On Asymptotic Behavior of Bell Polynomials and High Moments of Vertex Degree of Random Graphs*

O. Khorunzhiy
Université de Versailles - Saint-Quentin
45, Avenue des Etats-Unis, 78035 Versailles, FRANCE
e-mail: oleksiy.khorunzhiy@uvsq.fr

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Abstract

We study the asymptotic behavior of single variable Bell polynomials $\mathcal{B}_k(x)$ in the limit of infinite k and $x > 0$. We discuss our results in relations with the spectral theory of large random matrices and the properties of the vertex degree of large Erdős-Rényi random graphs.

1 Binomial distributions and Bell polynomials

This paper is motivated by the studies of the spectral properties of random matrices determined by the Ihara zeta function of random graphs of infinitely increasing dimension, $n \to \infty$. The Erdős-Rényi ensemble of random graphs $\{\Gamma_n^{(\rho)}\}$, $0 < \rho < n$ that have n vertices and the edge probability ρ/n can be described by the family of adjacency matrices $\{A_n^{(\rho)}\}$, where $A_n^{(\rho)}$ is a real symmetric $n \times n$ matrix whose entries above the diagonal are given by the family of jointly independent Bernoulli random variables $a_{ij}^{(n,\rho)}$ that take zero value with the probability $1 - \rho/n$ (see monograph [1]). The Ihara zeta function of the graph $\Gamma_n^{(\rho)}$, in the determinant form, is given by relation [4, 9, 16]

$$Z_{\Gamma_n^{(\rho)}}(u) = \Big((1-u^2)^{r-1} \det H_n^{(\rho)}(u) \Big)^{-1}, \quad u \in \mathbb{C},$$

where

$$H_n^{(\rho)}(u) = (1-u^2)I + u^2 B_n^{(\rho)} - uA_n^{(\rho)}, \tag{1.1}$$

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$B_n^{(\rho)}$ is the diagonal matrix whose non-zero entries are given by the vertex degrees of $\Gamma_n^{(\rho)}$,

$$\left(B_n^{(\rho)}\right)_{ij} = b_i^{(n,\rho)}\delta_{ij}, \quad b_i^{(n,\rho)} = \sum_{l=1}^{n} a_{il}^{(n,\rho)} \ , \quad 1 \le i \le j \le n, \tag{1.2}$$

and $r-1 = \text{Tr}(B_n^{(\rho)} - 2I)/2$. Here and below we denote by δ_{ij} the Kronecker delta-symbol that is equal to one if $i = j$ and is zero otherwise.

The logarithm of $\det H_n^{(\rho)}(u)$ can be studied with the help of the normalized eigenvalue counting function of $H_n^{(\rho)}$. It is shown in [12] that this function, when regarded in the case of the Erdős-Rényi random graphs $\{\Gamma_n^{(\rho)}\}$, converges in the limit of infinite n and ρ under the condition that the real spectral parameter u is renormalized by the square root of ρ, $u = v/\sqrt{\rho}, v \in \mathbb{R}$. The proof is based on the convergence of the averaged value of the k-th moment $\text{Tr}\left(H_n^{(\rho)}(v/\sqrt{\rho})\right)^k$ as $n, \rho \to \infty$. Further considerations of the spectral properties of random matrices $H_n^{(\rho)}$ can require the knowledge of the asymptotic behavior of these moments in the limit when k infinitely increases at the same time as n and ρ tend to infinity. In particular, this is needed in the studies the maximal eigenvalue of random matrices by the moment method [8]. On the first stages, one could ask about the asymptotic behavior of the moments of the matrix $B_n^{(\rho)}$.

Slightly simplifying the definition of random variables $a_{ij}^{(n,\rho)}$ and $b_i^{(n,\rho)}$, we consider an ensemble of n jointly independent Bernoulli random variables

$$a_j^{(n,\rho)} = \begin{cases} 1, & \text{with probability } \rho/n, \\ 0, & \text{with probability } 1 - \rho/n \end{cases}, \quad 1 \le j \le n, \ \rho > 0. \tag{1.3}$$

and study the moments of the binomial random variable

$$X_n^{(\rho)} = \sum_{j=1}^{n} a_j^{(n,\rho)}$$

in the limit of infinite n. The probability distribution of $X_n^{(\rho)}$ converges to the Poisson one as $n \to \infty$ both for bounded ρ or for infinitely increasing ρ (see Section 3 for more details). The probability distribution of the centered random variables

$$\tilde{X}_n^{(\rho)} = X_n^{(\rho)} - \mathbb{E}X_n^{(\rho)} = X_n^{(\rho)} - \rho = \sum_{j=1}^{n} \tilde{a}_j^{(n,\rho)},$$

where $\mathbb{E}$ is the mathematical expectation with respect to the measure generated by the family $\mathcal{A}^{(n,\rho)} = \left\{ \{a_j^{(n,\rho)}\}_{1\le j\le n} \right\}$ converges to the centered Poisson probability distribution [13].

We are interested in the asymptotic behavior of the moments

$$\mathcal{M}_k^{(n,\rho)} = \mathbb{E}\left(X_n^{(\rho)}\right)^k \tag{1.4}$$

and

$$\tilde{\mathcal{M}}_k^{(n,\rho)} = \mathbb{E}\left(\tilde{X}_n^{(\rho)}\right)^k \tag{1.5}$$

in the limit of $n, k \to \infty$ when ρ is either finite or infinitely increasing. Let

$$\mathcal{B}_k(x) = \sum_{(l_1,l_2,\dots,l_k)} B_k(l_1, l_2, \dots, l_k)\, x^{l_1+l_2+\dots+l_k}, \quad k \ge 1,\ x \in \mathbb{R}, \tag{1.6}$$

where

$$B_k(l_1, l_2, \dots l_k) = \frac{k!}{(1!)^{l_1} l_1!\,(2!)^{l_2} l_2! \cdots (k!)^{l_k} l_k!} \tag{1.7}$$

and the sum in (1.6) runs over such integers $l_i \ge 0$ that $l_1+2l_2+\dots+kl_k = k$. The value $\mathcal{B}_k(1) = B_k$ is known as the k-th Bell number and the polynomials $\mathcal{B}_k(x)$ are referred to as a single variable Bell polynomials or simply as to the Bell polynomials [2, 3]. We also consider modified Bell polynomials

$$\tilde{\mathcal{B}}_k(x) = \sum_{(l_2,l_3,\dots,l_k)'} \tilde{B}_k(l_2, \dots, l_k)\, x^{l_2+l_3+\dots+l_k}, \quad k \ge 1,\ x \in \mathbb{R}, \tag{1.8}$$

where

$$\tilde{B}_k(l_2, l_3 \dots l_k) = \frac{k!}{(2!)^{l_2} l_2!(3!)^{l_3} l_3! \cdots (k!)^{l_k} l_k!} \tag{1.9}$$

and the sum in (1.8) runs over such $l_i \ge 0$ that $2l_2 + 3l_3 + \dots + kl_k = k$. We will refer to the numbers $\tilde{B}_k = \tilde{\mathcal{B}}_k(1)$ as to the restricted (or centered) Bell numbers and say that $\tilde{\mathcal{B}}_k(x)$ are the restricted Bell polynomials, or Bell-type polynomials. In Section 3 below, we prove the following statement.

