Counting self-conjugate $(s, s+1, s+2)$-core partitions

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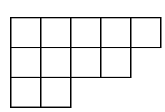
Abstract

We are concerned with counting self-conjugate $(s, s+1, s+2)$-core partitions. A Motzkin path of length n is a path from $(0,0)$ to $(n,0)$ which stays above the x-axis and consists of the up $U=(1,1)$, down $D=(1,-1)$, and flat $F=(1,0)$ steps. We say that a Motzkin path of length n is symmetric if its reflection about the line $x=n/2$ is itself. In this paper, we show that the number of self-conjugate $(s, s+1, s+2)$-cores is equal to the number of symmetric Motzkin paths of length s, and give a closed formula for this number.

1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of a positive integer n. The *Young diagram* of λ is a collection of n boxes in ℓ rows with λ_i boxes in row i. For example, the Young diagram for $\lambda = (5,4,2)$ is below.



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Let the leftmost column be column 1. The box in row i and column j is said to be in position (i,j). For the Young diagram of λ, the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$ is called the *conjugate* of λ, where λ'_j denotes the number of boxes in column j. A partition whose conjugate is equal to itself is called *self-conjugate*. For each box in its Young diagram, we define its *hook length* by counting the number of boxes directly to its right or below, including the box itself. Equivalently, for the box in position (i,j), the hook length of λ is defined by

$$h(i,j) = \lambda_i + \lambda'_j - i - j + 1.$$

For a positive integer t, a partition λ is called a t-*core* if none of its hook lengths are multiples of t. We use the notation of a $(t_1, ..., t_p)$-core if it is simultaneously a t_1-core,. . . , and a t_p-core. See for details [1, 2, 3, 5, 6, 7, 10].

For a set S of positive integers, we say that a is generated by S if a can be written as a non-negative linear combination of the elements of S. Let $P = P_S$ be the set of elements which are not generated by S, and let $(P, <_P)$ be a poset by defining the cover relation so that a covers b if and only if $a - b \in S$. For example, see Figure 1 for the poset $P_{\{8,9,10\}}$. For the detailed explanation of poset, we refer the reader to [1, 9, 11].

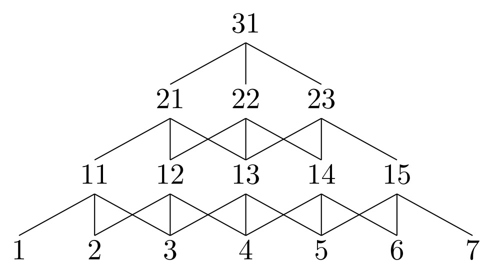

Figure 1: The Hasse diagram of $P_{\{8,9,10\}}$

For a poset $(P, <_P)$, a set $I \subset P$ is called a *lower ideal* of P if $a <_P b$ and $b \in I$ implies $a \in I$. In [2], Anderson gave a natural bijection between t-cores and lower ideals of a poset $P_{\{t\}}$. Moreover, she proved that for relatively prime positive integers s and t, the number of (s,t)-cores has a nice closed formula by finding a bijection between (s,t)-cores and lattice paths from $(0,0)$ to (s,t) consisting of north and east steps which stay above the diagonal.

Theorem 1.1. *[2] For relatively prime positive integers s and t, the number of (s,t)-cores is*

$$\frac{1}{s+t}\binom{s+t}{s}.$$

Since the work of Anderson, the topic counting simultaneous cores has received growing attention. In [4], Ford, Mai, and Sze proved the following analog of Anderson's work.

Theorem 1.2. *[4] For relatively prime positive integers s and t, the number of self-conjugate (s,t)-cores is*

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}.$$

An (s,k)-*generalized Dyck path* is a path from $(0,0)$ to (s,s) which stays above the diagonal and consists of the steps $N_k = (0,k)$, $E_k = (k,0)$, and $D_i = (i,i)$ for $1 \le i \le k-1$. For example, an $(s,1)$-generalized Dyck path is a (classical) *Dyck path of order* s. We say that an (s,k)-generalized Dyck path is *symmetric* if its reflection about the line $y = s - x$ is itself. It is often observed that counting the number of simultaneous cores can sometimes be described as counting the number of different paths.

