

2-regular Digraphs of the Lovelock Lagrangian

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The manuscripts tabulates arc lists of the 1, 1, 3, 8, 25, 85, 397 . . . unlabeled 2-regular digraphs on $n = 0, 1, 2, \dots, 9$ nodes, including disconnected graphs, graphs with multiarcs and/or graphs with loops. Each of these graphs represents one term of the Lagrangian of Lovelock's type — a contraction of a product of n Riemann tensors — once the 2 covariant and 2 contravariant indices of a tensor are associated with the in-edges and out-edges of a node.

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I. REGULAR DIGRAPHS

A. Nomenclature

In the common language of graph theory, digraphs (directed graphs) are graphs where the edges (called arcs) are oriented, i.e., the two nodes of an arc are distinguished to be tail and head of the arc. The indegree of a node is the number of arcs that have heads at a node; the outdegree is the number of arcs that have tails at a node. k -regular digraphs are digraphs where the indegree and outdegree at each node is the same k [1, 2]. Loops are arcs where head and tail are the same node. (Each loop increases the indegree and outdegree of the node by one.) Multiarcs are multisets of two or more arcs that share a head and tail node.

The underlying simple graph of a digraph is obtained by reducing the arcs to undirected edges, replacing multiedges by single edges, and removing loops. We will be only interested in these simple graphs to classify the digraphs by the number of components of their underlying simple graphs. (That means according to *weak* connectivity).

(Vertex) labeled graphs have distinct labels at their nodes, usually taken to be the positive integers while handling graphs on computers, or letters from a onwards. We define the Adjacency Matrix of a digraph of nodes labeled $1, 2, \dots$ as the square matrix which contains in row r and column c the number of arcs which start at node r and end at node c . Construction of the labeled k -regular digraphs admitting multiarcs and loops is therefore equivalent to constructing the $n \times n$ matrices with non-negative integer entries where all row sums and all column sums are k . These are registered in Table A008300 in the Online Encyclopedia of Integer Sequences for the cases where multiarcs are *not* admitted, and in Table A257493 if multiarcs are admitted [3].

Denote the number of unlabeled and labeled k -regular digraphs with n nodes and c components by $U_k(n, c)$ and $L_k(n, c)$.

The unlabeled graphs with $c > 1$ components can be derived from the connected graphs quickly by the Multiset Transformation, collecting arrangements over all partitions of n into c parts [4, Theor. I.1][5, (4.2.3)]:

$$U_k(n, c) = \sum_{\substack{n=n_1+2n_2+\dots+cn_c \\ n_i \geq 0}} \prod_{i=1}^c \binom{U_k(i, 1) + n_i - 1}{n_i}. \quad (1)$$

The number of unlabeled and labeled k -regular digraphs with n nodes is

$$U_k(0) = 1; \quad U_k(n) \equiv \sum_{c=1}^n U_k(n, c); \quad (2)$$

$$L_k(0) = 1; \quad L_k(n) \equiv \sum_{c=1}^n L_k(n, c). \quad (3)$$

The case of the unlabeled 1-regular digraphs is simple: the connected 1-regular digraphs are cycles (unicycles), including the case with 1 node and its loop:

$$U_1(n, 1) = 1. \quad (4)$$

So the number of unlabeled, not necessarily connected, 1-regular digraphs on n nodes, $U_1(n)$, is the number of partitions of n [3, A000041], and the $U_1(n, c)$ are the partition numbers [3, A008284].

B. Symmetries

The key part of this work is to identify the Automorphism Group of the labeled 2-regular digraphs, which is the group of permutations of the labels which keeps the Adjacency Matrix of a graph the same. This bundles a set of one or more labeled digraphs which are mapped onto each other by the permutations of the group, and each such set is represented by a single unlabeled 2-regular digraph. This is one way of erasing/forgetting any particular order on the nodes while maintaining the connectivity and structure of the graph.