Lemma 1.1. *The moments (1.4) and (1.5) verify the following asymptotic relations,*

$$\mathcal{M}_k^{(n,\rho)} = \mathcal{B}_k(\rho)(1 + o(1)), \quad n, k \to \infty \tag{1.10}$$

and

$$\tilde{\mathcal{M}}_k^{(n,\rho)} = \tilde{\mathcal{B}}_k(\rho)(1 + o(1)), \quad n, k \to \infty, \tag{1.11}$$

where k and n are such that $k = o(n^{2/3})$ and ρ is finite of infinite such that $\rho = o(n)$.

The main results of the present paper are related with the asymptotic behavior of polynomials $\mathcal{B}_k(x)$ and $\tilde{\mathcal{B}}_k(x)$ with positif x in the limit when k tends to infinity. The asymptotic properties of the Bell polynomials $\mathcal{B}_k(x)$ as $k \to \infty$ are well studied (see [6, 7, 19] and references therein). Up to our knowledge, the limiting behavior of the Bell polynomials $\mathcal{B}_k(x)$ for infinite k and x has not been considered, while this kind of limiting transition is fairly natural from the point of view of random graphs and random matrices. The asymptotic behavior of the Bell polynomials in the case of restricted Bell numbers $\tilde{\mathcal{B}}_k(x)$ has not yet been studied as well.

It should be pointed out that the Bell polynomials (1.6) determine the moments of the Poisson distribution. The moments of the centered Poisson distribution are given by the restricted Bell numbers $\tilde{B}_k$ and corresponding Bell-type polynomials $\tilde{\mathcal{B}}_k(x)$. Therefore our results can be interesting also from this point of view.

In Section 2, we prove our main statements and use them to determine the asymptotic behavior of high moments of the diagonal matrix $B_n^{(\rho)}$ (1.2). As a consequence, we estimate the deviation probability of the maximal vertex degree of large Erdős-Rényi random graphs. In Section 3, we prove auxiliary statements and present supplementary facts about the convergence of random variables $X_n^{(\rho)}$. We also discuss further generalizations of the Bell polynomials inspired by relations with the theory of sparse random matrices.

2 Asymptotic behavior of $\mathcal{B}_k(x)$ and $\tilde{\mathcal{B}}_k(x)$

In this section, we study the asymptotic behavior of the Bell polynomials $\mathcal{B}_k(x)$ (1.6) and polynomials of restricted Bell numbers $\tilde{\mathcal{B}}_k(x)$ (1.8) in the limit $k \to \infty$ for given values of $x > 0$ and in the asymptotic regime

$$k, x \to \infty, \quad x = \chi k \tag{2.1}$$

that we denote for simplicity by $(k, x)_\chi \to \infty$, where $\chi > 0$ is finite or infinitely increasing. Here, we mainly follow the probabilistic approach presented in paper [18], where the asymptotic behavior of the Bell numbers B_k, $k \to \infty$ is considered. We give the full proofs in the case of the Bell polynomials $\mathcal{B}_k(x)$, $x > 0$ and then briefly describe the results obtained for the Bell polynomials of the restricted Bell numbers $\tilde{\mathcal{B}}_k(x)$, $x > 0$.

2.1 Auxiliary random variables and Central Limit Theorem

Regarding the sequence of Bell polynomials, $\mathfrak{B} = (\mathcal{B}_k(x))_{k\geq 0}$ with $x > 0$, let us introduce an auxiliary random variables $Z^{(x,u)}$ given by the probability distribution

$$P(Z^{(x,u)} = k) = p_k^{(x,u)} = \mathcal{B}_k(x)\frac{u^k}{k!\,G(x,u)}, \quad k \geq 0,\ u > 0, \tag{2.2}$$

where $G(x,u)$ is the exponential generating function of the sequence $\mathfrak{B}$,

$$G(x,u) = \sum_{k=0}^{\infty} \mathcal{B}_k(x)\frac{u^k}{k!}.$$

It is easy to prove the following equality (see for instance [5])

$$G(x,u) = \exp\{x(e^u - 1)\}. \tag{2.3}$$

The generating function of the probability distribution of $Z^{(x,u)}$ is given by

$$F_{x,u}(z) = \sum_{k=0}^{\infty} P(Z^{(x,u)} = k)\, z^k = \sum_{k=0}^{\infty} \mathcal{B}_k(x)\frac{u^k}{k!G(x,u)}\, z^k = \frac{G(x,zu)}{G(x,u)}. \tag{2.4}$$

Elementary computations show that

$$F'_{x,u}(1) = xue^u \quad \text{and} \quad F''_{x,u}(1) = xu^2e^u + x^2u^2e^{2u}.$$

Denoting by $\mathbf{E}$ the mathematical expectation with respect to the distribution (2.2), we deduce from the last two equalities that

$$\mathbf{E}Z^{(x,u)} = xue^u \quad \text{and} \quad Var(Z^{(x,u)}) = xu(u+1)e^u. \tag{2.5}$$

In what follows, we will omit the superscripts (x,u) when no confusion can arise.

Let us consider a random variable $Y = (Z - \mathbf{E}Z)/\sigma_Z$, where $\sigma_Z^2 = Var(Z)$ and introduce its characteristic function $\Phi_Y(t) = \mathbf{E}e^{-itY}$.

Lemma 2.1. *For any given $t \in \mathbb{R}$, the characteristic function $\Phi_{Y^{(x,u)}}(t)$ of the random variable $Y^{(x,u)}$ converges to the one of the standard normal distribution,*

$$\Phi_{Y^{(x,u)}}(t) = e^{-t^2/2}(1 + o(1)), \quad xue^u \to \infty. \tag{2.6}$$

Proof. We rewrite the characteristic function $\Phi_Y(t)$ in the following form,

$$\Phi_Y(t) = e^{-it\,\mathbf{E}Z/\sigma_Z}\, F_{u,x}\left(e^{it/\sigma_z}\right). \tag{2.7}$$

Relations (2.3) and (2.4) imply equality

$$F_{x,u} = \exp\left\{xe^u\left(e^{u\Delta} - 1\right)\right\},$$

where we denoted $\Delta = e^{it/\sigma_Z} - 1$. Using the following asymptotic expansion

$$u\Delta = \frac{iut}{\sigma_Z} + \frac{u}{2}\left(\frac{it}{\sigma_Z}\right)^2 + O\left(\frac{ut^3}{\sigma_Z^3}\right)$$

that holds in the limit (2.6) and taking into account convergence

$$\frac{u}{\sigma_Z} = \frac{1}{\sqrt{xe^u}} \to 0, \tag{2.8}$$

we can write that

$$\ln F_{x,u}\left(e^{it/\sigma_Z}\right) = xe^u\left(\frac{iut}{\sigma_Z} - \frac{(u+u^2)t^2}{2\sigma_Z^2} + O\left(\frac{u^3t^3}{\sigma_Z^3}\right)\right).$$

Using the second relation of (2.5) and remembering (2.7), we see that the following equality holds in the limit (2.6),

$$\Phi_Y(t) = \exp\left\{-\frac{t^2}{2} + O\left(\frac{xu^3e^ut^3}{\sigma_Z^3}\right)\right\} = e^{-t^2/2}\left(1 + O\left(\frac{t^3}{\sqrt{xe^u}}\right)\right).$$

This relation together with (2.8) completes the proof of Lemma 2.1. □

2.2 Local Limit Theorem

Definition (2.2) can be rewritten as follows,

$$\mathcal{B}_k(x) = P(Z^{(x,u)} = k)\frac{k!G(x,u)}{u^k}. \tag{2.9}$$

One of the main ingredients of the method proposed in [18] is to use the following statement that can be regarded as a version of the discrete local limit theorem.

