A Laplacian to compute intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and correlation functions in NCQFT

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Abstract
Let $F_g(t)$ be the generating function of intersection numbers on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of complex curves of genus g. As by-product of a complete solution of all non-planar correlation functions of the renormalised Φ^3-matrical QFT model, we explicitly construct a Laplacian Δ_t on the space of formal parameters t_i satisfying $\exp(\sum_{g\geq 2} N^{2-2g}F_g(t)) = \exp((-\Delta_t + F_2(t))/N^2)1$ for any $N > 0$. The result is achieved via Dyson-Schwinger equations from noncommutative quantum field theory combined with residue techniques from topological recursion. The genus-g correlation functions of the Φ^3-matricial QFT model are obtained by repeated application of another differential operator to $F_g(t)$ and taking for t_i the renormalised moments of a measure constructed from the covariance of the model.

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1. Advertisement

This paper completes the reverse engineering of a special quantum field theory on noncommutative geometries. The final step could be of interest in other areas of mathematics:

Theorem 1.1. *Let*

$$F_g(t_0, t_2, t_3, \dots, t_{3g-2}) := \sum_{(k)} \frac{\langle \tau_2^{k_2} \tau_3^{k_2} \dots \tau_{3g-2}^{k_{3g-2}} \rangle}{(1-t_0)^{2g-2+\sum_i k_i}} \prod_{i=2}^{3g-2} \frac{t_i^{k_i}}{k_i!}, \quad \sum_{i\geq 2}(i-1)k_i = 3g-3,$$

be the generating function of intersection numbers of ψ-classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of complex curves of genus g [Wit91, Kon92]. For any $N > 0$, the stable partition function satisfies

$$\boxed{\exp\Big(\sum_{g=2}^{\infty} N^{2-2g} F_g(t)\Big) = \exp\Big(-\frac{1}{N^2}\Delta_t + \frac{F_2(t)}{N^2}\Big)1} \tag{1.1}$$

where $F_2(t) = \dfrac{7}{240}\cdot\dfrac{t_2^3}{3!T_0^5} + \dfrac{29}{5760}\dfrac{t_2t_3}{T_0^4} + \dfrac{1}{1152}\dfrac{t_4}{T_0^3}$ *with* $T_0 := (1-t_0)$ *generates the intersection numbers of genus 2 and* $\Delta_t = -\sum_{i,j}\hat{g}^{ij}\partial_i\partial_j - \sum_i \hat{\Gamma}^i\partial_i$ *is a Laplacian on the formal parameters* $t_0, t_2, t_3, \dots$ *given by*

$$
\begin{aligned}
\Delta_t :=& -\Big(\frac{2t_2^3}{45T_0^3} + \frac{37t_2t_3}{1050T_0^2} + \frac{t_4}{210T_0}\Big)\frac{\partial^2}{\partial t_0^2} - \Big(\frac{2t_2^3}{27T_0^4} + \frac{1097t_2t_3}{12600T_0^3} + \frac{41t_4}{2520T_0^2}\Big)\frac{\partial}{\partial t_0} \\
&- \sum_{k=2}^{\infty}\Big(\Big(\frac{2t_2^2}{45T_0^3} + \frac{2t_3}{105T_0^2}\Big)t_{k+1} + \frac{t_2\mathcal{R}_{k+1}(t)}{2T_0} + \frac{3\mathcal{R}_{k+2}(t)}{2(3+2k)}\Big)\frac{\partial^2}{\partial t_k\partial t_0} \\
&- \sum_{k,l=2}^{\infty}\Big(\frac{t_2t_{k+1}t_{l+1}}{90T_0^2} + \frac{t_{k+1}\mathcal{R}_{l+1}(t)}{4T_0} + \frac{t_{l+1}\mathcal{R}_{k+1}(t)}{4T_0} + \frac{(1{+}2k)!!(1{+}2l)!!\mathcal{R}_{k+l+1}(t)}{4(1{+}2k{+}2l)!!}\Big)\frac{\partial^2}{\partial t_k\partial t_l} \\
&- \sum_{k=2}^{\infty}\Big(\Big(\frac{19t_2^2}{540T_0^4} + \frac{5t_3}{252T_0^3}\Big)t_{k+1} + \frac{t_2\mathcal{R}_{k+1}(t)}{48T_0^2} + \frac{\mathcal{R}_{k+2}(t)}{16(3+2k)T_0} + \frac{t_2t_{k+2}}{90T_0^3} + \frac{\mathcal{R}_{k+2}(t)}{2T_0}\Big)\frac{\partial}{\partial t_k}
\end{aligned}
$$

with $\mathcal{R}_m(t) := \dfrac{2}{3}\displaystyle\sum_{k=1}^{m}\frac{(2m{-}1)!!\,kt_{k+1}}{(2k{+}3)!!T_0}\sum_{l=0}^{m-k}\frac{l!}{(m-k)!}B_{m-k,l}\Big(\Big\{\frac{j!t_{j+1}}{(2j{+}1)!!T_0}\Big\}_{j=1}^{m-l+1}\Big).$

The $F_g(t)$ are recursively extracted from $\mathcal{Z}_g(t) := \frac{1}{(g-1)!}(-\Delta_t + F_2(t))^{g-1}1$ and

$$
\begin{aligned}
F_g(t) &= \mathcal{Z}_g(t) - \tfrac{1}{(g-1)!}\textstyle\sum_{k=2}^{g-1} B_{g-1,k}\big(\{h!F_{h+1}(t)\}_{h=1}^{g-k}\big) \\
&= \mathcal{Z}_g(t) - \tfrac{1}{(g-1)!}\textstyle\sum_{k=2}^{g-1}(-1)^{k-1}(k-1)!B_{g-1,k}\big(\{h!\mathcal{Z}_{h+1}(t)\}_{h=1}^{g-k}\big).
\end{aligned}
$$

Here and in Theorem 1.1, $B_{m,k}(\{x\})$ are the Bell polynomials (see Definition 4.9). These equations are easily implemented in any computer algebra system.

Theorem 1.1 seems to be closely related with $\exp(\sum_{g\geq 0}F_g) = \exp(\hat{W})1$ proved by Alexandrov [Ale11][1], where $\hat{W} := \frac{2}{3}\sum_{k=1}^{\infty}(k+\frac{1}{2})t_k\hat{L}_{k-1}$ involves the generators $\hat{L}_n$ of the Virasoro algebra. Including N and moving $\exp(N^2F_0 + F_1)$ to the other side, our Δ_t is in principle obtained via Baker-Campbell-Hausdorff formula from Alexandrov's equation. Of course, evaluating the necessary commutators is not viable.

Theorem 1.1 suggests several fascinating questions which we haven't studied yet:

- Is $\hat{\Gamma}^i$ a Levi-Civita connection for $\hat{g}^{ij}$, i.e. $\hat{\Gamma}^i = \sum_j \hat{g}^{ij}\sqrt{\det\hat{g}^{-1}}\partial_j(\frac{1}{\sqrt{\det\hat{g}^{-1}}})$? Here $\det\hat{g}^{-1}$ would be the determinant of $(\hat{g}^{ij})$, whatever this means.
- Is there a reasonable definition of a volume $\int dt\frac{1}{\sqrt{\det\hat{g}^{-1}(t)}}$? If so, is there any relation to the Weil-Petersson volumina which are deeply connected with intersection numbers [AC96, Mir07]?
- Is it possible to reconfirm, maybe also to improve, the asymptotic estimates of Weil-Petersson volumina and intersection numbers [MZ15]?
- Is it possible to prove that $\sum_{g=2}^{\infty}N^{2-2g}F_g(t)$ is Borel summable for $t_l < 0$?

[1] We thank Gaëtan Borot for bringing this reference to our attention.

Theorem 1.1 is a by-product of our effort to construct the non-planar sector of the renormalised Φ^3_D-matricial quantum field theory in any dimension $D \in \{0, 2, 4, 6\}$. These are closely related to the Kontsevich model [Kon92] so that a link to intersection numbers is obvious. After an overview about the context we proceed with our construction. It is clear that a more streamlined proof of Theorem 1.1 in a better adapted setting can be given. This will be done elsewhere.

2. Introduction

Matrix models have a huge scope of applications [BIPZ78, DF04], ranging from combinatorics over 2D quantum gravity [GM90, DFGZJ95] up to quantum field theory on noncommutative spaces [LS02, LSZ04, GW05, GS06b, GS06a, GS08, DGMR07, GW14]. Of particular interest is the Kontsevich model [Kon92], which was designed to prove Witten's conjecture [Wit91] that the generating function of the intersection numbers of stable complex curves satisfies the KdV equations. See also [Wit92, LZ04, Eyn16].

More recent investigations of matrix models led to the discovery of a universal structure called *topological recursion* [CEO06, EO07]. Topological recursion was subsequently identified in many different areas of mathematics and theoretical physics [Eyn14, Eyn16]. The Kontsevich model itself satisfies topological recursion, even though it is not a matrix model in the usual sense (but related via Miwa transformation; see e.g. [ACKM93]).

On the other hand, renormalisation of quantum field theories on noncommutative geometries generically leads to matrix models similar to the Kontsevich model. The crucial difference is that convergence of all (usually) formal sums is addressed, and achieved by renormalisation [GS06b, GS06a, GS08]. Renormalisation is sensitive to the dimension encoded in the covariance of the matrix model. For historic reasons, namely the perturbative renormalisation [GW05] of the Φ^4-model on Moyal space and its vanishing β-function [DGMR07], also the quartic analogue of the Kontsevich model was intensely studied. In [GW14] the simplest topological sector was reduced to a closed equation for the 2-point function. This equation was recently solved for the covariance of 2D-Moyal space [PW18]. All correlation functions with simplest topology can be explicitly described [dJHW19].

In [GSW17, GSW18] these methods developed for the Φ^4-model were reapplied to the cubic (Kontsevich-type) model. The new tools, together with the Makeenko-Semenoff solution [MS91] of a non-linear integral equation, permitted an exact solution of all planar (i.e. genus-0) renormalised correlation functions in dimension $D \in \{2, 4, 6\}$. In particular, exact (and surprisingly compact) formulae for planar correlation functions with $B \geq 2$ boundary components were obtained. The simplicity of the formulae [GSW17] for $B \geq 2$ suggests an underlying pattern. It is traced back to the universality phenomena captured by topological recursion[2]. We refer to the book [Eyn16].

In this article we give the complete description of the non-planar sector of the renormalised Φ^3_D-model. The notation defined in [GSW18] will be used and recalled in sec. 3. We borrow from topological recursion the notational simplification to complex variables z for the previous $\sqrt{X + c}$ and the vision that the correlation functions are holomorphic

[2] We thank Roland Speicher for the hint that there might be a relation between our work and topological recursion.

in $z \in \mathbb{C} \setminus \{0\}$. Knowing this, we proceed however in a different way. Our main tool is a boundary creation operator $\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}$ which, when applied to a genus-g correlation function $\mathcal{G}_g(z_1|\dots|z_{B-1})$ with $B-1$ boundaries labelled $z_1,\dots,z_{B-1}$, creates a B^{th} boundary labelled z_B. The existence of such an operator is suggested by the 'loop insertion operator' in topological recursion [Eyn16], but their precise relationship is not entirely clear to us.

We rely on the sequence $\{\varrho_l\}_{l\in\mathbb{N}}$ of moments of a measure arising from the renormalised planar 1-point function [GSW17, GSW18]. This sequence is uniquely defined by the renormalised covariance of the model, the renormalised coupling constant and the dimension $D \in \{2,4,6\}$. The boundary creation operator acts on Laurent polynomials in the z_i with coefficients in rational functions of the ϱ_l. The heart of this paper is a combinatorial proof, independent of topological recursion, that the boundary creation operator does what it should (Theorem 4.6, portioned into Lemmata proved in an appendix). It is then (to our taste) considerably easier compared with topological recursion to derive structural results about the $\mathcal{G}_g(z_1|\dots|z_B)$ such as the degree of the Laurent polynomials, the maximal number of occurring $\{\varrho_l\}$ and the weight of the rational function.

By Dyson-Schwinger techniques we derive an equation of type $\hat{K}_z\mathcal{G}_g(z) = f(\mathcal{G}_h(z), \hat{\mathrm{A}}^{\dagger g-1}_{z,z}\mathcal{G}_{g-1}(z))$ for $h<g$, where f is a second-order polynomial and $\hat{K}_z$ an integral operator. Thus, all $\mathcal{G}_g(z)$ can be recursively evaluated if $\hat{K}_z$ can be inverted. Topological recursion tells us that the inverse is a residue combined with a special kernel operator. We give a direct combinatorial proof that the same method works in our case.

We easily show that the $\mathcal{G}_g(z)$ arise for $g \geq 1$ by application of the boundary creation operator to a uniquely defined 'free energy' $F_g(\varrho)$. These $F_g(\varrho)$ are characterised by 'only' $p(3g-3)$ rational numbers, where $p(n)$ is the number of partitions of n. The above second-order polynomial f can be written as a second-order differential operator acting on $\exp(\sum_{g\geq 1} N^{2-2g}F_g)$ in which it is convenient to eliminate F_1. The result is Theorem 1.1 expressed in terms of $\varrho_0 = 1-t_0$ and $\varrho_l = -\frac{t_l+1}{(2l+1)!!}$. In other words, to construct the non-planar sector of the Φ^3_D-matricial QFT model one has to replace the formal parameters t_l in the generating function $F_g(t)$ of intersection numbers by precisely determined moments $\{\varrho_l\}$ resulting from the renormalisation of the planar sector of the model.

3. Summary of previous results

3.1. Setup

We are following the definitions in [GSW18] to generalise their results to higher genera in the same notation. Define the following tuple $\underline{n} = (n_1, n_2, .., n_{\frac{D}{2}}), n_i \in \mathbb{N}$ with the 1-norm $|\underline{n}| = n_1 + n_2 + .. + n_{\frac{D}{2}}$. The number of tuples $\underline{n}$ of norm $|\underline{n}|$ is $\binom{|\underline{n}|+\frac{D}{2}-1}{\frac{D}{2}-1}$. A further convention will be $\mathbb{N}^{D/2}_{\mathcal{N}} := \{\underline{m} \in \mathbb{N}^{D/2} : |\underline{m}| \leq \mathcal{N}\}$.

The action of the Φ^3_D-model is then defined by

$$S[\Phi] = V\Bigg(\sum_{\underline{n},\underline{m}\in\mathbb{N}^{D/2}_{\mathcal{N}}} Z\frac{H_{\underline{n}\underline{m}}}{2}\Phi_{\underline{n}\underline{m}}\Phi_{\underline{m}\underline{n}} + \sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}}(\kappa+\nu E_{\underline{n}}+\zeta E^2_{\underline{n}})\Phi_{\underline{n}\underline{n}} + \frac{\lambda_{bare}Z^{3/2}}{3}\sum_{\underline{n},\underline{m},\underline{k}\in\mathbb{N}^{D/2}_{\mathcal{N}}}\Phi_{\underline{n}\underline{m}}\Phi_{\underline{m}\underline{k}}\Phi_{\underline{k}\underline{n}}\Bigg), \tag{3.1}$$

where $H_{\underline{n}\underline{m}} := E_{\underline{n}} + E_{\underline{m}}$. The constant V is first of all a formal parameter; for a noncommutative quantum field theory model, $V = (\frac{\theta}{4})^{D/2}$ will be related to the deformation parameter of the Moyal plane. The parameters $\lambda_{bare}, \kappa, \nu, \zeta, Z$ and soon μ_{bare} are $\mathcal{N}$-dependent renormalisation parameters. They will be determined by normalisation conditions parametrised by physical mass μ and coupling constant λ. The matrices $(\Phi_{\underline{n}\underline{m}})$ are multi-indexed Hermitian matrices, $\Phi_{\underline{n}\underline{m}} = \overline{\Phi_{\underline{m}\underline{n}}}$. The external matrix $E = (E_{\underline{m}}\delta_{\underline{n},\underline{m}})$ can be assumed to be diagonal and has the eigenvalues $E_{\underline{n}} = \frac{\mu_{bare}^2}{2} + e\big(\frac{|\underline{n}|}{\mu^2 V^{2/D}}\big)$, where $e(x)$ is a monotonously increasing differentiable function with $e(0) = 0$ (on the noncommutative Moyal plane, $e(x) = x$).

