A GEOMETRICAL SUMMATION METHOD FOR THE RIEMANN Z TA FUNCTION

A PREPRINT

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March 27, 2019

ABSTRACT

Disclaimer: *I am no professional mathematician. Though, it seems to me that the intuition presented in this preprint has not yet been explored by the mathematical community. This intuition seems very consistent and makes predictions, these predictions can be veri ed numerically. Because this article is a preliminary work, some parts of the proof are not complete. They will be explicitly indicated.*

In this paper, we introduce a geometrical summation method that makes the original Riemann series converge over the critical strip. This method gives an analytical function, that coincides with z ta. This point of view allows us to introduce a quantity of interest that seems to give a characterization of the non-trivial zeros of the Riemann z ta function.
For: $z = x + iy, x \in \mathbb{R}_+^*, y \in \mathbb{R}^*$, z ta can be de ned as:

$$\boxed{\zeta(z) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^z} + \frac{1}{(N+1)^z} \frac{1}{1-z} \left(1 - x - \frac{y}{\tan(y \ln(\frac{N+2}{N+1}))}\right)}$$

From here, we can show that if z is a non trivial zero of z ta, the following quantity converges to a relative integer seemingly even. Though the reciprocal is false, it is natural to compute it for the known non-trivial zeros of z ta:

$$\boxed{\zeta(z) = 0 \Rightarrow \lim_{n\to\infty} \frac{-y \ln(n+1) - \underset{sum}{\arg}(\zeta_n(z)) + \arctan(\frac{y}{1-x})}{\pi} = \mathcal{U}_z, \mathcal{U}_z \in \mathbb{Z}}$$

$$\underset{sum}{\arg}(\zeta_n(z)) = \sum_{m=1}^{n} \arg\left(\frac{\sum_{l=1}^{m+1} \frac{1}{l^z}}{\sum_{l=1}^{m} \frac{1}{l^z}}\right)$$

At the end of this document you can nd the table of these values computed for the rst 30 know zeros of z ta. This last identity seems to be deeply correlated with the Riemann hypothesis.
In this paper, we are taking full advantage of formal calculation algorithms, especially to compute some asymptotic expansions. Here is the GitHub link to all codes used for this work:

https://github.com/UlysseREGLADE/Zeta

Keywords Riemann · Z ta · Summation method

1 Introduction

For convenience, we de ne for $z \in \Omega$:

$$\begin{aligned} \zeta_n(z) &= \sum_{j=1}^{n} \frac{1}{j^z}, n \in \mathbb{N}^* \\ \zeta_0(z) &= 0 \end{aligned} \tag{1}$$

In the whole paper, z is de ned as a complex number of Ω an open set of $\mathbb{C}$. x and y are de ned with no ambiguity as its real and complex parts:

$$\begin{aligned} x &= \Re z \\ y &= \Im z \end{aligned} \tag{2}$$

We are not going to make a real difference between the complex representation of z and its vectorial form (x, y). Therefore, we will allow ourselves to use these representations quite freely.

Let's now discuss the intuition that gave rise to this paper. I'm uncomfortable with the usual way of de ning the z ta function [1].

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, x > 1 \tag{3}$$

In this notation, information is lost on how the partials sums converge to z ta. To me, a more meaningful, but still not perfect way to de ne z ta would be something like:

$$\zeta(z) = \overline{\{\zeta_n(z)\}_{n \in \mathbb{N}}} \tag{4}$$

This de nition is not equivalent to the usual one, it is indeed de ned even if $x = 1$. Though it is possible to do much better. What we are going to show is that, in a way, the z ta function can be de ned as the center of a logarithmic asymptotic spiral described by the partial sums of the Riemann series $\{\zeta_n(z)\}_{n \in \mathbb{N}}$. In fact, this sequence describes a converging spiral for $x > 1$, a circle for $x = 1$, and a diverging spiral for $x < 1$. In each case, the value of z ta is the center of these spirals. We are going to take advantage from this fact to de ne zeta for: $x > 0$.

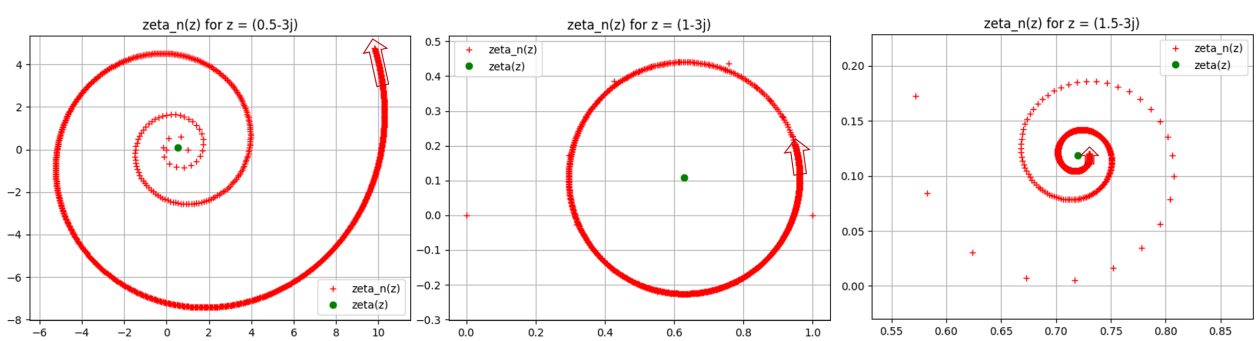

Figure 1: Plot of the Riemann series, for: $z = \frac{1}{2} - 3i$, $z = 1 - 3i$, ans $z = \frac{3}{2} - 3i$

2 The radial convergence, a geometrical summation method for the Riemann series

2.1 The radial convergence for the special case $z \in \{1 + iy, y \in \mathbb{R}^*\}$.

In this subsection, Ω is de ned as:

$$\Omega = \{1 + iy, y \in \mathbb{R}^*\} \tag{5}$$

Let us de ne:

$$\Delta_n(z) = \left\{\zeta_n(z) + \frac{te^{i\frac{\pi}{2}}}{(n+1)^z}, t \in \mathbb{R}\right\}, z \in \Omega, n \in \mathbb{N} \tag{6}$$