The unlabeled 2-regular digraphs might also be obtained by starting from the cubic QED vacuum polarization diagrams [3, A170946][6], coalescing each edge

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that represents an undirected photon line and its two incident nodes into a single node such that only the directed fermion lines connect to the remaining nodes, plus a cleanup that eliminates duplicates.

We shall keep track of the Automorphism Group \mathcal{A} of each graph by writing down the Cycle Index of the group of the node permutations [5, §2]. Since \mathcal{A} is a subgroup of the group of all permutations of n , its order, $|\mathcal{A}|$, is a divisor of $n!$. The denominator of the cycle index polynomial is $|\mathcal{A}|$, so we can recover the number of labeled digraphs by dividing $n!$ through the denominator of the Cycle Index [5, (1.1.3)]. For $n = 3$ nodes, for example, $U_2(3) = 8$ unlabeled 2-regular digraphs exist with 4 different Cycle Indices:

- 1 graph with cycle index $(t_1^3)/1$,
- 3 graphs with cycle index $(t_1^3 + t_1 t_2)/2$,
- 2 graphs with cycle index $(t_1^3 + 2t_3)/3$,
- and 2 graphs with cycle index $(t_1^3 + 3t_1 t_2 + 2t_3)/6$,

and $1 \times 3!/1 + 3 \times 3!/2 + 2 \times 3!/3 + 2 \times 3!/6 = 21 = L_2(3)$ is the number of labeled graphs on 3 nodes [3, A000681]. As we are admitting loops, there is always at least one graph on n nodes (the one consisting of n isolated nodes with two loops each) that has the maximum symmetry here, $|\mathcal{A}| = n!$.

The labeled graphs with $c > 1$ components are deduced from the weakly connected labeled graphs by a Bell transformation, summing over all compositions of n into positive parts weighted by multinomial coefficients [7, EFJ]:

$$L_k(n, c) = \frac{1}{c!} \sum_{\substack{n=n_1+n_2+\dots+n_c \\ n_i \geq 1}} \binom{n}{n_1, n_2, \dots, n_c} \prod_{i=1}^c L_k(n_i, 1). \quad (5)$$

If

$$L_k(x, 1) \equiv \sum_{n \geq 1} \frac{L_k(n, 1)}{n!} x^n \quad (6)$$

denotes the exponential generating function of the weakly connected labeled graphs, the bivariate exponential generating function of the labeled graphs is [8]

$$L_k(x, t) \equiv \sum_{n, c \geq 0} \frac{L_k(n, c)}{n!} x^n t^c = \exp[tL_k(x, 1)]. \quad (7)$$

The corresponding derivation starting from $L_1(n, 1) = 1$ demonstrates that $L_1(n, c)$ are the Stirling Numbers of the Second Kind [3, A008277] and $L_1(n)$ the Bell Numbers [3, A000110].

For L_2 , the number of labeled digraphs refined by the number of weakly connected components is summarized in Table I.

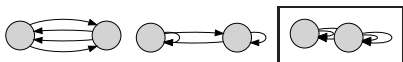
II. GALLERY OF UNLABELED 2-REGULAR DIGRAPHS

The unlabeled 2-regular digraphs on $n \leq 5$ nodes are illustrated in the following sections. For easier visual recognition, the graphs with more than one component are surrounded by a frame.

