n} \rangle_g \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{1}{x_i^{a_i+1}} \\
&= 2 \sum_{m, a_1, \dots, a_n \geq 1} \sum_{j=1}^n a_j \cdot \langle p_{2a_1} \cdots p_{2a_j+2m-2} \cdots p_{2a_n} \rangle_g \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{1}{x_i^{a_i+1}} \\
&+ \sum_{m \geq 2, a_1, \dots, a_n \geq 1} 2t \langle p_{2m-2} \cdot p_{2a_1} \cdots p_{2a_n} \rangle_0 \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{1}{x_i^{a_i+1}} \\
&+ 2 \sum_{k=1}^{m-2} \langle p_{2k} p_{2m-2-2k} \cdot p_{2a_1} \cdots p_{2a_n} \rangle_{g-1} \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{1}{x_i^{a_i+1}} \\
&+ 2 \sum_{k=1}^{m-2} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2=[n]}} \langle p_{2k} \cdot \prod_{i \in I_1} p_{a_i} \rangle_{g_1}^c \cdot \langle p_{2m-2-2k} \cdot \prod_{i \in I_2} p_{2a_i} \rangle_{g_2}^c \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{1}{x_i^{a_i+1}} \\
&- \frac{1}{8} \delta_{g,1} \delta_{n,0} \frac{1}{x_0^2}.
\end{aligned}$$

Denote by I , II , III , IV and V the five lines on the right-hand side of the second equality respectively.

We now rewrite I in the following fashion. Note

$$\begin{aligned}
& \sum_{m, a_j \geq 1} a_j \cdot \langle p_{2a_1} \cdots p_{2a_j+2m-2} \cdots p_{2a_n} \rangle_g \frac{1}{x_0^{m+1}} \frac{1}{x_j^{a_j+1}} \\
&= \sum_{b_j \geq 1} \sum_{\substack{m, a_j \geq 1 \\ m+a_j=b_j+1}} a_j \cdot \langle p_{2a_1} \cdots p_{2b_j} \cdots p_{2a_n} \rangle_g \frac{1}{x_0^{m+1}} \frac{1}{x_j^{a_j+1}}
\end{aligned}$$

This leads us to an operator

$$(48) \quad \frac{1}{x_j^{l+1}} \mapsto \sum_{\substack{m, n \geq 1 \\ m+n=l+1}} n \cdot \frac{1}{x_0^{m+1}} \frac{1}{x_j^{n+1}}$$

For $l \geq 0$. Because we have:

$$\begin{aligned}
& \sum_{\substack{m, n \geq 1 \\ m+n=l+1}} n \cdot \frac{1}{x_0^{m+1}} \frac{1}{x_j^{n+1}} \\
&= \sum_{n=1}^l \frac{n}{x_0^{l+2-n}} \frac{1}{x_j^{n+1}} = \frac{1}{x_0^{l+1} x_j^2} \sum_{n=1}^l \frac{n}{x_0^{1-n}} \frac{1}{x_j^{n-1}} \\
&= \frac{1}{x_0^{l+1} x_j^2} \left(\frac{1 - (\frac{x_0}{x_j})^l}{(1 - \frac{x_0}{x_j})^2} - \frac{l(\frac{x_0}{x_j})^l}{1 - \frac{x_0}{x_j}} \right) \\
&= \frac{1}{x_0(x_0 - x_j)^2} \left(\frac{1}{x_0^l} - \frac{1}{x_j^l} \right) + \frac{1}{x_0(x_0 - x_j)} \cdot \frac{l}{x_j^{l+1}},
\end{aligned}$$

this operator can be realized by:

$$(49) \quad D_{x_0, x_j} f(x_j) = \frac{x_0 f(x_0) - x_j f(x_j)}{x_0(x_0 - x_j)^2} - \frac{1}{x_0(x_0 - x_j)} \frac{d}{dx_j} (x_j f(x_j)).$$

Now we examine

$$III = \sum_{k=1}^{m-2} \langle p_{2k} p_{2m-2-2k} \cdot p_{a_1} \cdots p_{a_n} \rangle_{g-1}^c(t) \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{1}{x_i^{a_i+1}}.$$

This leads us to the operator:

$$(50) \quad \frac{1}{u^{k+1}} \frac{1}{v^{l+1}} \mapsto \frac{1}{x_0^{k+l+2}}$$

This operator can be realized by taking the limit:

$$(51) \quad E_{x_0, u, v} f(u, v) = \lim_{u \rightarrow v} f(u, v)|_{v=x_0}.$$

We combine *II* with the terms in *IV* with $g_1 = 0$ and $|I_1| = 0$, or $g_2 = 0$ and $|I_2| = 0$. These together give us $4W_{0,1}(x_0)G_{0,n+1}(x_0, x_1, \dots, x_n)$. The rest of the terms in *IV* give us

$$\sum_{\substack{g_1+g_2=g \\ I_1 \amalg I_2=[n]}} 'E_{x_0, u, v} \left(G_{g_1}(u, x_{I_1}) \cdot G_{g_2}(v, x_{I_2}) \right).$$

To summarize, we obtain the following identity:

$$\begin{aligned}
& G_g(x_0, x_1, \dots, x_n) \\
&= 2 \sum_{j=1}^n D_{x_0, x_j} G_g(x_1, \dots, x_n) + 2E_{x_0, u, v} G_{g-1}(u, v, x_1, \dots, x_n) \\
&+ 2 \sum_{\substack{g_1+g_2=g \\ I_1 \amalg I_2=[n]}} 'E_{x_0, u, v} \left(G_{g_1}(u, x_{I_1}) \cdot G_{g_2}(v, x_{I_2}) \right) \\
&+ 4W_{0,1}(x_0)G_{0,n+1}(x_0, x_1, \dots, x_n) - \frac{1}{8} \delta_{g,1} \delta_{n,0} \frac{1}{x_0^2}.
\end{aligned}$$

From this we derive the following:

Proposition 3.2. Define the renormalized operators \tilde{D} and \tilde{E} as follows:

$$(52) \quad \tilde{D}_{x_0, x_j} = \frac{2}{1 - 4W_{0,1}(x_0)} D_{x_0, x_j}, \quad \tilde{E}_{x_0, u, v} = \frac{2}{1 - 4W_{0,1}(x_0)} E_{x_0, u, v}.$$

Then one has:

$$(53) \quad \begin{aligned} & G_{g, n+1}(x_0, x_1, \dots, x_n) \\ &= \sum_{j=1}^n \tilde{D}_{x_0, x_j} G_{g, n}(x_1, \dots, x_n) + \tilde{E}_{x_0, u, v} G_{g-1, n+2}(u, v, x_1, \dots, x_n) \\ &+ \sum_{\substack{g_1 + g_2 = g \\ I_1 \amalg I_2 = [n]}} \tilde{E}_{x_0, u, v} \left(G_{g_1, |I_1|+1}(u, x_{I_1}) \cdot G_{g_2, |I_2|+1}(v, x_{I_2}) \right) \\ &- \frac{\delta_{g,1} \delta_{n,0}}{8(1 - 4W_{0,1}(x_0))} \frac{1}{x_0^2}. \end{aligned}$$

3.6. Examples. We now present some sample computations of $G_{g,n}$ using (53).

3.6.1. Three-point function in genus zero.

$$\begin{aligned} & G_0(x_0, x_1, x_2) \\ &= \sum_{j=1}^2 \tilde{D}_{x_0, x_j} G_0(x_1, x_2) + 2\tilde{E}_{x_0, u, v} \left(G_0(u, x_1) \cdot G_0(v, x_2) \right) \\ &= \sum_{j=1}^2 \tilde{D}_{x_0, x_j} \left(\frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2}}{2(x_1 - x_2)^2 \sqrt{(1 - \frac{4t}{x_1})(1 - \frac{4t}{x_2})}} - \frac{1}{2(x_1 - x_2)^2} \right) \\ &+ 2\tilde{E}_{x_0, u, v} \left(\left(\frac{1 - \frac{2t}{u} - \frac{2t}{x_1}}{2(u - x_1)^2 \sqrt{(1 - \frac{4t}{u})(1 - \frac{4t}{x_1})}} - \frac{1}{2(u - x_1)^2} \right) \right. \\ &\quad \left. \cdot \left(\frac{1 - \frac{2t}{v} - \frac{2t}{x_2}}{2(v - x_2)^2 \sqrt{(1 - \frac{4t}{v})(1 - \frac{4t}{x_2})}} - \frac{1}{2(v - x_2)^2} \right) \right). \end{aligned}$$

After a complicated computation with the help of Maple, the following simple formula is obtained:

$$(54) \quad G_0(x_0, x_1, x_2) = \frac{4t^2}{x_0^2 x_1^2 x_2^2 ((1 - 4t/x_0)(1 - 4t/x_1)(1 - 4t/x_2))^{3/2}}.$$

After expanding this in Taylor series in t :

$$G_0(x_0, x_1, x_2) = \frac{4t^2}{x_0^2 x_1^2 x_2^2} \prod_{j=0}^2 \sum_{m_j=0}^{\infty} \frac{(2m_j + 3)! (2t)^{m_j}}{m_j! x_j^{m_j}},$$

we get:

$$(55) \quad \langle p_{2n_1} p_{2n_2} p_{2n_3} \rangle_0^c = (2t)^{\sum_{j=1}^3 n_j - 1} \prod_{j=1}^3 \frac{(2n_j + 1)!!}{n_j!}.$$

3.6.2. *Four-point function in genus zero.* In this case (53) takes the following form:

$$\begin{aligned} G_0(x_0, x_1, x_2, x_3) &= \sum_{j=1}^3 \tilde{D}_{x_0, x_j} G_0(x_1, x_2, x_3) \\ &+ 2\tilde{E}_{x_0, u, v} \left(G_0(u, x_1) \cdot G_0(v, x_2, x_3) \right. \\ &\left. + G_0(u, x_2) \cdot G_0(v, x_1, x_3) + G_0(u, x_3) \cdot G_0(v, x_1, x_2) \right). \end{aligned}$$

A calculation shows that:

$$(56) \quad G_{0,4}(x_0, \dots, x_3) = 24t^2 \frac{e_4 - 2e_3t + 32e_1t^3 - 256t^4}{\prod_{j=0}^3 x_j^3 (1 - \frac{4t}{x_j})^{5/2}},$$

where e_j denotes the j -th elementary symmetric polynomial in x_0, \dots, x_3 .

$$\begin{aligned} G_{0,4}(x_0, \dots, x_3) &= \frac{24t^2}{\prod_{j=0}^3 x_j^3} (e_4 - 2e_3t + 32e_1t^3 - 256t^4) \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+3)!!}{3 \cdot m_j!} \frac{(2t)^{m_j}}{x_j^{m_j}} \\ &= 24t^2 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+3)!!}{3 \cdot m_j!} \frac{(2t)^{m_j}}{x_j^{m_j+2}} \\ &\quad - 48t^3 \sum_{k=0}^3 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+3)!!}{3 \cdot m_j!} \frac{(2t)^{m_j}}{x_j^{m_j+2+\delta_{j,k}}} \\ &\quad + 768t^5 \sum_{k=0}^3 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+3)!!}{3 \cdot m_j!} \frac{(2t)^{m_j}}{x_j^{m_j+3-\delta_{j,k}}} \\ &\quad - 6144t^6 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+3)!!}{3 \cdot m_j!} \frac{(2t)^{m_j}}{x_j^{m_j+3}} \\ &= 24t^2 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+3)!!}{3 \cdot m_j!} \frac{(2t)^{m_j}}{x_j^{m_j+2}} \\ &\quad - 24t^2 \sum_{k=0}^3 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j - 2\delta_{j,k} + 3)!!}{3 \cdot (m_j - \delta_{j,k})!} \frac{(2t)^{m_j}}{x_j^{m_j+2}} \\ &\quad + 96t^2 \sum_{k=0}^3 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j + 2\delta_{j,k} + 1)!!}{3 \cdot (m_j - 1 + \delta_{j,k})!} \frac{(2t)^{m_j}}{x_j^{m_j+2}} \\ &\quad - 384t^2 \prod_{j=0}^3 \sum_{m_j=0}^{\infty} \frac{(2m_j+1)!!}{3 \cdot (m_j-1)!} \frac{(2t)^{m_j}}{x_j^{m_j+2}}. \end{aligned}$$