$\Delta_n(z)$ is a straight line of the complex plan. Considering $\frac{1}{n^z}$ as a vector, it is clearly not collinear to $\frac{1}{(n+1)^z}$. Indeed: $\arg(\frac{n}{n+1}^z) = y\ln(\frac{n}{n+1}) \neq 0$. Therefore, $\Delta_n(z)$ and $\Delta_{n+1}(z)$ must have an unique intersection point. Let us call it $c_n(z)$:

$$c_n(z) = \Delta_n(z) \cap \Delta_{n+1}(z), n \in \mathbb{N} \tag{7}$$

Let us adopt the following notation:

$$\delta_n(z) = c_n(z) - c_{n-1}(z), n \in \mathbb{N}^* \tag{8}$$

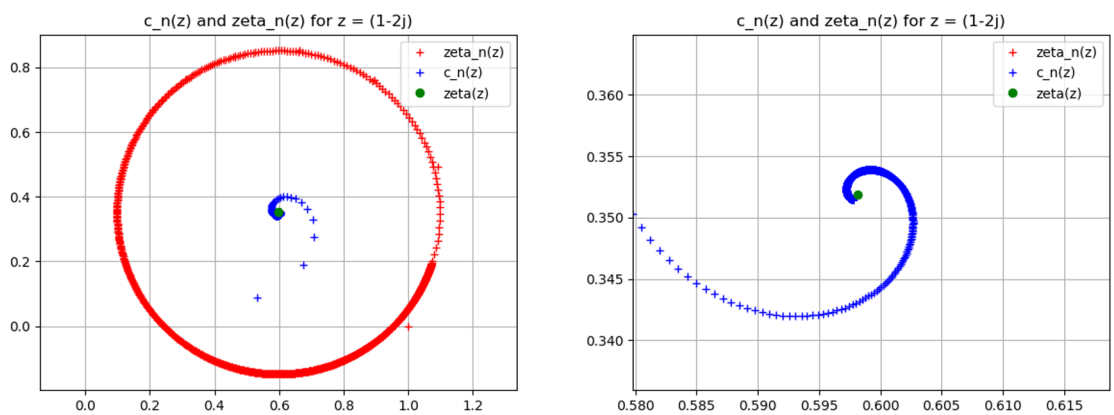

Figure 2: Plot of $\{c_n(z)\}_{n\in\mathbb{N}}$ for: $z = 1 - 2i$

Figure 2 is a plot $\{c_n(z)\}_{n\in\mathbb{N}}$, and we can enunciate our rst theorem:

Theorem 2.1. *For $z \in \Omega$, $\sum \delta_n(z)$ converges by Riemann sommation, and we can de ne with no ambiguity:*

$$c(z) = \lim_{n\to\infty} c_n(z) \tag{9}$$

Proof. Using geometrical relationships, we can write from the **Figure 3**:

$$|\delta_n(z)| = \left|\frac{1}{(n+1)\tan(-y\ln(\frac{n+2}{n+1}))} - \sqrt{\frac{1}{n^2} + \frac{1}{n^2\tan(y\ln(\frac{n+1}{n}))^2}}\right| \tag{10}$$

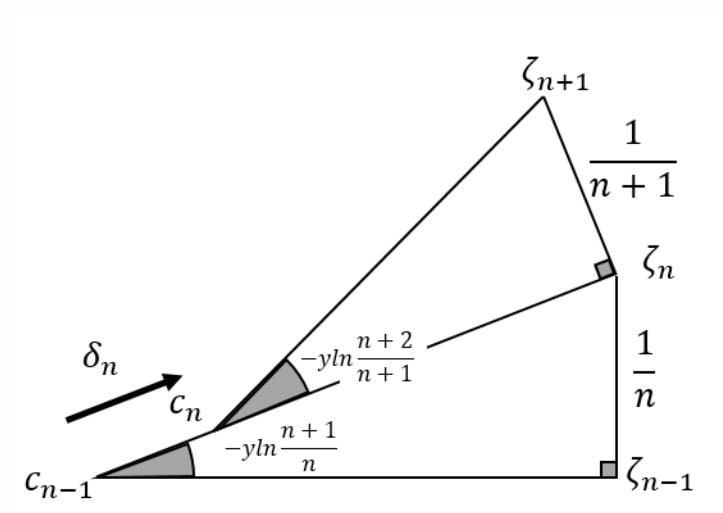

Figure 3: Geometrical view of the situation

From **Equation 10**, we can calculate the asymptotic expansion of $\delta_n(z)$. This calculation is quite tedious, but at the end of the day one can show that:

$$\delta_n(z) \underset{n\to\infty}{=} \frac{1}{2n^2}\left|y+\frac{1}{y}\right| + o(\frac{1}{n^2}) \tag{11}$$

The series converges by Riemann sommation. This proves the theorem.

□

In addition, we can observe that:

$$|c_n(z)-\zeta_n(z)| = \frac{1}{(n+1)\tan(|y|\ln(\frac{n+2}{n+1}))} \underset{n\to\infty}{=} \frac{1}{|y|} + o(1) \tag{12}$$

Which leads us to our second theorem:

Theorem 2.2 (Asymptotic circle)**.** *Lets note $\mathcal{C}_b^a$ the circle of radius b and center a. We have:*

$$\overline{\{\zeta_n\}_{n\in\mathbb{N}}} \subset \mathcal{C}^{c(z)}_{\frac{1}{|y|}} \tag{13}$$

Proof. By triangular inequality, we can write:

$$|\zeta_n(z)-c_n(z)| - |c_n(z)-c(z)| \leq |\zeta_n(z)-c(z)| \leq |\zeta_n(z)-c_n(z)| + |c_n(z)-c(z)| \tag{14}$$

Let us denote $d(C^{c(z)}_{\frac{1}{|y|}}, \zeta_n(z))$ the distance between the circle and the sequence:

$$d(C^{c(z)}_{\frac{1}{|y|}}, \zeta_n(z)) = \left||\zeta_n(z)-c(z)| - \frac{1}{|y|}\right| \tag{15}$$

We have:

$$\begin{aligned} |\zeta_n(z)-c_n(z)| &= \frac{1}{|y|} + o(1) \\ |c_n(z)-c(z)| &= o(1) \end{aligned} \tag{16}$$

From **Equation 14** we can conclude that:

$$\lim_{n\to\infty} d(\mathcal{C}^{c(z)}_{\frac{1}{|y|}}, \zeta_n(z)) = 0 \tag{17}$$

which nishes the proof.

