

Verifying the Firoozbakht, Nicholson, and Farhadian conjectures up to the 81st maximal prime gap.

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ABSTRACT: The Firoozbakht, Nicholson, and Farhadian conjectures can be phrased in terms of increasingly powerful conjectured bounds on the prime gaps $g_n := p_{n+1} - p_n$.

$$g_n \leq p_n (p_n^{1/n} - 1) \quad (n \geq 1; \textit{Firoozbakht}).$$

$$g_n \leq p_n ((n \ln n)^{1/n} - 1) \quad (n > 4; \textit{Nicholson}).$$

$$g_n \leq p_n \left(\left(p_n \frac{\ln n}{\ln p_n} \right)^{1/n} - 1 \right) \quad (n > 4; \textit{Farhadian}).$$

While a general proof of any of these conjectures is far out of reach I shall show that all three of these conjectures are unconditionally and explicitly verified for all primes below the location of the 81st maximal prime gap, certainly for all primes $p < 2^{64}$. For the Firoozbakht conjecture this is a very minor improvement on currently known results, for the Nicholson and Farhadian conjectures this may be more interesting.

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1 Introduction

The Firoozbakht, Nicholson, and Farhadian conjectures would, if proved to be true, impose increasingly strong constraints on the distribution of the primes; this distribution being a fascinating topic that continues to provide many subtle and significant open questions [1–23]. The Firoozbakht conjecture [24–28] is normally phrased as follows.

Conjecture 1. (*Firoozbakht conjecture, two most common versions*)

$$(p_{n+1})^{\frac{1}{n+1}} \leq (p_n)^{\frac{1}{n}}; \quad \text{equivalently} \quad \frac{\ln p_{n+1}}{n+1} \leq \frac{\ln p_n}{n}; \quad (n \geq 1). \quad (1.1)$$

To see why this conjecture might be somewhat plausible, use the standard inequalities $n \ln n < p_n < n \ln p_n$, which hold for $n \geq 1$ and $n \geq 4$ respectively, and observe that

$$\frac{\ln(n \ln n)}{n} \leq \frac{\ln p_n}{n} \leq \frac{\ln^2 p_n}{p_n}; \quad (n \geq 1; n \geq 4). \quad (1.2)$$

Now $\frac{\ln(n \ln n)}{n}$ is monotone decreasing for $n \geq 5$, and $\frac{\ln^2 p_n}{p_n}$ is monotone decreasing for $p_n > 7$. So for $n \geq 5$, corresponding to $p_n \geq 11$, the function $\frac{\ln p_n}{n}$ is certainly bounded between two monotone decreasing functions; the overall trend is monotone decreasing. The stronger conjecture that $\frac{\ln p_n}{n}$ is itself monotone decreasing depends on fluctuations in the distribution of the primes p_n ; fluctuations which can be rephrased in terms of the prime gaps $g_n := p_{n+1} - p_n$.

Indeed, Kourbatov [26] using results on first occurrence prime gaps has recently verified Firoozbakht’s conjecture to hold for all primes $p < 4 \times 10^{18}$. Furthermore Kourbatov [27] has also derived a *sufficient* condition for the Firoozbakht conjecture to hold:

$$g_n \leq \ln^2 p_n - \ln p_n - 1.17; \quad (n \geq 10; p_n \geq 29). \quad (1.3)$$

Using tables of first occurrence prime gaps and maximal prime gaps Kourbatov has now extended this discussion [28], and subsequently verified that Firoozbakht’s conjecture holds for all primes $p < 1 \times 10^{19}$. More recently (2018), two additional maximal prime gaps have been found [30], so that Kourbatov’s arguments now certainly verify the Firoozbakht conjecture up to the 80th maximal prime gap — more precisely, for all primes below currently unknown location of the 81st maximal prime gap — though we do now know (September 2018) that $p_{81}^* > 2^{64}$ [29], see also [28]. So certainly the Firoozbakht conjecture holds for all primes $p < 2^{64} = 18,446,744,073,709,551,616 \approx 1.844 \times 10^{19}$. Note that this automatically verifies a strong form of Cramér’s conjecture

$$g_n \leq \ln^2 p_n; \quad (n \geq 5; p_n \geq 11), \quad (1.4)$$

at least for all primes $p < 2^{64} \approx 1.844 \times 10^{19}$.