Remark 1.3. *Let s be a positive integer.*

1. *The number of $(s,s+1)$-cores is the sth Catalan number $C_s = \frac{1}{s+1}\binom{2s}{s}$ which counts the number of Dyck paths of order s.*
2. *The number of self-conjugate $(s,s+1)$-cores is $\binom{s}{\lfloor s/2 \rfloor}$ which counts the number of* symmetric Dyck paths *of order s.*

In [1], Amdeberhan and Leven expand Anderson's result to $(s,s+1,\dots,s+k)$-cores.

Theorem 1.4. *[1] The followings are equinumerous:*

(a) The number of $(s,s+1,\dots,s+k)$-cores.

(b) The number of (s,k)-generalized Dyck paths.

(c) The number of lower ideals in $P_{\{s,s+1,\dots,s+k\}}$.

We note that $(s,2)$-generalized Dyck paths are equivalent to Motzkin paths of length s. From Theorem 1.4, one can obtain the following corollary.

Corollary 1.5. *For a positive integer s, the number of $(s,s+1,s+2)$-cores is*

$$M_s = \sum_{i=0} \frac{1}{i+1}\binom{s}{2i}\binom{2i}{i},$$

the sth Motzkin number which counts the number of Motzkin paths of length s.

We note that Yang, Zhong, and Zhou [11] proved Corollary 1.5 independently.

It is natural to ask whether the number of self-conjugate $(s,s+1,s+2)$-cores and the number of symmetric Motzkin paths of length s are equinumerous from Remark 1.3 and Corollary 1.5. In this paper, we prove that these two quantities are equal by showing that they satisfy the same recurrence relation which will be proved in Seciton 3. Furthermore, we give a closed formula for these numbers.

2 Poset structure for self-conjugate $(s, s+1, s+2)$-cores

In this section, we construct a poset whose lower ideals with some restrictions are corresponding to self-conjugate $(s, s+1, s+2)$-cores, and then give a simple diagram to visualize that poset.

For a partition λ, let $MD(\lambda)$ denote the set of main diagonal hook lengths. Therefore, $MD(\lambda)$ is a set of distinct odds when λ is self-conjugate. In [4], authors gave a useful result for determining self-conjugate t-cores.

Proposition 2.1. *[4] Let λ be a self-conjugate partition. Then λ is a t-core partition if and only if both of the following hold:*

(a) If $h \in MD(\lambda)$ with $h > 2t$, then $h - 2t \in MD(\lambda)$.

(b) If $h_1, h_2 \in MD(\lambda)$, then $h_1 + h_2 \not\equiv 0 \pmod{2t}$.

For a positive integer s, we consider an induced subposet of $P = P_{\{2s,2s+1,\ldots,2s+4\}}$,

$$\tilde{P}_{\{s,s+1,s+2\}} = \{h \in P \ : \ s \not\leq_P h, \ s+1 \not\leq_P h, \ s+2 \not\leq_P h, \text{ and } h \text{ is odd}\}.$$

We note that the poset $\tilde{P}_{\{s,s+1,s+2\}}$ is the disjoint union of two posets, say Q and R, where Q is the maximal induced subposet of P of which minimal elements are odd integers less than s, and R is the maximal induced subposet of P of which minimal elements are odd integers x such that $s+2 < x < 2s$. See Figures 2 and 3 for example.

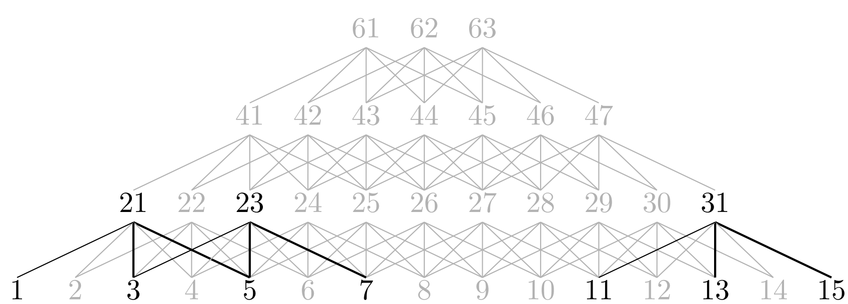

Figure 2: The Hasse diagram of the induced subposet $\tilde{P}_{\{8,9,10\}}$ of $P_{\{16,17,18,19,20\}}$

Now, we restate Proposition 2.1 by using the poset we constructed.