After simplification we get

$$G_{0,4}(x_0, \dots, x_3) = 8t^2 \sum_{m_0, \dots, m_3=0}^{\infty} \left(\sum_{j=0}^3 m_j + 3 \right) \prod_{j=0}^3 \frac{(2m_j+1)!!}{m_j!} \frac{(2t)^{m_j}}{x_j^{m_j}}.$$

In terms of correlators,

$$(57) \quad \langle p_{2n_1} \cdots p_{2n_4} \rangle_0^c = 2^{\sum_{j=1}^4 n_j - 1} t^{\sum_{j=1}^4 n_j - 2} (n_1 + \cdots + n_4 - 1) \cdot \prod_{j=1}^4 \frac{(2n_j - 1)!!}{(n_j - 1)!}.$$

3.6.3. *One-point function in genus one.* In this case (53) takes the form:

$$G_{1,1}(x_0) = -\frac{1}{8x_0^2(1-4t/x_0)^{1/2}} + \tilde{E}_{x_0,u,v} G_{0,2}(u,v).$$

It is easy to see that

$$\begin{aligned} E_{x_0,x,y} W_0^{(2)}(x,y) &= \frac{2}{\sqrt{1-\frac{4t}{x_0}}} \lim_{u \rightarrow v} \left(\frac{1 - \frac{2t}{u} - \frac{2t}{v}}{2(u-v)^2 \sqrt{(1-\frac{4t}{u})(1-\frac{4t}{v})}} - \frac{1}{2(u-v)^2} \right) \Big|_{v \rightarrow x_0} \\ &= \frac{2t^2}{x_0^4(1-4t/x_0)^{5/2}}. \end{aligned}$$

Therefore,

$$(58) \quad G_{1,1}(x_0) = -\frac{1}{8x_0^2(1-4t/x_0)^{1/2}} + \frac{2t^2}{x_0^4(1-4t/x_0)^{5/2}}.$$

By expanding into Taylor series in t ,

$$\begin{aligned} G_{1,1}(x_0) &= -\frac{1}{8x_0^2} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{m!} \frac{(2t)^m}{x_0^m} + \frac{2t^2}{3x_0^4} \sum_{n=0}^{\infty} \frac{(2n+3)!!}{n!} \frac{(2t)^n}{x_0^n} \\ &= -\frac{1}{8x_0^2} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{m!} \frac{(2t)^m}{x_0^m} + \frac{1}{6x_0^2} \sum_{m=0}^{\infty} m(m-1) \frac{(2m-1)!!}{m!} \frac{(2t)^m}{x_0^m}, \end{aligned}$$

we get

$$\begin{aligned} \langle p_{2n} \rangle_1 &= -\frac{1}{8} \frac{(2n-3)!!}{(n-1)!} (2t)^{n-1} + \frac{1}{6} \frac{(2n-3)!!}{(n-3)!} (2t)^{n-1} \\ &= \frac{(2n-5) \cdot (2n-1)!!}{24 \cdot (n-1)!} (2t)^{n-1}. \end{aligned}$$

A crucial observation which will play an important role below is that one can rewrite $G_{1,1}(x_0)$ as follows:

$$(59) \quad G_{1,1}(x_0) = -\frac{1}{8x_0^2(1-4t/x_0)^{3/2}} + \frac{t}{2x_0^3(1-4t/x_0)^{5/2}}.$$

3.7. **General structure of $G_{g,n}(x_1, \dots, x_n)$.** By induction one easily sees that

Proposition 3.3. *When $2g - 2 + n > 0$, $G_{g,n}(p_1, \dots, p_n)$ have the following form:*

$$(60) \quad \sum_{\substack{a_1, \dots, a_n \geq 2 \\ b_1, \dots, b_n \in \mathbb{Z}}} A_{a_1, \dots, a_n; b_1 \geq a_1 - 1, 1 \leq i \leq n}^{g,n}(t) x_1^{-a_1} \cdots x_n^{-a_n} y_1^{2b_1+1} \cdots y_n^{2b_n+1},$$

where y_i is defined by

$$(61) \quad y_i = -\frac{1}{4} \sqrt{1 - \frac{4t}{x_i}}.$$

In particular, they only have poles of odd orders at $y_j = 0$.

Proof. We have verified the cases of $G_{0,3}$, $G_{1,1}$ and $G_{0,4}$. We now use (53) to inductively finish the proof. We first consider the terms $\tilde{E}_{x_0,u,v}G_{g-1,n+2}(u, v, x_1, \dots, x_n)$. They involve

$$\tilde{E}_{x_0,u,v}(u^{-a_1}y(u)^{-2b_1-1} \cdot v^{-a_2}y(v)^{-2b_2-1}) = -\frac{1}{2}x_0^{-a_1-a_2}y(x_0)^{-2b_1-2b_2-3},$$

where we have used the fact that

$$\frac{2}{1-4W_{0,2}(x_0)} = -\frac{1}{2y_0}.$$

Because $b_1 \geq a_1 - 1$, $b_2 \geq a_2 - 1$, we have

$$b_1 + b_2 + 1 \geq (a_1 - 1) + (a_2 - 1) + 1 = a_1 + a_2 - 1.$$

Similarly, when $(g_1, |I_1|+1) \neq (0, 2)$ and $(g_2, |I_2|+1) \neq (0, 2)$, $\tilde{E}_{x_0,u,v}\left(G_{g_1,|I_1|+1}(u, x_{I_1}) \cdot G_{g_2,|I_2|+1}(v, x_{I_2})\right)$ can be treated in the same way. The rest of the terms are of the following form:

$$\sum_{j=1}^n \left[\tilde{D}_{x_0,x_j}G_{g,n}(x_1, \dots, x_n) + 2\tilde{E}_{x_0,u,v}\left(G_{0,2}(u, x_j) \cdot G_{g,n}(v, x_1, \dots, \hat{x}_j, \dots, x_n)\right) \right],$$

and so they involve:

$$\begin{aligned} & D_{x_0,x_j}(x_j^{-a}y_j^{-2b-1}) + 2\mathcal{E}_{x_0,u,v}(G_{0,2}(x_j, u)v^{-a}y(v)^{-2b-1}) \\ = & \frac{x_0 \cdot x_0^{-a}y_0^{-2b-1} - x_j \cdot x_j^{-a}y_j^{-2b-1}}{x_0(x_0 - x_j)^2} - \frac{1}{x_0(x_0 - x_j)} \cdot \frac{d}{dx_j}\left(x_j \cdot x_j^{-a}y_j^{-2b-1}\right) \\ + & 2\left(\frac{1 - \frac{2t}{x_0} - \frac{2t}{x_j}}{2(x_0 - x_j)^2(1 - \frac{4t}{x_0})^{1/2}(1 - \frac{4t}{x_j})^{1/2}} - \frac{1}{2(x_0 - x_j)^2}\right) \cdot x_0^{-a}y_0^{-2b-1} \\ = & \frac{x_0 \cdot x_0^{-a}y_0^{-2b-1} - x_j \cdot x_j^{-a}y_j^{-2b-1}}{x_0(x_0 - x_j)^2} \\ - & \frac{1}{x_0(x_0 - x_j)} \cdot \left((-a+1)x_j^{-a}y_j^{-2b-1} - (2b+1)x_j^{-a+1}y_j^{-2b-2} \cdot \frac{t}{8x_j^2y_j} \right) \\ + & \frac{(1 - \frac{2t}{x_0} - \frac{2t}{x_j})x_0^{-a}y_0^{-2b-1}}{16(x_0 - x_j)^2y_0y_j} - \frac{x_0^{-a}y_0^{-2b-1}}{(x_0 - x_j)^2} \\ = & \frac{1}{x_0(x_0 - x_j)^2} \left[-x_j^{-a+1}y_j^{-2b-1} \right. \\ - & (x_0 - x_j) \cdot \left((-a+1)x_j^{-a}y_j^{-2b-1} - (2b+1)x_j^{-a-1}y_j^{-2b-3} \cdot \frac{t}{8} \right) \\ + & \left. \frac{1}{16}\left(1 - \frac{2t}{x_0} - \frac{2t}{x_j}\right)x_0^{-a+1}y_0^{-2b-2}y_j^{-1}\right]. \end{aligned}$$

We rewrite the right-hand side as follows:

$$\begin{aligned}
& \frac{x_j^{-a-1}y_j^{-2b-3}x_0^{-a-1}y_0^{-2b-2}}{x_0(x_0-x_j)^2} \left[-x_j^2y_j^2x_0^{a+1}y_0^{2b+2} \right. \\
& - (x_0-x_j) \cdot \left((-a+1)x_jy_j^2 - (2b+1) \cdot \frac{t}{8} \right) x_0^{a+1}y_0^{2b+2} \\
& \left. + \frac{1}{16} \left(1 - \frac{2t}{x_0} - \frac{2t}{x_j} \right) x_0^2x_j^{a+1}y_j^{2b+2} \right] \\
& = \frac{x_j^{-a-1}y_j^{-2b-3}x_0^{-a-1}y_0^{-2b-2}}{2^{8b+12}x_0(x_0-x_j)^2} \left[-x_j^2 \cdot \left(1 - \frac{4t}{x_j} \right) x_0^{a+1} \left(1 - \frac{4t}{x_0} \right)^{b+1} \right. \\
& - (x_0-x_j) \cdot \left((-a+1)x_j \cdot \left(1 - \frac{4t}{x_j} \right) - (2b+1) \cdot 2t \right) x_0^{a+1} \left(1 - \frac{4t}{x_0} \right)^{b+1} \\
& \left. + \left(1 - \frac{2t}{x_0} - \frac{2t}{x_j} \right) x_0^2x_j^{a+1} \left(1 - \frac{4t}{x_j} \right)^{b+1} \right] \\
& = \frac{x_j^{-a-1}y_j^{-2b-3}x_0^{-a-1}y_0^{-2b-2}}{2^{8b+12}x_0(x_0-x_j)^2} \left[-x_j \cdot (x_j-4t)x_0^{a-b}(x_0-4t)^{b+1} \right. \\
& - (x_0-x_j) \cdot \left((-a+1)(x_j-4t) - (2b+1) \cdot 2t \right) x_0^{a-b}(x_0-4t)^{b+1} \\
& \left. + (x_0x_j-2tx_0-2tx_j)x_0x_j^{a-b-1}(x_j-4t)^{b+1} \right] \\
& = \frac{y_j^{-2b-3}y_0^{-2b-2}}{2^{8b+12}x_0^{b+2}x_j^{b+2}(x_0-x_j)^2} \left[-x_j^{b-a+2} \cdot (x_j-4t)(x_0-4t)^{b+1} \right. \\
& - (x_0-x_j) \cdot \left((-a+1)(x_j-4t) - (2b+1) \cdot 2t \right) x_j^{b-a+1}(x_0-4t)^{b+1} \\
& \left. + (x_0x_j-2tx_0-2tx_j)x_0^{b-a+1}(x_j-4t)^{b+1} \right] \\
& = \frac{y_j^{-2b-3}y_0^{-2b-2}}{2^{8b+12}x_0^{b+2}x_j^{b+2}} \sum_{\substack{p+q+r=2b-a+2 \\ 0 \leq p, q \leq b}} \alpha_{p,q,r} x_0^p x_j^q t^r.
\end{aligned}$$

By this we complete the proof. \square

4. EMERGENT GEOMETRY OF MODIFIED HERMITIAN ONE-MATRIX MODEL

In last Section we have defined the n -point function $G_{g,n}$ and derive an algorithm to compute them by the operators \tilde{D} and \tilde{E} . This algorithm is based on the Virasoro constraints. Inspired by Eynard-Orantin [6], we will reformulate the n -point functions as multilinear differentials on a plane algebraic curve, satisfying the Eynard-Orantin topological recursions in the next Section. In this Section we show how spectral curve and its geometry naturally appear from the point of view of emergent geometry.