□

From the **Figure 2**, one can expect $\{\zeta_n(z)\}_{n\in\mathbb{N}}$ to populate densely the circle, which is a reasonable conjecture. It would lead to an equality and not a simple inclusion in **Theorem 2.2**. Though, the demonstration of this conjecture is not necessary for the rest of the proof.

An other reasonable hypothesis is that c is actually ζ. But as **Equation 10** only applies for $x = 1$, we need rst to extend the de nition of c to a bigger open set and show that it is analytical and coincides with the Riemann z ta function.

2.2 The radial convergence for $z \in \{x + iy, x > 0, y \in \mathbb{R}^*\}$

For the rest this section, we de ne Ω and ω as:

$$\begin{aligned} \Omega &= \{x + iy, x > 0, y \in \mathbb{R}^*\} \\ \omega &= \{x + iy, x > 1, y \in \mathbb{R}^*\} \end{aligned} \tag{18}$$

To understand the rest of the proof, we rst need to make the following observation:

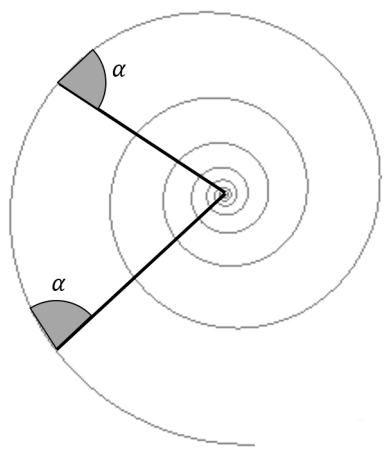

Figure 4: Representation of a logarithmic spirale

Let us consider the integral test for convergence of $\zeta_n(z)$:

$$\begin{aligned} \int n^{-z} dn = \frac{n^{1-z}}{1-z} = \frac{1}{1-z} e^{(1-x)\ln(n)} e^{-iy\ln(n)} = \frac{1}{1-z} e^{\frac{x-1}{y}\theta} e^{i\theta} \\ \theta = -y\ln(n) \end{aligned} \tag{19}$$

We recognize the de nition of a logarithmic spiral [2] in the complex plan. Such a spiral intersects its radius with a constant angle α, like it is shown in **Firgure 4**. In this case, we can compute this value, and we nd:

$$\alpha(z) = \begin{cases} \frac{\pi}{2} - \arctan(\frac{1-x}{y}) & y > 0 \\ \frac{3\pi}{2} - \arctan(\frac{1-x}{y}) & y < 0 \end{cases} \tag{20}$$

We can see that $x = 1$ implies $\alpha = \frac{\pi}{2}[\pi]$. Therefore, it is natural to give a new de nition to $\Delta_n(z)$:

$$\Delta_n(z) = \left\{ \zeta_n(z) + \frac{te^{i\alpha}}{(n+1)^z}, t \in \mathbb{R} \right\}, z \in \Omega, n \in \mathbb{N} \tag{21}$$

For the same reason than in the rst section, we can de ne with no ambiguity the sequences $\{c_n(z)\}_{n\in\mathbb{N}}$ and $\{\delta_n(z)\}_{n\in\mathbb{N}^*}$.

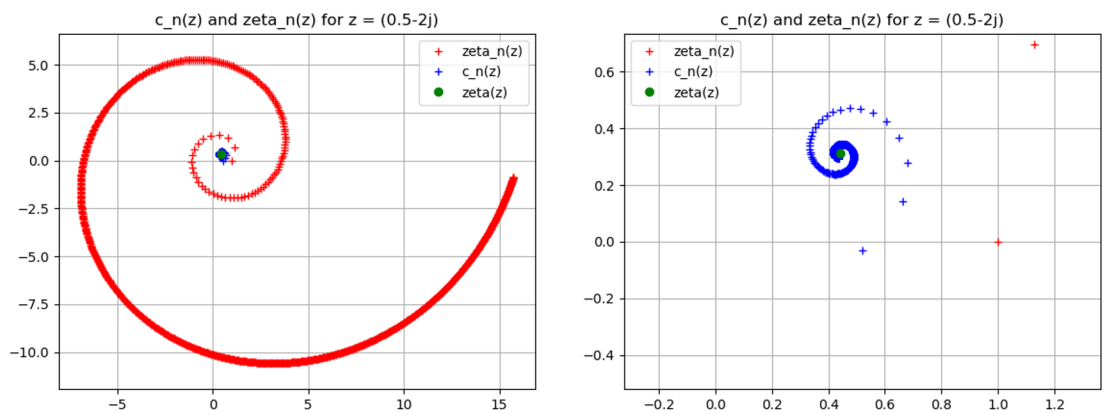

Figure 5: Plot of the convergence for $z = 0.5 - 2i$

From the **Figure 5**, it seems reasonable to expect $\{c_n(z)\}_{n\in\mathbb{N}}$ to converge over the critical strip. It can even already conjecture that their limit coincides with $\zeta(z)$.

2.3 Existence and continuity of $c : \Omega \to \mathbb{C}$

First, let us remind the de nition of a domination function:

De nition 2.1. *$D_d : \mathbb{R} \to \mathbb{R}$ dominates $f_n : \mathbb{C} \to \mathbb{C}$ over Ω_d if and only if:*

$$\exists n_0 \in \mathbb{N} | \forall z \in \Omega_d, \forall n > n_0, D_d(n) > |f_n(z)| \tag{22}$$

We are now ready to formulate the weak version of what will be called from now the radial convergence for the Riemann z ta function.

Theorem 2.3. *For $z \in \Omega$, the sequence $\{c_n(z)\}_{n\in\mathbb{N}}$ converges uniformly to $c(z)$. $c : \Omega \to \mathbb{C}$ is continuous over Ω, and we have:*

$$c(z) = \zeta(z), z \in \omega \tag{23}$$

c is complex-differentiable (i.e. holomorphic) over $\omega \subset \Omega$, and it coincides with ζ over this open set.