Proposition 2.2. *Let λ be a self-conjugate partition. Then λ is an $(s, s+1, s+2)$-core partition if and only if the set $MD(\lambda)$ is a lower ideal of $\tilde{P}_{\{s,s+1,s+2\}}$ with no elements h_1, h_2 such that $h_1 + h_2 \in \{2s, 2s+2, 2s+4\}$.*

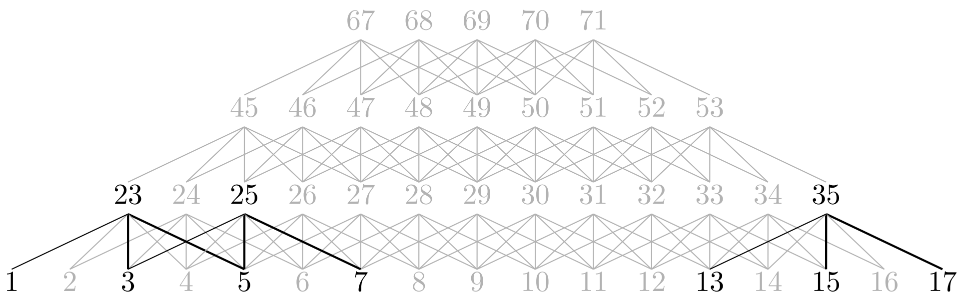

Figure 3: The Hasse diagram of the induced subposet $\tilde{P}_{\{9,10,11\}}$ of $P_{\{18,19,20,21,22\}}$

Example 2.3. *For a self-conjugate* $(8,9,10)$*-core partition* $\lambda = (6,3,3,1,1,1)$*, the set* $MD(\lambda) = \{11,3,1\}$ *of main diagonal hook lengths is a lower ideal of* $\tilde{P}_{\{8,9,10\}}$ *with no elements* h_1, h_2 *such that* $h_1 + h_2 \in \{16, 18, 20\}$.

For convenience, we add dotted edges connecting elements h_1, h_2 in the Hasse diagram of $\tilde{P}_{\{s,s+1,s+2\}}$ with $h_1 + h_2 \in \{2s, 2s+2, 2s+4\}$ so that at most one end point of each dotted edge can be selected for the lower ideal corresponding to an $(s, s+1, s+2)$-core. From now on, we use the *modified diagram* for $\tilde{P}_{\{s,s+1,s+2\}}$ as in Figure 4.

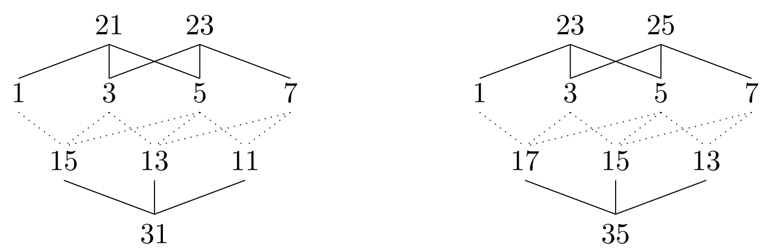

Figure 4: The modified diagrams of $\tilde{P}_{\{8,9,10\}}$ and $\tilde{P}_{\{9,10,11\}}$

We note that

$$Q \cong P_{\{\lfloor \frac{s}{2} \rfloor+1, \lfloor \frac{s}{2} \rfloor+2, \lfloor \frac{s}{2} \rfloor+3\}} \quad \text{and} \quad R \cong P_{\{\lfloor \frac{s}{2} \rfloor, \lfloor \frac{s}{2} \rfloor+1, \lfloor \frac{s}{2} \rfloor+2\}},$$

and therefore, $\tilde{P}_{\{2s,2s+1,2s+2\}}$ is equivalent to $\tilde{P}_{\{2s+1,2s+2,2s+3\}}$. Moreover, it is not hard to notice that the modified diagrams of $\tilde{P}_{\{2s,2s+1,2s+2\}}$ and $\tilde{P}_{\{2s+1,2s+2,2s+2\}}$ are also equivalent. Thus, we have the following proposition.