The next step is to define the partition function

$$\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}\Phi \exp\left(-S[\Phi] + V\mathrm{Tr}(J\Phi)\right)\\
&= K \exp\Big(- \frac{\lambda_{bare} Z^{3/2}}{3V^2} \sum_{\underline{n},\underline{m},\underline{k}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^3}{\partial J_{\underline{n}\underline{m}}\partial J_{\underline{m}\underline{k}}\partial J_{\underline{k}\underline{n}}}\Big)\mathcal{Z}_{\text{free}}[J], \qquad (3.2)\\
\mathcal{Z}_{\text{free}}[J] &:= \exp\left(V \sum_{\underline{n},\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{(J_{\underline{n}\underline{m}} - (\kappa + \nu E_{\underline{n}} + \zeta E_{\underline{n}}^2)\delta_{\underline{m},\underline{n}})(J_{\underline{m}\underline{n}} - (\kappa + \nu E_{\underline{n}} + \zeta E_{\underline{n}}^2)\delta_{\underline{m},\underline{n}})}{2ZH_{\underline{n}\underline{m}}}\right),\\
K &:= \int \mathcal{D}\Phi \exp\Big(- VZ \sum_{\underline{n},\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{H_{\underline{n}\underline{m}}}{2}\Phi_{\underline{n}\underline{m}}\Phi_{\underline{m}\underline{n}}\Big),
\end{aligned}$$

where the source $(J_{\underline{n}\underline{m}})$ is a multi-indexed Hermitian matrix of rapidly decaying entries.

The correlation functions are defined as moments of the partition function. It turns out by earlier work [GW14] that the correlation functions expand into multi-cyclic contributions. It is therefore convenient to work with $\mathbb{J}_{\underline{p}_1\dots\underline{p}_{N_\beta}} := \prod_{j=1}^{N_\beta} J_{\underline{p}_j\underline{p}_{j+1}}$ with $\underline{p}_{N_\beta+1} \equiv \underline{p}_1$. Taking into account that genus-g correlation functions scale with V^{-2g} [BIPZ78, GW14], the following expansion of the partition function is obtained:

$$\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} =: \sum_{B=1}^{\infty}\sum_{1\le N_1\le\dots\le N_B}^{\infty}\sum_{\underline{p}_1^1,\dots,\underline{p}_{N_B}^B\in\mathbb{N}_{\mathcal{N}}^{D/2}}\sum_{g=0}^{\infty} V^{2-B-2g}\frac{G^{(g)}_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}}{S_{(N_1,\dots,N_B)}}\prod_{\beta=1}^{B}\frac{\mathbb{J}_{\underline{p}_1^\beta\dots\underline{p}_{N_\beta}^\beta}}{N_\beta}. \qquad (3.3)$$

We call the moment $G^{(g)}_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}$ an $(N_1 + \dots + N_B)$-point function of genus g; when the N_β do not matter, a correlation function of genus g with B boundary components.

Finally, we recall from [GSW18] the Ward-Takahashi identity for $|\underline{q}| \neq |\underline{p}|$

$$\begin{aligned}
\sum_{\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^2}{\partial J_{\underline{q}\underline{m}}\partial J_{\underline{m}\underline{p}}}\mathcal{Z}[J] = &\sum_{\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{V}{(E_{\underline{q}} - E_{\underline{p}})Z}\Big(J_{\underline{m}\underline{q}}\frac{\partial}{\partial J_{\underline{m}\underline{p}}} - J_{\underline{p}\underline{m}}\frac{\partial}{\partial J_{\underline{q}\underline{m}}}\Big)\mathcal{Z}[J]\\
&- \frac{V}{Z}(\nu + \zeta H_{\underline{p}\underline{q}})\frac{\partial\mathcal{Z}[J]}{\partial J_{\underline{q}\underline{p}}}. \qquad (3.4)
\end{aligned}$$

It arises from invariance of the partition function under unitary transformation $\Phi \mapsto U^\dagger\Phi U$ of the integration variable [DGMR07], or directly from the structure of $\mathcal{Z}_{free}[J]$ [HW18].

3.2. Dyson-Schwinger equation for $B = 1$

The Dyson-Schwinger equations are determined in [GSW18] for $g = 0$ and solved for all planar correlation functions. To be safe in using the results of [GSW18], we define

$$G_{|\underline{p}_1^1 \ldots \underline{p}_{N_1}^1|\ldots|\underline{p}_1^B \ldots \underline{p}_{N_B}^B|} := \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{|\underline{p}_1^1 \ldots \underline{p}_{N_1}^1|\ldots|\underline{p}_1^B \ldots \underline{p}_{N_B}^B|} \tag{3.5}$$

and the shifted 1-point function

$$\begin{aligned} \frac{W^{(g)}_{|\underline{p}|}}{2\lambda} &:= G^{(g)}_{|\underline{p}|} + \delta_{g,0}\frac{F_{\underline{p}}}{\lambda}, \quad W_{|\underline{p}|} := \sum_{g=0}^{\infty} V^{-2g} W^{(g)}_{|\underline{p}|}, \\ F_{\underline{p}} &:= E_{\underline{p}} - \frac{\lambda\nu}{2} = \frac{\mu^2}{2} + e\Big(\frac{|\underline{p}|}{\mu^2 V^{2/D}}\Big). \end{aligned} \tag{3.6}$$

Three relations between renormalisation parameters are immediate [GSW18]: $\lambda = Z^{1/2}\lambda_{bare}$, $\frac{\lambda\zeta}{Z} = 1 - \frac{1}{Z}$ and $\mu^2_{bare} = \mu^2 + \lambda\nu$. Now we can use all the Dyson-Schwinger equations evaluated in [GSW18]. The 1-point function satisfies [GSW18, eq. (3.12)]

$$(W_{|\underline{p}|})^2 + 2\lambda\nu W_{|\underline{p}|} + \frac{2\lambda^2}{V} \sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{W_{|\underline{p}|} - W_{|\underline{n}|}}{F_{\underline{p}}^2 - F_{\underline{n}}^2} + \frac{4\lambda^2}{V^2} G_{|\underline{p}|\underline{p}|} = \frac{4F_{\underline{p}}^2}{Z} + C, \tag{3.7}$$

where $C := -\frac{\lambda^2\nu^2(1+Z)+4\kappa\lambda}{Z}$. The convention (3.5) immediately gives the genus expansion:

$$\sum_{h+h'=g} W^{(h)}_{|\underline{p}|} W^{(h')}_{|\underline{p}|} + 2\lambda\nu W^{(g)}_{|\underline{p}|} + \frac{2\lambda^2}{V} \sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{W^{(g)}_{|\underline{p}|} - W^{(g)}_{|\underline{n}|}}{F_{\underline{p}}^2 - F_{\underline{n}}^2} + 4\lambda^2 G^{(g-1)}_{|\underline{p}|\underline{p}|} = \delta_{0,g}\Big(\frac{4F_{\underline{p}}^2}{Z} + C\Big). \tag{3.8}$$

We choose here that C has no expansion in $\frac{1}{V}$ because non-planar correlation functions do not need to be renormalised. There would be no problem treating a non-vanishing rhs in (3.8) at any g; only the formulae become clumsy.

3.3. Integral equations

Introducing the measure

$$\varrho(X) := \frac{2(2\lambda)^2}{V} \sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \delta(X - 4F_{\underline{n}}^2), \tag{3.9}$$

we can rewrite (3.8) as an integral equation. The measure has support in $[4F_{\underline{0}}^2, \Lambda_{\mathcal{N}}^2]$ where $\Lambda_{\mathcal{N}}^2 = \max(4F_{\underline{n}}^2 \ : \ |\underline{n}| = \mathcal{N})$. For quantum field theory it is necessary to take a large-$\mathcal{N}$ limit. In general this produces divergences which need renormalisation. Optionally the large-$\mathcal{N}$ limit can be entangled with a limit $V \to \infty$ which, supposing the $F_{\underline{n}}$ scale down with V (as e.g. in (3.6)), can be designed to let $\varrho(X)$ converge to a continuous function.

We also pass to mass-dimensionless quantities via multiplication by specified powers of μ [GSW18]. This amounts to choose the mass scale as $\mu = 1$.

To keep maximal flexibility we consider a measure ϱ with support in $[1, \Lambda^2]$ of which a limit $\Lambda \to \infty$ has to be taken for quantum field theory. As already observed in [MS91], for $g = 0$ the resulting integral equation extends to a closed equation for a sectionally holomorphic function $W_0(X)$ from which one extracts $W^{(0)}_{|\underline{p}|} = W_0(4F^2_{\underline{p}})$. The same reformulation of (3.8) can be done for any genus:

$$\sum_{h+h'=g} W_h(X)W_{h'}(X) + 2\lambda\nu W_g(X) + \int_1^{\Lambda^2} dY\, \varrho(Y) \frac{W_g(X) - W_g(Y)}{X - Y} \tag{3.10}$$
$$= -4\lambda^2 G_{g-1}(X|X) + \delta_{0,g}\Big(\frac{X}{Z} + C\Big),$$

where a similar extension of the $1 + 1$-point function is assumed. In general, the original correlation functions are recovered from the continuous formulation via

$$G^{(g)}_{|\underline{p}^1_1\dots\underline{p}^1_{N_1}|\dots|\underline{p}^B_1\dots\underline{p}^B_{N_B}|} = G_g\big(4F^2_{\underline{p}^1_1}, \dots, 4F^2_{\underline{p}^1_{N_1}}|\dots|4F^2_{\underline{p}^B_1}, \dots, 4F^2_{\underline{p}^B_{N_B}}\big),$$
$$W^{(g)}_{|\underline{p}|} = W_g(4F^2_{\underline{p}}).$$

Using techniques for boundary values of sectionally holomorphic functions [MS91], easily adapted to include $Z - 1, \nu, C \neq 0$ [GSW18], one obtains the following solution of (3.10) for the 1-point function at genus $g = 0$:

$$W_0(X) = \frac{\sqrt{X + c}}{\sqrt{Z}} - \lambda\nu + \frac{1}{2}\int_1^{\Lambda^2} dY \frac{\varrho(Y)}{(\sqrt{X + c} + \sqrt{Y + c})\sqrt{Y + c}}. \tag{3.11}$$

Here, the finite parameter c and the (for $\Lambda^2 \to \infty$) possibly divergent Z, ν are determined by renormalisation conditions depending on the dimension:

$$\underbrace{W_0(1) = 1}_{D\geq 2}, \qquad \underbrace{W_0'(X) = \frac{1}{2}}_{D\geq 4}, \qquad \underbrace{W_0''(1) = -\frac{1}{4}}_{D=6},$$

together with the convention $Z = 1$ for $D \in \{2, 4\}$ and $\nu = 0$ for $D = 2$. For given coupling constant λ as the only remaining parameter, these equations can be solved for c:

$$(1 - \sqrt{1 + c})(1 + \sqrt{1 + c})^{\delta_{D,6}} = \frac{1}{2}\int_1^{\Lambda^2} dY \frac{\varrho(Y)}{(\sqrt{1 + c} + \sqrt{Y + c})^{D/2}\sqrt{Y + c}}. \tag{3.12}$$

By the implicit function theorem, (3.12) has a smooth solution in an inverval $-\lambda_c < \lambda < \lambda_c$, in any dimension $D \in \{0, 2, 4, 6\}$. The Lagrange inversion theorem gives the expansion of c in λ^2:

$$c = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^{n-1}}{dw^{n-1}}\bigg|_{w=0} \left(\frac{\frac{w}{2}\int_1^{\Lambda^2} dY \frac{\varrho(Y)}{(\sqrt{1+w}+\sqrt{Y+w})^{D/2}\sqrt{Y+w}}}{(1 - \sqrt{1 + w})(1 + \sqrt{1 + w})^{\delta_{D,6}}}\right)^n.$$

After that renormalisation procedure the limit $\Lambda^2 \to \infty$ is safe in all correlation function and any dimension $D \in \{2, 4, 6\}$.

3.4. Dyson-Schwinger equation for $B > 1$

An $(N_1 + ... + N_B)$-point function of genus g is given by the $(1+1+...+1)$-point function of genus g with B boundary components through the explicit formula [GSW18, Prop. 4.1]

$$\begin{aligned}
&G_g(X_1^1, ..., X_{N_1}^1|...|X_1^B, ..., X_{N_B}^B) \\
&= \lambda^{N_1+...+N_B-B} \sum_{k_1=1}^{N_1} ... \sum_{k_B=1}^{N_B} G_g(X_{k_1}^1|...|X_{k_B}^{N_B}) \prod_{\beta=1}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta\neq k_\beta}}^{N_\beta} \frac{4}{X_{k_\beta}^\beta - X_{l_\beta}^\beta}. \qquad (3.13)
\end{aligned}$$

Furthermore, a $(1+1+\cdots+1)$-point function $G_g(X_1|X_{\triangleleft J})$ with $B > 1$ boundary components and genus g fulfils the linear integral equation [GSW18, eq. (4.5)]

$$\begin{aligned}
0 =&\lambda G_{g-1}(X_1|X_1|X_{\triangleleft J}) + \lambda \sum_{h+h'=g} \sum_{\substack{I\subset J \\ 1\leq|I|\leq B-2}} G_h(X_1|X_{\triangleleft I})G_{h'}(X_1|X_{\triangleleft J\setminus I}) \\
&+ \sum_{h+h'=g} W_h(X_1)G_{h'}(X_1|X_{\triangleleft J}) + \lambda\nu G_g(X_1|X_{\triangleleft J}) + \lambda \sum_{\beta\in J} G_g(X_1, X_\beta, X_\beta|X_{\triangleleft J\setminus\{\beta\}}) \\
&+ \frac{1}{2}\int_1^{\Lambda^2} dY\, \varrho(Y) \frac{G_g(X_1|X_{\triangleleft J}) - G_g(Y|X_{\triangleleft J})}{X_1 - Y}, \qquad (3.14)
\end{aligned}$$

where $J = \{2, 3, .., B\}$ and $G_g(X_{\triangleleft I}) := G_g(X_{i_1}|X_{i_2}|...|X_{i_p})$ if $I = \{i_1, ..., i_p\}$. The difference to the planar sector ($g = 0$) is the first term indexed $g - 1$ which only contributes if $g \geq 1$. Furthermore, the entire sector of genus $h < g$ contributes to the genus-g sector.

The equations (3.14) for $g = 0$ have been solved in [GSW17]:

$$\begin{aligned}
G_0(X|Y) =&\frac{4\lambda^2}{\sqrt{X+c}\sqrt{Y+c}(\sqrt{X+c}+\sqrt{Y+c})^2}, \\
G(X^1|\dots|X^B) =& \frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}\sqrt{X^1+c-2t}^3\cdots\sqrt{X^B+c-2t}^3}\Big)\Big|_{t=0}, \quad B\geq 3, \qquad (3.15) \\
\text{where}\quad R(t) :=& \lim_{\Lambda^2\to\infty}\Big(\frac{1}{\sqrt{Z}} - \int_1^{\Lambda^2} \frac{dT\rho(T)}{\sqrt{T+c}} \frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\Big).
\end{aligned}$$

Note that multiple t-derivatives of $R(t)$ at $t = 0$ produce renormalised moments of the measure (3.9):

$$\varrho_l := \lim_{\Lambda^2\to\infty}\Big(\frac{\delta_{l,0}}{\sqrt{Z}} - \frac{1}{2}\int_1^{\Lambda^2} \frac{dT\,\rho(T)}{(\sqrt{T+c})^{3+2l}}\Big). \qquad (3.16)$$

In fact the proof of (3.15) consists in a resummation of an ansatz which involves Bell polynomials (see Definition 4.9) in the $\{\varrho_l\}$.

The next goal is to find solutions for (3.10) and (3.14) at any genus by employing techniques of complex analysis. The moments (3.16) will be of paramount importance for that. We will find that all solutions are universal in terms of $\{\varrho_l\}$. The concrete model characterised by the sequence $E_{\underline{n}}$, coupling constant λ and the dimension D only affects the values of $\{\varrho_l\}$ via the measure (3.9) and the D-dependent solution c of (3.12).