Proof. First, we observe from **Figure 6** that we can write $\delta_n(z)$ as:

$$\delta_n(z) = \frac{1}{n^x \sqrt{1 + (\frac{1-x}{y})^2}} \left(\frac{1}{\sin(-y \ln \frac{n+1}{n})} - \frac{1}{(1+\frac{1}{n})^x} \left(\frac{1}{\tan(-y \ln(\frac{n+2}{n+1}))} + \frac{1-x}{y} \right) \right) e^{i(\alpha(z) - y \ln(n+1))} \tag{24}$$

To obtain this result, we have rst computed the algebrical module of $\delta_n(z)$ only using geometrical relations in the **Figure 6**. Then, we compute the algebrical argument of $\delta_n(z)$ by induction and show that it is in fact: $\alpha(z) - y \ln(n+1)$.

We also work out the expression of $c_0(z)$ by hand. It can be neatly written as:

$$c_0(z) = 1 - \frac{1}{1-z} \frac{y2^{-iy}}{\sin(y \ln 2)} \tag{25}$$

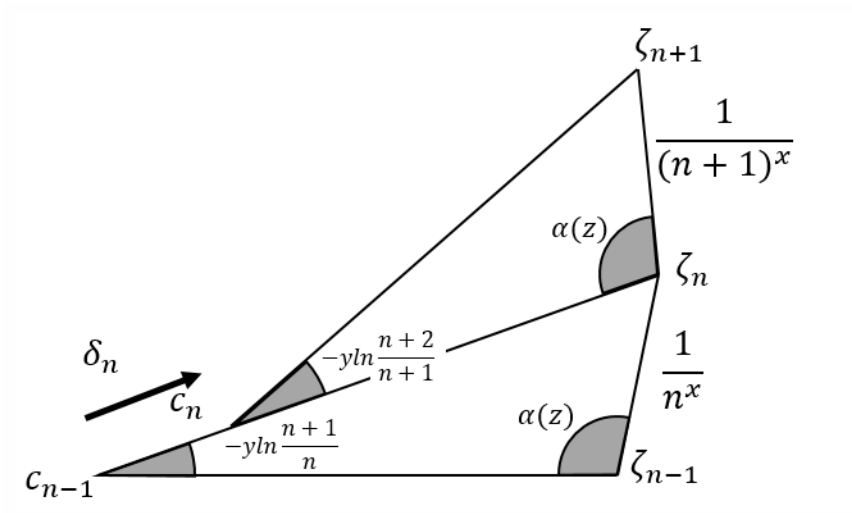

Figure 6: Geometrical view of the situation for $\alpha \in]0, \pi[\cup]\pi, 2\pi[$

This enable us to write:

$$c_n(z) = c_0(z) + \sum_{i=1}^{n} \delta_i(z) \tag{26}$$

Then, just like in the rst section, we compute the asymptotic expansion of $|\delta_n(z)|$ as n goes to in nity:

$$|\delta_n(z)| = \frac{1}{n^x\sqrt{1+(\frac{1-x}{y})^2}}\left|\frac{1}{\sin(y\ln\frac{n+1}{n})} - \frac{1}{(1+\frac{1}{n})^x}\left(\frac{1}{\tan(y\ln(\frac{n+2}{n+1}))} + \frac{x-1}{y}\right)\right| \tag{27}$$

We nd:

$$|\delta_n(z)| \underset{n\to\infty}{=} \frac{1}{n^{x+1}}\frac{x^2+y^2}{2\sqrt{(1-x)^2+y^2}} = \frac{1}{n^{x+1}}\frac{|z|^2}{2|1-z|} + o(\frac{1}{n^{1+x}}) \tag{28}$$

$c_0(z)$ and $\delta_n(z)$ are continuous with respect to z over Ω. It can be even say right away that they are real-differentiable.

Now, a domination function needs to be chosen. Though, it is not that easy in this case. Indeed, for large values of y, because of the presence of $\frac{1}{\sin(y\ln\frac{n+1}{n})}$ and $\frac{1}{\tan(y\ln\frac{n+2}{n+1})}$ in the expression of $|\delta_n|$, we can expect $|\delta_n|$ to explode for the rst values of n.

From **Equation 27**, we can give the following expression for $|\delta_n|$:

$$|\delta_n(z)| = \frac{1}{n^x|1-z|}\left|\frac{y}{\sin(y\ln\frac{n+1}{n})} - \frac{y}{(1+\frac{1}{n})^x\tan(y\ln(\frac{n+2}{n+1}))} + \frac{1-x}{(1+\frac{1}{n})^x}\right| \tag{29}$$

We introduce: $d_{x,y} : \mathbb{R}_+^* \to \mathbb{R}$, the function de ned by:

$$d_{x,y}(n) = n\left(\frac{y}{\sin(y\ln\frac{n+1}{n})} - \frac{y}{(1+\frac{1}{n})^x\tan(y\ln(\frac{n+2}{n+1}))} + \frac{1-x}{(1+\frac{1}{n})^x}\right) \tag{30}$$

The limit of $d_{x,y}$ is:

$$\lim_{n\to\infty} d_{x,y}(n) = \frac{x^2+y^2}{2} \tag{31}$$

The detailed study of $d_{x,y}$ is necessary in order to be able to properly dominate δ_n.