What is trickier with Kourbatov's techniques is to say anything useful about the slightly stronger Nicholson [33] and Farhadian [34, 35] conjectures, and it is this issue we shall address below.

2 Firoozbakht, Nicholson, and Farhadian

When comparing the Firoozbakht conjecture with the slightly stronger Nicholson and Farhadian conjectures it is useful to work with the ratio of successive primes, p_{n+1}/p_n .

Conjecture 2. (*Firoozbakht/Nicholson/Farhadian conjectures; successive primes*)

$$(p_{n+1}/p_n)^n \leq p_n \quad (n \geq 1; \text{Firoozbakht}). \quad (2.1)$$

$$(p_{n+1}/p_n)^n \leq n \ln n \quad (n > 4; \text{Nicholson}). \quad (2.2)$$

$$(p_{n+1}/p_n)^n \leq p_n \frac{\ln n}{\ln p_n} \quad (n > 4; \text{Farhadian}). \quad (2.3)$$

When phrased in this way the standard inequalities $n \ln n < p_n < n \ln p_n$ show that Farhadian \implies Nicholson \implies Firoozbakht. To study the numerical evidence in favour of these conjectures it is useful to convert them into statements about the prime gaps $g_n := p_{n+1} - p_n$.

Conjecture 3. (*Firoozbakht/Nicholson/Farhadian conjectures; prime gap version*)

$$g_n \leq p_n (p_n^{1/n} - 1) \quad (n \geq 1; \text{Firoozbakht}). \quad (2.4)$$

$$g_n \leq p_n ((n \ln n)^{1/n} - 1) \quad (n > 4; \text{Nicholson}). \quad (2.5)$$

$$g_n \leq p_n \left(\left(p_n \frac{\ln n}{\ln p_n} \right)^{1/n} - 1 \right) \quad (n > 4; \text{Farhadian}). \quad (2.6)$$

This can further be rephrased as:

$$g_n \leq p_n \left(\exp \left(\frac{\ln p_n}{n} \right) - 1 \right) \quad (n \geq 1; \text{Firoozbakht}). \quad (2.7)$$

$$g_n \leq p_n \left(\exp \left(\frac{\ln(n \ln n)}{n} \right) - 1 \right) \quad (n > 4; \text{Nicholson}). \quad (2.8)$$

$$g_n \leq p_n \left(\exp \left(\frac{1}{n} \ln \left(p_n \frac{\ln n}{\ln p_n} \right) \right) - 1 \right) \quad (n > 4; \text{Farhadian}). \quad (2.9)$$

These inequalities are all of the form $g_n \leq f(p_n, n)$, with $f(p_n, n)$ a function of both p_n and n .

While p_n and n are both monotone increasing, unfortunately $f(p_n, n)$ is not guaranteed to be monotone increasing, so one would have to check each individual n independently. So our strategy will be to seek to find suitable *sufficient* conditions for the Firoozbakht/Nicholson/Farhadian conjectures of the form $g_n \leq f(n)$, with the function $f(n)$ being some monotone function of its argument. Once this has been achieved we can develop an argument using maximal prime gaps.

3 Sufficient condition for the Nicholson/Firoozbakht conjectures

Using the fact that $e^x - 1 > x$ we deduce a sufficient condition for the Nicholson conjecture (which is then automatically also sufficient for the Firoozbakht conjecture).

Sufficient condition 1. (*Nicholson/Firoozbakht*)

$$g_n < \frac{p_n \ln(n \ln n)}{n}; \quad (n > 4; n \geq 1). \quad (3.1)$$

Now use Dusart's result [14] that for $n \geq 2$ we have $p_n > n(\ln(n \ln n) - 1)$ to deduce the stronger sufficient condition

Sufficient condition 2. (*Nicholson/Firoozbakht*)

$$g_n < f(n) = (\ln(n \ln n) - 1) \ln(n \ln n); \quad (n > 4; n \geq 2). \quad (3.2)$$

A posteriori we shall verify that this last condition is strong enough to be useful, and weak enough to be true over the domain of interest.