4.1. **Emergence of the spectral curve.** Recall the Virasor constraints in genus zero are the following sequence of differential equations:

$$(62) \quad \frac{1}{2} \frac{\partial F_0}{\partial s_2} = \sum_{k \geq 1} k s_{2k} \frac{\partial F_0}{\partial s_{2k}} + \frac{t^2}{4},$$

$$(63) \quad \frac{1}{2} \frac{\partial F_0}{\partial s_{2n+2}} = \sum_{k=1}^{n-1} \frac{\partial F_0}{\partial s_{2k}} \frac{\partial F_0}{\partial s_{2n-2k}} + t \frac{\partial F_0}{\partial s_{2n}} + \sum_{k \geq 1} k s_{2k} \frac{\partial F_0}{\partial s_{2k+2n}}, \quad n \geq 1.$$

In earlier work the author has developed the theory of emergent geometry of spectral curves, associated with Virasoro constraints, see [15, 16, 17]. The starting point is to consider a suitable generating series of the first derivatives of F_0 in all coupling constants. In this case, we consider:

$$(64) \quad y := \frac{1}{2} \sum_{k=1}^{\infty} k (s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} + \frac{t}{2x} + \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \frac{\partial F_0}{\partial s_{2k}}.$$

Then we have:

$$\begin{aligned} y^2 &= \frac{1}{4} \left(\sum_{k=1}^{\infty} k (s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} \right)^2 + \frac{t^2}{4x^2} + \left(\sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \frac{\partial F_0}{\partial s_{2k}} \right)^2 \\ &+ \frac{t}{2} \sum_{k=1}^{\infty} k (s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-2} + t \sum_{k=1}^{\infty} \frac{1}{x^{k+2}} \frac{\partial F_0}{\partial s_{2k}} + \sum_{k,l \geq 1} k (s_{2k} - \frac{1}{2} \delta_{k,1}) \frac{\partial F_0}{\partial s_{2l}} x^{k-l-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} (y^2)_- &= \left(\frac{1}{2} t (s_2 - \frac{1}{2}) + \sum_{l \geq 1} (l+1) s_{2l+2} \frac{\partial F_0}{\partial s_{2l}} \right) x^{-1} \\ &+ \left(\sum_{k \geq 1} k (s_{2k} - \frac{1}{2} \delta_{k,1}) \frac{\partial F_0}{\partial s_{2k}} + \frac{t^2}{4} \right) x^{-2} \\ &+ \sum_{n=1}^{\infty} \frac{1}{x^{n+2}} \left(\sum_{k=1}^{n-1} \frac{\partial F_0}{\partial s_{2k}} \frac{\partial F_0}{\partial s_{2n-2k}} + t \frac{\partial F_0}{\partial s_{2n}} + \sum_{k \geq 1} k (s_{2k} - \frac{1}{2} \delta_{k,1}) \frac{\partial F_0}{\partial s_{2n+2k}} \right). \end{aligned}$$

So by the Virasoro constraints above:

$$(65) \quad (y^2)_- = \left(\frac{1}{2} t (s_2 - \frac{1}{2}) + \sum_{l \geq 1} (l+1) s_{2l+2} \frac{\partial F_0}{\partial s_{2l}} \right) x^{-1},$$

and so

$$\begin{aligned} y^2 &= \frac{1}{4} \left(\sum_{k=1}^{\infty} k (s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} \right)^2 \\ &+ \frac{t}{2} \sum_{k=1}^{\infty} k (s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-2} + \sum_{l \geq 1} \sum_{k \geq l+1} k s_{2k} \frac{\partial F_0}{\partial s_{2l}} x^{k-l-2}. \end{aligned}$$

It follows that when $s_{2k} = 0$,

$$(66) \quad y^2 = \frac{1}{16} - \frac{t}{4x}.$$

This defines an algebraic curve on \mathbb{C}^2 . We refer to this curve as the *spectral curve* of the modified Hermitian one-matrix model. We say that (64) defines a *special deformation* of the spectral curve. From (66) we get:

$$(67) \quad x = \frac{4t}{1 - 16y^2}.$$

We call the right-hand side the LG superpotential function of the modified Hermitian one-matrix model.

4.2. Uniqueness of special deformation. For a formal power series $a(x) = \sum_{n \in \mathbb{Z}} a_n z^n$, let

$$(68) \quad a_{<-1} = \sum_{n < -1} a_n z^n.$$

The following result is very easy to prove:

Theorem 1. *For a series of the form*

$$(69) \quad y = \frac{1}{2} \frac{\partial S(x; \mathbf{s})}{\partial x} + \frac{t}{2x} + \sum_{n \geq 0} w_n x^{-n-2},$$

where S is the universal action defined by:

$$(70) \quad S(x; \mathbf{s}) = -\frac{1}{2}x + \sum_{n \geq 1} s_{2n} x^n,$$

and each $w_n \in \mathbb{C}[[s_2, s_4, \dots]]$, the equation

$$(71) \quad (y^2)_{<-1} = 0$$

has a unique solution given by:

$$y = \frac{1}{2} \sum_{n=0}^{\infty} (s_{2n} - \frac{1}{2} \delta_{n,1}) x^n + \frac{t}{2x} + \sum_{n=1}^{\infty} \frac{1}{x^{n+1}} \frac{\partial F_0(t)}{\partial s_{2n}}.$$

4.3. Quantization. The Virasoro constraints are given by the following differential operators:

$$(72) \quad L_0^{even} = \sum_{k \geq 1} k (s_{2k} - \frac{1}{2} \delta_{k,1}) \frac{\partial}{\partial s_{2k}} + \frac{t^2}{4g_s^2} - \frac{1}{16},$$

$$(73) \quad L_n^{even} = g_s^2 \sum_{k=1}^{n-1} \frac{\partial^2}{\partial s_{2k} \partial s_{2n-2k}} + t \frac{\partial}{\partial s_{2n}} + \sum_{k \geq 1} k (s_{2k} - \frac{1}{2} \delta_{k,1}) \frac{\partial}{\partial s_{2k+2n}}, \quad n \geq 1.$$

Consider the generating series of these Virasoro operators:

$$\begin{aligned}
& \sum_{n \geq 0} L_0^{even} x^{-n-2} \\
= & \left(\sum_{k \geq 1} k(s_{2k} - \frac{1}{2}\delta_{k,1}) \frac{\partial}{\partial s_{2k}} + \frac{t^2}{4g_s^2} - \frac{1}{16} \right) x^{-2} \\
+ & \sum_{n \geq 1} \left(g_s^2 \sum_{k=1}^{n-1} \frac{\partial^2}{\partial s_{2k} \partial s_{2n-2k}} + t \frac{\partial}{\partial s_{2n}} + \sum_{k \geq 1} k(s_{2k} - \frac{1}{2}\delta_{k,1}) \frac{\partial}{\partial s_{2k+2n}} \right) \frac{1}{x^{n+2}} \\
= & \sum_{k \geq 1} \sum_{n \geq 0} \beta_{-k} \beta_{k+n} \frac{1}{x^{n+k}} + \frac{\beta_0^2}{x^2} - \frac{1}{16x^2} + \sum_{n \geq 1} \left(\sum_{k=1}^{n-1} \beta_k \beta_{n-k} + 2\beta_0 \beta_n \right) \frac{1}{x^{n+2}}
\end{aligned}$$

where

$$(74) \quad \beta_{-k} = \frac{1}{2} g_s^{-1} k(s_{2k} - \frac{1}{2}\delta_{k,1}), \quad \beta_0 = \frac{t}{2g_s}, \quad \beta_k = g_s \frac{\partial}{\partial s_{2k}},$$

$k > 0$. They satisfy the Heisenberg commutation relations:

$$(75) \quad [\beta_k, \beta_l] = \frac{k}{2} \delta_{k,-l}.$$

As usual we take $\{\beta_k\}_{k>0}$ to be annihilators and take $\{\beta_{-k}\}_{k>0}$ to be creators, and one can then define normally ordered products. With these notations,

$$(76) \quad \sum_{n \geq 0} L_0^{even} x^{-n-2} =: \hat{y}(x)^2 : - \frac{1}{16} x^{-2},$$

where

$$(77) \quad \hat{y}(x) := \sum_{k=1}^{\infty} \beta_{-k} x^{k-1} + \frac{\beta_0}{x} + \sum_{k=1}^{\infty} \frac{\beta_k}{x^{k+1}},$$

Note we have the following well-known OPE:

$$(78) \quad \hat{y}(z)\hat{y}(w) =: \hat{y}(z)\hat{y}(w) : + \frac{1/2}{(z-w)^2}.$$

To account for the extra term $-\frac{1}{16}x^{-2}$, we use the idea of regularized product in [15]. In this case one needs to consider the twisted field

$$(79) \quad \hat{y}^{twist}(x) := x^{1/2} \cdot \hat{y}(x).$$

We have the following OPE for the twisted field:

$$(80) \quad \hat{y}^{twist}(z)\hat{y}^{twist}(w) =: \hat{y}^{twist}(z)\hat{y}^{twist}(w) : + \frac{z^{1/2}w^{1/2}}{2(z-w)^2}.$$

In particular,

$$\begin{aligned}
& \hat{y}^{twist}(x+\epsilon)\hat{y}^{twist}(x) \\
= & : \hat{y}^{twist}(x+\epsilon)\hat{y}^{twist}(x) : + \frac{(x+\epsilon)^{1/2}x^{1/2}}{2\epsilon^2} \\
= & : \hat{y}^{twist}(x+\epsilon)\hat{y}^{twist}(x) : + \frac{x}{2\epsilon^2} + \frac{1}{4\epsilon} - \frac{1}{16x} + \dots,
\end{aligned}$$

where we have the following expansion:

$$(x + \epsilon)^{1/2} x^{1/2} = x(1 + \epsilon/x)^{1/2} = x \left(1 + \frac{1}{2} \frac{\epsilon}{x} - \frac{1}{8} \frac{\epsilon^2}{x^2} + \dots \right).$$

We defined the *regularized product* $\hat{y}^{twist}(x)^{\odot 2} = \hat{y}^{twist}(x) \odot \hat{y}^{twist}(x)$ by:

$$(81) \quad \hat{y}^{twist}(x)^{\odot 2} := \lim_{\epsilon \rightarrow 0} \left(\hat{y}^{twist}(x + \epsilon) \hat{y}^{twist}(x) - \left(\frac{x}{2\epsilon^2} + \frac{1}{4\epsilon} \right) \right),$$

then we have

$$(82) \quad \hat{y}^{twist}(x)^{\odot 2} =: \hat{y}^{twist}(x) \hat{y}^{twist}(x) : - \frac{1}{16x} = x \left(: \hat{y}(x) \hat{y}(x) : - \frac{1}{16x} \right),$$

and so the Virasoro constraints can be reformulated in the following form:

$$(83) \quad \left(\hat{y}^{twist}(x)^{\odot 2} Z \right)_- = 0.$$

Remark 1. *It seems to be more natural to consider instead:*

$$(84) \quad y^{twist}(x) := x^{1/2} y(x) = \frac{1}{2} \sum_{k=1}^{\infty} k(s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1/2} + \frac{t}{2x} + \sum_{k=1}^{\infty} \frac{1}{x^{k+1/2}} \frac{\partial F_0}{\partial s_{2k}}.$$

With this choice (65) becomes:

$$(85) \quad ((y^{twist})^2)_- = \left(\frac{1}{2} t (s_2 - \frac{1}{2}) + \sum_{l \geq 1} (l+1) s_{2l+2} \frac{\partial F_0}{\partial s_{2l}} \right) x^{-1},$$

and so

$$\begin{aligned} (y^{twist})^2 &= \frac{x}{4} \left(\sum_{k=1}^{\infty} k(s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} \right)^2 \\ &+ \frac{t}{2} \sum_{k=1}^{\infty} k(s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} + \sum_{l \geq 1} \sum_{k \geq l+1} k s_{2k} \frac{\partial F_0}{\partial s_{2l}} x^{k-l-1}. \end{aligned}$$

When $s_{2k} = 0$,

$$(86) \quad (y^{twist})^2 = \frac{x}{16} - \frac{t}{4}.$$

However, algebraic considerations force us to choose y . It is well-known that if

$$(87) \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{(m^3 - m)c}{12} \delta_{m,-n},$$

then the field

$$(88) \quad L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

satisfies the OPE:

$$(89) \quad L(z)L(w) \sim \frac{L'(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}.$$

In other words, $\sum_{n \geq 0} L_n^{even} x^{-n-2} =: \hat{y}(x)^2 : - \frac{1}{16x^2}$. We are then forced by these algebraic considerations to make the seemingly unnatural choice of taking $\hat{y}(x)$ in the quantum case and $y(x)$ in the classical case.