Let us denote Ω_d a bounded open set of Ω:

$$\begin{aligned} a &= \underset{z\in\Omega_d}{inf}\, x \\ A &= \underset{z\in\Omega_d}{sup}\, x \\ b &= \underset{z\in\Omega_d}{inf}\, |y| \\ B &= \underset{z\in\Omega_d}{sup}\, |y| \end{aligned} \tag{32}$$

Theorem 2.4. *Let $\mathcal{O}_b(\Omega)$ the set of the bounded open sets of Ω. We can de ne:*

$$n_0 : \left| \begin{array}{l} \mathcal{O}_b(\Omega) \to \mathbb{N} \\ \Omega_d \mapsto n_0(\Omega_d) \end{array} \right. \tag{33}$$

This function is such as:

$$\forall \Omega_d \in \mathcal{O}_b(\Omega), \forall z \in \Omega_d, n > n_0(\Omega_d) \Rightarrow \frac{1}{4}(x^2+y^2) < d_{x,y}(n) < \frac{3}{4}(x^2+y^2) \tag{34}$$

Proof. This proof is very technical, and is not detailed in this paper. But here are the mains ideas:

- First, we observe that the sign of $d_{x,y}(n)$ does not depend on the sign of y. We can limit our study to $y > 0$.
- Then, we observe that for $n > n_B = \frac{2-e^{\frac{\pi}{B}}}{e^{\frac{\pi}{B}}-1}$, the signs of $\sin(y\ln(\frac{n+1}{n}))$ and $\tan(y\ln(\frac{n+2}{n+1}))$ do not change anymore.
- We take advantage from the fact that the series expansion of $\sin(x)$, $cos(x)$, $\ln(x)$, and $\frac{1}{(1+x)^{\alpha>0}}$, when truncated, gives majorations or minorations of these function for $x > 0$.
- From here, we de ne two multivariate rational fractions that give upper and lower bounds for $d_{x,y}(n)$ starting from n_B, **Figure 7**.
- The dominant coef cient of the denominator of these two fractions is an integer.

Using this last argument, it is possible to show that from a certain value of n, $d_{x,y}(n)$ must lie in a certain range around its limit. The code used to produce these upper and lower bounds of $d_{x,y}$ is accessible via the GitHub link at the beginning of the document.

□

From here, it will be assumed such a domination function exists:

$$\begin{aligned} D_{\Omega_d} &: \mathbb{R}_+^* \to \mathbb{R} \\ D_d(n) &= \frac{1}{n^{1+a}} \frac{3(A^2+B^2)}{4|a-1|} \end{aligned} \tag{35}$$

This function is integrable by Riemann sommation over any Ω_d in Ω. We can even specify from when this domination is effective for Ω_d. It is from $n_0(\Omega_d)$, which does not depend on the choice of z in Ω_d.

To summarize, we have:

- $\delta_n : \Omega_d \to \mathbb{C}$ is continuous.
- $\sum|\delta_n(z)|$ converges over Ω_d by Riemann sommation.

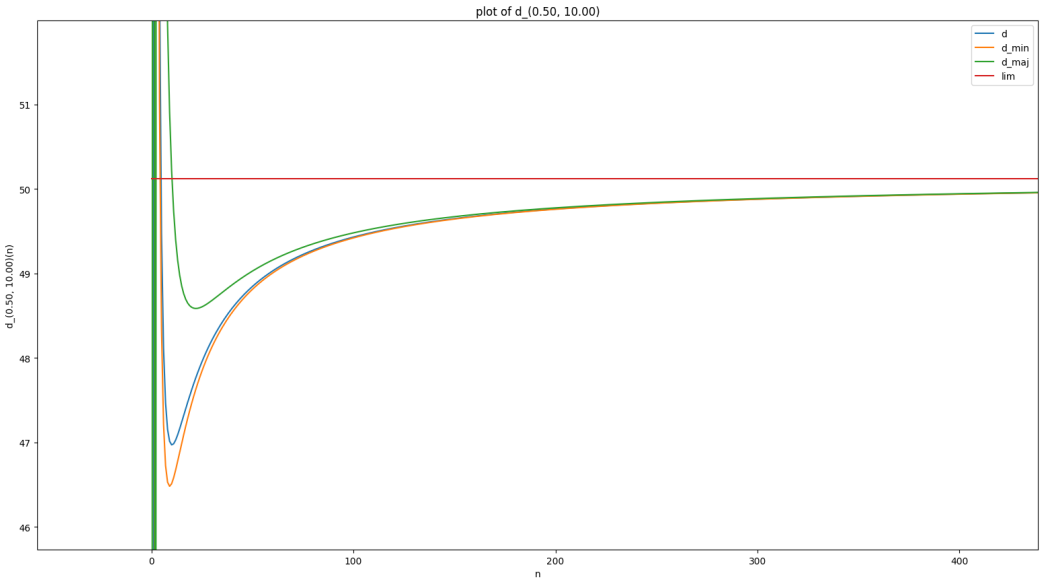

Figure 7: Graph of the framing for $d_{x,y}(n)$

- $D_d(n) : \mathbb{R}_+^* \to \mathbb{R}$ dominates $\delta_n : \Omega_d \to \mathbb{C}$ over Ω_d, and is integrable.

Therefore, $c_0 + \sum \delta_n$ converges normally and in consequence uniformly to c over Ω_d. c is continuous over any Ω_d, so c is continuous over Ω.

Finally, we have to prove that this expression coincides with ζ on $\omega \subset \Omega$. It comes from the fact that:

$$\zeta_n(z) \underset{n\to\infty}{\to} \zeta(z), z \in \omega \tag{36}$$

From **Figure 6**, we can show that:

$$\begin{aligned} &|\zeta_n(z) - c_n(z)| \underset{n\to\infty}{=} \frac{1}{|1-z|n^{x-1}} + o(\frac{1}{n^{x-1}}) \\ &\frac{1}{|1-z|n^{x-1}} \underset{n\to\infty}{\to} 0, z \in \omega \end{aligned} \tag{37}$$

Therefore:

$$z \in \omega \Rightarrow c(z) = \zeta(z) \tag{38}$$

This implies that c is complex-differentiable over ω. We now need to show that c is complex-differentiable over Ω, in particular on the critical strip.

□

2.4 Real-differenciability of $c : \Omega \to \mathbb{C}$

Showing that c is complex-differentiable over the critical strip is quite challenging because c_0 and δ_n are not holomorphic functions. Therefore, the rst step is to show that the Jacobian of δ_n is well de ned and continuous (real-differentiability). Only then we can verify it satis es the Cauchy-Riemann equation (complex-differentiability).