Proposition 2.4. *For a positive integer* s*, the number of self-conjugate* $(2s, 2s+1, 2s+2)$*-cores are equal to the number of self-conjugate* $(2s+1, 2s+2, 2s+3)$*-cores.*

3 Counting self-conjugate simultaneous core partitions

In this section, we give a formula for the number of symmetric Motzkin paths, and then show that the number of self-conjugate $(2s, 2s+1, 2s+2)$-cores and the number of symmetric Motzkin paths of length s satisfy the same recurrence relation.

3.1 Counting symmetric Motzkin paths

For a fixed i, there are $C_i\binom{n}{2i}$ Motzkin paths with exactly i up steps since there are C_i Dyck paths and there are $\binom{n}{2i}$ ways to insert $n-2i$ flat steps into a Dyck path with i up steps. We say that a Motzkin path of length n is *symmetric* if its reflection about the line $x = \frac{n}{2}$ is itself. Let S_n denote the number of symmetric Motzkin paths of length n. For exmaple, $S_0 = 1, S_1 = 1, S_2 = 2, S_3 = 2, S_4 = 5$.

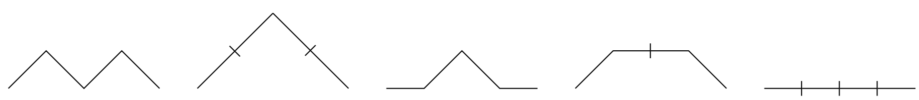

Figure 5: Symmetric Motzkin paths of length 4

We note that the $(n+1)$st step of any symmetric Motzkin path of length $2n+1$ must be a flat step, and therefore, there is a natural bijection between symmetric Motzkin paths of length $2n+1$ and that of length $2n$ so that $S_{2n+1} = S_{2n}$.

Now, we count the number of symmetric Motzkin paths.

Proposition 3.1. *The number of symmetric Motzkin paths of length n is*

$$S_n = \sum_{i\geq 0} \binom{\lfloor \frac{n}{2} \rfloor}{i}\binom{i}{\lfloor \frac{i}{2} \rfloor}.$$

Proof. It is enough to enumerate symmetric Motzkin paths of length $2n$. Suppose we are given a symmetric Dyck path with i up steps, so that it has $2n-2i$ flat steps. To obtain a symmetric Motzkin path of length $2n$ with i up steps, it is enough to consider inserting $n-i$ flat steps into the first half of the given symmetric Dyck path. Since there are $\binom{i}{\lfloor i/2 \rfloor}$ symmetric Dyck paths with i up steps as in Remark 1.3, and there are $\binom{n}{i}$ ways to insert flat steps, the number of symmetric Motzkin paths of length $2n$ with i up steps is $\binom{n}{i}\binom{i}{\lfloor i/2 \rfloor}$. Therefore, $S_{2n} = S_{2n+1} = \sum_{i\geq 0}\binom{n}{i}\binom{i}{\lfloor i/2 \rfloor}$. □