5. EMERGENCE OF TOPOLOGICAL RECURSIONS

In this Section we show that the recursion relations in terms of operators \tilde{D} and \tilde{E} are Eynard-Orantin topological recursions for the spectral curve discussed in last Section.

5.1. Genus zero one-point function and the spectral curve. Let us take $s_{2k} = 0$ in (64) to get:

$$(90) \quad y = -\frac{1}{4} + \frac{t}{2x} + \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \frac{\partial F_0}{\partial s_{2k}} \Big|_{s_{2n}=0, n \geq 1}.$$

By the definition of correlators and $G_{0,1}(x)$,

$$(91) \quad y = -\frac{1}{4} + \frac{t}{2x} + G_{0,1}(x).$$

By the formula (36) for $G_{0,1}$,

$$(92) \quad y = -\frac{1}{4} \sqrt{1 - \frac{4t}{x}},$$

and so

$$(93) \quad y^2 = \frac{1}{16} - \frac{t}{4x}.$$

This recovers the formula (66) for the spectral curve of the modified Hermitian one-matrix model derived in last Section.

The coordinates of a point p on the spectral curve is given by two holomorphic functions $x = x(p)$ and $y = y(p)$. But this curve is a rational curve in the plane, it has a global coordinate given by y , and

$$(94) \quad x = \frac{4t}{1 - 16y^2}.$$

There is a natural hyperelliptic structure on this curve: One can define an involution $p \mapsto \sigma(p)$ by

$$(95) \quad \sigma(x, y) = (x, -y).$$

5.2. Correlation functions as functions on the spectral curve. With the introduction of the spectral curve, one can regard the genus g n -point correlation functions $G_{g,n}(x_1, \dots, x_n)$ as functions on it. We understand x and y as meromorphic function on the spectral curve. For a point p_j on it, we write

$$(96) \quad x_j = x(p_j), \quad y_j = y(p_j).$$

In §3 we have used local coordinates x_1, \dots, x_n . The concrete examples computed there can now be translated into functions in y_1, \dots, y_n :

$$\begin{aligned}
G_{0,1}(y_1) &= \frac{1}{4} - \frac{t}{2x_1} + y_1 = \frac{1}{8} + y_1 + 2y_1^2 = \frac{1}{8}(4y_1 + 1)^2, \\
G_{0,2}(y_1, y_2) &= \frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2}}{2 \cdot 4^2 \cdot (x_1 - x_2)^2 y_1 y_2} - \frac{1}{2(x_1 - x_2)^2} \\
&= \frac{1}{2^{14}} \frac{(1 - 16y_1^2)^2 (1 - 16y_2^2)^2}{t^2 y_1 y_2 (y_1 + y_2)^2}, \\
G_{0,3}(y_1, y_2, y_3) &= -\frac{t^2}{2^{16} \cdot x_1^2 x_2^2 x_3^2 (y_1 y_2 y_3)^3} \\
&= -\frac{1}{2^{28} t^4} \frac{(1 - 16y_1^2)^2 (1 - 16y_2^2)^2 (1 - 16y_3^2)^2}{y_1^3 y_2^3 y_3^3}, \\
G_{0,4}(y_1, \dots, y_4) &= 24t^2 \frac{e_4 - 2e_3 t + 32e_1 t^3 - 256t^4}{\prod_{j=0}^3 x_j^3 (4y_j)^5}, \\
G_{1,1}(y_1) &= \frac{1}{2^5 x_1^2 y_1} - \frac{t^2}{2^9 x_1^4 y_1^5} \\
&= \frac{(1 - 16y_1^2)^2}{2^9 y_1} - \frac{1}{2^{17}} \frac{(1 - 16y_1^2)^4}{t^2 y_1^5}.
\end{aligned}$$

Now we translate (53) into the following form:

$$\begin{aligned}
&G_{g,n+1}(y_0, y_1, \dots, y_n) \\
(97) \quad &= \sum_{j=1}^n \mathcal{D}_{y_0, y_j} G_{g,n}(y_1, \dots, y_n) + \mathcal{E}_{y_0, y, y'} G_{g-1, n+2}(y, y', y_1, \dots, y_n) \\
&+ \sum_{\substack{g_1 + g_2 = g \\ I_1 \amalg I_2 = [n]}} \mathcal{E}_{y_0, y, y'} \left(G_{g_1, |I_1|+1}(y, y_{I_1}) \cdot G_{g_2, |I_2|+1}(y', y_{I_2}) \right) - \frac{\delta_{g,1} \delta_{n,0}}{32y_0} \frac{1}{x_0^2},
\end{aligned}$$

where \mathcal{D}_{p_0, p_j} and $\mathcal{E}_{p_0, q, r}$ are some operators to be determined below. Recall

$$\begin{aligned}
\tilde{D}_{x_0, x_j} f(x_j) &= \frac{2}{1 - 4W_{0,1}(x_0)} \left(\frac{x_0 f(x_0) - x_j f(x_j)}{x_0(x_0 - x_j)^2} - \frac{1}{x_0(x_0 - x_j)} \frac{d}{dx_j} (x_j f(x_j)) \right), \\
\tilde{E}_{x_0, u, v} f(u, v) &= \frac{2}{1 - 4W_{0,1}(x_0)} \lim_{u \rightarrow v} f(u, v)|_{v=x_0}.
\end{aligned}$$

It is clear that the operator \tilde{E} can be translated into the following operator acting on functions in y :

$$(98) \quad \mathcal{E}_{y_0, y, y'} g(y, y') = -\frac{1}{2y_0} \lim_{y' \rightarrow y} f(y, y')|_{y=y_0}.$$

On the other hand,

$$\begin{aligned}
& \frac{2}{1-4W_{0,1}(x_0)} \left(\frac{x_0 f(x_0) - x_j f(x_j)}{x_0(x_0 - x_j)^2} - \frac{1}{x_0(x_0 - x_j)} \frac{d}{dx_j} (x_j f(x_j)) \right) \\
&= -\frac{1}{2y_0} \left[\frac{\frac{4t}{1-16y_0^2} g(y_0) - \frac{4t}{1-16y_j^2} g(y_j)}{\frac{4t}{1-16y_0^2} \left(\frac{4t}{1-16y_0^2} - \frac{4t}{1-16y_j^2} \right)^2} \right. \\
&\quad \left. - \frac{1}{\frac{4t}{1-16y_0^2} \left(\frac{4t}{1-16y_0^2} - \frac{4t}{1-16y_j^2} \right)} \cdot \frac{1}{\frac{dx_j}{dy_j}} \cdot \frac{d}{dy_j} \left(\frac{4t}{1-16y_j^2} g(y_j) \right) \right] \\
&= -\frac{(1-16y_0^2)^2(1-16y_j^2)^2}{16384t^2y_0y_j} \cdot \frac{2y_j(g(y_0) - g(y_j)) - (y_0^2 - y_j^2)g'(y_j)}{(y_0^2 - y_j^2)^2},
\end{aligned}$$

and so

(99)

$$\mathcal{D}_{y_0, y_j} g(y_j) = -\frac{(1-16y_0^2)^2(1-16y_j^2)^2}{2^{14}t^2y_0y_j} \cdot \frac{2y_j(g(y_0) - g(y_j)) - (y_0^2 - y_j^2)g'(y_j)}{(y_0^2 - y_j^2)^2}.$$

Note

$$\begin{aligned}
& \mathcal{E}_{y_0, y, y'} \left(x^{-a} y^{-2b-1} \cdot x'^{-a'} y'^{-2b'-1} \right) = -\frac{1}{2} x^{-a-a'} y^{-2(b+b')-3}, \\
& \mathcal{D}_{y_0, y_j} \left(x^{-a} y_j^{-2b-1} \right) = \frac{(1-16y_0^2)^2(1-16y_j^2)^2}{2^{14}t^2y_0y_j} \\
& \quad \cdot \frac{2y_j \left(\left(\frac{1-16y_0^2}{4t} \right)^a y_0^{-2b-1} - \left(\frac{1-16y_j^2}{4t} \right)^a y_j^{-2b-1} \right) - (y_0^2 - y_j^2) \frac{d}{dy_j} \left[\left(\frac{1-16y_j^2}{4t} \right)^a y_j^{-2b-1} \right]}{(y_0^2 - y_j^2)^2}.
\end{aligned}$$

5.3. Examples. We now present some sample computations of $G_{g,n}$ using (97).

5.3.1. *Three-point function in genus zero.*

$$\begin{aligned}
& G_0(y_0, y_1, y_2) \\
&= \sum_{j=1}^2 \mathcal{D}_{y_0, y_j} G_0(y_1, y_2) + 2\mathcal{E}_{y_0, y, y'} \left(G_0(y, y_1) \cdot G_0(y', y_2) \right) \\
&= \sum_{j=1}^2 \tilde{D}_{y_0, y_j} \left(\frac{1}{2^{14}} \frac{(1-16y_1^2)^2(1-16y_2^2)^2}{t^2y_1y_2(y_1+y_2)^2} \right) \\
&\quad + 2\tilde{E}_{y_0, y, y'} \left(\frac{1}{2^{14}} \frac{(1-16y^2)^2(1-16y_1^2)^2}{t^2yy_1(y+y_1)^2} \right. \\
&\quad \left. \cdot \frac{1}{2^{14}} \frac{(1-16y'^2)^2(1-16y_2^2)^2}{t^2y'y_2(y'+y_2)^2} \right).
\end{aligned}$$

After a complicated computation with the help of Maple, the following simple formula is obtained:

$$(100) \quad G_0(x_0, x_1, x_2) = \frac{4t^2}{x_0^2x_1^2x_2^2((1-4t/x_0)(1-4t/x_1)(1-4t/x_2))^{3/2}}.$$

5.3.2. *One-point function in genus one.* In this case (97) takes the form:

$$\begin{aligned} G_{1,1}(x_0) &= \frac{1}{32x_0^2y_0} + \mathcal{E}_{y_0,y,y'}G_{0,2}(y,y') \\ &= \frac{1}{32x_0^2y_0} - \frac{1}{2y_0} \lim_{y' \rightarrow y} \left(\frac{1}{2^{14}} \frac{(1-16y^2)^2(1-16y'^2)^2}{t^2yy'(y+y')^2} \right) \Big|_{y \rightarrow y_0} \\ &= \frac{(1-16y_0^2)^2}{2^9t^2y_0} - \frac{1}{2^{17}} \frac{(1-16y_0^2)^4}{t^2y_0^5}. \end{aligned}$$

5.4. **Multilinear differential forms.** Instead of the functions $G_{g,n}(p_1, \dots, p_n)$, one can also consider the multilinear differential forms:

$$(101) \quad W_{g,n}(p_1, \dots, p_n) = \hat{G}_{g,n}(y_1, \dots, y_n) dx_1 \dots dx_n,$$

where $\hat{G}_{g,n}(y_1, \dots, y_n) = G_{g,n}(y_1, \dots, y_n)$ except for the following two exceptional cases:

$$(102) \quad \hat{G}_{0,1}(y_1) = -\frac{1}{4} + \frac{t}{2x_1} + G_{0,1}(y_1),$$

$$(103) \quad \hat{G}_{0,2}(y_1, y_2) = \frac{1}{(x_1 - x_2)^2} + G_{0,2}(y_1, y_2).$$