A way to show that c is indeed real-differentiable is to show that $\sum J_{\delta_n}$ converges normally for the in nity norm. This works, but it leads to very tedious calculations that are not described here. Again, the domination is by far the most technical aspect of this proof, though the formal calculation of J_{δ_n} gives immediately the following result:

$$\begin{aligned}
\frac{\partial \delta_{n_x}}{\partial x} &= O(\frac{\ln(n)}{n^{1+x}}) \\
\frac{\partial \delta_{n_x}}{\partial y} &= O(\frac{\ln(n)}{n^{1+x}}) \\
\frac{\partial \delta_{n_y}}{\partial x} &= O(\frac{\ln(n)}{n^{1+x}}) \\
\frac{\partial \delta_{n_y}}{\partial y} &= O(\frac{\ln(n)}{n^{1+x}})
\end{aligned} \tag{39}$$

Therefore:

$$\|J_{\delta_n}\|_\infty = O(\frac{\ln(n)}{n^{1+x}}) \tag{40}$$

We can conclude that $\sum J_{\delta_n}$ converges for any (x, y) in Ω. We now need to nd a domination function for J_{δ_n}, for the in nity norm over Ω_d.

It's possible to reuse the technique described in the last section to nd a domination function for J_{δ_n}, but I think there must be a more elegant solution to this problem. For the rest of the proof, we are just going to assume that we have the following sum to converge normally for the in nity norm:

$$J_c((x, y)) = J_{c_0}((x, y)) + \sum_{i=1}^{\infty} J_{\delta_n}((x, y)), x > 0, y \in \mathbb{R}^* \tag{41}$$

2.5 Complex-differenciability of $c : \Omega \to \mathbb{C}$, equality with $\zeta : \Omega \to \mathbb{C}$

First, we can rewrite **Equation 24** as:

$$\delta_n(z) = y\frac{1-\overline{z}}{|1-z|^2}\left(\frac{2i}{n^z}\frac{1}{\frac{n+1}{n}^{2iy}-1} - \frac{i}{(n+1)^z}\frac{\frac{n+2}{n+1}^{2iy}+1}{\frac{n+2}{n+1}^{2iy}-1} + \frac{1}{(n+1)^z}\frac{1-x}{y}\right) \tag{42}$$

The series $\sum \delta_n$ converges normally, therefore we can rearrange the terms in the sum. More precisely we observe this sum is actually telescopic. We can write that:

$$\sum_{n=1}^{N} \delta_n(z) = \sum_{n=2}^{N} \frac{1}{n^z} + \frac{1-\overline{z}}{|1-z|^2}\left(\frac{y2^{-iy}}{\sin(y\ln 2)} - \frac{y}{(N+1)^z}\frac{1}{\tan(y\ln(\frac{N+2}{N+1}))} + (1-x)\frac{1}{(N+1)^z}\right) \tag{43}$$

Therefore, from **Equation 25**, we can write:

$$\begin{aligned}
c_N(z) &= c_0 + \sum_{n=1}^{N} \delta_n \\
&= \zeta_N(z) + \frac{1}{(N+1)^z}\frac{1-\overline{z}}{|1-z|^2}\left(1 - x - \frac{y}{\tan(y\ln(\frac{N+2}{N+1}))}\right)
\end{aligned} \tag{44}$$

Now, let us introduce:

$$\begin{aligned} g_n(z) &= \frac{1}{(n+1)^z} \\ f_n(z) &= \frac{1-\overline{z}}{|1-z|^2}\left(1-x-\frac{y}{\tan(y\ln(\frac{N+2}{N+1}))}\right) \\ r_n(z) &= g_n(z)f_n(z) \end{aligned} \tag{45}$$

Because $\sum J_{\delta_n}$ converges normally, this rearrangement is also true for the jacobian of $c_n(z)$:

$$J_{c_n}((x,y)) = J_{\zeta_n}((x,y)) + J_{r_n}((x,y)) \tag{46}$$

$\zeta_n(z)$ is a sum of holomorphic functions. Therefore, what we have to show is that $J_{r_n}((x,y))$ tends to satisfy the Cauchy-Riemann equation as n goes to in nity for any $(x,y) \in \Omega$.

First, we compute the following quantities:

$$\begin{aligned} \frac{\partial f_{nx}}{\partial x} - \frac{\partial f_{ny}}{\partial y} &\underset{n\to\infty}{=} \frac{1-x}{|1-z|^2} + o(\frac{1}{n}) \\ \frac{\partial f_{nx}}{\partial y} + \frac{\partial f_{ny}}{\partial x} &\underset{n\to\infty}{=} \frac{1-x}{|1-z|^2} + o(\frac{1}{n}) \end{aligned} \tag{47}$$

Now, we can write:

$$\begin{aligned} \frac{\partial r_{nx}}{\partial x} &= \frac{\partial}{\partial x}\left(f_{nx}g_{nx} - f_{ny}g_{ny}\right) \\ &= \frac{\partial f_{nx}}{\partial x}g_{nx} + \frac{\partial g_{nx}}{\partial x}f_{nx} - \frac{\partial f_{ny}}{\partial y}g_{ny} - \frac{\partial g_{ny}}{\partial y}f_{ny} \\ \frac{\partial r_{ny}}{\partial y} &= \frac{\partial}{\partial y}\left(f_{nx}g_{ny} + f_{ny}g_{nx}\right) \\ &= \frac{\partial f_{nx}}{\partial y}g_{ny} + \frac{\partial g_{ny}}{\partial y}f_{nx} + \frac{\partial f_{ny}}{\partial y}g_{nx} + \frac{\partial g_{nx}}{\partial y}f_{ny} \end{aligned} \tag{48}$$

Then, we take adventage from the fact that g_n is complex-differentiable:

$$\begin{aligned} \frac{\partial g_{nx}}{\partial x} - \frac{\partial g_{ny}}{\partial y} &= 0 \\ \frac{\partial g_{nx}}{\partial y} + \frac{\partial g_{ny}}{\partial x} &= 0 \end{aligned} \tag{49}$$

Therefore we nd:

$$\left|\frac{\partial r_{nx}}{\partial x} - \frac{\partial r_{ny}}{\partial y}\right| \leq \left|\frac{\partial f_{nx}}{\partial x} - \frac{\partial f_{ny}}{\partial y}\right||g_{nx}| + \left|\frac{\partial f_{nx}}{\partial y} + \frac{\partial f_{ny}}{\partial x}\right||g_{ny}| \tag{50}$$