Lemma 2.2. *Relation*

$$P(Z^{(x,u)} = k) - \frac{1}{\sqrt{2\pi}\sigma_Z}\exp\left\{-\frac{\left(k - \mathbf{E}Z^{(x,u)}\right)^2}{2\sigma_Z^2}\right\} = o\left(\sigma_Z^{-1}\right),$$

holds in the limit (2.6) for all k such that $|k - xe^u| = O(\sigma_Z)$. In particular, if $|k - xue^u| = o(1)$, then

$$P(Z^{(x,u)} = k) = \frac{1}{\sqrt{2\pi}\sigma_Z} + o\left(\sigma_Z^{-1}\right), \quad xue^u \to \infty. \tag{2.10}$$

Proof. To prove Lemma 2.2, we adapt the arguments given by T. Tao [17]. Regarding the mathematical expectation of the both parts of the identity

$$\mathbf{I}_{\{Z=k\}}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{isZ}e^{-isk}ds,$$

we get by the use of the Fubini's theorem that

$$\begin{aligned} P(Z=k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{Z-\mathbf{E}Z}(s)e^{is\,\mathbf{E}Z}e^{-isk}ds \\ &= \frac{1}{2\pi\sigma_Z}\int_{-\pi\sigma_Z}^{\pi\sigma_Z} \Phi_Y(y)e^{-iy(k-\mathbf{E}Z)/\sigma_Z}dy, \end{aligned}$$

where $Y=(Z-\mathbf{E}Z)/\sigma_Z$. Standard computations show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iy(k-\mathbf{E}Z)/\sigma_Z - y^2/2}dy = \frac{1}{\sqrt{2\pi}}e^{-(k-\mathbf{E}Z)^2/(2\sigma_Z^2)}.$$

Observing that

$$\Big|\int_{|y|>\pi\sigma_Z} e^{-y^2/2+i\alpha y}\Big|dy \le \int_{|y|>\pi\sigma_Z} e^{-y^2/2}dy$$

for any α, we conclude that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iy\left(\frac{k-\mathbf{E}Z}{\sigma_Z}\right)-\frac{y^2}{2}}dy - \frac{1}{2\pi}\int_{-\pi\sigma_Z}^{\pi\sigma_Z} e^{iy\left(\frac{k-\mathbf{E}Z}{\sigma_Z}\right)-\frac{y^2}{2}}dy = o(1), \quad \sigma_Z\to\infty \tag{2.11}$$

uniformly in k. Let us show that

$$\int_{-\pi\sigma_Z}^{\pi\sigma_Z} \Phi_Y(y)e^{-iy(k-\mathbf{E}Z)/\sigma_Z}dy - \int_{-\pi\sigma_Z}^{\pi\sigma_Z} e^{iy\left(\frac{k-\mathbf{E}Z}{\sigma_Z}\right)-\frac{y^2}{2}}dy = o(1), \quad \sigma_Z\to\infty \tag{2.12}$$

This relation will follow from Lemma 2.1 as soon as we bound the difference

$$\Phi_Y(y) - e^{-y^2/2}, \quad y\in[-\pi\sigma_Z, \pi\sigma_Z]$$

by an absolutely integrable function. We consider first the interval $|y|\le\sigma_Z\varepsilon$ with some $0<\varepsilon<1$. It is not hard to deduce from (2.4) that

$$\mu_3 = \mathbf{E}(Z-\mathbf{E}Z)^3 = xu^3e^u\left(1+\frac{3}{u}+\frac{1}{u^2}\right)$$

and

$$\mu_4 = \mathbf{E}(Z - \mathbf{E}Z)^4 = x^2u^4e^{2u}\left(3 + \frac{1}{xe^u} + \frac{6}{u} + \frac{6}{xue^u} + \frac{7}{xu^2e^u} + \frac{1}{u^2} + \frac{12}{xu^3e^u}\right).$$

We see that the Taylor expansion of the fourth order exists,

$$\Phi_{Z-\mathbf{E}Z}(t) = 1 - \frac{\mu_2 t^2}{2} - \frac{i\mu_3 t^3}{3!} + \frac{t^4}{4!}\frac{d^4}{dt^4}\Phi_{Z-\mathbf{E}Z}(\theta), \tag{2.13}$$

where $\mu_2 = \sigma_Z^2 = Var(Z)$ and $|\theta| < |t|$. Taking into account the upper bound

$$|\frac{d^4}{dt^4}\Phi_{Z-\mathbf{E}Z}(\theta)| \leq \mu_4$$

and relations

$$\frac{\mu_3}{Var(Z)}t = o(1) \quad \text{and} \quad \frac{\mu_4}{Var(Z)}t^2 = o(1), \quad \sigma_Z t \to 0,$$

we deduce from (2.13) that

$$\Phi_{Z-\mathbf{E}Z}(t) = 1 - \frac{\sigma_Z^2 t^2}{2} + o(\sigma_Z^2 t^2), \quad \sigma_Z t \to 0.$$

Therefore it exists $\varepsilon > 0$ such that

$$|\Phi_{Z-\mathbf{E}Z}(y/\sigma_Z)| = |\Phi_Y(y)| \leq 1 - \frac{y^2}{4} \leq e^{-y^2/4}$$

for all $|y| \leq \varepsilon\sigma_Z$.

Let us consider the values of $t = y/\sigma_Z$ such that $|t| > \varepsilon$. It follows from (2.4) that

$$|\Phi_Z(t)| = \exp\left\{xe^u\left(e^{u(\cos t - 1)}\cos(u\sin t) - 1\right)\right\}. \tag{2.14}$$

There exists $\delta > 0$ such that

$$|e^{u(\cos t-1)}\cos(u\sin t)| \leq e^{-\delta u}, \quad \text{for all} \ \ |t| \geq \varepsilon.$$

It is not hard to deduce from (2.14) that for sufficiently large u such that $e^{-\delta u} - 1 \leq -1/2$ the following upper bound holds,

$$|\Phi_Z(t)| \leq \exp\{-xe^u/2\}, \quad \text{for all} \ \ |t| \geq \varepsilon.$$

This inequality completes the proof of Lemma 2.2. □

2.3 Asymptotic behavior of Bell polynomials

Remembering equality (2.9) and using (2.3) together with (2.5) and (2.10), we conclude that the following equality

$$\mathcal{B}_k(x) = \frac{1}{\sqrt{2\pi x u(u+1)e^u}} \exp\{x(e^u - 1)\} \frac{k!}{u^k} (1 + o(1)) \tag{2.15}$$

holds in the asymptotic regime of infinite k described in Lemma 2.2. In particular, we can say that (2.15) holds for $k \to \infty$, where u is determined by k and $x > 0$ by equality

$$ue^u = k/x. \tag{2.16}$$

Solution of (2.16) is known as the Lambert's function (see [6] and references therein). Using the Stirling formula,

$$k! = \frac{1}{\sqrt{2\pi k}} \left(\frac{k}{e}\right)^k (1 + o(1)), \quad k \to \infty$$

we can write that

$$\mathcal{B}_k(x) = \frac{x^k}{\sqrt{u+1}} \exp\left\{xu(u-1)e^u + x(e^u - 1)\right\} (1 + o(1)), \quad k \to \infty$$

and that

$$\mathcal{B}_k(x) = \frac{x^k}{\sqrt{u+1}} \exp\left\{k\left(u - 1 + \frac{1}{u}\right) - x\right\} (1 + o(1)), \quad k \to \infty, \tag{2.17}$$

where u is determined by (2.16) Let us note that relation (2.17) considered with $x = 1$ coincides with that obtained by E. G. Tsylova and E.Ya. Ekgauz [18]. Similar to (2.17) expression has been obtained by D. Dominici [6]. However, in the asymptotic equality for $\mathcal{B}_k(x)$ obtained in [6], the factor x^k from the right-hand side of (2.17) is absent.