Since $\hat{G}_{0,1}(y_1) = y_1$, so we have:

$$(104) \quad W_{0,1}(p_1) = y_1 dx_1.$$

By the following computations

$$\begin{aligned} W_{0,2}(p_1, p_2) &= \left(\frac{1 - \frac{1}{2}(1-16y_1^2) - \frac{1}{2}(1-16y_2^2)}{32 \left(\frac{4t}{1-16y_1^2} - \frac{4t}{1-16y_2^2} \right)^2} y_1 y_2 + \frac{1}{2 \left(\frac{4t}{1-16y_1^2} - \frac{4t}{1-16y_2^2} \right)^2} \right) \\ &\quad \cdot d \frac{4t}{1-16y_1^2} d \frac{4t}{1-16y_2^2} \\ &= \frac{dy_1 dy_2}{(y_1 - y_2)^2}, \end{aligned}$$

we get:

$$(105) \quad W_{0,2}(p_1, p_2) = \frac{dy_1 dy_2}{(y_1 - y_2)^2}.$$

5.5. **The computation of the recursion kernel.** We use $W_{0,2}$ as the Bergman kernel. Then

$$\begin{aligned} \int_{q=\sigma(p_2)}^{p_2} B(p_1, q) &= \int_{y=-y_2}^{y_2} \frac{dy_1 dy_2}{(y_1 - y)^2} \\ &= \frac{dy_1}{y_1 - y} \Big|_{y=-y_2}^{y_2} = \frac{2y_2 dy_1}{y_1^2 - y_2^2}. \end{aligned}$$

It follows that

$$(106) \quad K(p_0, p) = \frac{dy_0}{2(y_0^2 - y^2)dx} = \frac{(1-16y^2)^2}{2^8ty(y_0^2 - y^2)} \frac{dy_0}{dy}.$$

This has poles at $y = 0$ and $y = \pm y_0$. To understand its behavior at $y = \infty$, let $y = 1/z$. Then

$$(107) \quad K(p_0, p) = \frac{z(z^2 - 16)^2}{2^8 t(1 - y_0^2 z^2)} \frac{dy_0}{dz}.$$

5.6. **Eynard-Orantin topological recursions.** Note

$$(108) \quad dy = \frac{tdx}{8x^2y}, \quad dx = \frac{2^7 tydy}{(1 - 16y^2)^2}.$$

Let us carry out the first few calculations of Eynard-Orantin recursion for the spectral curve (93) with $\omega_{0,1} = W_{0,1}$ and $\omega_{0,2} = W_{0,2}$ given by (104) and (105) respectively.

$$\begin{aligned} & \omega_{0,3}(p_0, p_1, p_2) \\ = & \operatorname{Res}_{p \rightarrow p_+} K(p_0, p) [W_{0,2}(p, p_1)W_{0,2}(\sigma(p), p_2) + W_{0,2}(p, p_2)W_{0,2}(\sigma(p), p_1)] \\ = & \operatorname{Res}_{y \rightarrow 0} \frac{(1 - 16y^2)^2 dy_0}{2^8 ty(y_0^2 - y^2)dy} \left(\frac{dydy_1}{y^2 - y_1^2} \cdot \frac{-dydy_2}{y^2 - y_2^2} + \frac{dydy_2}{y^2 - y_2^2} \cdot \frac{-dydy_1}{y^2 - y_1^2} \right) \\ = & -\frac{dy_0 dy_1 dy_2}{2^7 ty_0^2 y_1^2 y_2^2} = -\frac{t^2 dx_0 dx_1 dx_2}{2^{16} x_0^2 x_1^2 x_2^2 y_0^3 y_1^3 y_2^3} \\ = & W_{0,3}(p_0, p_2, p_2). \end{aligned}$$

$$\begin{aligned} \omega_{1,1}(p_0) &= \operatorname{Res}_{p \rightarrow p_+} K(p_0, p) W_{0,2}(p, \sigma(p)) \\ &= \operatorname{Res}_{y \rightarrow 0} \frac{(1 - 16y^2)^2 dy_0}{2^8 ty(y_0^2 - y^2)dy} \cdot \frac{-(dy)^2}{4y^2} \\ &= -\frac{1 - 2^5 y_0^2}{2^{10} ty_0^4} dy_0 = -\frac{1 - 2^5 y_0^2}{2^{10} ty_0^4} \cdot \frac{t}{8x_0^2 y_0} dx_0 \\ &= \left(\frac{1}{2^8 x_0^2 y_0^3} - \frac{1}{2^{13} x_0^2 y_0^5} \right) dx_0 \\ &= \left(\frac{1}{2^9 x_0^2 y_0^3} - \frac{t}{2^{11} x_0^3 y_0^5} \right) dx_0 \\ &= W_{1,1}(p_0). \end{aligned}$$

$$\begin{aligned}
& \omega_{0,4}(p_0, p_1, p_2, p_4) \\
&= \text{Res}_{p \rightarrow p_+} K(p_0, p) [W_{0,2}(p, p_1)W_{0,3}(\sigma(p), p_2, p_3) \\
&+ W_{0,2}(p, p_2)W_{0,2}(\sigma(p), p_1, p_3) + W_{0,2}(p, p_3)W_{0,2}(\sigma(p), p_1, p_2) \\
&+ W_{0,3}(p, p_1, p_2)W_{0,2}(\sigma(p), p_3) + W_{0,3}(p, p_1, p_3)W_{0,2}(\sigma(p), p_2) \\
&+ W_{0,3}(p, p_1, p_3)W_{0,2}(\sigma(p), p_2)] \\
&= \text{Res}_{p \rightarrow p_+} \frac{dy_0}{(y_0^2 - y^2)dx} \left[\frac{dydy_1}{(y - y_1)^2} \cdot \frac{-t^2 dx dx_2 dx_3}{1024 x^2 x_2^2 x_3^2 (-y)^3 y_2^3 y_3^3} + \text{perm.} \right. \\
&+ \left. \frac{-t^2 dx dx_2 dx_3}{2^{10} x^2 x_2^2 x_3^2 y^3 y_2^3 y_3^3} \cdot \frac{d(-y)dy_1}{(-y - y_1)^2} \cdot + \text{perm.} \right] \\
&= \text{Res}_{y \rightarrow 0} \frac{dy_0}{(y_0^2 - y^2)} \left[\frac{dydy_1}{(y - y_1)^2} \cdot \frac{t^2 dx_2 dx_3}{1024 \left(\frac{2t}{1-4y^2}\right)^2 x_2^2 x_3^2 y^3 y_2^3 y_3^3} + \text{perm.} \right. \\
&+ \left. \frac{t^2 dx_2 dx_3}{1024 \left(\frac{2t}{1-4y^2}\right)^2 x_2^2 x_3^2 y^3 y_2^3 y_3^3} \cdot \frac{dydy_1}{(y + y_1)^2} + \text{perm.} \right] \\
&= 2 \cdot \frac{dy_0 dy_1 dx_2 dx_3}{2^{12} x_2^2 x_3^2 y_2^3 y_3^3} \cdot \frac{-8y_0^2 y_1^2 + 3y_0^2 + y_1^2}{y_0^4 y_1^4} + \text{perm.} \\
&= \frac{t^2 dx_0 dx_1 dx_2 dx_3}{2^{15} x_0^2 x_1^2 x_2^2 x_3^2 y_0^5 y_1^5 y_2^5 y_3^5} \cdot y_2^2 y_3^2 (-8y_0^2 y_1^2 + 3y_0^2 + y_1^2) + \text{perm.} \\
&= 3t^2 \cdot \frac{e_4 - e_3 t + 4e_1 t^3 - 16t^4}{2^{20} \prod_{j=0}^3 x_j^3 y_j^5} dx_1 \cdots dx_4.
\end{aligned}$$

Theorem 2. *The multi-linear differential forms $W_{g,n}(p_1, \dots, p_n)$ defined by (101) satisfy the Eynard-Orantin topological recursions given by the spectral curve*

$$(109) \quad y^2 = \frac{1}{16} - \frac{t}{4x}.$$

I.e., we have

$$(110) \quad W_{g,n}(p_1, \dots, p_n) = \omega_{g,n}(p_1, \dots, p_n).$$

Proof. We have explicitly check the case of $\omega_{0,3}$ and $\omega_{1,1}$. We now show that other case can be checked by induction. By (4) and the induction hypothesis,

$$\begin{aligned}
& \omega_{g,n+1}(p_0, p_1, \dots, p_n) = \text{Res}_{y \rightarrow 0} K(y_0, y) \\
& \cdot \left[\hat{G}_{g-1, n+2}(y, -y, y_{[n]}) dx dx \right. \\
& + \left. \sum_{h=0}^g \sum_{I \subset [n]} \hat{G}_{h, |I|+1}(y, y_I) \hat{G}_{g-h, n-|I|+1}(-y, y_{[n]-I}) dx dx \right] dx_1 \cdots dx_n.
\end{aligned}$$

By (97) and (101),

$$\begin{aligned}
& W_{g,n+1}(y_0, \dots, y_n) \\
&= G_g(y_0, y_1, \dots, y_n) dx_0 \cdots dx_n \\
&= \sum_{j=1}^n \mathcal{D}_{y_0, y_j} G_g(y_1, \dots, y_n) \cdot dx_0 \cdots dx_n \\
&+ \mathcal{E}_{y_0, y, y'} G_{g-1}(y, y', y_1, \dots, y_n) \cdot dx_0 \cdots dx_n \\
&+ \sum_{\substack{g_1+g_2=g \\ I_1 \amalg I_2=[n]}} \mathcal{E}_{y_0, y, y'} \left(G_{g_1}(y, y_{I_1}) \cdot G_{g_2}(y', y_{I_2}) \right) \cdot dx_0 \cdots dx_n \\
&- \frac{\delta_{g,1} \delta_{n,0}}{32y_0} \frac{1}{x_0^2} dx_0,
\end{aligned}$$

By comparing these two recursion, we see that it suffices to show that

$$\begin{aligned}
(111) \quad & \text{Res}_{y \rightarrow 0} K(y_0, y) \cdot \left[G_{g-1, n+2}(y, -y, y_{[n]}) dx dx \right] \\
&= \mathcal{E}_{y_0, y, y'} G_{g-1}(y, y', y_1, \dots, y_n) \cdot dx_0,
\end{aligned}$$

and when $(h, |I| + 1) \neq (0, 2)$ and $(g - h, n - |I| + 1) \neq (0, 2)$,

$$\begin{aligned}
(112) \quad & \text{Res}_{y \rightarrow 0} K(y_0, y) \left[G_{h, |I|+1}(y, y_I) G_{g-h, n-|I|+1}(-y, y_{[n]-I}) dx dx \right] \\
&= \mathcal{E}_{y_0, y, y'} \left[G_{h, |I|+1}(y, y_I) G_{g-h, n-|I|+1}(y', y_{[n]-I}) \right] \cdot dx_0,
\end{aligned}$$

and furthermore

$$\begin{aligned}
(113) \quad & \text{Res}_{y \rightarrow 0} K(y_0, y) \left[\hat{G}_{0,2}(y, y_j) dx dx_j \cdot G_{g,n}(-y, y_1, \dots, \hat{y}_j, \dots, y_n) dx \right] \\
&+ \text{Res}_{y \rightarrow 0} K(y_0, y) \left[\hat{G}_{0,2}(-y, y_j) dx dx_j G_{g,n}(y, y_1, \dots, \hat{y}_j, \dots, y_n) dx \right] \\
&= \left(\mathcal{D}_{y_0, y_j} \left[G_{g,n}(y_1, \dots, y_n) \right] \right. \\
&+ \mathcal{E}_{y_0, y, y'} \left[G_{0,2}(y, y_j) G_{g,n}(y', y_1, \dots, \hat{y}_j, \dots, y_n) \right. \\
&\quad \left. \left. + G_{g,n}(y, y_1, \dots, \hat{y}_j, \dots, y_n) G_{0,2}(y', y_j) \right] \right) \cdot dx_0 dx_j.
\end{aligned}$$