4 Verifying the Firoozbakht and Nicholson conjectures for all primes $p < 2^{64}$

This is a variant of the argument given for the Andrica conjecture in references [22, 23]. Consider the maximal prime gaps: Following a minor modification of the notation of references [22, 23], let the quartet (i, g_i^*, p_i^*, n_i^*) denote the i^{th} maximal prime gap; of width g_i^* , starting at the n_i^{th} prime $p_i^* = p_{n_i^*}$. (See the sequences A005250, A002386, A005669, A000101, A107578.) As of April 2019, some 80 such maximal prime gaps are known [30–32], up to $g_{80}^* = 1550$ and

$$p_{80}^* = 18,361,375,334,787,046,697 > 1.836 \times 10^{19}, \quad (4.1)$$

which occurs at

$$n_{80}^* = 423, 731, 791, 997, 205, 041 \approx 423 \times 10^{15}. \quad (4.2)$$

One now considers the interval $[p_i^*, p_{i+1}^* - 1]$, from the lower end of the i^{th} maximal prime gap to just below the beginning of the $(i + 1)^{\text{th}}$ maximal prime gap. Then everywhere in this interval

$$\forall p_n \in [p_i^*, p_{i+1}^* - 1] \quad g_n \leq g_i^*; \quad f(n_i^*) \leq f(n). \quad (4.3)$$

Therefore, if the sufficient condition for the Nicholson/Firoozbakht conjectures holds at the beginning of the interval $p_n \in [p_i^*, p_{i+1}^* - 1]$, then it certainly holds on the entire interval. (Note that for the Nicholson/Firoozbakht conjectures, in addition to knowing the p_i^* , it is also essential to know all the $n_i^* = \pi(p_i^*)$ in order for this particular verification procedure to work; for the Andrica conjecture one can quietly discard the $n_i^* = \pi(p_i^*)$ and only work with the p_i^* [22, 23].)

Explicitly checking a table of maximal prime gaps [30–32], both of the Nicholson and Firoozbakht conjectures certainly hold on the interval $[p_5^*, p_{81}^* - 1]$, from $p_5^* = 89$ up to just before the beginning of the 81^{st} maximal prime gap, $p_{81}^* - 1$, even if we do not yet know the value of p_{81}^* . Then explicitly checking the primes below 89 the Firoozbakht conjecture holds for all primes p less than p_{81}^* , while the Nicholson conjecture holds for all primes p less than p_{81}^* , except $p \in \{2, 3, 5, 7\}$. Since we do not explicitly know p_{81}^* , (though an exhaustive search has now verified that $p_{81}^* > 2^{64}$ [29], see also [28]), a safe fully explicit statement is that both the Firoozbakht and Nicholson conjectures are verified for all primes $p < 2^{64} \approx 1.844 \times 10^{19}$.

5 Sufficient condition for the Farhadian conjecture

The Farhadian conjecture is a little trickier to deal with. Again using the fact that $e^x - 1 > x$ we can deduce a sufficient condition.

Sufficient condition 3. (*Farhadian*)

$$g_n < \frac{p_n \ln \left(p_n \frac{\ln n}{\ln p_n} \right)}{n} = \frac{p_n (\ln p_n + \ln \ln n - \ln \ln p_n)}{n}; \quad (n > 4). \quad (5.1)$$

Now inside the brackets use the lower bound $p_n \geq n \ln n$ (valid for $n \geq 1$), and the upper bound $p_n \leq n \ln(n \ln n)$ (valid for $n \geq 6$). This gives a new slightly stronger sufficient condition.

Sufficient condition 4. (*Farhadian*)

$$g_n < \frac{p_n (\ln(n \ln n) + \ln \ln n - \ln \ln(n \ln(n \ln n)))}{n}; \quad (n > 6). \quad (5.2)$$

Now use Dusart's result [14] that for $n \geq 2$ we have $p_n > n(\ln(n \ln n) - 1)$ to deduce another yet even slightly stronger sufficient condition.