Now we will examine the variable $\mathcal{B}_k(x)$ in the following three asymptotic regimes: $x \ll k$, $x = O(k)$ and $x \gg k$.

Theorem 2.1 *The Bell polynomials $\mathcal{B}_k(x)$ show the following asymptotic behavior:*
a) if $0 < x \ll k$, then

$$\mathcal{B}_k(x) = \left(\frac{k}{e(\ln k - \ln x)}(1 + o(1))\right)^k; \tag{2.18}$$

b) if $x = O(k)$ as $k \to \infty$, then

$$\mathcal{B}_k(x) = \left(k\chi e^{v}\left(1 + o(1)\right)\right)^k, \quad k \to \infty, \tag{2.19}$$

where $v = u - 1 + u^{-1} - (ue^u)^{-1}$ and u is such that $ue^u = \chi = x/k$;

c) if $k \to \infty$ and $\chi = x/k \to \infty$, then

$$\mathcal{B}_k(x) = \left(k\chi(1 + o(1))\right)^k, \quad \chi \to \infty, \ k \to \infty. \tag{2.20}$$

Proof. Let us consider the following auxiliary variable,

$$\mathcal{H}_k(x) = \frac{1}{k}\ln\left(\frac{\mathcal{B}_k(x)}{x^k}\right).$$

It follows from (2.17) that

$$\mathcal{H}_k(x) = u - 1 + \frac{1}{u} - \frac{1}{ue^u} - \frac{1}{2k}\ln(u+1) + o(k^{-1}) \tag{2.21}$$

in the limit $k \to \infty$ and u determined by (2.16).

a) If $\chi = x/k \to 0$, then the right-hand side of equality (2.16) tends to infinity. It is not hard to see that the solution u of the transcendent equation (2.16) verifies the following relation [5]

$$u = \ln\left(\frac{k}{x}\right) - \ln\ln\left(\frac{k}{x}\right) + O\left(\frac{\ln\ln(k/x)}{\ln(k/x)}\right), \quad \frac{k}{x} \to \infty. \tag{2.22}$$

Substituting this expression into the right-hand side of (2.21), we get the following asymptotic equality,

$$\mathcal{H}_k(x) = \ln\left(\frac{k}{x}\right) - \ln\ln\left(\frac{k}{x}\right) - 1 + O\left(\frac{\ln\ln(k/x)}{\ln(k/x)}\right), \quad \frac{k}{x} \to \infty, \quad k \to \infty.$$

Returning to the variable $\mathcal{B}_k(x)$, we can write that

$$\mathcal{B}_k(x) = x^k \exp\left\{k\ln\left(\frac{k}{x}\right) - k\ln\ln\left(\frac{k}{x}\right) - k + O\left(\frac{k\ln\ln(k/x)}{\ln(k/x)}\right)\right\}.$$

Then (2.18) follows.

Regarding the last equality in the particular case $x = 1$, we get the following asymptotic relation for the Bell numbers $B_k = \mathcal{B}_k(1)$,

$$\frac{\ln B_k}{k} = \ln k - \ln\ln k - 1 + O\left(\frac{\ln\ln k}{\ln k}\right), \quad k \to \infty. \tag{2.23}$$

The first three terms of the right-hand side of (2.23) coincide with those obtained by N. G. de Bruijn [5] for the asymptotic expansion of the Bell numbers. Relation (2.23) is also equivalent to the results obtained by E. G. Tsylova and E. Ya. Ekgauz [18].

b) If $\chi = x/k = Const$ as $k \to \infty$ (cf. (2.1)), then relation (2.21) takes the form

$$\mathcal{H}_k(x) = u - 1 + \frac{1}{u} - \frac{1}{ue^u} + o(1), \quad (k,x)_\chi \to \infty.$$

This asymptotic equality implies (2.19).

c) Consider the last asymptotic regime characterized by $x \gg k$. Then $\chi = x/k \to \infty$ and relation (2.16) mean that in this case $u \to 0$ at the same time when $k \to \infty$ and $x \to \infty$. In fact,

$$u = \frac{1}{\chi} - \frac{4}{\chi^2} + o(\chi^{-2}).$$

Then we can write that

$$\mathcal{H}_k(x) = \frac{1}{u}\left(1 - \frac{1}{e^u}\right) + u - 1 + o(1/k) = \frac{u}{2} + o(u^2) + o(1/k), \quad (k,x)_\chi \to \infty,\ \chi \to \infty.$$

Then

$$\mathcal{H}_k(x) = \frac{1}{2\chi} - \frac{2}{\chi^2} + o(\chi^{-2}) + o(k^{-1})$$

and therefore

$$\mathcal{B}_k(x) = \left(k\chi e^{1/(2\chi) - 2/\chi^2 + o(1/k)}\right)^k, \quad (k,x)_\chi \to \infty,\ \chi \to \infty.$$

Then (2.20) follows. Theorem 2.1 is proved. □

2.4 Restricted Bell polynomials

In Section 3, we show that the exponential generating function of the restricted Bell numbers $\tilde{\mathcal{B}}_k(x)$ is given by

$$\tilde{G}(x,u) = \sum_{k=0}^{\infty} \tilde{\mathcal{B}}_k(x) \frac{u^k}{k!} = \exp\{x(e^u - u - 1)\}. \tag{2.24}$$

Regarding the random variable $\tilde{Z}^{(x,u)}$ with the probability distribution

$$P(\tilde{Z}^{(x,u)} = k) = \tilde{\mathcal{B}}_k(x) \frac{u^k}{k!\tilde{G}(x,u)},$$

we can easily deduce from (2.24) that

$$\mathbb{E}\tilde{Z}^{(x,u)} = xu(e^u - 1) \quad \text{and} \quad Var(\tilde{Z}^{(x,u)}) = xu((u+1)e^u - 1).$$

Introducing a centered and rescaled random variable

$$\tilde{Y} = \frac{\tilde{Z} - \mathbb{E}\tilde{Z}}{\tilde{\sigma}},$$

where $\tilde{\sigma}^2 = (\sigma_{\tilde{Z}})^2 = Var(\tilde{Z})$, we can write that

$$\Phi_{\tilde{Y}}(y) = e^{-iy\mathbb{E}\tilde{Z}/\tilde{\sigma}} \tilde{F}(e^{iy/\tilde{\sigma}}).$$

By using the same arguments as before, it is not hard to show that in the limit

$$xu((u+1)e^u - 1) \to \infty, \tag{2.25}$$

the following relation holds for any given y,

$$\ln \tilde{F}(e^{iy/\tilde{\sigma}}) = \frac{iy}{\tau\sigma}\mathbb{E}\tilde{Z} - \frac{y^2}{2} + O\left(\frac{xu^3e^uy^3}{\tilde{\sigma}^3}\right). \tag{2.26}$$

Then, in complete analogy with Lemma 3.1,

$$\Phi_{\tilde{Y}}(y) \to e^{-y^2/2}$$

in the limit (2.25). The analogue of the Lemma 3.2 holds and we can write that

$$P(\tilde{Z} = k) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} + o\left(\frac{1}{\tilde{\sigma}}\right), \quad \tilde{\sigma} = \sigma_{\tilde{Z}} \to \infty$$

for any u such that $u(e^u - 1) = k/x$. For these values of k, we have

$$\tilde{\mathcal{B}}_k(x) = \frac{1}{\sqrt{2\pi xu((u+1)e^u - 1)}}\sqrt{2\pi k}\left(\frac{k}{eu}\right)^k e^{x(e^u - u - 1)}(1 + o(1)). \tag{2.27}$$

Now we can formulate our second main result.