4. Solution of the non-planar sector

4.1. Change of variables

As already mentioned, the equation for W_0 and its solution holomorphically extend to (certain parts of) the complex plane. The corresponding techniques have been brought to perfection by Eynard. We draw a lot of inspiration from the exposition given in [Eyn16]. Starting point is another change of variables:

$$\begin{aligned} z :=& \sqrt{X+c}, \\ \mathcal{G}_g(z(X)) :=& W_g(X) \qquad \text{for the 1-point function,} \\ \mathcal{G}_g(z_1^1(X_1^1),...,z_{N_1}^1(X_{N_1}^1)|...|z_1^B(X_1^B),...,z_{N_B}^B(X_{N_B}^B)) :=& G_g(X_1^1,...,X_{N_1}^1|...|X_1^B,...,X_{N_B}^B). \end{aligned}$$

In the beginning, z is defined to be positive; nevertheless all correlation functions have an analytic continuation. We define them by the complexification of the equations (3.10) and (3.14), where we assume that the complex variables fulfil the equations if they lie on the interval $[\sqrt{1+c}, \sqrt{1+\Lambda^2}]$. By recursion hypothesis each correlation function is analytic for non-vanishing imaginary part of the complex variables z_i, possibly with the exception of diagonals $z_i = \pm z_j$.

We rephrase some of the earlier results in this setup. The solutions (3.10), (3.15) and the formula for the $(1+1+1)$-point function given in [GSW17] are easily translated into

$$\mathcal{G}_0(z) = \frac{z}{\sqrt{Z}} - \lambda\nu + \frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} dy \frac{\tilde{\varrho}(y)}{(z+y)y}, \qquad \tilde{\varrho}(y) := 2y\varrho(\sqrt{y^2-c}), \quad (4.1)$$

$$\mathcal{G}_0(z_1|z_2) = \frac{4\lambda^2}{z_1z_2(z_1+z_2)^2}, \qquad \mathcal{G}_0(z_1|z_2|z_3) = -\frac{32\lambda^5}{\rho_0 z_1^3z_2^3z_3^3}.$$

Note that $\tilde{\varrho}(y)$ has support in $[\sqrt{1+c}, \sqrt{\Lambda^2+c}] \subset \mathbb{R}_+$ because of $c > -1$ [GSW18]. Furthermore, $\mathcal{G}_0(z)$ extends to a sectionally holomorphic function with branch cut along $[-\sqrt{1+\Lambda^2}, -\sqrt{1+c}]$, the $(1+1)$-point function of genus zero is holomorphic outside $z_i = 0$ and the diagonals $z_1 = -z_2$, whereas the $(1+1+1)$-point function (and all higher-B functions) at genus 0 are meromorphic with only pole at $z_i = 0$.

Definition 4.1. *Let $\hat{K}_z$ be the integral operator of the linear integral equation,*

$$\hat{K}_z f(z) := \mathcal{G}_0(z)f(z) + \lambda\nu f(z) + \frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} dy\, \tilde{\varrho}(y)\frac{f(z)-f(y)}{z^2-y^2},$$

where $\mathcal{G}_0(z)$ is given by (4.1).

In this notation, (3.10) takes the form

$$\hat{K}_z\mathcal{G}_g(z) = -\frac{1}{2}\sum_{h=1}^{g-1}\mathcal{G}_h(z)\mathcal{G}_{g-h}(z) - 2\lambda^2\mathcal{G}_{g-1}(z|z). \qquad (4.2)$$

We will heavily rely on:

Lemma 4.2. *The operator $\hat{K}_z$ defined in Definition 4.1 satisfies*

$$\hat{K}_z\Big(\frac{1}{z}\Big) = \frac{1}{\sqrt{Z}}, \qquad \hat{K}_z\Big(\frac{1}{z^{3+2n}}\Big) = \sum_{k=0}^{n} \frac{\varrho_k}{z^{2n+2-2k}}.$$

Proof. This is a reformulation of [GSW17, Lemma (5.5)]. □

The first step beyond [GSW17] is to determine the 1-point function at genus 1:

Proposition 4.3. *The solution of* (4.2) *for* $g = 1$ *is*

$$\mathcal{G}_1(z) = \frac{2\lambda^4\varrho_1}{\varrho_0^2 z^3} - \frac{2\lambda^4}{\varrho_0 z^5},$$

where the ϱ_l *are given in* (3.16).

Proof. From (4.1) we have $-2\lambda^2\mathcal{G}_0(z|z) = -\frac{2\lambda^4}{z^4}$. Lemma 4.2 suggests the ansatz $\mathcal{G}_1(z) = \frac{\beta}{z^3} + \frac{\gamma}{z^5}$ with $\hat{K}_z\mathcal{G}_1(z) = \frac{\beta\varrho_0}{z^2} + \frac{\gamma\varrho_0}{z^4} + \frac{\gamma\varrho_1}{z^2}$. Comparison of coefficients yields the assertion. □

The N-point function of genus 1 is given by the explicit formula [GSW18, Prop. 3.1], which holds for every genus g. In complex variables it reads

$$\mathcal{G}_1(z_1, ..., z_N) = \sum_{k=1}^{N} \frac{\mathcal{G}_1(z_k)}{2\lambda} \prod_{l=1, l\neq k}^{N} \frac{4\lambda}{z_k^2 - z_l^2}.$$

Next we express equation (3.14) in the new variables. To find more convenient results we use (3.13) to write with $J = \{2, .., B\}$

$$\begin{aligned}
\mathcal{G}_g(z_1, z_\beta, z_\beta|z\triangleleft_{J\backslash\{\beta\}}) &= \lim_{z_\alpha\to z_\beta} \mathcal{G}_g(z_1, z_\alpha, z_\beta|z\triangleleft_{J\backslash\{\beta\}}) \\
=&16\lambda^2 \left[\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\backslash\{\beta\}})}{(z_1^2 - z_\beta^2)^2} - \lim_{z_\alpha\to z_\beta} \frac{\frac{\mathcal{G}_g(z_\alpha|z\triangleleft_{J\backslash\{\beta\}})}{z_1^2 - z_\alpha^2} - \frac{\mathcal{G}_g(z_\beta|z\triangleleft_{J\backslash\{\beta\}})}{z_1^2 - z_\beta^2}}{(z_\alpha + z_\beta)(z_\alpha - z_\beta)}\right] \\
=&16\lambda^2 \frac{\partial}{2z_\beta\partial z_\beta} \frac{\mathcal{G}_g(z_1|z\triangleleft_{J\backslash\{\beta\}}) - \mathcal{G}_g(z_\beta|z\triangleleft_{J\backslash\{\beta\}})}{z_1^2 - z_\beta^2}
\end{aligned} \tag{4.3}$$

and $\mathcal{G}_g(z_1, z_2, z_2) = 8\lambda \frac{\partial}{2z_2\partial z_2} \frac{\mathcal{G}_g(z_1) - \mathcal{G}_g(z_2)}{z_1^2 - z_2^2}$ for one boundary component.

Inserting (4.3) into (3.14) gives with Definition 4.1 the following formula for $\mathcal{G}_g(z_1|z\triangleleft_J)$, for $J = \{2, .., B\} \neq \varnothing$:

$$\begin{aligned}
0 = \hat{K}_{z_1}\mathcal{G}_g(z_1|z\triangleleft_J) + \lambda\mathcal{G}_{g-1}(z_1|z_1|z\triangleleft_J) + \lambda\sum_{h=0}^{g} \sum_{\substack{I\subset J \\ 1\leq|I|\leq B-2}} \mathcal{G}_h(z_1|z\triangleleft_I)\mathcal{G}_{g-h}(z_1|z\triangleleft_{J\backslash I}) \\
+ \sum_{h=1}^{g} \mathcal{G}_h(z_1)\mathcal{G}_{g-h}(z_1|z\triangleleft_J) + \frac{(2\lambda)^3}{(2\lambda)^{\delta_{B,2}}} \sum_{\beta\in J} \frac{\partial}{z_\beta\partial z_\beta} \frac{\mathcal{G}_g(z_1|z\triangleleft_{J\backslash\{\beta\}}) - \mathcal{G}_g(z_\beta|z\triangleleft_{J\backslash\{\beta\}})}{z_1^2 - z_\beta^2}.
\end{aligned} \tag{4.4}$$

4.2. Boundary creation operator

We are going to construct an operator which plays the rôle of the formal $T_{\underline{n}} := \frac{1}{E_{\underline{n}}}\frac{\partial}{\partial E_{\underline{n}}}$ applied to the logarithm of the partition function $\mathcal{Z}[0]$ given in (3.2). In dimension $D = 0$ where $Z - 1 = \kappa = \nu = \zeta = 0$ and $\mu_{bare} = \mu$, $\lambda_{bare} = \lambda$ we formally have

$$\begin{aligned}T_{\underline{n}} \log\Big(\int d\Phi\; e^{-\mathrm{tr}(E\Phi^2+\frac{\lambda}{3}\Phi^3)}\Big) &= -\frac{1}{\mathcal{Z}[0]}\int d\Phi \sum_{\underline{m}} \frac{\Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{m}}}{E_{\underline{n}}} e^{-\mathrm{tr}(E\Phi^2+\frac{\lambda}{3}\Phi^3)} \\ &= \frac{1}{\lambda\mathcal{Z}[0]}\int d\Phi\; \frac{1}{E_{\underline{n}}}\Big(\frac{\partial}{\partial \Phi_{\underline{n}\underline{n}}} + 2E_{\underline{n}}\Phi_{\underline{n}\underline{n}}\Big)e^{-\mathrm{tr}(E\Phi^2+\frac{\lambda}{3}\Phi^3)} \\ &= \frac{2}{\lambda\mathcal{Z}[0]}\int d\Phi\; \Phi_{\underline{n}\underline{n}} e^{-\mathrm{tr}(E\Phi^2+\frac{\lambda}{3}\Phi^3)} = \frac{2}{\lambda}G_{|\underline{n}|}. \end{aligned} \tag{4.5}$$

By repeated application of $T_{\underline{n}_i}$ we formally produce an $(1 + \cdots + 1)$-point function. Of course, these operations are not legitimate: In dimensions $D \in \{2, 4, 6\}$ we have to include for renormalisation the Φ-linear terms in (3.1), and the partition function has no chance to exist for real λ.

Nevertheless, we are able to show that $T_{\underline{n}_i}$ admits a rigorous replacement which we call the *boundary creation operator*. It will be our main device:

Definition 4.4. *For $J = \{1, \dots, p\}$ let $|J| := p$ and $z_J := (z_1, \dots, z_p)$. Then*

$$\hat{\mathrm{A}}^{\dagger g}_{z_J,z} := \sum_{l=0}^{3g-3+|J|}\Big(-\frac{(3+2l)\varrho_{l+1}}{\varrho_0 z^3} + \frac{3+2l}{z^{5+2l}}\Big)\frac{\partial}{\partial\varrho_l} + \sum_{i\in J}\frac{1}{\varrho_0 z^3 z_i}\frac{\partial}{\partial z_i}. \tag{4.6}$$

Note that the last variable z in $\hat{\mathrm{A}}^{\dagger g}_{z_J,z}$ plays a very different rôle than the z_J!

Lemma 4.5. *The differential operators $\hat{\mathrm{A}}^{\dagger g}_{z_J,z}$ are commutative,*

$$\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p,z_q}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p} = \hat{\mathrm{A}}^{\dagger g}_{z_J,z_q,z_p}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_q}.$$

Proof. Being a derivative, it is enough to verify $\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p,z_q}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p}(\varrho_k) = \hat{\mathrm{A}}^{\dagger g}_{z_J,z_q,z_p}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_q}(\varrho_k)$ for any k and $\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p,z_q}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p}(z_i) = \hat{\mathrm{A}}^{\dagger g}_{z_J,z_q,z_p}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_q}(z_i)$ for any $i \in J$. This is guaranteed by

$$\begin{aligned}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p,z_q}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p}(\varrho_k) &= \frac{(3+2k)(5+2k)\varrho_{k+2}}{\varrho_0^2 z_q^3 z_p^3} - \frac{(3+2k)(5+2k)}{\varrho_0 z_q^{7+2k} z_p^3} - \frac{3(3+2k)\varrho_{k+1}\varrho_1}{\varrho_0^2 z_q^3 z_p^3} \\ &\quad + \frac{3(3+2k)\varrho_{k+1}}{\varrho_0^2 z_p^5 z_p^3} + \frac{3(3+2k)\varrho_{k+1}}{\varrho_0^2 z_q^3 z_p^5} - \frac{(3+2k)(5+2k)}{\varrho_0 z_q^3 z_p^{7+2k}}, \\ \hat{\mathrm{A}}^{\dagger g}_{z_J,z_p,z_q}\hat{\mathrm{A}}^{\dagger g}_{z_J,z_p}(z_i) &= \frac{3\varrho_1}{\varrho_0^2 z_q^3 z_p^3 z_i} - \frac{3}{\varrho_0^2 z_q^5 z_p^3 z_i} - \frac{3}{\varrho_0^2 z_q^3 z_p^5 z_i} - \frac{1}{\varrho_0^2 z_q^3 z_p^3 z_i^3}.\end{aligned}$$

∎

This shows that boundary components labelled by z_i behave like bosonic particles at position z_i. The creation operator $(2\lambda)^3\hat{\mathrm{A}}^{\dagger g}_{z_J,z}$ adds to a $|J|$-particle state another particle at position z. The $|J|$-particle state is precisely given by $\mathcal{G}_g(z\triangleleft_J)$:

Theorem 4.6. *Assume that $\mathcal{G}_g(z)$ is, for $g \geq 1$, an odd function of $z \neq 0$ and a rational function of $\varrho_0, \ldots, \varrho_{3g-2}$ (true for $g = 1$). Then the $(1+1+\ldots+1)$-point function of genus $g \geq 1$ and B boundary components of the remormalised Φ^3_D-matricial QFT model in dimension $D \in \{2, 4, 6\}$ has the solution*

$$\mathcal{G}_g(z_1|\ldots|z_B) = (2\lambda)^{3B-4} \hat{\mathrm{A}}^{\dagger g}_{z_1,\ldots,z_B} \big(\hat{\mathrm{A}}^{\dagger g}_{z_1,\ldots,z_{B-1}} \big(\cdots \hat{\mathrm{A}}^{\dagger g}_{z_1,z_2} \mathcal{G}_g(z_1) \ldots \big)\big), \qquad z_i \neq 0, \tag{4.7}$$

where $\mathcal{G}_g(z_1)$ is the 1-point function of genus $g \geq 1$ and the boundary creation operator $\hat{\mathrm{A}}^{\dagger g}_{z_J}$ is defined in Definition 4.4. For $g = 0$ the boundary creation operators act on the $(1+1)$-point function

$$\mathcal{G}_0(z_1|\ldots|z_B) = (2\lambda)^{3B-6} \hat{\mathrm{A}}^{\dagger 0}_{z_1,\ldots,z_B} \big(\hat{\mathrm{A}}^{\dagger 0}_{z_1,\ldots,z_{B-1}} \big(\cdots \hat{\mathrm{A}}^{\dagger 0}_{z_1,z_2,z_3} \mathcal{G}_0(z_1|z_2) \ldots \big)\big).$$

Proof. We rely on several Lemmata proved in Appendix A. Regarding (4.7) as a *definition*, we prove in Lemma A.6 an equivalent formula for the linear integral equation (4.4). This expression is satisfied because Lemma A.3 and Lemma A.5 add up to 0. Consequently, the family of functions (4.7) satisfies (4.4). This solution is unique because of uniqueness of the perturbative expansion. ☐

Corollary 4.7. *Let $J = \{2, \ldots, B\}$. Assume that $z \mapsto \mathcal{G}_g(z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$ with $\mathcal{G}_g(z) = -\mathcal{G}_g(-z)$ for all $z \in \mathbb{C} \setminus \{0\}$ and $g \geq 1$. Then all $\mathcal{G}_g(z_1|z_{\triangleleft J})$ with $2 - 2g - B < 0$*

1. *are holomorphic in every $z_i \in \mathbb{C} \setminus \{0\}$*
2. *are odd functions in every z_i, i.e. $\mathcal{G}_g(-z_1|z_{\triangleleft J}) = -\mathcal{G}_g(z_1|z_{\triangleleft J})$ for all $z_1, z_i \in \mathbb{C} \setminus \{0\}$.*

Proof. The boundary creation operator $\hat{\mathrm{A}}^{\dagger g}_{z_J,z}$ of Definition 4.4 preserves holomorphicity in $\mathbb{C} \setminus \{0\}$ and maps odd functions into odd functions. Thus only the initial conditions need to be checked. They are fulfilled for $\mathcal{G}_0(z_1|z_2|z_3)$ and $\mathcal{G}_1(z_1)$ according to (4.1); for $g \geq 2$ by assumption. ☐

The assumption will be verified later in Proposition 4.13.