We now prove (111) and (112) by Cauchy's residue theorem. Indeed,

$$\begin{aligned}
& \operatorname{Res}_{y \rightarrow 0} K(y_0, y) \cdot \left[G_{g-1, n+2}(y, -y, y_{[n]}) dx dx \right] \\
&= \operatorname{Res}_{y \rightarrow 0} \frac{(1-16y^2)^2}{2^8 t y (y_0^2 - y^2)} \frac{dy_0}{dy} G_{g-1, n+2}(y, -y, y_{[n]}) dx dx \\
&= \operatorname{Res}_{y \rightarrow 0} \left[\frac{(1-16y^2)^2}{2^8 t y (y_0^2 - y^2)} \frac{dy_0}{dy} \right. \\
&\quad \cdot \sum_{\substack{a, a', a_1, \dots, a_n \geq 2 \\ b, b', b_1, \dots, b_n}} A_{a, a', a_1, \dots, a_n; b, b', b_1, \dots, b_n}^{g-1, n+2}(t) x^{-a} y^{-2b-1} \cdot x^{-a'} (-y)^{-2b'-1} \\
&\quad \cdot \prod_{i=1}^n x_i^{-a_i} y_i^{-2b_i-1} \cdot \left(\frac{2^7 t y dy}{(1-16y^2)^2} \right)^2 \left. \right] \\
&= - \sum_{\substack{a, a', a_1, \dots, a_n \geq 2 \\ b, b', b_1, \dots, b_n}} A_{a, a', a_1, \dots, a_n; b, b', b_1, \dots, b_n}^{g-1, n+2}(t) \cdot \prod_{i=1}^n x_i^{-a_i} y_i^{-2b_i-1} \\
&\quad \cdot \frac{dy_0}{2^{2a+2a'-6} t^{a+a'-1}} \operatorname{Res}_{y \rightarrow 0} \left[\frac{(1-16y^2)^{a+a'-2}}{y^{2b+2b'+1} (y_0^2 - y^2)} dy \right],
\end{aligned}$$

where $b \geq a-1$, $b' \geq a'-1$, $b_i \geq a_i-1$, $i = 1, \dots, n$ by Proposition 3.3. By Cauchy residue theorem, the residue at $y = 0$ can be computed by the residue at $y = \pm y_0$ and the residue at $y = \infty$. Because

$$\begin{aligned}
\frac{(1-16y^2)^{a+a'-2}}{y^{2b+2b'+1} (y_0^2 - y^2)} dy &= \frac{(1-16/z^2)^{a+a'-2}}{z^{-(2b+2b'+1)} (y_0^2 - \frac{1}{z^2})} d\frac{1}{z} \\
&= z^{2b+2b'-2a-2a'+5} \frac{(z^2-16)^{a+a'-2}}{1-y_0^2 z^2} dz,
\end{aligned}$$

and $b \geq a-1$, $b' \geq a'-1$, the residue at $y = \infty$ vanishes. We also have

$$\begin{aligned}
& \operatorname{Res}_{y \rightarrow y_0} \left[\frac{(1-16y^2)^{a+a'-2}}{y^{2b+2b'+1} (y_0^2 - y^2)} dy \right] + \operatorname{Res}_{y \rightarrow -y_0} \left[\frac{(1-16y^2)^{a+a'-2}}{y^{2b+2b'+1} (y_0^2 - y^2)} dy \right] \\
&= \frac{(1-16y_0^2)^{a+a'-2}}{y_0^{2b+2b'+2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \operatorname{Res}_{y \rightarrow 0} K(y_0, y) G_{g-1, n+2}(y, -y, y_{[n]}) dx dx \\
&= - \sum_{\substack{a, a', a_1, \dots, a_n \geq 2 \\ b, b', b_1, \dots, b_n \geq 0}} A_{a, a', a_1, \dots, a_n; b, b', b_1, \dots, b_n}^{g-1, n+2}(t) \cdot \prod_{i=1}^n x_i^{-a_i} y_i^{-2b_i-1} \\
&\quad \cdot \frac{dy_0}{2^{2a+2a'-6} t^{a+a'-1}} \frac{(1-16y_0^2)^{a+a'-2}}{y_0^{2b+2b'+2}} \\
&= - \sum_{\substack{a, a', a_1, \dots, a_n \geq 2 \\ b, b', b_1, \dots, b_n \geq 0}} A_{a, a', a_1, \dots, a_n; b, b', b_1, \dots, b_n}^{g-1, n+2}(t) \cdot \prod_{i=1}^n x_i^{-a_i} y_i^{-2b_i-1} \\
&\quad \cdot \frac{dx_0}{2x_0^{a+a'} y_0^{2b+2b'+3}} \\
&= \mathcal{E}_{y_0, y, y'} G_{g-1}(y, y', y_1, \dots, y_n) \cdot dx_0.
\end{aligned}$$

In the same fashion one can prove (112).

We now only need to show that

$$\begin{aligned}
& \operatorname{Res}_{y \rightarrow 0} K(y_0, y) \left[\hat{G}_{0,2}(y, y_j) G_{g,n}(-y, y_1, \dots, \hat{y}_j, \dots, y_n) dx dx \right] dx_j \\
&+ \operatorname{Res}_{y \rightarrow 0} K(y_0, y) \left[\hat{G}_{0,2}(-y, y_j) G_{g,n}(y, y_1, \dots, \hat{y}_j, \dots, y_n) dx dx \right] dx_j \\
&= \left(\mathcal{D}_{y_0, y_j} \left[G_{g,n}(y_1, \dots, y_n) \right] + \mathcal{E}_{y_0, y, y'} \left[G_{0,2}(y, y_j) G_{g,n}(y', y_1, \dots, \hat{y}_j, \dots, y_n) \right. \right. \\
&\quad \left. \left. + G_{g,n}(y, y_1, \dots, \hat{y}_j, \dots, y_n) G_{0,2}(y', y_j) \right] \right) \cdot dx_0 dx_j.
\end{aligned}$$

Indeed, the left-hand side of (113) is

$$\begin{aligned}
& \operatorname{Res}_{y \rightarrow 0} \left[\frac{(1-16y^2)^2}{2^8 t y (y_0^2 - y^2)} \frac{dy_0}{dy} \cdot \frac{dy dy_j}{(y - y_j)^2} \right. \\
&\quad \cdot \sum_{\substack{a_1, \dots, a_n \geq 2 \\ b_1, \dots, b_n \geq 0}} A_{a_1, \dots, a_n; b_1, \dots, b_n}^{g,n}(t) x^{-a_j} (-y)^{-2b_j-1} \cdot \prod_{i \neq j} x_i^{-a_i} y_i^{-2b_i-1} \cdot \left. \frac{2^7 t y dy}{(1-16y^2)^2} \right] \\
&+ \operatorname{Res}_{y \rightarrow 0} \left[\frac{(1-16y^2)^2}{2^8 t y (y_0^2 - y^2)} \frac{dy_0}{dy} \cdot \frac{d(-y) dy_j}{(-y - y_j)^2} \right. \\
&\quad \cdot \sum_{\substack{a_1, \dots, a_n \geq 2 \\ b_1, \dots, b_n \geq 0}} A_{a_1, \dots, a_n; b_1, \dots, b_n}^{g,n}(t) x^{-a_j} y^{-2b_j-1} \cdot \prod_{i \neq j} x_i^{-a_i} y_i^{-2b_i-1} \cdot \left. \frac{2^7 t y dy}{(1-16y^2)^2} \right] \\
&= -\frac{1}{2} \sum_{\substack{a_1, \dots, a_n \geq 2 \\ b_1, \dots, b_n \geq 0}} A_{a_1, \dots, a_n; b_1, \dots, b_n}^{g,n}(t) \prod_{i \neq j} x_i^{-a_i} y_i^{-2b_i-1} \\
&\quad \cdot \operatorname{Res}_{y \rightarrow 0} \left[\frac{x^{-a_j} y^{-2b_j-1}}{(y_0^2 - y^2)} \cdot \frac{dy}{(y - y_j)^2} + \frac{x^{-a_j} y^{-2b_j-1}}{(y_0^2 - y^2)} \cdot \frac{dy}{(-y - y_j)^2} \right] dy_j dy_0,
\end{aligned}$$

so the computation of the left-hand is reduced to the computation of

$$A = -\frac{1}{2} \operatorname{Res}_{y \rightarrow 0} \left[\frac{x^{-a_j} y^{-2b_j-1}}{(y_0^2 - y^2)} \cdot \frac{dy}{(y - y_j)^2} + \frac{x^{-a_j} y^{-2b_j-1}}{(y_0^2 - y^2)} \cdot \frac{dy}{(-y - y_j)^2} \right].$$

In the same way, the right-hand side is reduced to

$$\begin{aligned} B &= \left(\mathcal{D}_{y_0, y_j} \left[x_j^{-a_j} y_j^{-2b_j-1} \right] + \mathcal{E}_{y_0, y, y'} \left[G_{0,2}(y, y_j) x'^{-a_j} y'^{-2b_j-1} \right. \right. \\ &\quad \left. \left. + x^{-a_j} y^{-2b_j-1} G_{0,2}(y', y_j) \right] \right) \cdot \frac{dx_0}{dy_0} \cdot \frac{dx_j}{dy_j} \\ &= \left(\mathcal{D}_{y_0, y_j} \left[x_j^{-a_j} y_j^{-2b_j-1} \right] \right. \\ &\quad \left. + \mathcal{E}_{y_0, y, y'} \left[\frac{1}{2^{14}} \frac{(1 - 16y^2)^2 (1 - 16y_j^2)^2}{t^2 y y_j (y + y_j)^2} \cdot x'^{-a_j} y'^{-2b_j-1} \right. \right. \\ &\quad \left. \left. + x^{-a_j} y^{-2b_j-1} \cdot \frac{1}{2^{14}} \frac{(1 - 16y'^2)^2 (1 - 16y_j'^2)^2}{t^2 y' y_j (y' + y_j)^2} \right] \right) \cdot \frac{dx_0}{dy_0} \cdot \frac{dx_j}{dy_j} \\ &= \left(-\frac{(1 - 16y_0^2)^2 (1 - 16y_j^2)^2}{2^{14} t^2 y_0 y_j} \right. \\ &\quad \left. \frac{2y_j \left(x_0^{-a_j} y_0^{-2b_j-1} - x_j^{-a_j} y_j^{-2b_j-1} \right) - (y_0^2 - y_j^2) \frac{d}{dy_j} \left[x_j^{-a_j} y_j^{-2b_j-1} \right]}{(y_0^2 - y_j^2)^2} \right. \\ &\quad \left. - \frac{2}{2y_0} \frac{1}{2^{14}} \frac{(1 - 16y_0^2)^2 (1 - 16y_j^2)^2}{t^2 y_0 y_j (y_0 + y_j)^2} \cdot x_0^{-a_j} y_0^{-2b_j-1} \right) \cdot \frac{2^{14} t^2 y_0 y_j}{(1 - 16y_0^2)^2 (1 - 16y_j^2)^2} \\ &= -\frac{2y_j \left(x_0^{-a_j} y_0^{-2b_j-1} - x_j^{-a_j} y_j^{-2b_j-1} \right) - (y_0^2 - y_j^2) \frac{d}{dy_j} \left[x_j^{-a_j} y_j^{-2b_j-1} \right]}{(y_0^2 - y_j^2)^2} \\ &\quad - \frac{1}{y_0 (y_0 + y_j)^2} \cdot x_0^{-a_j} y_0^{-2b_j-1}. \end{aligned}$$

and so we only need to show that

$$(114) \quad A = B.$$

This can be reduced to the following identity:

$$\begin{aligned} &\frac{1}{2} \operatorname{Res}_{y \rightarrow 0} \left[\frac{y^{-2n-1}}{(y_0^2 - y^2)} \cdot \frac{dy}{(y - y_j)^2} + \frac{y^{-2n-1}}{(y_0^2 - y^2)} \cdot \frac{dy}{(-y - y_j)^2} \right] \\ &= \frac{2y_j \left(y_0^{-2n-1} - y_j^{-2n-1} \right) - (y_0^2 - y_j^2) \frac{d}{dy_j} \left[y_j^{-2n-1} \right]}{(y_0^2 - y_j^2)^2} + \frac{1}{y_0 (y_0 + y_j)^2} \cdot y_0^{-2n-1}, \end{aligned}$$