Theorem 2.2. *The Bell polynomials of restricted Bell numbers $\tilde{\mathcal{B}}_k(x)$ show the following asymptotic behavior:*

a) if $1 < x \ll k$, then

$$\tilde{\mathcal{B}}_k(x) = \left(\frac{k}{e(\ln k - \ln x)}(1 + o(1))\right)^k; \tag{2.28}$$

b) if $x = O(k)$ as $k \to \infty$, then

$$\tilde{\mathcal{B}}_k(x) = \left(k\chi e^{\tilde{v}}(1+o(1))\right)^k, \quad k \to \infty, \tag{2.29}$$

where $\tilde{v} = u - 1 + u^{-1} + \ln(e^u - 1) - \ln u$ and u is such that $u(e^u - 1) = \chi$;

c) if $k \to \infty$ and $\chi = x/k \to \infty$, then

$$\tilde{\mathcal{B}}_k(x) = \left(k\sqrt{\chi}(1+o(1))\right)^k. \tag{2.30}$$

The proof of this theorem is given by an analysis of relation (2.27) in the three asymptotic regimes indicated. This analysis is similar to that performed in the proof of Theorem 2.1. To prove (2.28), we use an observation that the asymptotic expansion (2.22) holds as $k/x \to \infty$ also in the the case when u is determined by equality

$$u(e^u - 1) = k/x. \tag{2.31}$$

(see Section 3 for the proof). Relation (2.30) follows from the fact that

$$\tilde{\mathcal{H}}_k(x) = \frac{1}{k}\ln\left(\frac{\tilde{\mathcal{B}}_k(x)}{x^k}\right) = \ln(e^u - 1) - 1 + \frac{x}{k}\left(e^u - u - 1\right) + o(k^{-1}),$$

where u is determined by equality (2.31). In this asymptotic regime $k/x \to 0$ and therefore $u \to 0$ and $u = \sqrt{k/x}(1+o(1))$. Then

$$\ln(e^u - 1) - 1 = \ln\left(u + \frac{u^2}{2} + o(u^2)\right) - 1 = \ln u - 1 - u/2 + o(u)$$

and

$$\frac{x}{k}\left(e^u - u - 1\right) = \frac{1}{2} + o(1).$$

Then

$$\tilde{\mathcal{H}}_k(x) = \frac{1}{2}\left(\ln\left(\frac{k}{x}\right) - 1\right) + o(1), \quad k/x \to 0,\ k \to \infty,$$

$$\tilde{\mathcal{B}}_k(x) = \left(x \exp\left\{-\frac{\ln(k/x) + 1}{2} + o(1)\right\}\right)^k$$

and (2.30) follows.

2.5 High moments of $B_n^{(\rho)}$

Let us return to the random Erdős-Rényi graphs $\Gamma_n^{(\rho)}$ determined by the adjacency matrices $A_n^{(\rho)}$ whose elements above the diagonal are given by relation

$$\left(A_n^{(\rho)}\right)_{ij} = a_{ij}^{(n,\rho)} = \begin{cases} \delta_{ij}, & \text{with probability } \rho/n, \\ 0, & \text{with probability } 1-\rho/n, \end{cases} \qquad 1 \le i \le j \le n;$$

the elements $\left(A_n^{(\rho)}\right)_{ij}$ with $1 \le j < i \le n$ are determined by the symmetry condition. The random variables $\{a_{ij}^{(n,\rho)}, 1 \le i \le j \le n\}$ are jointly independent and we assume that for known values $\rho = \phi(n)$ the triangle-type array of random variables $\mathcal{A}' = \left\{\{a_{ij}^{(n,\rho)}, 1 \le i \le j \le n\}_{n\ge 1}\right\}$ is determined on the same probability space. We denote the mathematical expectation with respect to the measure generated by $\mathcal{A}'$ by $\mathbb{E}'$.

Let us consider the diagonal matrix $B_n^{(\rho)}$ (1.2) whose non-zero elements represent the degrees of vertices of the graph $\Gamma_n^{(\rho)}$. We consider also the diagonal matrices with centered elements $\tilde{B}_n^{(\rho)}$,

$$\left(\tilde{B}_n^{(\rho)}\right)_{ij} = \delta_{ij}\tilde{b}_i^{(n,\rho)}, \quad 1 \le i \le j \le n,$$

where

$$\tilde{b}_i^{(n,\rho)} = b_i^{(n,\rho)} - \mathbb{E}'\, b_i^{(n,\rho)} = b_i^{(n,\rho)} - \frac{(n-1)\rho}{n}.$$

We are interested in the asymptotic behavior of the normalized moments

$$\tilde{M}_k^{(n,\rho)} = \frac{1}{n}\mathbb{E}' \operatorname{Tr}\left(\tilde{B}_n^{(\rho)}\right)^k = \frac{1}{n}\mathbb{E}' \sum_{i=1}^{n} \left(\tilde{b}_i^{(n,\rho)}\right)^k$$

as n, ρ and k infinitely increase. The random variables $b_i^{(n,\rho)}, 1 \le i \le n$ are identically distributed and therefore

$$\tilde{M}_k^{(n,\rho)} = \mathbb{E}'\left(\tilde{b}_1^{(n,\rho)}\right)^k = \mathbb{E}\left(\sum_{j=1}^{n-1} \tilde{a}_j^{(n,\rho)}\right)^k, \qquad (2.32)$$

where random variables $a_j^{(n,\rho)}$ are determined by equality (1.3). By similar arguments, we have

$$M_k^{(n,\rho)} = \frac{1}{n}\mathbb{E}' \operatorname{Tr}\left(B_n^{(\rho)}\right)^k = \mathbb{E}\left(\sum_{j=1}^{n-1} a_j^{(n,\rho)}\right)^k.$$

A natural analogue of Lemma 1.1 is given by the following relations,

$$M_k^{(n,\rho)} = \mathcal{B}_k(\rho)(1+o(1)), \quad n,k\to\infty \tag{2.33}$$

and

$$\tilde M_k^{(n,\rho)} = \tilde{\mathcal{B}}_k(\rho)(1+o(1)), \quad n,k\to\infty \tag{2.34}$$

that are true for all k and n such that $k=o(\sqrt{n})$ (see Section 3 for the proof). These relations together with Theorems 2.1 and 2.2 determine the asymptotic behavior of the moments $M_k^{(n,\rho)}$ and $\tilde M_k^{(n,\rho)}$ as n,ρ and k tend to infinity.