Corollary 4.8. *The boundary creation operator $\hat{\mathrm{A}}^{\dagger g}_{z_J,z_1}$ acting on an $(N_1 + \ldots + N_B)$-point function of genus g gives the following $(1 + N_1 + \ldots + N_B)$-point function of genus g*

$$\mathcal{G}_g(z_1|z^1_1, .., z^1_{N_1}|..|z^B_1, .., z^B_{N_B}) = (2\lambda)^3 \hat{\mathrm{A}}^{\dagger g}_{z^1_1,..,z^1_{N_1},\ldots,z^B_1,..,z^B_{N_B},z_1} (\mathcal{G}_g(z^1_1, .., z^1_{N_1}|..|z^B_1, .., z^B_{N_B})).$$

Proof. This follows from the change to complex variables in equation (3.13) and $\hat{\mathrm{A}}^{\dagger g}_{z_J,z_1}(\frac{1}{z_i^2 - z_j^2}) = 0$ for $1 \neq i \neq j \neq 1$. ☐

4.3. Solution of the 1-point function for $g \geq 1$

It remains to check that the 1-point function $\mathcal{G}_g(z)$ at genus $g \geq 1$ satisfies the assumptions of Theorem 4.6 and Corollary 4.7, namely:

1. $\mathcal{G}_g(z)$ depends only on the moments $\varrho_0, \ldots, \varrho_{3g-2}$ of the measure,
2. $z \mapsto \mathcal{G}_g(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and an odd function of z.

We establish these properties by solving (4.2) via a formula for the inverse of $\hat{K}_z$. This formula is inspired by topological recursion, see e.g. [Eyn16]. We give a few details in section 4.4.

Definition 4.9. *The Bell polynomials are defined by*

$$B_{n,k}(x_1,...,x_{n-k+1}) = \sum \frac{n!}{j_1!j_2!...j_{n-k+1}!}\Big(\frac{x_1}{1!}\Big)^{j_1}\Big(\frac{x_2}{2!}\Big)^{j_2}...\Big(\frac{x_{n-k+1}}{(n-k+1)!}\Big)^{j_{n-k+1}}$$

for $n \geq 1$, where the sum is over non-negative integers $j_1,...,j_{n-k+1}$ with $j_1+...+j_{n-k+1} = k$ and $1j_1+...+(n-k+1)j_{n-k+1} = n$. Moreover, one defines $B_{0,0} = 1$ and $B_{n,0} = B_{0,k} = 0$ for $n, k > 0$.

An important application is Faà di Bruno's formula, the n-th order chain rule:

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x))\, B_{n,k}(g'(x), g''(x), ..., g^{(n-k+1)}(x)). \tag{4.8}$$

Proposition 4.10. *Let $f(z) = \sum_{k=0}^{\infty} \frac{a_{2k}}{z^{2k}}$ be an even Laurent series about $z = 0$ bounded at ∞. Then the inverse of the integral operator $\hat{K}_z$ of Definition 4.1 is given by the residue formula*

$$\Big[z^2\hat{K}_z\frac{1}{z}\Big]^{-1} f(z) = -\operatorname*{Res}_{z'\to 0}\,[K(z,z')\,f(z')dz'], \quad K(z,z') := \frac{2}{(\mathcal{G}_0(z') - \mathcal{G}_0(-z'))(z'^2 - z^2)}.$$

Proof. The formulae (4.1) give rise to the series expansion

$$\frac{1}{2}(\mathcal{G}_0(z') - \mathcal{G}_0(-z')) = \sum_{l=0}^{\infty} \varrho_l (z')^{2l+1}, \tag{4.9}$$

where the ϱ_l are given in (3.16). The series of its reciprocal is found using (4.8):

$$\frac{2}{(\mathcal{G}_0(z') - \mathcal{G}_0(-z'))} = \frac{1}{z'\varrho_0}\sum_{m=0}^{\infty}\frac{(z')^{2m}}{m!}S_m, \tag{4.10}$$

$$S_m := \frac{d^m}{d\tau^m}\Big|_{\tau=0}\Big(\sum_{l=0}^{\infty}\frac{\varrho_l}{\varrho_0}\tau^l\Big)^{-1} = \sum_{i=0}^{m}\frac{(-1)^i i!}{\varrho_0^i}B_{m,i}(1!\varrho_1, 2!\varrho_2, ..., (m-i+1)!\varrho_{m-i+1}).$$

Multiplication by the geometric series gives

$$K(z,z') = -\frac{1}{z^2 z'\varrho_0}\sum_{n,m=0}^{\infty}\frac{(z')^{2m+2n}}{m! z^{2n}}S_m. \tag{4.11}$$

The residue of a monomial in $f(z') = \sum_{k=0}^{\infty}\frac{a_{2k}}{(z')^{2k}}$ is then

$$\mathrm{Res}_{z'\to 0}\Big[K(z,z')\frac{dz'}{(z')^{2k}}\Big] = -\frac{1}{\varrho_0}\sum_{j=0}^{k}\frac{S_j}{j! z^{2k-2j+2}}. \tag{4.12}$$

In the next step we apply the operator $z^2\hat{K}\frac{1}{z}$ to (4.12), where Lemma 4.2 is used:

$$
\begin{aligned}
z^2\hat{K}_z\Big(\frac{1}{z}\frac{(-1)}{\varrho_0}\sum_{j=0}^{k}\frac{S_j}{j!z^{2k-2j+2}}\Big) &= -\frac{z^2}{\varrho_0}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\frac{S_j\varrho_i}{j!z^{2k-2j-2i+2}} \\
&= -\sum_{j=0}^{k}\frac{S_{k-j}}{(k-j)!z^{2j}} - \frac{1}{\varrho_0}\sum_{i=0}^{k-1}\sum_{j=i+1}^{k}\frac{S_{k-j}\varrho_{j-i}}{(k-j)!z^{2i}}.
\end{aligned}
\tag{4.13}
$$

The last sum over j is treated as follows, where the Bell polynomials are inserted for S_m:

$$
\begin{aligned}
&\sum_{j=i+1}^{k}\frac{S_{k-j}\varrho_{j-i}}{(k-j)!} = \sum_{j=1}^{k-i}\frac{S_{k-j-i}\varrho_j}{(k-j-i)!} \\
&= \sum_{j=1}^{k-i}\sum_{l=0}^{k-j-i}\frac{(-1)^l l!}{(k-j-i)!\varrho_0^l}\,\varrho_j B_{k-j-i,l}(1!\varrho_1,...,(k-j-i-l+1)!\varrho_{k-j-i-l+1}) \\
&= \sum_{l=0}^{k-j-i}\frac{(-1)^l l!}{\varrho_0^l(k-i)!}\sum_{j=1}^{k-i-l}\binom{k-i}{j}j!\,\varrho_j B_{k-j-i,l}(1!\varrho_1,...,(k-j-i-l+1)!\varrho_{k-j-i-l+1}) \\
&= \sum_{l=0}^{k-i}\frac{(-1)^l(l+1)!}{\varrho_0^l(k-i)!}B_{k-i,l+1}(1!\varrho_1,...,(k-i-l)!\varrho_{k-i-l}) \\
&= -\varrho_0\frac{S_{k-i}}{(k-i)!}.
\end{aligned}
$$

We have used $B_{n,0}=0$ and $B_{0,n}=0$ for $n>0$ to eliminate some terms, changed the order of sums, and used the following identity for the Bell polynomials [GSW17, Lemma 5.9]

$$
\sum_{j=1}^{n-k}\binom{n}{j}x_j B_{n-j,k}(x_1,...,x_{n-j-k+1}) = (k+1)B_{n,k+1}(x_1,...,x_{n-k}).
\tag{4.14}
$$

Inserted back we find that (4.13) reduces to the $(j=k)$-term of the first sum in the last line of (4.13), i.e.

$$
z^2\hat{K}_z\Big(\frac{1}{z}\operatorname*{Res}_{z'\to 0}\Big[K_B(z,z')\frac{dz'}{(z')^{2k}}\Big]\Big) = -\frac{1}{z^{2k}}.
$$

This finishes the proof. □

Theorem 4.11. *For any $g\geq 1$ and $z\in\mathbb{C}\setminus\{0\}$ one has*

$$
\mathcal{G}_g(z) = \frac{1}{2z}\operatorname*{Res}_{z'\to 0}\Bigg[K(z,z')\Big\{\sum_{h=1}^{g-1}\mathcal{G}_h(z')\mathcal{G}_{g-h}(z') + (2\lambda)^2\mathcal{G}_{g-1}(z'|z')\Big\}(z')^2dz'\Bigg].
\tag{4.15}
$$

Proof. The formula arises when applying Proposition 4.10 to (4.2) and holds if the function in $\{\ \}$ is an even Laurent polynomial in z' bounded in ∞. This is the case for $g=1$ where only $\mathcal{G}_0(z'|z') = \frac{\lambda^2}{(z')^4}$ contributes. Evaluation of the residue reconfirms Proposition 4.3. We proceed by induction in $g\geq 2$, assuming that all $\mathcal{G}_h(z')$

with $1 \leq h < g$ on the rhs of (4.15) are odd Laurent polynomials bounded in ∞; their product is even. The induction hypothesis also verifies the assumption of Theorem 4.6 so that $\mathcal{G}_{g-1}(-z'|-z') = -\mathcal{G}_g(z'|-z') = \mathcal{G}_g(z'|z')$ is even and, because of $\mathcal{G}_{g-1}(z'|z'') = (2\lambda)^3 \hat{\mathrm{A}}^{\dagger g}_{z'',z'} \mathcal{G}_{g-1}(z'')$, inductively a Laurent polynomial bounded in ∞. Thus, equation (4.15) holds for genus $g \geq 2$ and, as consequence of (4.12), $\mathcal{G}_g(z)$ is again an odd Laurent polynomial bounded in ∞. Equation (4.15) is thus proved for all $g \geq 1$, and the assumption of Theorem 4.6 is verified. □

A more precise characterisation can be given. It relies on

Definition 4.12. *A polynomial $P(x_1, x_2, \dots)$ is called n-weighted if* $\sum_{k=1}^{\infty} k x_k \frac{\partial}{\partial x_k} P(x_1, x_2, \dots) = nP(x_1, x_2, \dots)$.

The Bell polynomials $B_{n,k}(x_1, \dots, x_{n-k+1})$ are n-weighted. The number of monomials in an n-weighted polynomial is $p(n)$, the number of partitions of n. The sequence $p(n)$ is OEIS A000041 and starts with $(1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, \dots)$. The product of an n-weighted by an m-weighted polynomial is $(m + n)$-weighted.

Proposition 4.13. *For $g \geq 1$ one has*

$$\mathcal{G}_g(z) = (2\lambda)^{4g} \sum_{k=0}^{3g-2} \frac{P_{3g-2-k}(\varrho)}{\varrho_0^{2g-1} z^{2k+3}},$$

where $P_0 \in \mathbb{Q}$ and the $P_j(\varrho)$ with $j \geq 1$ are j-weighted polynomials in $\{\frac{\varrho_1}{\varrho_0}, \dots, \frac{\varrho_j}{\varrho_0}\}$ with rational coefficients.

Proof. The case $g = 1$ is directly checked. We proceed by induction in g for both terms in $\{\ \}$ in (4.15). The hypothesis gives $\mathcal{G}_h(z')\mathcal{G}_{g-h}(z') = (2\lambda)^{4g} \sum_{k=0}^{3g-4} \frac{P_{3g-4-k}(\varrho)}{\varrho_0^{2g-2}(z')^{2k+6}}$. In the second term in $\{\ \}$, $\mathcal{G}_{g-1}(z|z) = (2\lambda)^2 \hat{\mathrm{A}}^{\dagger g-1}_{z,z} \mathcal{G}_{g-1}(z)$, the three types of contributions in the boundary creation operator act as follows:

$$\begin{aligned}
(2\lambda)^4 \hat{\mathrm{A}}^{\dagger g-1}_{z',z'} \mathcal{G}_{g-1}(z') &= \frac{(2\lambda)^{4g}}{\varrho_0^{2g-2}} \sum_{k=0}^{3g-5} \Big(\sum_{l=0}^{3g-5-k} \Big(\frac{\varrho_{l+1}}{\varrho_0 (z')^3} + \frac{1}{(z')^{5+2l}} \Big) \frac{P_{3g-5-k-l}(\varrho)}{(z')^{2k+3}} + \frac{1}{(z')^4} \frac{P_{3g-5-k}(\varrho)}{(z')^{2k+4}} \Big) \\
&= \frac{(2\lambda)^{4g}}{\varrho_0^{2g-2}} \sum_{k=0}^{3g-5} \Big(\frac{P_{3g-4-k}(\varrho)}{(z')^{2k+6}} + \frac{P_{3g-5-k}(\varrho)}{(z')^{2k+8}} \Big),
\end{aligned}$$

which has the same structure as $\mathcal{G}_h(z')\mathcal{G}_{g-h}(z')$. Application of (4.12) yields

$$\begin{aligned}
\frac{1}{z} \operatorname*{Res}_{z' \to 0} \Big[K(z, z') dz' (z')^2 (2\lambda)^{4g} \sum_{k=0}^{3g-4} \frac{P_{3g-4-k}(\varrho)}{\varrho_0^{2g-2} (z')^{2k+6}} \Big] &= (2\lambda)^{4g} \sum_{k=0}^{3g-4} \sum_{j=0}^{k+2} \frac{P_{3g-4-k}(\varrho) S_j(\varrho)}{\varrho_0^{2g-1} z^{2k+7-2j}} \\
&= (2\lambda)^{4g} \sum_{k=0}^{3g-2} \frac{P_{3g-2-k}(\varrho)}{\varrho_0^{2g-1} z^{2k+3}},
\end{aligned}$$

because $S_j(\varrho)$ is also a j-weighted polynomial by (4.10). □

In particular, this proves the assumption of Theorem 4.6, namely that $\mathcal{G}_g(z)$ depends only on $\{\varrho_0, \dots, \varrho_{3g-2}\}$. To be precise, we reciprocally increase the genus in Theorem 4.6 and Proposition 4.13.

4.4. Remarks on topological recursion

A (1+1+...+1)-point function of genus g with B boundary components fulfils a universal structure called *topological recursion*. To introduce it here we have to define the following functions:

Definition 4.14. *The function $\omega_{g,B}$ is defined by*

$$\omega_{g,B}(z_1,...,z_B) := \left(\prod_{i=1}^{B} z_i\right)\left(\mathcal{G}_g(z_1|...|z_B) + 16\lambda^2 \frac{\delta_{g,0}\delta_{2,B}}{(z_1^2 - z_2^2)^2}\right), \qquad B > 1$$

$$\omega_{g,1}(z) := \frac{z\mathcal{G}_g(z)}{2\lambda}$$

and the spectral curve $y(x)$ by $x(z) = z^2$ and

$$y(z) := \frac{\mathcal{G}_0(z)}{2\lambda} = \frac{z}{2\lambda\sqrt{Z}} - \frac{\nu}{2} + \frac{1}{4\lambda}\int_{\sqrt{1+c}}^{\sqrt{1+\Lambda^2}} dt \frac{\tilde{\varrho}(t)}{t(t+z)}.$$

It can be checked that with these definitions, up to trivial redefinitions by powers of 2λ, the theorems proved in topological recursion [Eyn16] apply. These determine all $\omega_{g,B}$ with $2-2g-B<0$ out of the initial data $y(z)$ and $\omega_{0,2}$:

Theorem 4.15 ([Eyn16, Thm. 6.4.4])**.** *For $2-2g-(1+B)<0$ and $J=\{1,...,B\}$ the function $\omega_{g,B+1}(z_0,...,z_B)$ is given by topological recursion*

$$\begin{aligned}&\omega_{g,B+1}(z_0,...,z_B)\\ &= \operatorname*{Res}_{z\to 0}\left[K(z_0,z)\,dz\Big(\omega_{g-1,B+2}(z,-z,z_{\triangleleft J}) + \sideset{}{'}\sum_{\substack{h+h'=g\\ I\uplus I'=J}} \omega_{h,|I|+1}(z,z_{\triangleleft I})\omega_{h',|I'|+1}(-z,z_{\triangleleft I'})\Big)\right],\end{aligned}$$

where $K(z_0,z) = \frac{1}{(z^2-z_0^2)(y(z)-y(-z))}$ and the sum $\sum'$ excludes $(h,I)=(0,\emptyset)$ and $(h,I)=(g,J)$.