Exactly the same phenomenon happens for $\left|\frac{\partial r_{nx}}{\partial y} + \frac{\partial r_{nx}}{\partial y}\right|$, and we can write:

$$\begin{aligned} \left|\frac{\partial r_{nx}}{\partial x} - \frac{\partial r_{ny}}{\partial y}\right| &\underset{n\to\infty}{=} \frac{1}{(n+1)^x}\left|\frac{1-x}{1-z}\right| + o(\frac{1}{(n+1)^x}) \\ \left|\frac{\partial r_{nx}}{\partial y} + \frac{\partial r_{nx}}{\partial y}\right| &\underset{n\to\infty}{=} \frac{1}{(n+1)^x}\left|\frac{1-x}{1-z}\right| + o(\frac{1}{(n+1)^x}) \end{aligned} \tag{51}$$

This is enough to conclude that c in holomorphic over Ω. In addition, it coincides with ζ over ω which is an open set, therefore we nally have:

$$c(z) = \zeta(z), z \in \Omega \tag{52}$$

2.6 Discussion one the radial convergence

This summation procedure can remind the reader the link between the Riemann z ta function, and the Dirichlet ta function [3]:

$$\begin{aligned} \eta(z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, x > 0 \\ \zeta(z) &= \frac{1}{1-2^{1-z}} \eta(z), z \neq 1, x > 0 \end{aligned} \tag{53}$$

Like the radial convergence, this de nition is valid for $x > 0$. However, there is a much more interesting observation to make here.

Indeed, applying Ces ro [4] summation on the original Dirichlet series makes it converge for $x > -1$. Applying it again will have the series to converge for $x > -2$. From here, one can show by inference that the Dirichlet series can be analytically extended to the whole complex plan by applying Ces ro summation in nitely many times.

It seems that we have exactly the same kind of property for the radial convergence applied on the Riemann series. This result is purely numerical and will not be proven here. But it appears that applying the radial convergence in nitely many times on the Riemann series makes it converge over the whole complex plan, private from the real axis.

An observation that is consistent with this idea is the following fact: For $x = 0$, one can observe that the sequence $\{c_n(z)\}_{z \in \mathbb{N}}$ converges to an asymptotic circle.

3 Link with the non-trivial zeros of the Riemann z ta function

In this section, Ω is now the critical strip:

$$\Omega = \{x + iy, 0 < x < 1, y \in \mathbb{R}^*\} \tag{54}$$

We are now going to change our point of view on $c_n(z)$. Because it is the intersection of two straight lines in the complex plan, we can see $c_n(z)$ as the solution of the following system:

$$\begin{bmatrix} \Re c_n(z) \\ \Im c_n(z) \end{bmatrix} A_n = B_n \tag{55}$$

With:

$$A_n = \begin{bmatrix} \Im \frac{e^{i\alpha}}{(n+1)^z} & -\Re \frac{e^{i\alpha}}{(n+1)^z} \\ \Im \frac{e^{i\alpha}}{(n+2)^z} & -\Re \frac{e^{i\alpha}}{(n+2)^z} \end{bmatrix}, B_n = \begin{bmatrix} \Re \zeta_n(z) \Im \frac{e^{i\alpha}}{(n+1)^z} - \Im \zeta_n(z) \Re \frac{e^{i\alpha}}{(n+1)^z} \\ \Re \zeta_{n+1}(z) \Im \frac{e^{i\alpha}}{(n+2)^z} - \Im \zeta_{n+1}(z) \Re \frac{e^{i\alpha}}{(n+2)^z} \end{bmatrix} \tag{56}$$

We can now see explicitly that this system always has solutions:

$$det(A_n) = \frac{\sin(-y \ln(\frac{n+2}{n+1}))}{((n+1)(n+2))^x} \neq 0 \tag{57}$$

The complete expression of A_n^{-1} is:

$$A_n^{-1} = \frac{((n+1)(n+2))^x}{\sin(-y \ln(\frac{n+2}{n+1}))} \begin{bmatrix} -\Re \frac{e^{i\alpha}}{(n+2)^z} & \Re \frac{e^{i\alpha}}{(n+1)^z} \\ -\Im \frac{e^{i\alpha}}{(n+2)^z} & \Im \frac{e^{i\alpha}}{(n+1)^z} \end{bmatrix} \tag{58}$$

We then rewrite B_n as:

$$B_n(z) = \begin{bmatrix} b_n(z) \\ b_{n+1}(z) \end{bmatrix} \tag{59}$$

With:

$$b_n = \frac{|\zeta_n(z)|}{(n+1)^x} \sin\left(\arg\left(\frac{e^{i\alpha}}{(n+1)^z}\right) - \arg(\zeta_n(z))\right) \tag{60}$$

For $z \in \Omega$, it has already been proven that $|\zeta_n(z)|$ is unbounded. Therefore, considering the **Equation 55**, the following assumption seems very reasonable, though it would deserve a explicit proof.

If z is a zero of ζ over Ω, then we have:

$$\left|\sin\left(\arg\left(\frac{e^{i\alpha}}{(n+1)^z}\right) - \arg(\zeta_n(z))\right)\right| \underset{n\to\infty}{\to} 0 \tag{61}$$

Which with a bit of rearrangement can be formulated as the following theorem:

Theorem 3.1.

$$z \in \Omega, \zeta(z) = 0 \Rightarrow \lim_{n\to\infty} \frac{-y\ln(n+1) - \underset{sum}{\arg}(\zeta_n(z)) + \arctan(\frac{y}{1-x})}{\pi} = \mathcal{U}_z, \mathcal{U}_z \in \mathbb{Z} \tag{62}$$

With $\underset{sum}{\arg}(\zeta_n(z))$ *the cumulated argument of* $\zeta_n(z)$*:*

$$\underset{sum}{\arg}(\zeta_n(z)) = \sum_{j=1}^{n} \arg\left(\frac{\zeta_{j+1}(z)}{\zeta_j(z)}\right) \tag{63}$$

Again, the complete proof of this theorem is not given here, but the numerical results are very consistent.