Let us introduce random variables

$$V^{(n,\rho)} = \max_{1\le i\le n}\{b_i^{(n,\rho)}\} \quad \text{and} \quad \tilde V^{(n,\rho)} = \max_{1\le i\le n}\{\tilde b_i^{(n,\rho)}\}$$

that represent the maximal vertex degree of the graph $\Gamma_n^{(\rho)}$ and its centered value, respectively. Using elementary inequality

$$P(|\tilde V^{(n,\rho)}|>s) = P(\cup_{i=1}^n\{|\tilde b_i^{(n,\rho)}|>s\}) \le \sum_{i=1}^n P(\{\tilde b_i^{(n,\rho)}|>s\}),$$

and taking into account the upper bound

$$P(|\tilde b_1^{(n,\rho)}|>s)\le s^{-2km}\,\mathbb{E}'\left(\tilde b_1^{(n,\rho)}\right)^{2km}, \quad m\in\mathbb{N}$$

and relation (2.34), we obtain that

$$P(|\tilde V^{(n,\rho)}|>s)\le n s^{-2km}\,\tilde{\mathcal{B}}_{2km}(\rho)(1+o(1)), \quad n,k\to\infty \tag{2.35}$$

for all n and k such that $2km=o(\sqrt{n})$.

Let us consider the asymptotic regime $\rho=\chi k$ (2.1) when $k=\lfloor \ln n\rfloor$, $n\to\infty$ and $\chi>0$ is given. With the help of Theorem 2.2, we deduce from (2.35) that

$$P\left(\frac{|V^{(n,\rho)}-\rho|}{\rho}>t\right)\le \exp\left\{\ln n\left(1-2m\ln(te^{-\tilde v})+o(1)\right)\right\}, \quad n\to\infty.$$

Then we can conclude that

$$P\left(\limsup_{n\to\infty}\frac{|V^{(n,\rho)}-\rho|}{\rho}>t'\right)=0, \quad \rho=\chi\ln n \tag{2.36}$$

for any t' such that

$$t' > \exp\left\{u - 1 + u^{-1} + \ln(e^u - 1) - \ln u\right\} \quad \text{and} \quad u(e^u - 1) = \chi.$$

Regarding the asymptotic regime (2.1) with $k = \lfloor \ln n \rfloor$ and $\chi \to \infty$, we deduce from (2.35) with the help of Theorem 2.2 that

$$P(|\tilde{V}^{(n,\rho)}| > s) \le \exp\left\{\ln n \left(1 - 2m \ln(s\sqrt{\chi}/\rho) + o(1)\right)\right\}. \tag{2.37}$$

It is clear that if $t'' > 1$, then for any $\beta > 0$ there exists m such that $2m \ln(t'') > 2 + \beta$ and therefore (2.37) implies the upper bound

$$P\left(\sqrt{\chi}|\tilde{V}^{(n,\rho)}| > t''\rho\right) \le n^{-1-\beta}, \quad n, \chi \to \infty, \ \rho = \chi \ln n.$$

Finally, we can conclude that

$$P\left(\limsup_{n,\chi\to\infty} \sqrt{\chi}\frac{|V^{(n,\rho)} - \rho|}{\rho} > 1\right) = 0, \quad \rho = \chi \ln n. \tag{2.38}$$

The random variable $V^{(n,\rho)}$ we considered represents the maximal vertex degree of the ensemble of Erdős-Rényi random graphs. While the properties of the vertex degree of the Erdős-Rényi random graphs are deeply studied, we hope that our results (2.36), (2.37) and (2.38) contribute to the knowledge already obtained.

3 Auxiliary facts and statements

In this section we collect the proofs of the statements we have used above and formulate some supplementary facts of interest.

3.1 Binomial and Poisson random variables

Let us describe convergence of random variables $X_n^{(\rho)}$ in the limit $n, \rho \to \infty$.

Lemma 3.1. *The characteristic function*

$$\Phi_{X_n}(t) = \mathbb{E}\exp\{itX_n^{(\rho)}\}, \quad X_n^{(\rho)} = \sum_{j=1}^{n} a_j^{(n,\rho)}$$

converges as $n, \rho \to \infty$ to the one of the Poisson random variable,

$$\Phi_{Y^{(\rho)}}(t) = \exp\{(e^{it} - 1)\rho\}$$

in the sense that for any $t \in \mathbb{R}$

$$\Phi_{n,\rho}(t)/\Phi_{Y_\rho}(t) \to 1 \tag{3.1}$$

as $n, \rho \to \infty,\ \rho = o(\sqrt{n})$*. Also*

$$P(X_n^{(\rho)} = k)/P(Y_\rho = k) \to 1 \tag{3.2}$$

in the limit of infinite n, k, ρ *such that* $k = o(n^{2/3})$ *and* $\rho = o(\sqrt{n})$*.*

Lemma 3.2. *The characteristic function of the random variable*

$$U_n^{(\rho)} = \frac{X_n^{(\rho)} - \rho}{\sqrt{\rho}}$$

converges for any given t *as* $n, \rho \to \infty$ *provided* $\rho = o(n^{2/3})$ *to the one of the standard normal distribution,*

$$\mathbf{E} \exp\{itU_n^{(\rho)}\} \to e^{-t^2/2}.$$

The proofs of relations (3.1) and (3.2) are based on simple use the Taylor expansions of characteristic functions. Indeed, assuming $\rho = o(n), n \to \infty$, we can write that

$$\mathbb{E} \exp\{itX_n^{(\rho)}\} = \left(e^{it}\frac{\rho}{n} + \left(1 - \frac{\rho}{n}\right)\right)^n$$

$$= \exp\left\{n \ln\left(1 + \frac{(e^{it}-1)\rho}{n}\right)\right\} = \exp\left\{(e^{it}-1)\rho + O(\rho^2/n)\right\}.$$

Then (3.1) follows.

Regarding the probability distribution of the binomial random variable $X_n^{(\rho)}$, we can write that

$$P(X_n^{(\rho)} = k) = R(k,n)\,\frac{\exp\{n\ln(1-\rho/n)\}}{\exp\{k\ln(1-\rho/n)\}} \cdot \frac{\rho^k}{k!}, \tag{3.3}$$

where we denoted

$$R(k,n) = \prod_{i=1}^{k-1} \frac{n-i}{n} = \prod_{i=1}^{k-1}\left(1 - \frac{i}{n}\right).$$

If $k/n \to 0$, then

$$\ln R(k,n) = \sum_{i=1}^{k-1} \ln\left(1 - \frac{i}{n}\right)$$

$$= \sum_{i=1}^{k-1} \left(-\frac{i}{n} + \frac{i^2}{2n^2} + O\left(\frac{i^3}{n^3}\right) \right) = -\frac{(k-1)k}{2n} + O\left(\frac{k^3}{n^2}\right). \qquad (3.4)$$

Using the Taylor expansion of $\ln(1-\rho/n)$, one can easily deduce from (3.3) with the help of (3.4) that

$$P(X_n^{(\rho)} = k)\frac{e^\rho k!}{\rho^k} = \left(1 + O\left(\frac{k^2}{n}\right)\right)\left(1 + O\left(\frac{\rho^2}{n}\right)\right)\left(1 + O\left(\frac{k\rho}{n}\right)\right).$$

This relation implies (3.2).

The proof of Lemma 3.2 is elementary and we do not present it here.