This theorem motivated our ansatz for an inverse of $\hat{K}_z$ as the residue involving $K(z,z')$. The case $J=\varnothing$ of Theorem 4.15 is essentially the same as Theorem 4.11. Both proofs are of comparable difficulty and length. For us there is no need to prove the general case because higher $\omega(z_{\triangleleft J})$ can be obtained from Theorem 4.6.

5. A Laplacian to compute intersection numbers

5.1. Free energy and boundary annihilation operator

Definition 5.1. *We introduce the operators*

$$\hat{\mathrm{A}}^\dagger_z := \sum_{l=0}^{\infty}\Big(-\frac{(3+2l)\varrho_{l+1}}{\varrho_0 z^3} + \frac{3+2l}{z^{5+2l}}\Big)\frac{\partial}{\partial\varrho_l}, \qquad \hat{\mathrm{N}} = -\sum_{l=0}^{\infty}\varrho_l\frac{\partial}{\partial\varrho_l},$$

$$\hat{\mathrm{A}}_{\check{z}} f(\bullet) := -\sum_{l=0}^{\infty}\operatorname*{Res}_{z\to 0}\left[\frac{z^{4+2l}\varrho_l}{3+2l}f(z)dz\right]. \tag{5.1}$$

We call $\hat{\mathrm{A}}_{\check{z}}$ a boundary annihilation operator *acting on Laurent polynomials f.*

Proposition 5.2. *There is a unique function F_g of $\{\varrho_l\}$ satisfying $\mathcal{G}_g(z) = (2\lambda)^4 \hat{\mathrm{A}}^\dagger_z F_g$,*

$$F_1 = -\frac{1}{24}\log\varrho_0, \qquad F_g = \frac{1}{(2g-2)(2\lambda)^4}\hat{\mathrm{A}}_{\check{z}}\mathcal{G}_g(\bullet) \text{ for } g \geq 1.$$

The F_g have for $g > 1$ a presentation as

$$F_g = (2\lambda)^{4g-4}\frac{P_{3g-3}(\varrho)}{\varrho_0^{2g-2}}, \tag{5.2}$$

where $P_{3g-3}(\varrho)$ is a $(3g-3)$-weighted polynomial in $\{\frac{\varrho_1}{\varrho_0}, \dots, \frac{\varrho_{3g-3}}{\varrho_0}\}$.

Proof. The case $g = 1$ is checked by direct comparison with (4.1). From Proposition 4.13 we conclude

$$\begin{aligned}\frac{1}{(2\lambda)^4}\hat{\mathrm{A}}_{\check{z}}\mathcal{G}_g(\bullet) &= -(2\lambda)^{4g-4}\operatorname*{Res}_{z\to 0}\Big[\sum_{l=0}^{\infty}\frac{\varrho_l z^{4+2l}}{(3+2l)}\sum_{k=0}^{3g-2}\frac{P_{3g-2-k}(\varrho)}{\varrho_0^{2g-1}z^{2k+3}}dz\Big] \\ &= (2\lambda)^{4g-4}\sum_{k=1}^{3g-2}\frac{\frac{\varrho_{k-1}}{\varrho_0}\cdot P_{3g-2-k}(\varrho)}{\varrho_0^{2g-2}} = (2\lambda)^{4g-4}\frac{P_{3g-3}(\varrho)}{\varrho_0^{2g-2}},\end{aligned}$$

which confirms (5.2). Observe that the total ϱ-counting operator $\hat{\mathrm{N}}$ applied to any polynomial in $\{\frac{\varrho_1}{\varrho_0}, \frac{\varrho_2}{\varrho_0}, \dots\}$ is zero. Therefore, for $g > 1$,

$$\hat{\mathrm{N}}\Big(\frac{1}{(2\lambda)^4}\hat{\mathrm{A}}_{\check{z}}\mathcal{G}_g(\bullet)\Big) = (2g-2)\cdot\Big(\frac{1}{(2\lambda)^4}\hat{\mathrm{A}}_{\check{z}}\mathcal{G}_g(\bullet)\Big).$$

The boundary annihilation operator is designed to satisfy $\hat{\mathrm{A}}_{\check{z}} \circ \hat{\mathrm{A}}^\dagger_\bullet = \hat{\mathrm{N}}$. Dividing the previous equation by $(2g-2)$ and inserting the ansatz for F_g given in the Proposition, we have

$$0 = \hat{\mathrm{N}}F_g - \frac{1}{(2\lambda)^4}\hat{\mathrm{A}}_{\check{z}}\mathcal{G}_g(\bullet) = \hat{\mathrm{A}}_{\check{z}}\Big(\hat{\mathrm{A}}^\dagger_\bullet F_g - \frac{1}{(2\lambda)^4}\mathcal{G}_g(\bullet)\Big).$$

Since $f(z) := \hat{\mathrm{A}}^\dagger_z F_g - \frac{1}{(2\lambda)^4}\mathcal{G}_g(z)$ is by (5.2) and Proposition 4.13 a Laurent polynomial bounded at ∞, application of $\hat{\mathrm{A}}_{\check{z}}$ can only vanish if $f(z) \equiv 0$. This finishes the proof. □

Remark 5.3. *Proposition 5.2 shows that the F_g provide the most condensed way to describe the non-planar sector of the Φ^3-matricial QFT model. All information about the genus-g sector is encoded in the $p(3g-3)$ rational numbers which form the coefficients in the $(3g-3)$-weighted polynomial in $\{\frac{\varrho_1}{\varrho_0}, \frac{\varrho_2}{\varrho_0}, \dots\}$. From these polynomials we obtain the $(1+\dots+1)$-point function with B boundary components via $\mathcal{G}_g(z) = (2\lambda)^4\hat{\mathrm{A}}^\dagger_z F_g$ followed by Theorem 4.6.*

Lemma 5.4. *Whenever $(2g+B-2) > 0$, the operator $\hat{\mathrm{N}}$ measures the Euler characteristics,*

$$\hat{\mathrm{N}}\mathcal{G}_g(z_1|\dots|z_B) = (2g+B-2)\mathcal{G}_g(z_1|\dots|z_B).$$

Proof. Both cases with $(2g+B-2)=1$ are directly checked. The general case follows by induction from $[\hat{\mathrm{N}},\hat{\mathrm{A}}^{\dagger g}_{z_J,z}]=\hat{\mathrm{A}}^{\dagger g}_{z_J,z}$ in combination with Theorem 4.6 and $\hat{\mathrm{N}}F_g=(2g-2)F_g$ for $g\geq 2$. □

Corollary 5.5.

$$\hat{\mathrm{A}}_{\check{z}}\mathcal{G}_g(\bullet|z_2|\dots|z_B)=(2\lambda)^{3+\delta_{B,1}}(2g+B-3)\mathcal{G}_g(z_2|\dots|z_B)$$

whenever $(2g+B-3)>0$.

Hence, up to a rescaling, $\hat{\mathrm{A}}_{\check{z}}$ indeed removes the boundary component previously located at z. We also have $\hat{\mathrm{A}}_{\check{z}}F_g=0$ for all $g\geq 1$ so that the F_g play the rôle of a vacuum. Note that $\mathcal{G}_0(z)$ cannot be produced by whatever F_0.

5.2. The Laplacian

We express in (4.2) $\mathcal{G}_g(z)=(2\lambda)^4\hat{\mathrm{A}}^\dagger_z F_g$ and $\mathcal{G}_g(z|z)=(2\lambda)^2(\hat{\mathrm{A}}^\dagger_z+\frac{1}{\varrho_0 z^4}\frac{\partial}{\partial z})(\mathcal{G}_g(z))$ and multiply by $\frac{2V^{4-2g}}{(2\lambda)^8}\mathcal{Z}^{np}_V$. Summation over $g\geq 1$ gives

$$0=\Big(\frac{2V^2}{(2\lambda)^4}\hat{K}_z\hat{\mathrm{A}}^\dagger_z+\Big(\hat{\mathrm{A}}^\dagger_z+\frac{1}{\varrho_0 z^4}\frac{\partial}{\partial z}\Big)\hat{\mathrm{A}}^\dagger_z+\frac{V^2}{4(2\lambda)^4 z^4}\Big)\mathcal{Z}^{np}_V,\qquad \mathcal{Z}^{np}_V:=\exp\Big(\sum_{g=1}^\infty V^{2-2g}F_g\Big). \tag{5.3}$$

We invert $\hat{K}_z$ via Proposition 4.10 and apply $\hat{\mathrm{A}}_{\check{z}}$ given by the residue in Proposition 5.2:

$$\begin{aligned}&\frac{2V^2}{(2\lambda)^4}\hat{\mathrm{N}}\mathcal{Z}^{np}_V\\&=-\sum_{\ell=0}^\infty\operatorname*{Res}_{z\to 0}\Big[dz\frac{z^{3+2\ell}\varrho_\ell}{(3+2\ell)}\operatorname*{Res}_{z'\to 0}\Big[dz'(z')^2K(z,z')\Big(\Big(\hat{\mathrm{A}}^\dagger_{z'}+\frac{1}{\varrho_0(z')^4}\frac{\partial}{\partial z'}\Big)\hat{\mathrm{A}}^\dagger_{z'}+\frac{V^2}{4(2\lambda)^4(z')^4}\Big)\Big]\Big]\mathcal{Z}^{np}_V.\end{aligned}$$

We insert $K(z,z')$ from Proposition 4.10, expand only the geometric series about $z'=0$ while keeping (4.9). Then the outer residue in z is immediate

$$\frac{2V^2}{(2\lambda)^4}\hat{\mathrm{N}}\mathcal{Z}^{np}_V=\operatorname*{Res}_{z'\to 0}\Big[(z')^3\,dz'\frac{\sum_{\ell=0}^\infty\frac{(z')^{2\ell}\varrho_\ell}{(3+2\ell)}}{\sum_{j=0}^\infty\varrho_j(z')^{2j}}\Big(\Big(\hat{\mathrm{A}}^\dagger_{z'}+\frac{1}{\varrho_0(z')^4}\frac{\partial}{\partial z'}\Big)\hat{\mathrm{A}}^\dagger_{z'}+\frac{V^2}{4(2\lambda)^4(z')^4}\Big)\Big]\mathcal{Z}^{np}_V.$$

We rename z' to z and introduce the function

$$\mathcal{R}(z)=\frac{\sum_{\ell=0}^\infty\frac{\varrho_\ell z^{2\ell}}{(3+2\ell)}}{\sum_{j=0}^\infty\varrho_j z^{2j}}=\sum_{m=0}^\infty\mathcal{R}_m(\varrho)\,z^{2m}.$$

The denominator is given by (4.10), without the $\frac{1}{z'\varrho_0}$ prefactor. It combines with the numerator to

$$\mathcal{R}_m(\varrho)=\frac{S_m(\varrho)}{3m!}-\sum_{k=1}^m\frac{\varrho_k}{(3+2k)\varrho_0}\frac{S_{m-k}(\varrho)}{(m-k)!}=-\frac{2}{3}\sum_{k=1}^m\frac{k\varrho_k}{(3+2k)\varrho_0}\frac{S_{m-k}(\varrho)}{(m-k)!}, \tag{5.4}$$

where we have used (4.14) for the first $S_m(\varrho)$ to achieve better control of signs.

The residue of $\frac{V^2}{4(2\lambda)^4z^4}$ is immediate and can be moved to the lhs:

$$\begin{aligned}
\frac{2V^2}{(2\lambda)^4}\Big(\hat{\mathrm{N}}-\frac{1}{24}\Big)\mathcal{Z}^{np}_V &= \sum_{m=0}^\infty \mathcal{R}_m(\varrho)\operatorname*{Res}_{z\to 0}\Big[z^{3+2m}\,dz\Big(\Big(\hat{\mathrm{A}}^\dagger_z+\frac{1}{\varrho_0 z^4}\frac{\partial}{\partial z}\Big)\hat{\mathrm{A}}^\dagger_z\Big]\Big]\mathcal{Z}^{np}_V \\
&= \Big[\sum_{k=0}^\infty\Big(-\frac{3(3+2k)\varrho_1\varrho_{k+1}\mathcal{R}_1(\varrho)}{\varrho_0^3}+\frac{3(3+2k)\varrho_{k+1}\mathcal{R}_2(\varrho)}{\varrho_0^2}\Big)\frac{\partial}{\partial\varrho_k} \\
&+\sum_{k,l=0}^\infty\frac{(3+2k)(3+2l)\mathcal{R}_1(\varrho)}{\varrho_0^2}\varrho_{l+1}\frac{\partial}{\partial\varrho_l}\varrho_{k+1}\frac{\partial}{\partial\varrho_k} \\
&-\sum_{k,l=0}^\infty\frac{(3+2k)(3+2l)\mathcal{R}_{l+2}(\varrho)}{\varrho_0}\Big(\varrho_{k+1}\frac{\partial}{\partial\varrho_k}\frac{\partial}{\partial\varrho_l}+\frac{\partial}{\partial\varrho_l}\varrho_{k+1}\frac{\partial}{\partial\varrho_k}\Big) \\
&+\sum_{k,l=0}^\infty(3+2k)(3+2l)\mathcal{R}_{k+l+3}(\varrho)\frac{\partial}{\partial\varrho_k}\frac{\partial}{\partial\varrho_l} \\
&+\sum_{k=0}^\infty\frac{3(3{+}2k)\varrho_{k+1}\mathcal{R}_2(\varrho)}{\varrho_0^2}\frac{\partial}{\partial\varrho_k}-\sum_{k=0}^\infty\frac{(3{+}2k)(5{+}2k)\mathcal{R}_{k+3}(\varrho)}{\varrho_0}\frac{\partial}{\partial\varrho_k}\Big]\mathcal{Z}^{np}_V.
\end{aligned}$$

Next we separate the ϱ_0-derivatives:

$$\begin{aligned}
&\frac{2V^2}{(2\lambda)^4}\Big(\hat{\mathrm{N}}-\frac{1}{24}\Big)\mathcal{Z}^{np}_V \\
&=\Big[\Big(\frac{9\mathcal{R}_1(\varrho)\varrho_1^2}{\varrho_0^2}-\frac{18\mathcal{R}_2(\varrho)\varrho_1}{\varrho_0}+9\mathcal{R}_3(\varrho)\Big)\frac{\partial^2}{\partial\varrho_0^2} \\
&+\Big(-\frac{9\varrho_1^2\mathcal{R}_1(\varrho)}{\varrho_0^3}+\frac{18\varrho_1\mathcal{R}_2(\varrho)}{\varrho_0^2}+\frac{15\mathcal{R}_1(\varrho)\varrho_2}{\varrho_0^2}-\frac{30\mathcal{R}_3(\varrho)}{\varrho_0}\Big)\frac{\partial}{\partial\varrho_0} \\
&+\sum_{k=1}^\infty 6(3+2k)\Big(\mathcal{R}_{k+3}(\varrho)-\frac{\mathcal{R}_2(\varrho)\varrho_{k+1}}{\varrho_0}-\frac{\mathcal{R}_{k+2}(\varrho)\varrho_1}{\varrho_0}+\frac{\mathcal{R}_1(\varrho)\varrho_{k+1}\varrho_1}{\varrho_0^2}\Big)\frac{\partial}{\partial\varrho_k}\frac{\partial}{\partial\varrho_0} \\
&+\sum_{k,l=1}^\infty(3+2k)(3+2l)\Big(\frac{\varrho_{l+1}\varrho_{k+1}\mathcal{R}_1(\varrho)}{\varrho_0^2}+\mathcal{R}_{k+l+3}(\varrho)-\frac{2\varrho_{k+1}\mathcal{R}_{l+2}(\varrho)}{\varrho_0}\Big)\frac{\partial}{\partial\varrho_k}\frac{\partial}{\partial\varrho_l} \\
&+\sum_{k=1}^\infty(3k+2)\Big(-\frac{3\varrho_1\varrho_{k+1}\mathcal{R}_1(\varrho)}{\varrho_0^3}+\frac{6\varrho_{k+1}\mathcal{R}_2(\varrho)}{\varrho_0^2} \\
&\qquad+\frac{(5+2k)\varrho_{k+2}\mathcal{R}_1(\varrho)}{\varrho_0^2}-\frac{2(5+2k)\mathcal{R}_{k+3}(\varrho)}{\varrho_0}\Big)\frac{\partial}{\partial\varrho_k}\Big]\mathcal{Z}^{np}_V.
\end{aligned}$$