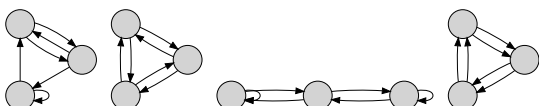
A. 1 graph on 1 node



B. 3 graphs (2 connected) on 2 nodes

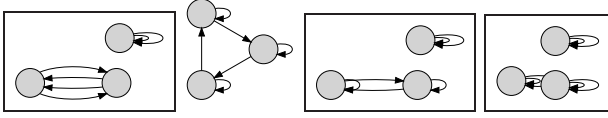


C. 8 graphs (5 connected) on 3 nodes

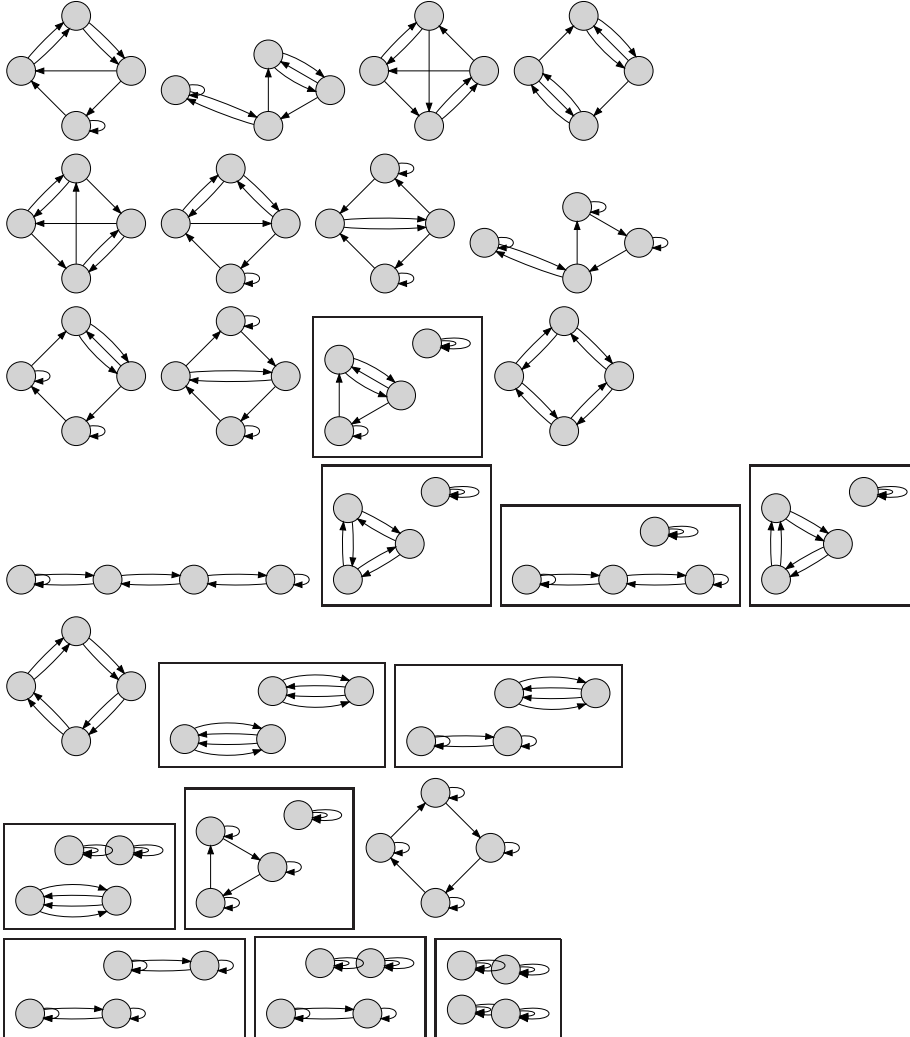


$n \setminus c$	1	2	3	4	5	6	7	$L_2(n)$
1	1							1
2	2	1						3
3	14	6	1					21
4	201	68	12	1				282
5	4704	1285	200	20	1			6210
6	160890	36214	4815	460	30	1		202410
7	7538040	1422288	160594	13755	910	42	1	9135630