3.2 Proof of relation (2.24)

Let us consider the following analogs of the Stirling numbers of the second kind,

$$\tilde{S}(k,r) = \frac{1}{r!} \sum_{(h_1,h_2,\ldots,h_k)'} \frac{k!}{h_1! h_2! \cdots h_r!} =$$

$$= \frac{1}{r!} \sum_{(h_1,h_2,\ldots,h_k)'} \binom{k}{h_1}\binom{k-h_1}{h_2} \cdots \binom{k-h_1-h_2 \cdots - h_{r-1}}{h_r}, \qquad (3.5)$$

where the sum over $(h_1, h_2, \ldots, h_k)'$ is such that $h_1 + \cdots + h_r = k$ and $h_i \geq 2, i = 1, \ldots, r$. It is easy to deduce from the last equality of (3.5) that

$$\sum_{k=r}^{\infty} \tilde{S}(k,r)\frac{t^k}{k!} = \frac{1}{r!}\left(e^t - t - 1\right)^r.$$

By definition (1.8), we have

$$\tilde{\mathcal{B}}_k(x) = \sum_{r=0}^{k} \tilde{S}(k,r) x^r$$

and therefore

$$\sum_{k=0}^{\infty} \tilde{\mathcal{B}}_k(x)\frac{t^k}{k!} = \sum_{r=0}^{\infty}\sum_{k=r}^{\infty} \tilde{S}(k,r) x^r \frac{t^k}{k!} = \exp\left\{x\left(e^t - t - 1\right)\right\},$$

where we have interchanged the order of summation by the standard arguments. The last equality completes the proof of (2.24).

3.3 Proof of Lemma 1.1

To study the moments $\mathcal{M}_k^{(n,\rho)}$ (1.4),

$$\mathcal{M}_k^{(n,\rho)} = \mathbf{E}\left(\sum_{j=1}^{n} a_j^{(n,\rho)}\right)^k \tag{3.6}$$

it is natural to represent the multiple sum of the right-hand side of (3.6) as the sum over the classes of equivalence $\mathcal{C}$, each class is associated with the partition of the set $\{j_1, j_2, \ldots, j_k\}$ into the blocks and the variables in each block are equal to the same value from the set $\{1, 2, \ldots, n\}$. It is easy to see that

$$\mathbb{E}\left(\sum_{j_1,j_2,\ldots,j_k=1}^{n} a_{j_1} a_{j_2} \cdots a_{j_k}\right)$$

$$= \sum_{\{\mathcal{C}\}} \prod_{i=1}^{k} \left(\mathbf{E} a_1^i\right)^{l_i} n(n-1)\cdots(n-|\mathcal{C}|+1), \tag{3.7}$$

where $|\mathcal{C}| = l_1 + l_2 + \cdots + l_k$ denotes the number of groups in the partition $\mathcal{C}$. Here and below we omit the superscripts (n, ρ). Since $\mathbb{E}a_1 = \rho/n$ and $|\mathcal{C}| \le k$, then the elementary estimate (see (3.4) for details)

$$\log \prod_{i=1}^{|\mathcal{C}|-1} \left(1 - \frac{i}{n}\right) = -\frac{|\mathcal{C}|(|\mathcal{C}|-1)}{n} + O(k^3/n^2) \tag{3.8}$$

shows that

$$\sum_{\{\mathcal{C}\}} \left(\frac{\rho}{n}\right)^{|\mathcal{C}|} n(n-1)\cdots(n-|\mathcal{C}|+1) = \sum_{\{\mathcal{C}\}} \rho^{|\mathcal{C}|}\,(1+o(1))$$

in the limit $n, k \to \infty$ such that $k = o(n^{2/3})$. Then in this limit,

$$\mathcal{M}_k^{(n,\rho)} = \sum_{\{\mathcal{C}\}} \rho^{|\mathcal{C}|}\,(1+o(1))$$

and relation (1.10) follows from the fact that the number of classes $\mathcal{C}$ with given $(l_1, l_2, \ldots, l_k)$ is equal to the number $B_k(l_1, l_2, \ldots, l_k)$ (1.7).

Let us consider the moments $\tilde{\mathcal{M}}_k^{(n,\rho)}$ (1.5). We have

$$\tilde{\mathcal{M}}_k^{(n,\rho)} = \mathbf{E}\left(\sum_{j_1,j_2,\ldots,j_k=1}^{n} \tilde{a}_{j_1} \tilde{a}_{j_2} \cdots \tilde{a}_{j_k}\right)$$

$$= \sum_{\{\mathcal{C}^*\}} \prod_{i=2}^{k} \left(\mathbf{E}\tilde{a}_1^i \right)^{l_i} n(n-1)\cdots(n-|\mathcal{C}^*|+1), \tag{3.9}$$

where the sum runs over the classes of equivalence $\mathcal{C}^*$ given by such partitions of the set $\{1, 2, \ldots, n\}$ that have no blocks of one element. It is easy to see that

$$\mathbf{E}\left(a_1 - \frac{\rho}{n}\right)^m = \frac{\rho}{n} Q_m(\rho/n),$$

where

$$Q_m(\rho/n) = \sum_{l=2}^{m} \binom{m}{l} \left(-\frac{\rho}{n}\right)^{m-l} + (m-1)\left(-\frac{\rho}{n}\right)^{m-1}.$$

It is clear that

$$Q_m(\rho/n) = \left(1 - \frac{\rho}{n}\right)^m + \left(\frac{-\rho}{n}\right)^{m-1}\left(1 + \frac{\rho}{n}\right) \le \left(1 + \frac{2\rho}{n}\right)^m. \tag{3.10}$$

Elementary analysis shows that the lower estimate

$$Q_m(\rho/n) \ge \left(1 - \frac{\rho}{n}\right)^m \left(1 - \frac{4\rho}{n-\rho}\right) \ge \left(1 - \frac{\rho}{n}\right)^m \left(1 - \frac{4\rho}{n-\rho}\right)^m \tag{3.11}$$

is true for $m \ge 2$ and sufficiently large n, ρ such that $\rho = o(n)$.

Relations (3.9) and (3.10) imply inequality

$$\tilde{M}_k^{(n,\rho)} \le \sum_{\{\mathcal{C}^*\}} \prod_{i=2}^{k} \left(\frac{\rho}{n}\left(1 + \frac{2\rho}{n}\right)^i \right)^{l_i} n(n-1)\cdots(n-|\mathcal{C}^*|+1)$$

$$\le \left(1 + \frac{2\rho}{n}\right)^k \sum_{\{\mathcal{C}^*\}} \frac{(n-1)(n-2)\cdots(n-|\mathcal{C}^*|+1)}{n^{|\mathcal{C}^*|-1}}.$$

Then the reasoning based on (3.9) shows that

$$\tilde{M}_k^{(n,\rho)} \le \sum_{\{\mathcal{C}^*\}} \rho^{|\mathcal{C}^*|}(1 + o(1)) = \tilde{\mathcal{B}}_k(\rho)(1 + o(1)) \tag{3.12}$$

in the limit $n, k, \rho \to \infty$ such that $k = o(n^{2/3})$ and $\rho = o(n)$. Here we have used the fact that the number of classes $\mathcal{C}^*$ with given $(l_2, \ldots, l_k)$ is equal to $\tilde{B}_k(l_2, \ldots, l_k)$ (1.9). Relation (3.11) means that the following lower bound holds in the same asymptotic regime,

$$\tilde{M}_k^{(n,\rho)} \ge \tilde{\mathcal{B}}_k(\rho)(1 + o(1)).$$

This inequality together with (3.12) completes the proof of Lemma 1.1.