We isolate F_1, i.e. $\mathcal{Z}^{np}_V=\varrho_0^{-\frac{1}{24}}\mathcal{Z}^{stable}_V$, where $\mathcal{Z}^{stable}_V=1+\sum_{g=2}^\infty V^{2-2g}\mathcal{Z}_g$. We commute the factor $\varrho_0^{-\frac{1}{24}}$ in front of [ ] and move it to the other side:

$$\begin{aligned}
\frac{2V^2}{(2\lambda)^4}\hat{\mathrm{N}}\mathcal{Z}^{stable}_V &= \Big[\Big(\frac{49\varrho_1^2\mathcal{R}_1(\varrho)}{64\varrho_0^4}-\frac{49\varrho_1\mathcal{R}_2(\varrho)}{32\varrho_0^3}-\frac{5\mathcal{R}_1(\varrho)\varrho_2}{8\varrho_0^3}+\frac{105\mathcal{R}_3(\varrho)}{64\varrho_0^2}\Big) \\
&+\Big(\frac{9\mathcal{R}_1(\varrho)\varrho_1^2}{\varrho_0^2}-\frac{18\mathcal{R}_2(\varrho)\varrho_1}{\varrho_0}+9\mathcal{R}_3(\varrho)\Big)\frac{\partial^2}{\partial\varrho_0^2}
\end{aligned}$$

$$+\Big(-\frac{39\varrho_1^2\mathcal{R}_1(\varrho)}{4\varrho_0^3}+\frac{39\varrho_1\mathcal{R}_2(\varrho)}{2\varrho_0^2}+\frac{15\mathcal{R}_1(\varrho)\varrho_2}{\varrho_0^2}-\frac{123\mathcal{R}_3(\varrho)}{4\varrho_0}\Big)\frac{\partial}{\partial\varrho_0}$$
$$+\sum_{k=1}^\infty 6(3{+}2k)\Big(\mathcal{R}_{k+3}(\varrho)-\frac{\mathcal{R}_2(\varrho)\varrho_{k+1}}{\varrho_0}-\frac{\mathcal{R}_{k+2}(\varrho)\varrho_1}{\varrho_0}+\frac{\mathcal{R}_1(\varrho)\varrho_{k+1}\varrho_1}{\varrho_0^2}\Big)\frac{\partial}{\partial\varrho_k}\frac{\partial}{\partial\varrho_0}$$
$$+\sum_{k,l=1}^\infty (3{+}2k)(3{+}2l)\Big(\frac{\varrho_{l+1}\varrho_{k+1}\mathcal{R}_1(\varrho)}{\varrho_0^2}+\mathcal{R}_{k+l+3}(\varrho)-\frac{2\varrho_{k+1}\mathcal{R}_{l+2}(\varrho)}{\varrho_0}\Big)\frac{\partial}{\partial\varrho_k}\frac{\partial}{\partial\varrho_l}$$
$$+\sum_{k=1}^\infty (3{+}2k)\Big(-\frac{\mathcal{R}_{k+3}(\varrho)}{4\varrho_0}+\frac{25\varrho_{k+1}\mathcal{R}_2(\varrho)}{4\varrho_0^2}+\frac{\mathcal{R}_{k+2}(\varrho)\varrho_1}{4\varrho_0^2}-\frac{13\varrho_1\varrho_{k+1}\mathcal{R}_1(\varrho)}{4\varrho_0^3}$$
$$+\frac{(5{+}2k)\varrho_{k+2}\mathcal{R}_1(\varrho)}{\varrho_0^2}-\frac{2(5{+}2k)\mathcal{R}_{k+3}(\varrho)}{\varrho_0}\Big)\frac{\partial}{\partial\varrho_k}\Big]\mathcal{Z}_V^{stable}.$$

Next observe

$$\hat{\mathrm{N}}\mathcal{Z}_V^{stable}=\sum_{g=2}^\infty (V^{-2})^{g-1}(2g-2)\mathcal{Z}_g=2V^{-2}\frac{d}{dV^{-2}}\sum_{g=2}^\infty (V^{-2})^{g-1}\mathcal{Z}_g=2V^{-2}\frac{d}{dV^{-2}}\mathcal{Z}_V^{stable}.$$

Consequently, we obtain a parabolic differential equation in V^{-2} which is easily solved. Inserting

$$\mathcal{R}_1(\varrho)=-\frac{2}{15}\frac{\varrho_1}{\varrho_0},\quad \mathcal{R}_2(\varrho)=\frac{2}{15}\frac{\varrho_1^2}{\varrho_0^2}-\frac{4}{21}\frac{\varrho_2}{\varrho_0},\quad \mathcal{R}_3(\varrho)=-\frac{2}{15}\frac{\varrho_1^3}{\varrho_0^3}+\frac{34}{105}\frac{\varrho_1\varrho_2}{\varrho_0^2}-\frac{2}{9}\frac{\varrho_3}{\varrho_0},$$

we have:

Theorem 5.6. *When expressed in terms of the moments of the measure ϱ, the stable partition function is given by*

$$\mathcal{Z}_V^{stable}:=\exp\Big(\sum_{g=2}^\infty V^{2-2g}F_g(\varrho)\Big)=\exp\Big(-\frac{(2\lambda)^4}{V^2}\Delta_\varrho+F_2(\rho)\Big)1,$$

where

$$F_2=\frac{(2\lambda)^4}{V^2}\Big(-\frac{21\varrho_1^3}{160\varrho_0^5}+\frac{29}{128}\frac{\varrho_1\varrho_2}{\varrho_0^4}-\frac{35}{384}\frac{\varrho_3}{\varrho_0^3}\Big),\tag{5.5}$$

$$\Delta_\varrho:=-\Big(-\frac{6\varrho_1^3}{5\varrho_0^3}+\frac{111\varrho_1\varrho_2}{70\varrho_0^2}-\frac{\varrho_3}{2\varrho_0}\Big)\frac{\partial^2}{\partial\varrho_0^2}-\Big(\frac{2\varrho_1^3}{\varrho_0^4}-\frac{1097\varrho_1\varrho_2}{280\varrho_0^3}+\frac{41\varrho_3}{24\varrho_0^2}\Big)\frac{\partial}{\partial\varrho_0}$$
$$-\sum_{k=1}^\infty (3+2k)\Big(\Big(-\frac{2\varrho_1^2}{5\varrho_0^3}+\frac{2\varrho_2}{7\varrho_0^2}\Big)\varrho_{k+1}-\frac{3\mathcal{R}_{k+2}(\varrho)\varrho_1}{2\varrho_0}+\frac{3\mathcal{R}_{k+3}(\varrho)}{2}\Big)\frac{\partial^2}{\partial\varrho_k\partial\varrho_0}$$
$$-\sum_{k,l=1}^\infty (3{+}2k)(3{+}2l)\Big(-\frac{\varrho_1\varrho_{l+1}\varrho_{k+1}}{30\varrho_0^2}-\frac{\varrho_{k+1}\mathcal{R}_{l+2}(\varrho)}{4\varrho_0}-\frac{\varrho_{l+1}\mathcal{R}_{k+2}(\varrho)}{4\varrho_0}+\frac{\mathcal{R}_{k+l+3}(\varrho)}{4}\Big)\frac{\partial^2}{\partial\varrho_k\partial\varrho_l}$$
$$-\sum_{k=1}^\infty (3+2k)\Big(\Big(\frac{19\varrho_1^2}{60\varrho_0^4}-\frac{25\varrho_2}{84\varrho_0^3}\Big)\varrho_{k+1}+\frac{\varrho_1\mathcal{R}_{k+2}(\varrho)}{16\varrho_0^2}-\frac{\mathcal{R}_{k+3}(\varrho)}{16\varrho_0}$$
$$-\frac{(5+2k)\varrho_1\varrho_{k+2}}{30\varrho_0^3}-\frac{(5+2k)\mathcal{R}_{k+3}(\varrho)}{2\varrho_0}\Big)\frac{\partial}{\partial\varrho_k}\tag{5.6}$$

and $\mathcal{R}_m(\varrho)$ given by (5.4).

Because we are essentially treating the Kontsevich model [Kon92], our F_g are nothing else than the generators of intersection numbers on the moduli space of complex curves [Wit91, Kon92, IZ92, Eyn16]. These free energies are listed in different conventions in the literature. The translation to e.g. [IZ92, Eyn16] is as follows:

$$\begin{aligned} &[\text{IZ92}]: && (1-I_1)=\varrho_0, && I_{k+1}=-(2k+1)!!\varrho_k \quad \text{for } k\geq 1,\\ &[\text{Eyn16}]: && (2-t_3)=\varrho_0, && t_{2k+3}=-\varrho_k, \quad \text{for } k\geq 1. \end{aligned}$$

It is clear that Theorem 5.6 translates into the same statement for the generating function of intersection numbers. We have given this formulation in the very beginning in Theorem 1.1. There we adopt the conventions in [IZ92] but rename $I_k \equiv t_k$ and $T_0 \equiv (1-I_1)$. The formula can easily be implemented in computer algebra[3] and quickly computes the free energies $F_g(t)$ to moderately large g. Several other implementations exist. We are aware of an implementation [Xu07] (up to $g=10$) in Maple and a powerful implementation [DSvZ18] in Sage which performs many more natural operations in the tautological ring. These were an important consistency check for us. Algorithms to compute κ, δ, λ-classes from ψ-classes are given in [Fab99]. For convenience we list

$$\begin{aligned} F_3 = {}& \frac{1225}{144}\cdot\frac{t_2^6}{6!T_0^{10}} + \frac{193}{288}\cdot\frac{t_2^4t_3}{4!T_0^9} + \frac{205}{3456}\cdot\frac{t_2^2t_3^2}{2!2!T_0^8} + \frac{53}{1152}\cdot\frac{t_2^3t_4}{3!T_0^8} + \frac{583}{96768}\cdot\frac{t_3^3}{3!T_0^7} + \frac{1121}{241920}\cdot\frac{t_2t_3t_4}{T_0^7}\\ &+ \frac{17}{5760}\cdot\frac{t_2^2t_5}{2!T_0^7} + \frac{607}{1451520}\cdot\frac{t_4^2}{2!T_0^6} + \frac{503}{1451520}\cdot\frac{t_3t_5}{T_0^6} + \frac{77}{414720}\cdot\frac{t_2t_6}{T_0^6} + \frac{1}{82944}\cdot\frac{t_7}{T_0^5} \end{aligned}$$

(already given in [IZ92, eq. (5.30)]) and

$$\begin{aligned} F_4 = {}& \frac{1816871}{48}\cdot\frac{t_2^9}{9!T_0^{15}} + \frac{3326267}{1728}\cdot\frac{t_2^7t_3}{7!T_0^{14}} + \frac{728465}{6912}\cdot\frac{t_2^5t_3^2}{5!2!T_0^{13}} + \frac{43201}{6912}\cdot\frac{t_2^3t_3^3}{3!3!T_0^{12}} + \frac{134233}{331776}\cdot\frac{t_2t_3^4}{4!T_0^{11}}\\ &+ \frac{70735}{864}\cdot\frac{t_2^6t_4}{6!T_0^{13}} + \frac{83851}{17280}\cdot\frac{t_2^4t_3t_4}{4!T_0^{12}} + \frac{26017}{82944}\cdot\frac{t_2^2t_3^2t_4}{2!2!T_0^{11}} + \frac{185251}{8294400}\cdot\frac{t_3^3t_4}{3!T_0^{10}} + \frac{5609}{23040}\cdot\frac{t_2^3t_4^2}{3!2!T_0^{11}}\\ &+ \frac{177}{10240}\cdot\frac{t_2t_3t_4^2}{2!T_0^{10}} + \frac{175}{165888}\cdot\frac{t_4^3}{3!T_0^9} + \frac{21329}{6912}\cdot\frac{t_2^5t_5}{5!T_0^{12}} + \frac{13783}{69120}\cdot\frac{t_2^3t_3t_5}{3!T_0^{11}} + \frac{1837}{129600}\cdot\frac{t_2t_3^2t_5}{2!T_0^{10}}\\ &+ \frac{7597}{691200}\cdot\frac{t_2^2t_4t_5}{2!T_0^{10}} + \frac{719}{829440}\cdot\frac{t_3t_4t_5}{T_0^9} + \frac{533}{967680}\cdot\frac{t_2t_5^2}{2!T_0^9} + \frac{2471}{23040}\cdot\frac{t_2^4t_6}{4!T_0^{11}} + \frac{7897}{1036800}\cdot\frac{t_2^2t_3t_6}{2!T_0^{10}}\\ &+ \frac{1997}{3317760}\cdot\frac{t_3^2t_6}{2!T_0^9} + \frac{1081}{2322432}\cdot\frac{t_2t_4t_6}{T_0^9} + \frac{487}{18579456}\cdot\frac{t_5t_6}{T_0^8} + \frac{4907}{1382400}\cdot\frac{t_2^3t_7}{3!T_0^{10}} + \frac{16243}{58060800}\cdot\frac{t_2t_3t_7}{T_0^9}\\ &+ \frac{1781}{92897280}\cdot\frac{t_4t_7}{T_0^8} + \frac{53}{460800}\cdot\frac{t_2^2t_8}{2!T_0^9} + \frac{947}{92897280}\cdot\frac{t_3t_8}{T_0^8} + \frac{149}{39813120}\cdot\frac{t_2t_9}{T_0^8} + \frac{1}{7962624}\cdot\frac{t_{10}}{T_0^7}. \end{aligned}$$

The first line agrees with [IZ92, Table II]. The notation is such that the intersection numbers are easily identified, e.g. $\langle\tau_2\tau_3^4\rangle = \frac{134233}{331776}$ or $\langle\tau_2^2\tau_4\tau_5\rangle = \frac{7597}{691200}$. The very last number is $\langle\tau_{3g-2}\rangle = \frac{1}{24^g\cdot g!}$ for $g=4$, in agreement with [IZ92, eq. (5.31)]. The arXiv version v1 of this paper also gave F_5 and F_6 in an appendix, but there is not really a need for them.

[3] A first implementation in Mathematica is provided via the arXiv page of this paper or via `http://wwwmath.uni-muenster.de/u/raimar/files/IntersectionNumbers.nb`
It takes less than 35 seconds on an office desktop to compute all intersection numbers up to $g=10$.