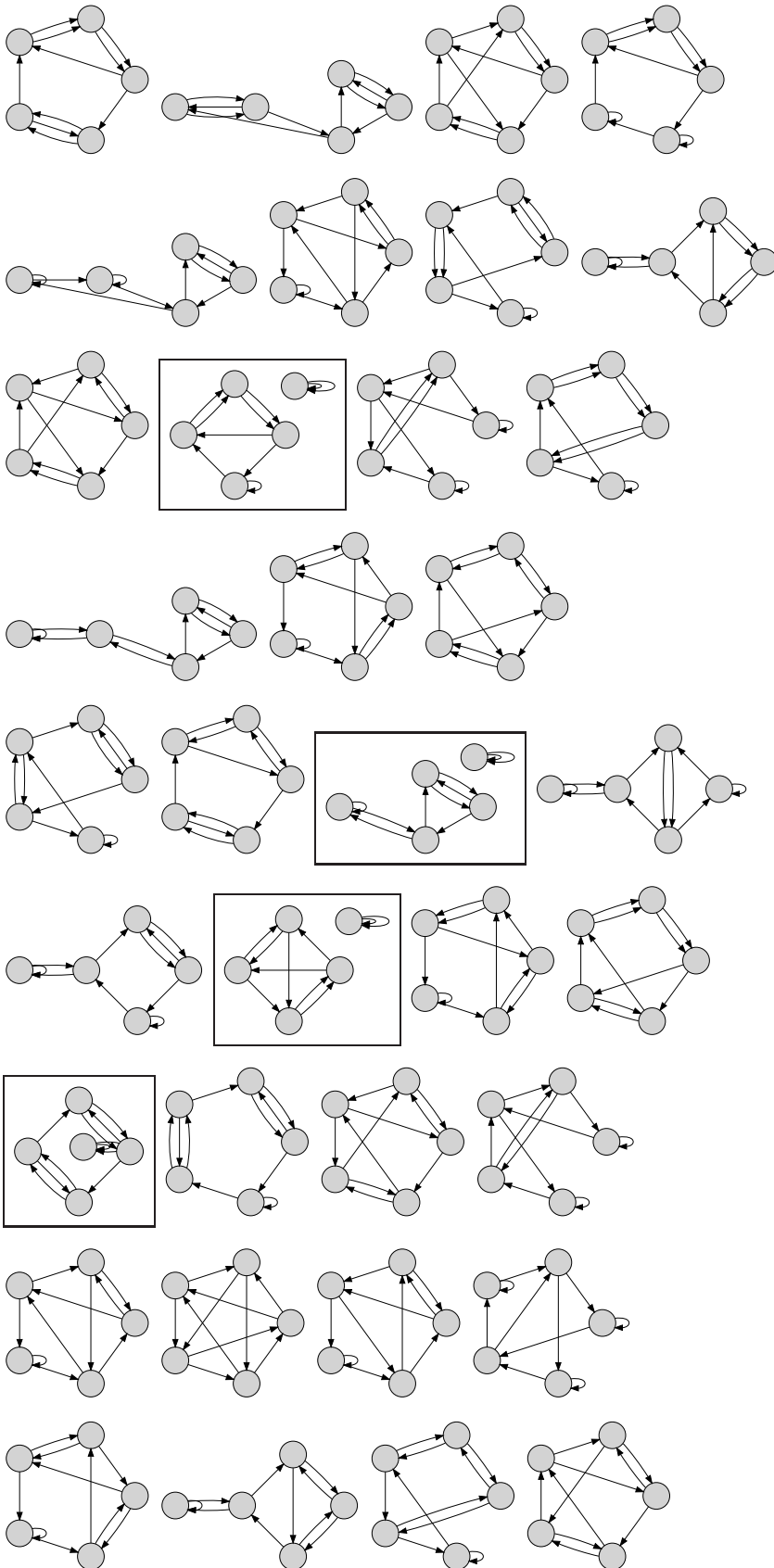
TABLE I. Labeled 2-regular digraphs $L_2(n, c)$ with n nodes and c (weak) components, allowing loops and multiarcs [3, A307804].