Sufficient condition 5. (*Farhadian*)

$$g_n < f(n) = (\ln(n \ln n) - 1) (\ln(n \ln n) + \ln \ln n - \ln \ln(n \ln(n \ln n))); \quad (n > 6). \quad (5.3)$$

It is now a somewhat tedious exercise in elementary calculus to verify that this function $f(n)$ is indeed monotone increasing as a function of n . *A posteriori* we shall verify that this last sufficient condition is strong enough to be useful, and weak enough to be true over the domain of interest.

6 Verifying the Farhadian conjecture for all primes $p < 2^{64}$

The logic is the same as for the Firoozbakht and Nicholson conjectures. If the sufficient condition for the Farhadian conjecture holds at the beginning of the interval $p_n \in [p_i^*, p_{i+1}^* - 1]$, then it certainly holds on the entire interval. Explicitly checking a table of maximal prime gaps [30–32], the Farhadian conjecture certainly holds on the interval $[p_5^*, p_{81}^* - 1]$, from $p_5^* = 89$ up to just before the beginning of the 81st maximal prime gap, $p_{81}^* - 1$, even if we do not yet know the value of p_{81}^* . Then explicitly checking the primes below $p_5^* = 89$ the Farhadian conjecture is verified to hold for all primes p less than p_{81} , except $p \in \{2, 3, 5, 7\}$. Since we do not explicitly know p_{81}^* , (though an exhaustive search has now verified that $p_{81}^* > 2^{64}$ [29], see also [28]), a safe fully explicit statement is that the Farhadian conjecture is verified for all primes $p < 2^{64} \approx 1.844 \times 10^{19}$.

7 Discussion

While Kourbatov's recent work [26–28] yields a useful and explicit domain of validity for the Firoozbakht conjecture, (ultimately, see [29] and [28], for all primes $p < 2^{64}$), the present article first slightly extends this domain of validity (all primes $p < p_{81}^*$), and second and more significantly obtains identical domains of validity for the related but slightly stronger Nicholson and Farhadian conjectures. The analysis has been presented in such a way that it can now be semi-automated.

Upon discovery, every new maximal prime gap g_i^* can, as long as one can also calculate the corresponding $n_i^* = \pi(p_i^*)$, see for instance [36], be used to push the domain of validity a little further.

Some cautionary comments are in order: Verification of these conjectures up to some maximal prime, however large, does not guarantee validity for all primes. Note that by the prime number theorem $\pi(n) \sim \text{li}(n)$ so

$$\frac{\ln(p_n)}{n} = \frac{\ln(p_n)}{\pi(p_n)} \sim \frac{\ln(p_n)}{\text{li}(p_n)}. \quad (7.1)$$

Now certainly $\ln(p)/\text{li}(p)$ is monotone decreasing, which is good. On the other hand $\pi(x) - \text{li}(x)$ changes sign infinitely often, (this is the Skewes phenomenon [37–40]), so that the monotone decreasing function $\ln(p_n)/\text{li}(p_n)$ both over-estimates and under-estimates the quantity of interest $\ln(p_n)/n$, which is not so good. Now this observation does not disprove the Firoozbakht conjecture, but it does indicate where there might be some potential difficulty.

On a more positive note, the Firoozbakht conjecture most certainly must hold when averaged over suitably long intervals. It is an elementary consequence of the Chebyshev theorems that $p_m p_n > p_{m+n}$, see [1–3]. But then $p_n^2 > p_{2n}$, and $p_n^3 > p_n p_{2n} > p_{3n}$. In general $(p_n)^m > p_{nm}$ and so $\ln p_n > \ln p_{nm}/m$. Consequently

$$\frac{\ln(p_n)}{n} > \frac{\ln(p_{nm})}{nm}. \quad (7.2)$$

This is much weaker than the usual Firoozbakht conjecture, but enjoys the merit of being unassailably true.

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