Now, we consider a recurrence relation of S_{2n} involving M_n. For a symmetric Motzkin path $P = P_1P_2\cdots P_{2n}$ of length $2n$, where P_i denotes the ith step, let $k \leq n$ be the largest number such that P meets x-axis at $(k, 0)$. We note that if $k = n$, then both of $P_1P_2\cdots P_n$ and $P_{n+1}P_{n+2}\cdots P_{2n}$ are Motzkin paths of length n which are symmetric to each other. On the other hand, if $k < n$, then $P_{k+1} = U$, $P_{2n-k} = D$, the subpath $P_{k+2}P_{k+3}\cdots P_{2n-k-1}$ is a symmetric Motzkin path of length $2n-2k-2$, and both of two subpaths $P_1P_2\cdots P_k$ and $P_{2n-k+1}P_{2n-k+2}\cdots P_{2n}$ are Motzkin paths of length k which are symmetric to each other. Hence, we have a relation between S_{2n} and M_n:

$$S_{2n} = M_n + \sum_{k=0}^{n-1} S_{2n-2k-2}M_k. \quad (1)$$

Equation (1) and a closed formula for S_{2n} can also be found in the OEIS as A005773 [8].

3.2 Counting self-conjugate $(2s, 2s+1, 2s+2)$-cores

The following lemma plays an important role to obtain a recurrence relation for the number of self-conjugate $(2s, 2s+1, 2s+2)$-cores.

Lemma 3.2. *Let s be a positive integer. The number of self-conjugate $(2s, 2s+1, 2s+2)$-cores λ with $2s-1 \in MD(\lambda)$ is equal to the number of self-conjugate $(2s-2, 2s-1, 2s)$-cores.*

Proof. By Proposition 2.2, there is a bijection between $(2s, 2s+1, 2s+2)$-cores λ with $2s-1 \in MD(\lambda)$ and lower ideals I of $\tilde{P}_{\{2s,2s+1,2s+2\}}$ containing $2s-1$ and no elements h_1, h_2 such that $h_1 + h_2 \in \{4s, 4s+2, 4s+4\}$. Thus, it is enough to consider lower ideals of the first diagram in Figure 6.

To prove the lemma, we construct a bijection ϕ between lower ideals I of the first diagram and lower ideals J of the second diagram in Figure 6, since the second diagram is equivalent to the modified diagram of $\tilde{P}_{\{2s-2,2s-1,2s\}}$.

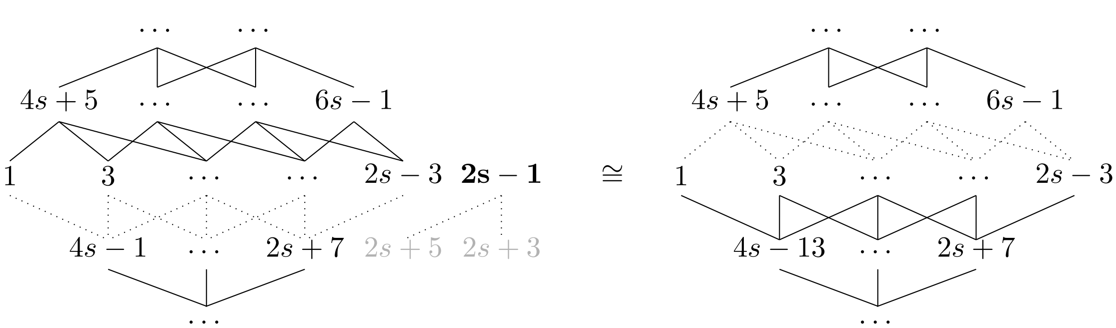

Figure 6: The modified diagram of $\tilde{P}_{\{2s,2s+1,2s+2\}}$ for ideals having $2s-1$

If I is a lower ideal of the first diagram, then I satisfies the following:

- I contains $2s-1$.

- For $4s+5 \le h \le 6s-1$, $h \in I$ implies $h-4s-4, h-4s-2, h-4s \in I$.
- For $2s+7 \le h \le 4s-1$, $h \in I$ implies $4s-h, 4s-h+2, 4s-h+4 \notin I$.

Now, we construct a corresponding set $\phi(I)$ from I as follows.

- For each $h \in I$ with $4s+5 \le h \le 6s-1$, delete $h-4s-4, h-4s-2, h-4s$ from I.
- For each $h \in I$ with $2s+7 \le h \le 4s-1$, add $4s-h, 4s-h+2, 4s-h+4$ to I.
- Delete $2s-1$ from the set.