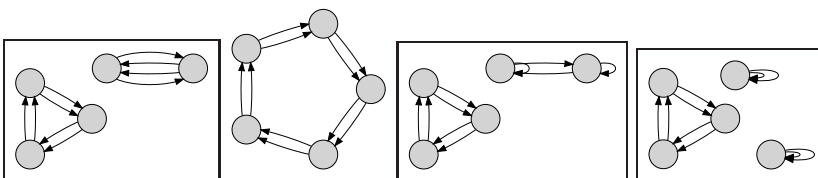
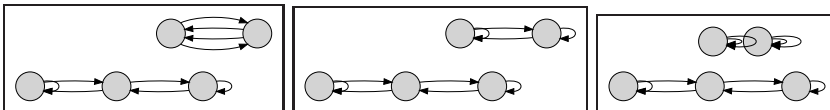
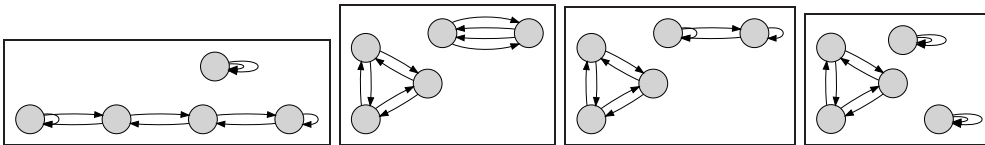
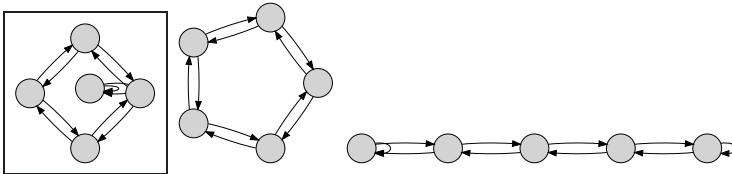
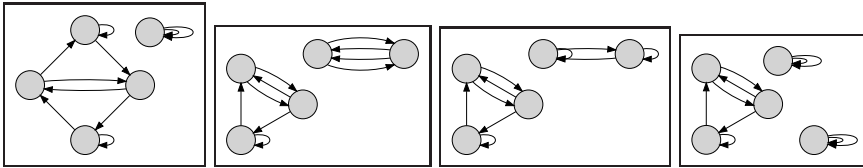
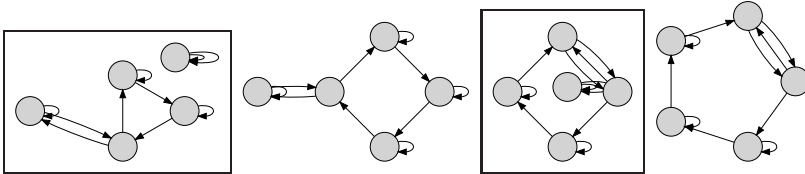
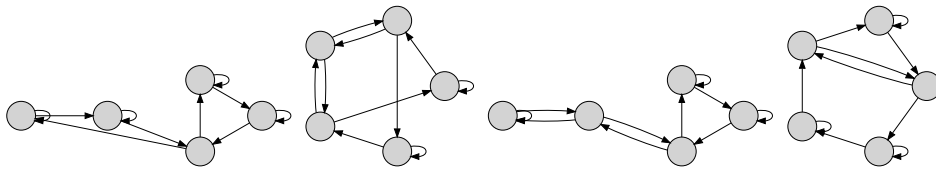
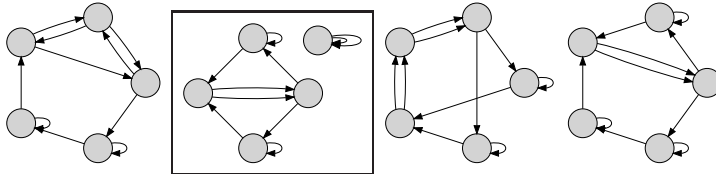
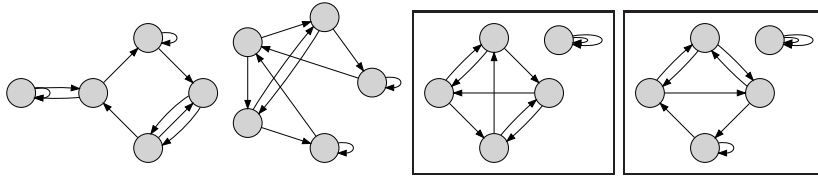
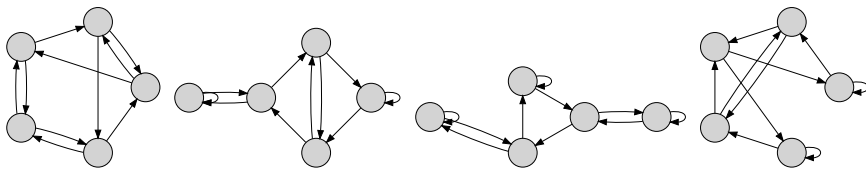