Relations (2.33) and (2.34) can be easily proved by the same arguments as above with the only difference that in relations (3.8) and (3.9) the values of $|\mathcal{C}|$ and $|\mathcal{C}^*|$ are replaced by $|\mathcal{C}|-1$ and $|\mathcal{C}^*|-1$, respectively.

3.4 Asymptotic expansion for the solution of (2.31)

In this subsection, we follow the reasoning by N. G. de Bruijn [5] used in the study of equation (2.16). We rewrite equality $u(e^u-1)=t$ as

$$\ln\left(e^u-1\right)=\ln t-\ln u. \tag{3.13}$$

Assuming that $t>e^2$, we deduce from (3.13) that $u>1$. In the opposite case, $0<u\leq 1$, we would get the upper bound $\ln(e^u-1)\leq\ln(e-1)<\ln 2$ that contradicts to (3.13). Since $u>1$, then $\ln\left(e^u-1\right)<\ln t$ and

$$0<\ln u<\ln\left(\ln t+1\right)$$

and therefore

$$\ln\left(e^u-1\right)=\ln t+O\left(\ln\ln t\right),\quad t\to\infty.$$

We denote $\ln t+O\left(\ln\ln t\right)=R$. Then

$$u=\ln\left(e^R+1\right)=R+\ln\left(1+\frac{1}{e^R}\right)=\ln t+O\left(\ln\ln t\right). \tag{3.14}$$

Taking logarithms of the both sides of (3.14), we see that

$$\ln u=\ln\left(\ln t+O\left(\ln\ln t\right)\right)=\ln\ln t+O\left(\frac{\ln\ln t}{\ln t}\right).$$

Now it follows from (3.13) that

$$\ln\left(e^u-1\right)=\ln t-\ln\ln t+O\left(\frac{\ln\ln t}{\ln t}\right),\quad t\to\infty$$

and that

$$u=\ln t-\ln\ln t+O\left(\frac{\ln\ln t}{\ln t}\right),\quad t\to\infty.$$

3.5 Restricted and strongly restricted Bell numbers

It is known that the single variable Bell polynomial $\mathcal{B}_k(\rho)$ represents the k-th moment of the Poisson probability distribution $\mathcal{P}(\rho)$. The Bell numbers $B_k = \mathcal{B}_k(1)$ represent the total number of partitions of k labeled elements. It is known that they can be determined by the following recurrence [5],

$$B_{k+1} = \sum_{l=0}^{k} \binom{k}{l} B_{k-l}, \ k \geq 0, \quad B_0 = 1.$$

The restricted Bell numbers $\tilde{B}_k = \tilde{\mathcal{B}}_k(1)$ determine the total number of ways to distribute k labeled elements into subsets such that the cardinality of each subset is greater that one. It is not hard to show that these numbers verify the following relation,

$$\tilde{B}_{k+1} = \sum_{l=1}^{k} \binom{k}{l} \tilde{B}_{k-l}, \ k \geq 1 \quad \tilde{B}_0 = 1, \tilde{B}_1 = 0. \tag{3.15}$$

The family $\{\tilde{\mathcal{B}}_k(\rho)\}_{k\geq 0}$ represents the moments of the centered Poisson distribution $\tilde{\mathcal{B}}_k(\rho) = \mathbb{E}(X - \mathbb{E}X)^k$, $X \sim \mathcal{P}(\rho)$ (see [13] for a general definition and [15] for combinatorial properties of the Poisson central moments).

One more modification of the Bell numbers and the restricted Bell numbers can be obtained in the study of the moments of random variables

$$Y_n^{(\rho)} = \sum_{j=1}^{n} a_j^{(n,\rho)} w_j^{(n)}, \tag{3.16}$$

where $a_j^{(n,\rho)}$ are determined by (1.3) and $w_j^{(n)}$ are jointly independent random variables also independent from $a_j^{(n,\rho)}$ and such that

$$w_j^{(n)} = \begin{cases} 1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases}$$

for all $1 \leq j \leq n$. The random variables of the form (3.16) arise as the matrix elements of dilute (or sparse) random matrices [10].

It is not hard to prove that the odd moments of $Y_n^{(\rho)}$ vanish while the even moments $\mathcal{L}_k^{(n,\rho)} = \mathbb{E}\left(Y_n^{(\rho)}\right)^{2k}$ are asymptotically close in the limit $n \to \infty$ to the variables $\hat{\mathcal{B}}_k(\rho)$,

$$\mathcal{L}_k^{(n,\rho)} = \hat{\mathcal{B}}_{2k}(\rho)(1 + o(1)), \quad n, k \to \infty,$$

such that

$$\hat{\mathcal{B}}_{2k}(x) = \sum_{(l_2,l_4,\dots,l_{2k})''} \hat{B}_{2k}(l_2,l_4,\dots,l_{2k})\, x^{l_2+\cdots+l_{2k}}, \quad k \ge 1,\ x \in \mathbb{R}, \quad (3.17)$$

where

$$\hat{B}_{2k}(l_2,l_4,\dots,l_{2k}) = \frac{(2k)!}{(2!)^{l_2}\, l_2!\, (4!)^{l_4} l_4! \cdots ((2k)!)^{l_{2k}}\, l_{2k}!}$$

and the sum runs over $l_i \ge 0$ such that $l_2 + 2l_4 + \cdots + kl_{2k} = k$. It is natural to refer to the numbers $\hat{B}_{2k} = \hat{\mathcal{B}}_{2k}(1)$ as to the strongly restricted Bell numbers and to say that the family $\{\hat{\mathcal{B}}_l(\rho)\}_{l\ge 0}$, where $\hat{\mathcal{B}}_{2k+1}(\rho) = 0$, represents strongly restricted Bell polynomials.

For an integer $k \ge 0$, $\hat{B}_{2k}$ gives the number of partitions of a set of $2k$ elements into non-empty subsets of even size. It is easy to see that the sequence $\hat{\mathrm{B}} = \{\hat{B}_{2k}\}_{k\ge 0}$ verifies the following recurrence,

$$\hat{B}_{2k+2} = 1 + \hat{B}_{2k} + \sum_{l=1}^{k} \binom{2k}{2l-1} \hat{B}_{2k+2-2l}, \quad \hat{B}_0 = 1,\ \hat{B}_2 = 1. \qquad (3.18)$$

It follows from (3.18) that $\hat{B}_4 = 4$, $\hat{B}_6 = 25$, $\hat{B}_8 = 262$ and $\hat{B}_{10} = 3991$. When preparing this paper, we could find no reference to $\hat{\mathrm{B}}$. As a result, a new sequence corresponding to (3.18) has been created by the staff of the Online Encyclopedia of Integer Sequences [14] (we gratefully thank them for the remarks that correct the value of $\hat{B}_{10}$ erroneously calculated by us).

It would be interesting to determine the asymptotic properties of the strongly restricted Bell polynomials $\hat{\mathcal{B}}_{2k}(\rho)$ (3.17) as $k \to \infty$ and $\rho > 0$ with the help of the approach described above. In particular, this could be useful in the studies of the asymptotic properties of the maximal value of the family of n independent random variables (3.16) in comparison with the results (2.36) and (2.38). Finally, let us note that the convergence of the maximum of n independent random variables of the form (3.16) can also be studied in the case of $\rho = O(\log n)$ with the help of the stochastic version of the Erdős-Rényi limit theorem [11].

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