Let us call this integer $\mathcal{U}_z$ when it is de ned. For the known zeros of the Riemann z ta function, we can numerically compute this value. Here are the rst 30 values for the 30 rst non trivial zeros:

We can observe these values are all even, But there are no clear pattern in their distribution despite this fact. The python code used to generate this table of values is on the GitHub. This sequence does not appear yet on: https://oeis.org/ [5].

Let us call:

$$\theta_n(z) = \arg\frac{\zeta_{n+1}(z)}{\zeta_n(z)} \tag{64}$$

In the gure **Figure 9**, we also introduce $\gamma_n(z)$ accordingly. One can show by induction that this $\gamma_n(z)$ value is actually:

$$\gamma_n(z) = -y\ln(n+1) - \underset{sum}{\arg}(\zeta_n(z)) \tag{65}$$

Now, using geometrical considerations, it is possible to give the following induction formula for $\sin(\theta_n(z))$:

$$\sin(\theta_{n+1}(z)) = \frac{\sin(-y\ln(\frac{n+2}{n+1}) + \gamma_n(z))}{(n+2)^x\sqrt{\frac{\sin(-y\ln(\frac{n+1}{n})+\gamma_{n-1}(z))^2}{(n+1)^{2x}\sin(\theta_n(z))^2} + \frac{1}{(n+2)^{2x}} + \frac{2\sin(-y\ln(\frac{n+1}{n})+\gamma_{n-1}(z))}{(n+1)^x(n+2)^x\sin(\theta_n(z))}cos(-y\ln(\frac{n+2}{n+1}) + \gamma_n(z))}} \tag{66}$$

This formula does not generalize nicely by induction. That is why the quantity $\underset{sum}{\arg}(\zeta_n(z))$ is not easy to compute.

However, from a geometrical point of view, because the Riemann series is notv bounded if z is in the critical strip, one can remark that we must have $\gamma_n(z)$ to converge to $\alpha(z)$ modulo 2π as n goes to in nity. This interpretation gives an

n	y(n)	U(1/2-i*y(n))
1	14.13472514173469379045725198356247 66	8
2	21.02203963877155499262847959389691 62	14
3	25.01085758014568876321379099256277 34	18
4	30.42487612585951321031189753058395 71	24
5	32.93506158773918969066236896407474 18	28
6	37.58617815882567125721776348070529 84	32
7	40.91871901214749518739812691463342 47	38
8	43.32707328091499951949612216540684 56	40
9	48.00515088116715972794247274942766 36	46
10	49.77383247767230218191678467856383 67	48
11	52.97032147771446064414729660888082 16	52
12	56.44624769706339480436775947670603 21	56
13	59.34704400260235307965364867499217 59	60
14	60.83177852460980984425990182452408 15	64
15	65.11254404808160666087505425318360 72	68
16	67.07981052949417371447882889652207 00	72
17	69.54640171117397925292685752655465 86	76
18	72.06715767448190758252210796982611 75	78
19	75.70469069908393316832691676203054 04	84
20	77.14484006887480537268266485630469 25	88
21	79.33737502024936792276359287711605 78	90
22	82.91038085408603018316483749477055 99	94
23	84.73549298051705010573531120682755 69	96
24	87.42527461312522940653166785091913 51	100
25	88.80911120763446542368234807950951 25	104
26	92.49189927055848429625972524181049 65	108
27	94.65134404051988696659792581520796 45	110
28	95.87063422824530975874102921924667 18	114
29	98.83119421819369223332442013862235 39	118
30	101.31785100573139122878544794029236 1	122

Figure 8: Values of $\mathcal{U}_z$ for the rst 30 non trivial zeros of zeta

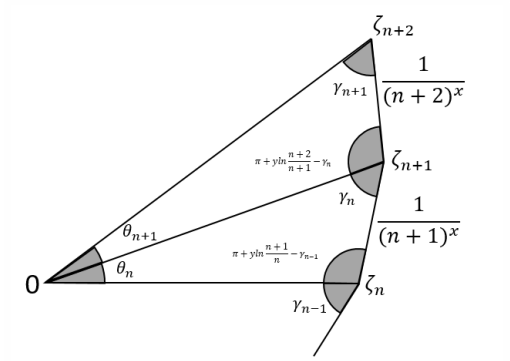

Figure 9: Geometrical view of the situation for $\theta_n(z)$

explanation to why $\mathcal{U}_z$ is always even. One can remark it is in reality well de ned all over the critical strip, though its convergence rate appears to be slower. From here, we can plot $\mathcal{U}_z$ for $x = \frac{1}{2}$ (**Figure 10**).

From this plot, we can clearly see that the zeros of the Riemann z ta function play a special role is the distribution of the values of $\mathcal{U}_z$.

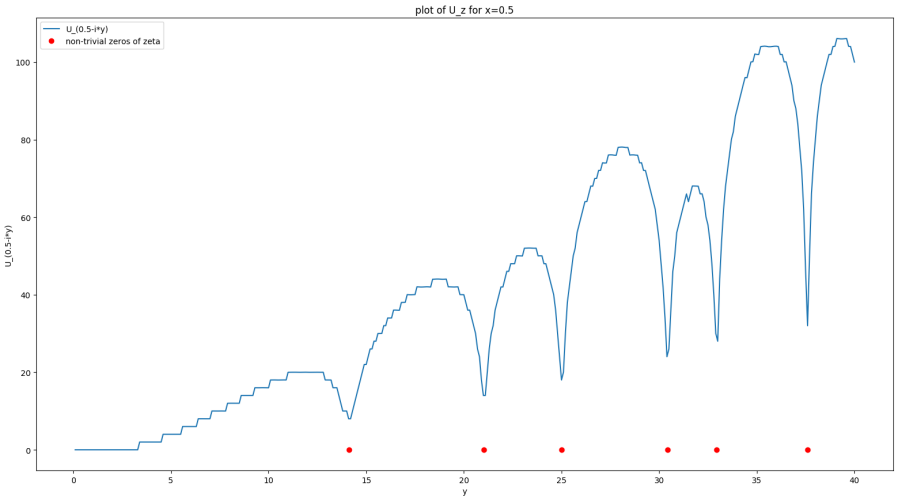

Figure 10: Plot of $\mathcal{U}_z$ for $z = \frac{1}{2} - y$

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