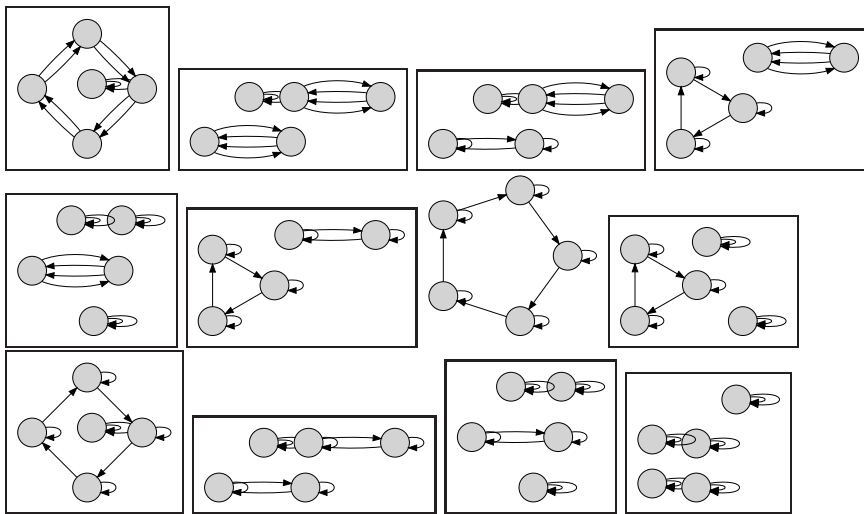
D. 25 graphs (14 connected) on 4 nodes



E. 85 graphs (50 connected) on 5 nodes







III. BIJECTION WITH LOVELOCK TERMS

Bogdanos has pointed at a graphical representation of the terms of Lovelock's Lagrange density [9–11]: A graph of a particular term which is a product of n Riemann Tensors is constructed as follows: (i) Start with a bottom row of n nodes plus a top row of another n nodes, where nodes are associated left-to-right with a left-to-right reading of the R -factors of the product. (ii) Add an edge from the b -th bottom node to the t -th top node if a covariant lower index of factor number b appears as a contravariant index of factor number t . This assignment works because the two contravariant indices are permutations of the covariant indices. Since the Riemann Tensor is (skew)-symmetric in the first and in the last two indices, the representation does not need to track the two edges at each node to distinguish a “first” from a “second.” Bogdanos' representation is a labeled, undirected 2-regular graph on $2n$ nodes.

Our graphical representation transforms these bipartite graphs once more [12]:

- Bogdanos' edges are turned into directed edges (arcs), always heading from a node of the bottom row to a node on the top row.
- Each pair of nodes associated with the same R -factor is coalesced into a single node, keeping all arcs fastened to their nodes. Loops appear if the R -factors were already contracted (Ricci tensors).
- Labels are erased, meaning that the left-to-right reading orders of the product of the R are all equivalent reflecting the usual commutative law for multiplications. The multiplicity may be recovered by examining the Automorphism Group of the new graph

This transformation of Bogdanos' bipartite graphs on $2n$ nodes to our 2-regular digraphs on n nodes is lossless (reversible).