Then $\phi(I)$ is a lower ideal of the poset structure defined by the second diagram in Figure 6 and it is easy to check that ϕ is a bijection. ☐

Example 3.3. *For self-conjugate* $(8,9,10)$*-cores* λ *such that* $7 \in MD(\lambda)$*, let* $I_1, I_2, \dots, I_{13}$ *be their corresponding lower ideals of* $\tilde{P}_{\{8,9,10\}}$*. Now, we list* I_i *and* $J_i = \phi(I_i)$ *for* $i = 1, 2, \dots, 13$*, where* ϕ *is the bijection defined in the proof of Lemma 3.2.*

$I_1 = \{7\}$	$J_1 = \emptyset$	$I_2 = \{1,7\}$	$J_2 = \{1\}$
$I_3 = \{3,7\}$	$J_3 = \{3\}$	$I_4 = \{5,7\}$	$J_4 = \{5\}$
$I_5 = \{1,3,7\}$	$J_5 = \{1,3\}$	$I_6 = \{1,5,7\}$	$J_6 = \{1,5\}$
$I_7 = \{3,5,7\}$	$J_7 = \{3,5\}$	$I_8 = \{1,3,5,7\}$	$J_8 = \{1,3,5\}$
$I_9 = \{1,3,5,7,21\}$	$J_9 = \{21\}$	$I_{10} = \{1,3,5,7,21,13\}$	$J_{10} = \{21,23\}$
$I_{11} = \{3,5,7,23\}$	$J_{11} = \{23\}$	$I_{12} = \{1,3,5,7,23\}$	$J_{12} = \{1,23\}$
$I_{13} = \{7,15\}$	$J_{13} = \{1,3,5,15\}$		

We note that for each i*,* I_i *is an ideal of the first diagram and* J_i *is an ideal of the second diagram of the following figure.*

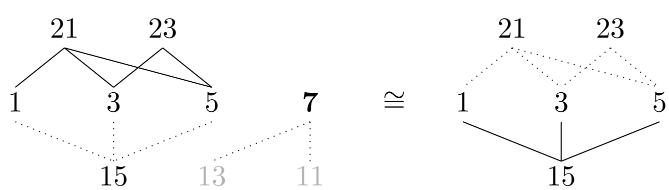

Figure 7: The modified diagram of $\tilde{P}_{\{8,9,10\}}$ for ideals I with $7 \in I$

The following proposition is a generalization of Lemma 3.2.

Proposition 3.4. *Let* s *and* k *be positive integers such that* $k \le s$*.*

(a) *The number of self-conjugate* $(2s, 2s+1, 2s+2)$*-cores* λ *satisfies that*

$$2k-1 \in MD(\lambda) \quad \textit{and} \quad 2k+1, 2k+3, \ldots, 2s-1 \notin MD(\lambda)$$

is the number of self-conjugate $(2k-2, 2k-1, 2k)$*-cores multiplied by* M_{s-k}.

(b) *The number of self-conjugate* $(2s, 2s+1, 2s+2)$*-cores* λ *with* $1, 3, \ldots, 2s-1 \notin MD(\lambda)$ *is* M_s.

Proof. We consider the modified diagram of $\tilde{P}_{\{2s,2s+1,2s+2\}}$ with restrictions in Figure 8.