for $n \geq -1$. But this is very easy to prove. Note

$$\begin{aligned} &\frac{1}{2} \left(\frac{1}{(y_0^2 - y^2)} \cdot \frac{1}{(y - y_j)^2} + \frac{1}{(y_0^2 - y^2)} \cdot \frac{1}{(-y - y_j)^2} \right) \\ &= \frac{1}{y_0^2 - y^2} \cdot \frac{y_j^2 + y^2}{(y_j^2 - y^2)^2} = \sum_{n=0}^{\infty} \frac{y^{2n}}{y_0^{2n+2} y_j^{2n+2}} \sum_{k=0}^n (2n+1-2j) y_0^{2n-2k} y_j^{2k}, \end{aligned}$$

therefore, the left-hand side is equal to

$$\frac{1}{y_0^{2n+2} y_j^{2n+2}} \sum_{k=0}^n (2n+1-2j) y_0^{2n-2k} y_j^{2k}.$$

The right-hand side is equal to

$$\begin{aligned} & \frac{2y_j \left(y_0^{-2n-1} - y_j^{-2n-1} \right) + (2n+1)(y_0^2 - y_j^2) y_j^{-2n-2}}{(y_0^2 - y_j^2)^2} + \frac{1}{y_0(y_0 + y_j)^2} \cdot y_0^{-2n-1} \\ &= \frac{1}{y_0^{2n+2} y_j^{2n+2} (y_0^2 - y_j^2)^2} \cdot \left(2y_0 y_j^2 (y_j^{2n+1} - y_0^{2n+1}) + (2n+1)(y_0^2 - y_j^2) y_0^{2n+2} \right. \\ &+ \left. (y_0 - y_j)^2 y_j^{2n+2} \right) \\ &= \frac{1}{y_0^{2n+2} y_j^{2n+2} (y_0^2 - y_j^2)^2} \cdot \left((2n+1) y_0^{2n+4} - (2n+3) y_j^2 y_0^{2n+2} \right. \\ &+ \left. y_0^2 y_j^{2n+2} + y_j^{2n+4} \right) \\ &= \frac{1}{y_0^{2n+2} y_j^{2n+2}} \sum_{k=0}^n (2n+1-2j) y_0^{2n-2k} y_j^{2k}. \end{aligned}$$

This completes the proof. \square

6. RELATIONSHIP WITH INTERSECTION NUMBERS

In this Section we relate the n -point functions of the modified GUE partition function with even couplings to intersection numbers.

6.1. Local Airy coordinate near the branchpoint. Recall the involution of the spectral curve is given by $\sigma : (x, y) \rightarrow (x, -y)$. It has only one fixed point: $(x, y) = (4t, 0)$. One can introduce the local Airy coordinate ζ so that

$$(115) \quad x = 4t + \zeta^2.$$

In other words,

$$(116) \quad \zeta^2 = x - 4t = \frac{64ty^2}{1 - 16y^2}.$$

We consider the following expansion of the corrected genus zero one-point function:

$$\begin{aligned} y &= - \left(\frac{1}{16} - \frac{t}{4(4t + \zeta^2)} \right)^{1/2} = \pm \left(\frac{\zeta^2}{16(4t + \zeta^2)} \right)^{1/2} = \frac{\zeta}{8t^{1/2}(1 + \zeta^2/(4t))^{1/2}} \\ &= \frac{1}{8t^{1/2}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{n!} \frac{\zeta^{2n+1}}{2^{3n} t^n}. \end{aligned}$$

This means the variables t_{2n+3} are given by

$$(117) \quad t_{2n+3} = (-1)^n \frac{(2n-1)!!}{n!} \frac{1}{2^{3n+3} t^{n+1/2}}.$$

For example,

$$t_3 = \frac{1}{2^3 t^{1/2}}, \quad t_5 = -\frac{1}{2^6 t^{3/2}}.$$

The conjugate variables \tilde{t}_k are then given by

$$(118) \quad e^{-\sum_k \tilde{t}_k u^{-k}} = \sum_{n \geq 0} (-1)^n \frac{(2n+1)!!(2n-1)!!}{n!} \frac{1}{2^{3n+3} t^{n+1/2}} u^{-n}.$$

6.2. Bergman kernel in local Airy coordinate. Next we consider the expansion of genus zero two-point function in the local Airy coordinate:

$$\begin{aligned} & B(p_1, p_2) - \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} \\ &= \left(\frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2}}{8(x_1 - x_2)^2 y_1 y_2} + \frac{1}{2(x_1 - x_2)^2} \right) dx_1 dx_2 - \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} \\ &= \left(\frac{1 - \frac{2t}{4t+\zeta_1^2} - \frac{2t}{4t+\zeta_2^2}}{8(\zeta_1^2 - \zeta_2^2)^2 \left(\frac{\zeta_1^2}{4(4t+\zeta_1^2)} \right)^{1/2} \left(\frac{\zeta_2^2}{4(4t+\zeta_2^2)} \right)^{1/2}} + \frac{1}{2(\zeta_1^2 - \zeta_2^2)^2} \right) d(4t + \zeta_1^2) d(4t + \zeta_2^2) \\ &- \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} \\ &= \left(\frac{(2t\zeta_1^2 + 2t\zeta_2^2 + \zeta_1^2 \zeta_2^2)}{2\zeta_1 \zeta_2 (\zeta_1^2 - \zeta_2^2)^2 (4t + \zeta_1^2)^{1/2} (4t + \zeta_2^2)^{1/2}} + \frac{1}{2(\zeta_1^2 - \zeta_2^2)^2} \right) 4\zeta_1 \zeta_2 d\zeta_1 d\zeta_2 \\ &- \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} \\ &= \frac{1}{(\zeta_1^2 - \zeta_2^2)^2} \left(\frac{2(2t\zeta_1^2 + 2t\zeta_2^2 + \zeta_1^2 \zeta_2^2)}{(4t + \zeta_1^2)^{1/2} (4t + \zeta_2^2)^{1/2}} - (\zeta_1^2 + \zeta_2^2) \right) d\zeta_1 d\zeta_2 \\ &= \frac{1}{(\zeta_1^2 - \zeta_2^2)^2} \left(\frac{(\zeta_1^2 + \zeta_2^2 + \zeta_1^2 \zeta_2^2 / (2t))}{(1 + \zeta_1^2 / (4t))^{1/2} (1 + \zeta_2^2 / (4t))^{1/2}} - (\zeta_1^2 + \zeta_2^2) \right) d\zeta_1 d\zeta_2. \end{aligned}$$

Now we use the same method used in the proof of (46). We let $s = 1/t$ and $\frac{\partial}{\partial s}$ on the right-hand side of the last equality to get:

$$\begin{aligned} & \frac{1}{8(1 + \zeta_1^2 s/4)^{3/2} (1 + \zeta_2^2 s/4)^{3/2}} \\ &= -\frac{1}{8} \sum_{m=0}^{\infty} \frac{(2m+1)!!}{2^m m!} (-\zeta_1^2 s/4)^m \cdot \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2^n n!} (-\zeta_2^2 s/4)^n, \end{aligned}$$

and so after integration we get:

$$(119) \quad B(p_1, p_2) = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} + \sum_{l=1}^{\infty} \frac{(-1)^l}{2^{3l} l^l} \sum_{m+n=l-1} \frac{(2m+1)!! \zeta_1^{2m}}{m!} \cdot \frac{(2n+1)!! \zeta_2^{2n}}{n!}.$$

In other words,

$$(120) \quad B_{2k, 2l} = \frac{(-1)^{k+l+1}}{2^{3k+3l+3} (k+l+1) t^{k+l+1}} \frac{(2k+1)!! (2l+1)!!}{k! l!}.$$

By (17),

$$(121) \quad \hat{B}_{k, l} = \frac{(-1)^{k+l+1}}{2^{4k+4l+4} (k+l+1) t^{k+l+1}} \frac{(2k+1)!! (2k-1)!! (2l+1)!! (2l-1)!!}{k! l!}.$$

6.3. **The variables ζ_n .** By (13) we then have

$$(122) \quad \zeta_k(z) = \frac{(2k-1)!!}{2^k} \left(\frac{1}{\zeta(z)^{2k+1}} - \sum_l \frac{(-1)^{k+l+1} (2k+1)!! (2l+1)!! \zeta(z)^{2l+1}}{2^{3k+3l+3} (k+l+1) k! l! t^{k+l+1} 2l+1} \right).$$

In particular,

$$(123) \quad \zeta_0(z) = \frac{1}{\zeta(z)} - \sum_l \frac{(-1)^{l+1} (2l+1)!! \zeta(z)^{2l+1}}{2^{3l+3} (l+1)! t^{l+1} 2l+1}.$$

The summation over l can be carried out easily to get:

$$(124) \quad \zeta_0(z) = \frac{(1 + \zeta(z)/(4t))^{1/2}}{\zeta(z)} = \frac{1}{8t^{1/2}y(z)} = -\frac{1}{2t^{1/2}\sqrt{1 - \frac{4t}{x}}}.$$

We then also have

$$(125) \quad d\zeta_0(z) = \frac{t^{1/2}}{x^2(1 - \frac{4t}{x})^{3/2}} dx.$$

In general, to take the summation over l in the expression for ζ_k , we introduce:

$$g_k(w, z) = \sum_l \frac{(-1)^{k+l+1} (2k+1)!! (2l+1)!! \zeta(z)^{2l+1}}{2^{3k+3l+3} (k+l+1) k! l! t^{k+l+1} 2l+1} w^{k+l+1}.$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial w} g_k(w, z) &= \sum_l \frac{(-1)^{k+l+1} (2k+1)!! (2l+1)!! \zeta(z)^{2l+1}}{2^{3k+3l+3} k! l! t^{k+l+1} 2l+1} w^{k+l} \\ &= \frac{(-1)^{k+1} (2k+1)!!}{2^{3k+3} k! t^{k+1}} \zeta(z) w^k \cdot \sum_l \frac{(-1)^l (2l+1)!! \zeta(z)^{2l}}{2^{3l} l! t^l 2l+1} w^l \\ &= \frac{(-1)^{k+1} (2k+1)!!}{2^{3k+3} k! t^{k+1}} \zeta(z) \cdot w^k \cdot (1 + w\zeta(z)^2/(4t))^{-1/2}. \end{aligned}$$

Now we integrate over w using Lemma 6.1 below

$$\begin{aligned} g_k(w, z) &= \frac{(-1)^{k+1} (2k+1)!!}{2^{3k+3} k! t^{k+1}} \zeta(z) \\ &\cdot \left(\frac{\sqrt{1 + w\zeta(z)^2/(4t)}}{(k+1) \binom{2k+2}{k+1} (\zeta(z)^2/(16t))^{k+1}} \sum_{j=0}^k (-1)^{k-j} \binom{2j}{j} (w\zeta(z)^2/(16t))^j \right. \\ &\quad \left. - \frac{(-1)^k}{(k+1) \binom{2k+2}{k+1} (\zeta(z)^2/(16t))^{k+1}} \right) \\ &= \frac{(-1)^{k+1} \sqrt{1 + w\zeta(z)^2/(4t)}}{\zeta(z)^{2k+1}} \sum_{j=0}^k (-1)^{k-j} \binom{2j}{j} (w\zeta(z)^2/(16t))^j + \frac{1}{\zeta(z)^{2k+1}}, \end{aligned}$$

and then set $w = 1$ to get:

$$(126) \quad \zeta_k(z) = \frac{(2k-1)!!}{2^k} \cdot \frac{\sqrt{1 + \zeta(z)^2/(4t)}}{\zeta(z)^{2k+1}} \sum_{j=0}^k (-1)^j \binom{2j}{j} \frac{\zeta(z)^{2j}}{2^{4j} t^j}.$$