5.3. A deformed Virasoro algebra

We return to (5.3), but instead of applying the inverse of $\hat{K}_z$ we directly take the residue

$$\tilde{L}_n := \mathrm{Res}_{z\to 0}\Big[z^{3+2n}\Big(\frac{2V^2}{(2\lambda)^4}\hat{K}_z\hat{\mathrm{A}}^\dagger_z + (\hat{\mathrm{A}}^\dagger_z)^2 + \frac{1}{\varrho_0 z^4}\frac{\partial \hat{\mathrm{A}}^\dagger_z}{\partial z} + \frac{V^2}{4(2\lambda)^4 z^4}\Big)dz\Big].$$

By construction, $\tilde{L}_n\mathcal{Z}^{np}_V = 0$. Recall that in the Kontsevich model one has $L_n\mathcal{Z}$ for the *full* partition function and generators L_n of a Virasoro algebra (or rather a Witt algebra). Surprisingly, our $\tilde{L}_n$ do not satisfy the commutataion relations of the Virasoro algebra exactly. An explicit expression is obtained from

$$\begin{aligned}
\hat{K}_z\hat{\mathrm{A}}^\dagger_z &= \sum_{l=0}^\infty\sum_{j=0}^{l}\frac{(3+2l)\varrho_{l-j}}{z^{4+2j}}\frac{\partial}{\partial\varrho_l},\\
\frac{1}{\varrho_0 z^4}\frac{\partial}{\partial z}\hat{\mathrm{A}}^\dagger_z &= \sum_{l=0}^\infty\Big(\frac{3(3+2l)\varrho_{l+1}}{\varrho_0^2 z^8} - \frac{(3+2l)(5+2l)}{\varrho_0 z^{10+2l}}\Big)\frac{\partial}{\partial\varrho_l},\\
\hat{\mathrm{A}}^\dagger_z\hat{\mathrm{A}}^\dagger_z &= \sum_{k=0}^\infty\Big(\frac{(5+2k)\varrho_{k+2}}{\varrho_0 z^3} - \frac{5+2k}{z^{7+2k}}\Big)\frac{(3+2k)}{\varrho_0 z^3}\frac{\partial}{\partial\varrho_k}\\
&\quad+\sum_{k=0}^\infty\Big(-\frac{3\varrho_1}{\varrho_0 z^3}+\frac{3}{z^5}\Big)\Big(\frac{(3{+}2k)\varrho_{k+1}}{\varrho_0^2 z^3}\Big)\frac{\partial}{\partial\varrho_k} + \sum_{l,k=0}^\infty\frac{(3{+}2l)(3{+}2k)\varrho_{k+1}\varrho_{l+1}}{\varrho_0^2 z^6}\frac{\partial^2}{\partial\varrho_l\partial\varrho_k}\\
&\quad-\sum_{l,k=0}^\infty\frac{2(3+2l)(3+2k)\varrho_{l+1}}{\varrho_0 z^{8+2k}}\frac{\partial^2}{\partial\varrho_l\partial\varrho_k} + \sum_{l,k=0}^\infty\frac{(3+2l)(3+2k)}{z^{10+2l+2k}}\frac{\partial^2}{\partial\varrho_l\partial\varrho_k}.
\end{aligned}$$

Evaluating the residues gives

$$\begin{aligned}
\tilde{L}_0 &= \frac{2V^2}{(2\lambda)^4}\Big(\sum_{l=0}^\infty(3+2l)\varrho_l\frac{\partial}{\partial\varrho_l}+\frac{1}{8}\Big),\\
\tilde{L}_1 &= \frac{2V^2}{(2\lambda)^4}\sum_{l=0}^\infty(5+2l)\varrho_l\frac{\partial}{\partial\varrho_{l+1}} + \Big(\sum_{k=0}^\infty\frac{(3+2k)}{\varrho_0^2}\varrho_{k+1}\frac{\partial}{\partial\varrho_k} - \frac{3\varrho_1}{\varrho_0^3}\Big)\sum_{l=0}^\infty(3+2l)\varrho_{l+1}\frac{\partial}{\partial\varrho_l}
\end{aligned}$$

and for $n\geq 2$:

$$\begin{aligned}
\tilde{L}_n &= \frac{2V^2}{(2\lambda)^4}\sum_{l=0}^\infty(3{+}2n{+}2l)\varrho_l\frac{\partial}{\partial\varrho_{n+l}} + \delta_{n,2}\sum_{l=0}^\infty\frac{6(3+2l)\varrho_{l+1}}{\varrho_0^2}\frac{\partial}{\partial\varrho_l} - \frac{2(2n{-}3)(2n{-}1)}{\varrho_0}\frac{\partial}{\partial\varrho_{n-3}}\\
&\quad+\sum_{l=0}^{n-3}(3+2l)(2n-2l-3)\frac{\partial^2}{\partial\varrho_l\partial\varrho_{n-3-l}} - \sum_{l=0}^\infty\frac{2(3+2l)(2n-1)\varrho_{l+1}}{\varrho_0}\frac{\partial^2}{\partial\varrho_{n-2}\partial\varrho_l}.
\end{aligned}$$

It is convenient to commute the factor $\exp(F_1)=\varrho_0^{-\frac{1}{24}}$ through the $\tilde{L}_n$ and to pass to

$$L_n := \frac{(2\lambda)^4}{4V^2}\varrho_0^{\frac{1}{24}}\tilde{L}_n\varrho_0^{-\frac{1}{24}}$$

The result is:

Lemma 5.7. *The stable partition* $\mathcal{Z}_V^{stable} = 1 + \sum_{g=2}^{\infty} V^{2-2g}\mathcal{Z}_g$ *satisfies the constraints* $L_n\mathcal{Z}_V^{stable} = 0$ *for all* $n \in \mathbb{N}$*, where*

$$
\begin{aligned}
L_0 &= \sum_{l=0}^{\infty} \frac{(3+2l)}{2}\varrho_l \frac{\partial}{\partial\varrho_l},\\
L_1 &= \sum_{l=0}^{\infty} \frac{(5+2l)}{2}\varrho_l \frac{\partial}{\partial\varrho_{l+1}} + \frac{(2\lambda)^4}{4V^2}\Big\{\sum_{k,l=0}^{\infty} \frac{(3+2k)(3+2l)\varrho_{k+1}\varrho_{l+1}}{\varrho_0^2}\frac{\partial^2}{\partial\varrho_k\partial\varrho_l}\\
&\quad + \sum_{k=0}^{\infty}(3+2k)\Big(-\frac{13\varrho_1\varrho_{k+1}}{4\varrho_0^3} + \frac{(5+2k)\varrho_{k+2}}{\varrho_0^2}\Big)\frac{\partial}{\partial\varrho_k} + \frac{49\varrho_1^2}{64\varrho_0^4} - \frac{5\varrho_2}{\varrho_0^3}\Big\}\\
L_2 &= \sum_{l=0}^{\infty} \frac{(7+2l)}{2}\varrho_l \frac{\partial}{\partial\varrho_{l+2}} + \frac{(2\lambda)^4}{4V^2}\Big\{-\sum_{k=0}^{\infty} \frac{6(3+2k)\varrho_{k+1}}{\varrho_0}\frac{\partial^2}{\partial\varrho_k\partial\varrho_0}\\
&\quad + \sum_{k=1}^{\infty}\frac{25(3+2k)\varrho_{k+1}}{4\varrho_0^2}\frac{\partial}{\partial\varrho_k} + \frac{39\varrho_1}{2\varrho_0^2}\frac{\partial}{\partial\varrho_0} - \frac{49\varrho_1}{32\varrho_0^3}\Big\}\\
L_3 &= \sum_{l=0}^{\infty} \frac{(9+2l)}{2}\varrho_l \frac{\partial}{\partial\varrho_{l+3}} + \frac{(2\lambda)^4}{4V^2}\Big\{9\frac{\partial^2}{\partial\varrho_0^2} - \sum_{k=0}^{\infty} \frac{10(3+2k)\varrho_{k+1}}{\varrho_0}\frac{\partial^2}{\partial\varrho_k\partial\varrho_1}\\
&\quad + \frac{5\varrho_1}{4\varrho_0^2}\frac{\partial}{\partial\varrho_1} - \frac{123}{4\varrho_0}\frac{\partial}{\partial\varrho_0} + \frac{105}{64\varrho_0^2}\Big\}
\end{aligned}
$$

and for $n \geq 4$

$$
\begin{aligned}
L_n &= \sum_{l=0}^{\infty} \frac{(3{+}2n{+}2l)}{2}\varrho_l \frac{\partial}{\partial\varrho_{n+l}} + \frac{(2\lambda)^4}{4V^2}\Big\{\sum_{l=0}^{n-3}(3+2l)(2n-2l-3)\frac{\partial^2}{\partial\varrho_l\partial\varrho_{n-3-l}}\\
&- \sum_{l=0}^{\infty}\frac{2(3{+}2l)(2n{-}1)\varrho_{l+1}}{\varrho_0}\frac{\partial^2}{\partial\varrho_{n-2}\partial\varrho_l} - \frac{(2n{-}3)(16n{-}7)}{4\varrho_0}\frac{\partial}{\partial\varrho_{n-3}} + \frac{(2n{-}1)\varrho_1}{4\varrho_0^2}\frac{\partial}{\partial\varrho_{n-2}}\Big\}.
\end{aligned}
$$

We have $[L_m, L_n] = (m-n)L_{m+n} + C_{m,n}$ for a nonvanishing differential operator $C_{m,n}$ which, by construction, also annihilates the $\mathcal{Z}_V^{stable}$. It will be studied elsewhere.

6. Summary

The construction of the renormalised Φ^3_D-QFT model on noncommutative geometries of dimension $D \leq 6$ is now complete. After the previous solution of the planar sector in [GSW17, GSW18] we establish in this paper an algorithm to compute any correlation function $G^{(g)}_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}$ of genus $g \geq 1$:

1. Compute the free energy $F_g(t)$ via Theorem 1.1 and the note thereafter. It encodes the $p(3g{-}3)$ intersection numbers of ψ-classes on the moduli space of complex curves of genus g. Take $F_1 = -\frac{1}{24}\log T_0$ for $g = 1$. Alternatively, start from intersection numbers obtained by other methods (e.g. [DSvZ18]).
2. Change variables to $\varrho_0 = 1 - t_0$ and $\varrho_l = -\frac{t_l+1}{(2l+1)!!}$, where ϱ_l are given by (3.16) for the measure (3.9) and with c implicitly defined by (3.12).

3. Apply to the resulting $F_g(\varrho)$ according to Proposition 5.2 and Theorem 4.6 the boundary creation operators $\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B} \circ \dots \hat{\mathrm{A}}^{\dagger g}_{z_1,z_2} \circ \hat{\mathrm{A}}^{\dagger g}_{z_1}$ defined in Definition 4.14. Multiply by $(2\lambda)^{4g+3B-4+\delta_{B,1}}$ to obtain $\mathcal{G}_g(z_1|\cdots|z_B)$.

4. Pass to $\mathcal{G}_g(z_1^1...z_{N_1}^1|...|z_1^B...z_{N_B}^B)$ via difference quotients similar to (3.13) [$X^\beta_{k_\beta}$ stands for $(z^\beta_{k_\beta})^2-c$].

5. Specify to $z^\beta_{k_\beta} \mapsto (4F^2_{\underline{p}^\beta_{k_\beta}}+c)^{1/2}$ to obtain $G^{(g)}_{|\underline{p}_1^1...\underline{p}_{N_1}^1|...|\underline{p}_1^B...\underline{p}_{N_B}^B|}$, where $F_{\underline{p}}$ arises by mass-renormalisation from the $E_{\underline{p}}$ in the initial action (3.1) of the model.

Our work was essentially a reverse engineering in opposite order. The last step 5. was given to us by the formal partition function of the model. From there we had to climb up to the formula for the intersection numbers.

We remark that, in spite of the relation to the integrable Kontsevich model, this Φ^3_D-model provides a fascinating toy model for a quantum field theory which shows many facets of renormalisation. Our exact formulae can be expanded about $\lambda = 0$ via (3.12) and agree with the usual perturbative renormalisation which in $D = 6$ needs Zimmermann's forest formula [Zim69] (see [GSW18]). Also note that at fixed genus g one expects $\mathcal{O}(n!)$ graphs with n vertices so that a convergent summation at fixed g cannot be expected a priori. Moreover, in $D = 6$ the β-function of the coupling constant is positive for real λ, which in this particular case poses not the slightest problem for summation.

What remains to understand is the resummation in the genus, i.e. $\sum_{g=2}^\infty V^{2-2g}\mathcal{G}_g(z)$ or $\sum_{g=2}^\infty N^{2-2g}F_g(t)$. All intersection numbers are positive for $t_l > 0$, which corresponds to $\varrho_l < 0$ for $l \geq 1$. Because of the λ^2-prefactor in front of (3.9) and the definition (3.16) of the ϱ_l, we have $t_l > 0$ for real λ. Therefore, the sum over the genus must diverge for $\lambda \in \mathbb{R}$, which is not surprising because in this case the action (3.1) is unbounded from below. In contrast, it was observed in [GSW18] that for the planar sector it is better to take $\lambda \in \mathbb{R}$. The final challenge of this model is to establish that $\sum_{g=2}^\infty N^{2-2g}F_g(t)$ is Borel summable for $t_l < 0$, which would achieve convergence of the genus expansion in two disks in the complex λ-plane tangent from above and below the real axis at $\lambda = 0$.

The deformed Virasoro algebra also deserves investigation.

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A. Lemmata relevant for Theorem 4.6

Assumption A.1. *We assume that $\mathcal{G}_g(z)$ is, for $g \geq 1$, a function of z and of $\varrho_0, \dots, \varrho_{3g-2}$ (true for $g = 1$). We take eq. (4.7) and in particular $\mathcal{G}_g(z|z\triangleleft_J) := (2\lambda)^3 \hat{\mathrm{A}}^{\dagger g}_{z_J,z}\mathcal{G}_g(z\triangleleft_J)$ as a* definition *of a family of functions $\mathcal{G}_g(z_1|z_J)$ and derive equations for that family.*

Lemma A.2. *Let $J = \{2, ..., B\}$. Then under Assumption A.1 and with Definition 4.1 of the operator $\hat{K}_{z_1}$ one has*

$$\hat{K}_{z_1}\mathcal{G}_g(z_1|z\triangleleft_J) = \frac{8\lambda^3}{z_1^2}\Bigg(\sum_{l=0}^{3g-3+|J|}(3+2l)\frac{\partial \mathcal{G}_g(z\triangleleft_J)}{\partial \varrho_l}\sum_{k=0}^{l}\frac{\varrho_k}{z_1^{2+2l-2k}} + \sum_{\beta\in J}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\mathcal{G}_g(z\triangleleft_J)\Bigg).$$

Proof. Take Definition 4.4 for $\hat{\mathrm{A}}^{\dagger g}_{z_J,z}\mathcal{G}_g(z\triangleleft_J)$ and apply Lemma 4.2. □

Lemma A.3. *Let $J = \{2, ..., B\}$. Then under Assumption A.1 one has*

$$\begin{aligned}
&\frac{8\lambda^3}{z_\beta}\frac{\partial}{\partial z_\beta}\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}}) - \mathcal{G}_g(z_\beta|z\triangleleft_{J\setminus\{\beta\}})}{z_1^2 - z_\beta^2} + 2\lambda\mathcal{G}_0(z_1|z_\beta)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}})\\
&= (2\lambda)^6\Bigg[\sum_{l=0}^{3g-4+|J|}\Bigg(-\sum_{n=0}^{1}\frac{(3+2l)(1+2n)\varrho_{l+1}}{\varrho_0 z_1^{4-2n}z_\beta^{3+2n}} + \sum_{n=0}^{l+2}\frac{(3+2l)(1+2n)}{z_1^{6+2l-2n}z_\beta^{3+2n}}\Bigg)\frac{\partial \mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial \varrho_l}\\
&+ \sum_{i\in J\setminus\{\beta\}}\sum_{n=0}^{1}\frac{1+2n}{\varrho_0 z_i z_1^{4-2n}z_\beta^{3+2n}}\frac{\partial \mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial z_i}\Bigg].
\end{aligned}$$

Proof. Definition 4.4 gives with $\frac{\frac{1}{z_1^{3+2j}} - \frac{1}{y^{3+2j}}}{z_1^2 - y^2} = -\sum_{l=0}^{2j+2}\frac{z_1^l y^{2j+2-l}}{z_1^{3+2j}y^{3+2j}(z+y)}$ for the first term

$$\begin{aligned}
&\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}}) - \mathcal{G}_g(z_\beta|z\triangleleft_{J\setminus\{\beta\}})}{(2\lambda)^3(z_1^2 - z_\beta^2)}\\
&= \sum_{l=0}^{3g-4+|J|}\Bigg(-\frac{(3+2l)\varrho_{l+1}}{\varrho_0}\Bigg(\frac{\frac{1}{z_1^3} - \frac{1}{z_\beta^3}}{z_1^2 - z_\beta^2}\Bigg) + (3+2l)\Bigg(\frac{\frac{1}{z_1^{5+2l}} - \frac{1}{z_\beta^{5+2l}}}{z_1^2 - z_\beta^2}\Bigg)\Bigg)\frac{\partial \mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial \varrho_l}\\
&+ \sum_{i\in J\setminus\{\beta\}}\frac{1}{\varrho_0 z_i}\Bigg(\frac{\frac{1}{z_1^3} - \frac{1}{z_\beta^3}}{z_1^2 - z_\beta^2}\Bigg)\frac{\partial \mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial z_i}\\
&= \sum_{l=0}^{3g-4+|J|}\Bigg(\frac{(3+2l)\varrho_{l+1}}{\varrho_0}\frac{\sum_{n=0}^{2}z_1^n z_\beta^{2-n}}{z_1^3 z_\beta^3(z_1+z_\beta)} - (3+2l)\frac{\sum_{n=0}^{2l+4}z_1^n z_\beta^{2l+4-n}}{z_1^{5+2l}z_\beta^{5+2l}(z_1+z_\beta)}\Bigg)\frac{\partial \mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial \varrho_l}\\
&- \sum_{i\in J\setminus\{\beta\}}\frac{1}{\varrho_0 z_i}\frac{\sum_{n=0}^{2}z_1^n z_\beta^{2-n}}{z_1^3 z_\beta^3(z_1+z_\beta)}\frac{\partial \mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial z_i}.
\end{aligned}$$