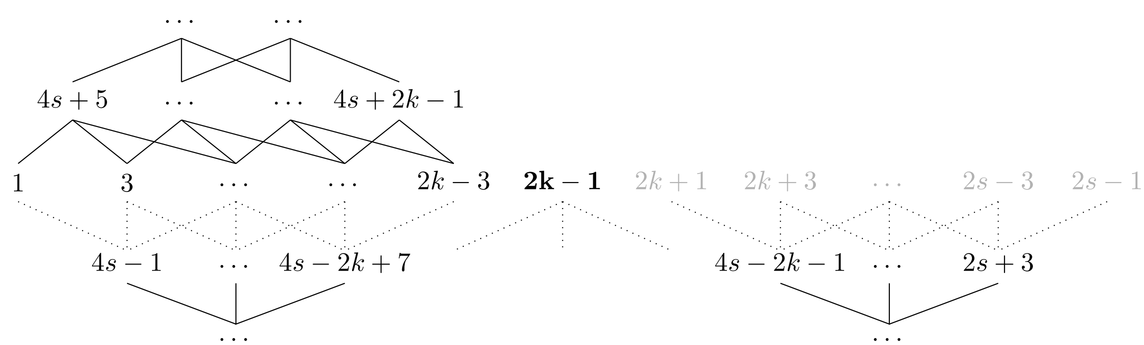

Figure 8: The modified diagram of $\tilde{P}_{\{2s,2s+1,2s+2\}}$ with restrictions

(a) On the modified diagram with restrictions, the left-hand side of $2k-1$ is equivalent to the modified diagram of $\tilde{P}_{\{2k-2,2k-1,2k\}}$ as we showed in Lemma 3.2, and the right-hand side of $2k-1$ is equivalent to the Hasse diagram of the poset $P_{\{2s-2k-2,2s-2k-1,2s-2k\}}$. Hence, by Corollary 1.5, the number of lower ideals I satisfying $2k-1 \in I$ and $2k+1, 2k+3, \ldots, 2s-1 \notin I$ is M_{s-k} times the number of self-conjugate $(2k-2, 2k-1, 2k)$-cores.

(b) If $1, 3, \ldots, 2s-1 \notin I$, then I is a lower ideal of a subposet of $\tilde{P}_{\{2s,2s+1,2s+2\}}$ which is equivalent to $P_{\{s,s+1,s+2\}}$. Hence, by Corollary 1.5, the number of lower ideals I with $1, 3, \ldots, 2s-1 \notin I$ is M_s.

□

Now, we are ready to prove our main theorem.

Theorem 3.5. *For a positive integer* s*, the number of self-conjugate* $(s, s+1, s+2)$*-cores is*

$$\sum_{i \geq 0} \binom{\lfloor \frac{s}{2} \rfloor}{i} \binom{i}{\lfloor \frac{i}{2} \rfloor},$$

which counts the number of symmetric Motzkin paths of length s.

Proof. Let a_s denote the number of self-conjugate $(s, s+1, s+2)$-cores. From Proposition 2.4, we have $a_{2s+1} = a_{2s}$. Hence, it is enough to show that $a_{2s} = S_{2s}$.

For $1 \le k \le s$, let A_k be the set of self-conjugate $(2s, 2s+1, 2s+2)$-cores λ that satisfies

$$2k-1 \in MD(\lambda) \quad \text{and} \quad 2k+1, 2k+3, \ldots, 2s-1 \notin MD(\lambda),$$

and let A_0 be the set of self-conjugate $(2s, 2s+1, 2s+2)$-cores λ with $2i-1 \notin MD(\lambda)$ for $1 \le i \le s$. Then, $A_0 \cup A_1 \cup \cdots \cup A_s$ is the set of self-conjugate $(2s, 2s+1, 2s+1)$-cores and

$$a_{2s} = |A_0| + |A_1| + \cdots + |A_s|.$$

From Proposition 3.4, we have $|A_0| = M_s$ and $|A_k| = a_{2k-2}M_{s-k}$ for $1 \le k \le s$, and therefore,

$$a_{2s} = M_s + \sum_{k=1}^{s} a_{2k-2}M_{s-k} = M_s + \sum_{k=0}^{s-1} a_{2s-2k-2}M_k.$$

Since the relation between a_{2s} and M_s is equivalent to (1) and $a_0 = S_0 = 1$, we have come to a conclusion that $a_{2s} = S_{2s} = \sum \binom{s}{i}\binom{i}{\lfloor i/2 \rfloor}$ by Proposition 3.1. $\square$

Encouraged by this success, we offer the following generalized conjecture.

Conjecture 3.6. *For given positive integers s and k, the number of self-conjugate $(s, s+1, \ldots, s+k)$-cores is equal to the number of symmetric (s, k)-generalized Dyck paths.*

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