IV. SUMMARY

Table II summarizes the bare counts $U_2(n, c)$.

$n \setminus c$	1	2	3	4	5	6	7	8	9	$U_2(n)$
1	1									1
2	2	1								3
3	5	2	1							8
4	14	8	2	1						25
5	50	24	8	2	1					85
6	265	93	28	8	2	1				397
7	1601	435	108	28	8	2	1			2183
8	11984	2486	507	113	28	8	2	1		15129
9	101884	17211	2811	527	113	28	8	2	1	122585

TABLE II. Unlabeled 2-regular digraphs $U_2(n, c)$ with n nodes and c (weak) components, allowing loops and multi-arcs [3, A306892, A006372].

The row sums 1, 3, 8, 25, ... count the graphs with any number of components. $U_2(4)$ and $U_2(5)$ have already been computed by Briggs [13, 14]. We observe that Briggs' extrapolations to more than 5 nodes [15] underestimate the true number of graphs for 6 – 9 nodes.

Appendix A: Machine Readable Tables

The ancillary directory contains the information of the unlabeled 2-regular digraphs in files named `Reg n .txt`, where n is the number of nodes. Due to file size constraints of the arXiv, `Reg9.txt` is not included. Each file contains two successive lines per graph:

1. An arc list for $2n$ arcs, referring to a representative of the labeled graphs created by the \mathcal{A} -group, with labels from 0 up to $n - 1$, in square brackets. Each bracket contains a pair of numbers, separated by

a comma; the first is the tail node and second the head node of the arc. Multiarcs are rendered by repeating such pairs.

2. A capital V , the number of multiarcs in the graph (i.e., the number of entries larger than 1 in the Adjacency Matrix), a blank, the trace of the Adjacency Matrix (i.e., the number of loops), a blank, and the Pólya Cycle Index (a multinomial in the free variables t_i).

The top-bottom order of the graphs in these files is the same as the top-down-left-right reading order of the pictures shown above.

Filtering the lines that start with $V0$ in these files we obtain 1, 3, 8, 27, . . . unlabeled, not necessarily connected, 2-regular digraphs with $n \geq 2$ nodes without multiarcs [3, A005641].

Filtering the lines that start with $V0_0$ in these files we obtain 1, 2, 5, 23, . . . unlabeled, loopless, not necessarily connected, 2-regular digraphs with $n \geq 3$ nodes without multiedges [3, A219889].

As a further check, filtering the lines that contain $_0$ we obtain 1, 2, 6, 15, 68, . . . graphs on $n \geq 2$ nodes without loops (which may have multiarcs) [3, A307180].

One application of this information yields the r -rooted unlabeled 2-regular digraphs by defining the generating function $r(x) = 1 + x$ for the number of ways of labelling a vertex as 0 (not marked) or 1 (marked), and then substituting $t_i \rightarrow r(x^i)$ in the cycle indices. Summation over the cycle indices of all graphs of fixed n generates Table III.

$n \setminus r$	0	1	2	3	4	5	6
1	1	1					
2	3	3	3				
3	8	13	13	8			
4	25	58	88	58	25		
5	85	310	588	588	310	85	
6	397	1909	4626	6035	4626	1909	397
7	2183	13843	40417	66471	66471	40417	13843
8	15129	114821	395324	782257	975715	782257	395324

TABLE III. Unlabeled 2-regular digraphs $U_2^{(r)}(n) = U_2^{(n-r)}(n)$ with n nodes and $0 \leq r \leq n$ rooted vertices, allowing loops and multiarcs. $U_2^{(0)}(n) = U_2(n)$.

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