The second term reads

$$\begin{aligned}&\frac{1}{(2\lambda)^3}\mathcal{G}_0(z_1|z_\beta)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}})\\&=-\frac{4\lambda^2}{z_1z_\beta}\frac{\partial}{\partial z_\beta}\frac{1}{(z_1+z_\beta)}\Bigg[\sum_{l=0}^{3g-4+|J|}\Big(\frac{-(3+2l)\varrho_{l+1}}{\varrho_0 z_1^3}+\frac{(3+2l)}{z_1^{5+2l}}\Big)\frac{\partial\mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial\varrho_l}\\&+\sum_{i\in J\setminus\{\beta\}}\frac{1}{\varrho_0 z_i z_1^3}\frac{\partial\mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial z_i}\Bigg].\end{aligned}$$

The denominator (z_1+z_β) cancels in the combination of interest:

$$\begin{aligned}&\frac{8\lambda^3}{z_\beta}\frac{\partial}{\partial z_\beta}\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}})-\mathcal{G}_g(z_\beta|z\triangleleft_{J\setminus\{\beta\}})}{z_1^2-z_\beta^2}+2\lambda\mathcal{G}_0(z_1|z_\beta)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}})\\&=\frac{(2\lambda)^6}{z_\beta}\frac{\partial}{\partial z_\beta}\Bigg[\sum_{l=0}^{3g-4+|J|}\Bigg(\frac{(3+2l)\varrho_{l+1}}{\varrho_0}\frac{z_1^2+z_\beta^2}{z_1^4z_\beta^3}-(3+2l)\frac{\sum_{n=0}^{l+2}z_1^{2n}z_\beta^{2l+4-2n}}{z_1^{6+2l}z_\beta^{5+2l}}\Bigg)\frac{\partial\mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial\varrho_l}\\&-\sum_{i\in J\setminus\{\beta\}}\frac{1}{\varrho_0 z_i}\frac{z_1^2+z_\beta^2}{z_1^4z_\beta^3}\frac{\partial\mathcal{G}_g(z\triangleleft_{J\setminus\{\beta\}})}{\partial z_i}\Bigg].\end{aligned}$$

The remaining z_β-derivative confirms the assertion. □

Lemma A.4. *Let $J=\{2,...,B\}$. Then under Assumption A.1 one has*

$$\begin{aligned}&\sum_{h=1}^{g}\mathcal{G}_h(z_1|z\triangleleft_J)\mathcal{G}_{g-h}(z_1)+\lambda\sum_{h=1}^{g-1}\sum_{\substack{I\subset J\\1\le|I|<|J|}}\mathcal{G}_h(z_1|z\triangleleft_I)\mathcal{G}_{g-h}(z_1|z\triangleleft_{J\setminus I})+\lambda G_{g-1}(z_1|z_1|z\triangleleft_J)\\&=-(2\lambda)^{3B-4}\hat{\text{A}}^{\dagger g}_{z_1,...,z_B}...\hat{\text{A}}^{\dagger g}_{z_1,z_2}\hat{K}_{z_1}\mathcal{G}_g(z_1).\end{aligned}$$

Proof. Equation (4.2) can be rewritten as

$$-\hat{K}_{z_1}\mathcal{G}_g(z_1)=\frac{1}{2}\sum_{h=1}^{g-1}\mathcal{G}_h(z_1)\mathcal{G}_{g-h}(z_1)+2\lambda^2\mathcal{G}_{g-1}(z_1|z_1).$$

Operating with $-(2\lambda)^{3B-4}\hat{\text{A}}^{\dagger g}_{z_1,...,z_B}...\hat{\text{A}}^{\dagger g}_{z_1,z_2}$ and taking the Leibniz rule into account, the assertion follows. □

Lemma A.5. *Let $J=\{2,...,B\}$. Then under Assumption A.1 one has*

$$\begin{aligned}&(2\lambda)^3[\hat{K}_{z_1},\hat{\text{A}}^{\dagger g}_{z_1,...,z_B}]\mathcal{G}_g(z_1|z\triangleleft_{J\setminus B})\\&=(2\lambda)^6\Bigg[\sum_{l=0}^{3g-4+|J|}\frac{3+2l}{z_1^2z_B^3}\Bigg(\frac{\varrho_{l+1}}{\varrho_0 z_1^2}+\frac{3\varrho_{l+1}}{\varrho_0 z_B^2}-\frac{1}{z_1^{4+2l}}-\frac{(5+2l)}{z_B^{4+2l}}\Bigg)\frac{\partial}{\partial\varrho_l}\\&-\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l}\frac{(3+2l)(3+2k)}{z_1^{4+2l-2k}z_B^{5+2k}}\frac{\partial}{\partial\varrho_l}-\sum_{i\in J\setminus\{B\}}\frac{1}{\varrho_0 z_1^2z_iz_B^3}\Big(\frac{1}{z_1^2}+\frac{3}{z_B^2}\Big)\frac{\partial}{\partial z_i}\Bigg]\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}}).\end{aligned}$$

Proof. The first term of the lhs, $\hat{K}_{z_1}\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}\mathcal{G}_g(z_1|z\triangleleft_{J\setminus B})$, is given by Lemma A.2 and $\mathcal{G}_g(z\triangleleft_J)=(2\lambda)^3\hat{\mathrm{A}}^{\dagger g}_{z_2,\dots,z_B}\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}})$ to

$$\begin{aligned}
&\hat{K}_{z_1}(\mathcal{G}_g(z_1|z\triangleleft_J))\\
&=\frac{(2\lambda)^6}{z_1^2}\Bigg[\sum_{l=0}^{3g-3+|J|}(3+2l)\sum_{k=0}^{l}\frac{\varrho_k}{z_1^{2+2l-2k}}\\
&\quad\times\frac{\partial}{\partial\varrho_l}\Bigg(\sum_{l'=0}^{3g-4+|J|}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_B^3}+\frac{3+2l'}{z_B^{5+2l'}}\Big)\frac{\partial}{\partial\varrho_{l'}}+\sum_{i\in J\setminus\{B\}}\frac{1}{\varrho_0 z_i z_B^3}\frac{\partial}{\partial z_i}\Bigg)\big(\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}})\big)\\
&+\sum_{\beta\in J}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\Bigg(\sum_{l'=0}^{3g-4+|J|}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_B^3}+\frac{3+2l'}{z_B^{5+2l'}}\Big)\frac{\partial}{\partial\varrho_{l'}}+\sum_{i\in J\setminus\{B\}}\frac{1}{\varrho_0 z_i z_B^3}\frac{\partial}{\partial z_i}\Bigg)\big(\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}})\big)\Bigg]\\
&=\frac{(2\lambda)^6}{z_1^2}\Bigg[\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l+1}\frac{-(5+2l)(3+2l)\varrho_k}{\varrho_0 z_1^{4+2l-2k}z_B^3}\frac{\partial}{\partial\varrho_l}+\sum_{l'=0}^{3g-4+|J|}\frac{3(3+2l')\varrho_0\varrho_{l'+1}}{\varrho_0^2 z_1^2 z_B^3}\frac{\partial}{\partial\varrho_{l'}}\\
&-\sum_{i\in J\setminus\{B\}}\frac{3\varrho_0}{\varrho_0^2 z_1^2 z_i z_B^3}\frac{\partial}{\partial z_i}+\sum_{l,l'=0}^{3g-4+|J|}\sum_{k=0}^{l}\frac{(3+2l)\varrho_k}{z_1^{2+2l-2k}}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_B^3}+\frac{3+2l'}{z_B^{5+2l'}}\Big)\frac{\partial^2}{\partial\varrho_l\partial\varrho_{l'}}\\
&+\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l}\sum_{i\in J\setminus\{B\}}\frac{(3+2l)\varrho_k}{\varrho_0 z_1^{2+2l-2k}z_i z_B^3}\frac{\partial^2}{\partial\varrho_l\partial z_i}\\
&+\sum_{\beta\in J\setminus\{B\}}\Bigg(\sum_{l'=0}^{3g-4+|J|}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_\beta z_B^3}+\frac{3+2l'}{z_\beta z_B^{5+2l'}}\Big)\frac{\partial^2}{\partial z_\beta\partial\varrho_{l'}}+\sum_{i\in J\setminus\{B\}}\frac{1}{\varrho_0 z_\beta z_B^3}\frac{\partial}{\partial z_\beta}\frac{1}{z_i}\frac{\partial}{\partial z_i}\Bigg)\\
&-\sum_{l'=0}^{3g-4+|J|}\Big(-\frac{3(3+2l')\varrho_{l'+1}}{\varrho_0 z_B^5}+\frac{(3+2l')(5+2l')}{z_B^{7+2l'}}\Big)\frac{\partial}{\partial\varrho_{l'}}-\sum_{i\in J\setminus\{B\}}\frac{3}{\varrho_0 z_i z_B^5}\frac{\partial}{\partial z_i}\Bigg]\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}}).
\end{aligned}$$

We have used that $\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}})$ can only depend on ϱ_l for $l\le 3g-4+|J|$. For the second term of the lhs, $\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}\hat{K}_{z_1}\mathcal{G}_g(z_1|z\triangleleft_{J\setminus B})$, Lemma A.2 can also be used with $B-1$ instead of B:

$$\begin{aligned}
&(2\lambda)^3\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}\hat{K}_{z_1}\mathcal{G}_g(z_1|z\triangleleft_{J\setminus B})\\
&=(2\lambda)^6\Bigg(\sum_{l'=0}^{3g-3+|J|}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_B^3}+\frac{3+2l'}{z_B^{5+2l'}}\Big)\frac{\partial}{\partial\varrho_{l'}}+\sum_{i\in J\setminus\{B\}}\frac{1}{\varrho_0 z_i z_B^3}\frac{\partial}{\partial z_i}+\frac{1}{\varrho_0 z_1 z_B^3}\frac{\partial}{\partial z_1}\Bigg)\\
&\quad\times\frac{1}{z_1^2}\Bigg[\sum_{l=0}^{3g-4-|J|}(3+2l)\sum_{k=0}^{l}\frac{\varrho_k}{z_1^{2+2l-2k}}\frac{\partial}{\partial\varrho_l}+\sum_{\beta\in J\setminus\{B\}}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\Bigg]\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}})\\
&=(2\lambda)^6\Bigg[\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l}\frac{(3+2l)}{z_1^{4+2l-2k}}\Big(-\frac{(3+2k)\varrho_{k+1}}{\varrho_0 z_B^3}+\frac{3+2k}{z_B^{5+2k}}\Big)\frac{\partial}{\partial\varrho_l}\\
&+\sum_{l,l'=0}^{3g-4-|J|}\sum_{k=0}^{l}\frac{(3+2l)\varrho_k}{z_1^{4+2l-2k}}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_B^3}+\frac{3+2l'}{z_B^{5+2l'}}\Big)\frac{\partial^2}{\partial\varrho_l\partial\varrho_{l'}}
\end{aligned}$$

$$+\sum_{i\in J\setminus\{B\}}\Big(\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l}\frac{(3+2l)\varrho_k}{\varrho_0 z_1^{4+2l-2k}z_iz_B^3}\frac{\partial^2}{\partial\varrho_l\partial z_i}+\sum_{\beta\in J\setminus\{B\}}\frac{1}{\varrho_0 z_1^2 z_i z_B^3}\frac{\partial}{\partial z_i}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\Big)$$
$$+\sum_{l'=0}^{3g-4+|J|}\sum_{\beta\in J\setminus\{B\}}\Big(-\frac{(3+2l')\varrho_{l'+1}}{\varrho_0 z_1^2 z_\beta z_B^3}+\frac{3+2l'}{z_1^2 z_\beta z_B^{5+2l'}}\Big)\frac{\partial^2}{\partial\varrho_{l'}\partial z_\beta}$$
$$-\frac{2}{\varrho_0 z_1^4 z_B^3}\Big(\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l}\frac{(3+2l)\varrho_k}{z_1^{2+2l-2k}}\frac{\partial}{\partial\varrho_l}+\sum_{\beta\in J\setminus\{B\}}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\Big)$$
$$-\frac{1}{\varrho_0 z_1^2 z_B^3}\sum_{l=0}^{3g-4+|J|}\sum_{k=0}^{l}\frac{(3+2l)(2+2l-2k)\varrho_k}{z_1^{4+2l-2k}}\frac{\partial}{\partial\varrho_l}\Big]\mathcal{G}_g(z\triangleleft_{J\setminus\{B\}}).$$

Subtracting the second from the first expression proves the Lemma. □

Lemma A.6. *Let* $J=\{2,...,B\}$*. The linear integral equation* (4.4) *is under Assumption A.1 and with Definition 4.4 equivalent to the expression*

$$\begin{aligned}0=&(2\lambda)^3[\hat{K}_{z_1},\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}]\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{B\}})+2\lambda\mathcal{G}_0(z_1|z_B)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{B\}})\\&+(2\lambda)^3\frac{1}{z_B}\frac{\partial}{\partial z_B}\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{B\}})-\mathcal{G}_g(z_B|z\triangleleft_{J\setminus\{B\}})}{z_1^2-z_B^2}.\end{aligned}$$

Proof. With Lemma A.4 we can rewrite the linear integral equation (4.4) in the form

$$\begin{aligned}0=&(2\lambda)^{3B-4}\hat{K}_{z_1}\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}\dots\hat{\mathrm{A}}^{\dagger g}_{z_1,z_2}\mathcal{G}_g(z_1)-(2\lambda)^{3B-4}\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}\dots\hat{\mathrm{A}}^{\dagger g}_{z_1,z_2}\hat{K}_{z_1}\mathcal{G}_g(z_1)\\&+\mathcal{G}_g(z_1)\mathcal{G}_0(z_1|z\triangleleft_J)+2\lambda\sum_{\substack{I\subset J\\1\le|I|<|J|}}\mathcal{G}_0(z_1|z\triangleleft_I)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus I})\\&+(2\lambda)^3\sum_{\beta\in J}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}})-\mathcal{G}_g(z_\beta|z\triangleleft_{J\setminus\{\beta\}})}{z_1^2-z_\beta^2}.\end{aligned}\tag{A.1}$$

By using this formula for $\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_{B-1}}\dots\hat{\mathrm{A}}^{\dagger g}_{z_1,z_2}\hat{K}_{z_1}\mathcal{G}_g(z_1)$ and inserting it back into (A.1) gives

$$\begin{aligned}0=&(2\lambda)^{3B-4}[\hat{K}_{z_1},\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}]\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_{B-1}}\dots\hat{\mathrm{A}}^{\dagger g}_{z_1,z_2}\mathcal{G}_g(z_1)\\&+\mathcal{G}_g(z_1)\mathcal{G}_0(z_1|z\triangleleft_J)-(2\lambda)^3\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}(\mathcal{G}_g(z_1)\mathcal{G}_0(z_1|z\triangleleft_{J\setminus B}))\\&+2\lambda\sum_{\substack{I\subset J\\1\le|I|<|J|}}\mathcal{G}_0(z_1|z\triangleleft_I)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus I})-(2\lambda)^4\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}\sum_{\substack{I\subset J\setminus\{B\}\\1\le|I|<|J|-1}}\mathcal{G}_0(z_1|z\triangleleft_I)\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{I,B\}})\\&+(2\lambda)^3\sum_{\beta\in J}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta\}})-\mathcal{G}_g(z_\beta|z\triangleleft_{J\setminus\{\beta\}})}{z_1^2-z_\beta^2}\\&-(2\lambda)^6\hat{\mathrm{A}}^{\dagger g}_{z_1,...,z_B}\sum_{\beta\in J\setminus\{B\}}\frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\frac{\mathcal{G}_g(z_1|z\triangleleft_{J\setminus\{\beta,B\}})-\mathcal{G}_g(z_\beta|z\triangleleft_{J\setminus\{\beta,B\}})}{z_1^2-z_\beta^2}.\end{aligned}$$

The second and third line break down to $2\lambda\mathcal{G}_0(z_1|z_B)\mathcal{G}_g(z_1|z_{\triangleleft J\setminus\{B\}})$. Therefore, the assertion follows if we can show that, in the fourth line, the part of the sum which excludes $\beta = B$ cancels with the fifth line. This is true because of

$$\Big[\hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}, \frac{1}{z_\beta}\frac{\partial}{\partial z_\beta}\Big] = 0 \quad \text{and} \quad \hat{\mathrm{A}}^{\dagger g}_{z_1,\dots,z_B}\frac{1}{z_1^2 - z_\beta^2} = 0.$$

Consequently, the linear integral equation can be written by operators of the form given in this Lemma. □

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