Using (116), we express ζ_k as a polynomial in y^{-1} :

$$(127) \quad \zeta_k(z) = \frac{(2k-1)!!}{2^k t^k} \cdot \frac{1}{8t^{1/2}y} \sum_{j=0}^k (-1)^j \binom{2j}{j} \frac{1}{2^{4j}} \left(\frac{1}{64y^2} - \frac{1}{4} \right)^{k-j},$$

and we can also express it as a power series in x^{-1} using:

$$(128) \quad \zeta_k(z) = -\frac{(2k-1)!!}{2^k} \cdot \sum_{j=0}^k (-1)^j \binom{2j}{j} \frac{1}{2^{4j+1} t^{j+1/2} x^{k-j} \left(1 - \frac{4t}{x}\right)^{k-j+1/2}}.$$

In particular,

$$\begin{aligned} \zeta_1(z) &= -\frac{1}{2} \cdot \sum_{j=0}^1 (-1)^j \binom{2j}{j} \frac{1}{2^{4j+1} t^{j+1/2} x^{1-j} \left(1 - \frac{4t}{x}\right)^{3/2-j}} \\ &= \frac{1}{2^5 t^{3/2} \left(1 - \frac{4t}{x}\right)^{1/2}} - \frac{1}{2^2 t^{1/2} x \left(1 - \frac{4t}{x}\right)^{3/2}}, \end{aligned}$$

and so

$$(129) \quad d\zeta_1(z) = \left(\frac{3}{16t^{1/2}x^2 \left(1 - \frac{4t}{x}\right)^{3/2}} + \frac{3t^{1/2}}{2x^3 \left(1 - \frac{4t}{x}\right)^{5/2}} \right) dx.$$

Lemma 6.1. For $n \geq 0$,

$$(130) \quad \int \frac{x^n}{\sqrt{1+4ax}} dx = \frac{\sqrt{1+4ax}}{(n+1) \binom{2n+2}{n+1} a^{n+1}} \sum_{j=0}^n (-1)^{n-j} \binom{2j}{j} a^j x^j.$$

Proof. This is very easy to verify: Simply take derivatives in x on both sides. The right-hand side gives us

$$\begin{aligned} & \frac{\sqrt{1+4ax}}{(n+1) \binom{2n+2}{n+1} a^{n+1}} \sum_{j=0}^n (-1)^{n-j} \binom{2j}{j} j a^j x^{j-1} \\ & + \frac{2a}{(n+1) \binom{2n+2}{n+1} a^{n+1} \sqrt{1+4ax}} \sum_{j=0}^n (-1)^{n-j} \binom{2j}{j} a^j x^j \\ & = \frac{1}{(n+1) \binom{2n+2}{n+1} a^{n+1} \sqrt{1+4ax}} \left(2a \sum_{j=0}^n (-1)^{n-j} \binom{2j}{j} a^j x^j \right. \\ & \quad \left. + (1+4ax) \sum_{j=0}^n (-1)^{n-j} \binom{2j}{j} j a^j x^{j-1} \right) \\ & = \frac{(-1)^n}{(n+1) \binom{2n+2}{n+1} a^{n+1} \sqrt{1+4ax}} \\ & \quad \cdot \left(\sum_{j=0}^n (-1)^j \binom{2j}{j} (2+4j) a^{j+1} x^j - \sum_{j=0}^{n-1} (-1)^j \binom{2j+2}{j+1} (j+1) a^{j+1} x^j \right) \\ & = \frac{x^n}{\sqrt{1+4ax}}. \end{aligned}$$

□

Remark 6.1. The numbers $n \binom{2n}{n}$ are the sequence A005430 on [12]. They are called the Apéry numbers.

6.4. Relationship with intersection numbers. By combining Theorem 2 with Eynard's result recalled in §2, we get the following:

Theorem 3. *For the modified partition function of Hermitian one-matrix model with even couplings,*

$$(131) \quad W_{g,n}(z_1, \dots, z_n) = 2^{3g-3+n} \sum_{d_1+\dots+d_n \leq 3g-3+n} \prod_i d\zeta_{d_i}(z_i) \left\langle e^{\frac{1}{2} \sum_\delta t_{\delta^*} \hat{B}(\psi, \psi')} e^{\sum_k \tilde{t}_k \kappa_k} \prod_i \psi_i^{d_i} \right\rangle_{g,n}$$

where \tilde{t}_k are given by:

$$(132) \quad e^{-\sum_k \tilde{t}_k u^{-k}} = \sum_{n \geq 0} (-1)^n \frac{(2n+1)!!(2n-1)!!}{n!} \frac{1}{2^{3n+3} t^{n+1/2}} u^{-n},$$

$\hat{B}_{k,l}$ are given by

$$(133) \quad \hat{B}_{k,l} = \frac{(-1)^{k+l+1}}{2^{4k+4l+4} (k+l+1) t^{k+l+1}} \frac{(2k+1)!!(2k-1)!!}{k!} \frac{(2l+1)!!(2l-1)!!}{l!},$$

and ζ_k are given by:

$$(134) \quad \zeta_k(z) = -\frac{(2k-1)!!}{2^k} \cdot \sum_{j=0}^k (-1)^j \binom{2j}{j} \frac{1}{2^{4j+1} t^{j+1/2} x^{k-j} (1 - \frac{4t}{x})^{k-j+1/2}}.$$

Let us now check some example. By (21) we have

$$\begin{aligned} \omega_{0,3}(z_1, z_2, z_3) &= \frac{1}{2t_3} d\zeta_0(z_1) d\zeta_0(z_2) d\zeta_0(z_3) \\ &= \frac{1}{2 \cdot \frac{1}{2^3 t^{1/2}}} \frac{t^{1/2}}{x_1^2 (1 - \frac{4t}{x_1})^{3/2}} dx_1 \cdot \frac{t^{1/2}}{x_2^2 (1 - \frac{4t}{x_2})^{3/2}} dx_2 \cdot \frac{t^{1/2}}{x_3^2 (1 - \frac{4t}{x_3})^{3/2}} dx_3 \\ &= \frac{4t^2}{x_0^2 x_1^2 x_2^2 ((1 - 4t/x_0)(1 - 4t/x_1)(1 - 4t/x_2))^{3/2}} dx_1 dx_2 dx_3. \end{aligned}$$

This matches with (54). By (22)

$$\begin{aligned} \omega_{1,1}(z) &= \frac{1}{24t_3} d\zeta_1(z) + \left(\frac{B_{0,0}}{4t_3} - \frac{t_5}{16t_3^2} \right) d\zeta_0(z) \\ &= \frac{1}{24 \cdot \frac{1}{2^3 t^{1/2}}} \cdot \left(\frac{3}{16t^{1/2} x^2 (1 - \frac{4t}{x})^{3/2}} + \frac{3t^{1/2}}{2x^3 (1 - \frac{4t}{x})^{5/2}} \right) dx \\ &+ \left(\frac{-\frac{1}{2^3 t}}{4 \cdot \frac{1}{2^3 t^{1/2}}} - \frac{-\frac{1}{2^6 t^{3/2}}}{16 \cdot (\frac{1}{2^3 t^{1/2}})^2} \right) \cdot \frac{t^{1/2}}{x^2 (1 - \frac{4t}{x})^{3/2}} dx \\ &= \left(\frac{1}{16x^2 (1 - \frac{4t}{x})^{3/2}} + \frac{t}{2x^3 (1 - \frac{4t}{x})^{5/2}} - \frac{3}{16x^2 (1 - \frac{4t}{x})^{3/2}} \right) dx \\ &= \left(-\frac{1}{8x^2 (1 - \frac{4t}{x})^{3/2}} + \frac{t}{2x^3 (1 - \frac{4t}{x})^{5/2}} \right) dx. \end{aligned}$$

This matches with (59).

$$\begin{aligned}
\omega_{0,4}(z_1, z_2, z_3, z_4) &= \frac{1}{2t_3^2} (d\zeta_1(z_1)d\zeta_0(z_2)d\zeta_0(z_3)d\zeta_0(z_4) + perm.) \\
&+ \frac{3}{4} \left(\frac{B_{0,0}}{t_3^2} - \frac{t_5}{t_3^3} \right) d\zeta_0(z_1)d\zeta_0(z_2)d\zeta_0(z_3)d\zeta_0(z_4) \\
&= \frac{1}{2 \cdot \left(\frac{1}{2^3 t^{1/2}}\right)^2} \cdot \left(\frac{3}{16t^{1/2}x_1^2(1-\frac{4t}{x_1})^{3/2}} + \frac{3t^{1/2}}{2x_1^3(1-\frac{4t}{x_1})^{5/2}} \right) dx_1 \\
&\quad \cdot \frac{t^{1/2}}{x_2^2(1-\frac{4t}{x_2})^{3/2}} dx_2 \cdot \frac{t^{1/2}}{x_3^2(1-\frac{4t}{x_3})^{3/2}} dx_3 \cdot \frac{t^{1/2}}{x_4^2(1-\frac{4t}{x_4})^{3/2}} dx_4 \\
&+ perm. \\
&+ \frac{3}{4} \left(\frac{-\frac{1}{2^3 t}}{\left(\frac{1}{2^3 t^{1/2}}\right)^2} - \frac{-\frac{1}{2^6 t^{3/2}}}{\left(\frac{1}{2^3 t^{1/2}}\right)^3} \right) \cdot \prod_{j=1}^4 \frac{t^{1/2}}{x_j^2(1-\frac{4t}{x_j})^{3/2}} dx_j \\
&= 6t^2(x_1+4t)(x_2-4t)(x_3-4t)(x_4-4t) \prod_{j=1}^4 \frac{dx_j}{x_j^3(1-\frac{4t}{x_j})^{5/2}} \\
&+ perm. \\
&= 24t^2 \frac{e_4 - 2e_3t + 32e_1t^3 - 256t^4}{\prod_{j=0}^3 x_j^3(1-\frac{4t}{x_j})^{5/2}} dx_1 \cdots dx_4,
\end{aligned}$$

where e_j denotes the j -th elementary symmetric polynomial in x_1, \dots, x_4 . This matches with (56).

Acknowledgements. The author is partly supported by NSFC grants 11661131005 and 11890662.

REFERENCES

- [1] B. Dubrovin, S.-Q. Liu, D. Yang, Y. Zhang, *Hodge-GUE correspondence and the discrete KdV equation*. arXiv:1612.02333.
- [2] R. Dijkgraaf, H. Verlinde, E. Verlinde, *Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity*. Nuclear Phys. B 348 (1991), no. 3, 435-456.
- [3] B. Eynard, *Recursion between Mumford volumes of moduli spaces*. Ann. Henri Poincaré 12(8), 1431-1447 (2011).
- [4] B. Eynard, *Intersection numbers of spectral curves*. arXiv:1104.0176
- [5] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*. Commun. Number Theory Phys. 8 (2014), no. 3, 541-588.
- [6] B. Eynard, N. Orantin *Invariants of algebraic curves and topological expansion*. Commun. Number Theory Phys. 1 (2007), no. 2, 347-452.
- [7] J. Harer, D. Zagier *The Euler characteristic of the moduli space of curves*. Inventiones Mathematicae, 1986, 85(3):457-485.
- [8] A. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary, Mosc.Math. J. 1 (2001), no. 4, 551-568.
- [9] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. 147 (1992), no. 1, 1-23.
- [10] K. Liu, H. Xu, *Recursion formulae of higher Weil-Petersson volumes*. Int. Math. Res. Not. 2009(5), 835-859.
- [11] M. Mulase, B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*. Indiana J. Math. 50(2008), 189-218.

- [12] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>, 2013.
- [13] R.C. Penner, *e m*Perturbative series and the moduli space of Riemann surfaces. *Journal of Differential Geometry*, 1988, 27(1988):35-53.
- [14] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, *Surveys in Differential Geometry*, vol.1, (1991) 243–310.
- [15] J. Zhou, *Topological recursions of Eynard-Orantin type for intersection numbers on moduli spaces of curves*. *Lett. Math. Phys.* 103 (2013), no. 11, 1191-1206.
- [16] J. Zhou, *Intersection numbers on Deligne-Mumford moduli spaces and quantum Airy curve*. arXiv:1206.5896.
- [17] J. Zhou, *Fat and thin emergent geometries of Hermitian one-matrix models*. arXiv:1810.03883.
- [18] J. Zhou, *In preparation*.

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, CHINA
E-mail address: jianzhou@mail.tsinghua.edu.cn