CATALAN INTERVALS AND UNIQUELY SORTED PERMUTATIONS

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Abstract. For each positive integer k, we consider five well-studied posets defined on the set of Dyck paths of semilength k. We establish bijections between uniquely sorted permutations that avoid various patterns and intervals in these posets. We end with several conjectures.

1. Introduction

A *Dyck path of semilength* k is a lattice path in the plane consisting of k $(1,1)$ steps (also called *up steps*) and k $(1,-1)$ steps (also called *down steps*) that starts at the origin and never passes below the horizontal axis. Letting U and D denote up steps and down steps, respectively, we can view a Dyck path of semilength k as a word over the alphabet $\{U, D\}$ that contains k copies of each letter and has the property that every prefix has at least as many U's as it has D's. The number of such paths is the k^{th} Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$; this is just one of the overwhelmingly abundant incarnations of these numbers.

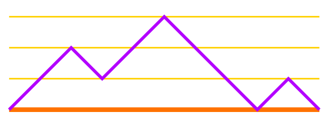

Figure 1. The Dyck path $UUDUUDDDUD$ of semilength 5.

Let $\mathbf{D}_k$ be the set of Dyck paths of semilength k. We obtain a natural partial order $\leq_S$ on $\mathbf{D}_k$ by declaring that $\Lambda \leq_S \Lambda'$ if Λ lies below or is equal to Λ'. Alternatively, we have $\Lambda_1 \cdots \Lambda_{2k} \leq_S \Lambda'_1 \cdots \Lambda'_{2k}$ if and only if the number of U's in $\Lambda_1 \cdots \Lambda_i$ is at most the number of U's in $\Lambda'_1 \cdots \Lambda'_i$ for every $i \in \{1, \ldots, 2k\}$. The poset $(\mathbf{D}_k, \leq_S)$ turns out to be a distributive lattice; it is known as the k^{th} *Stanley lattice* and is denoted by $\mathcal{L}_k^S$ [4]. The upper left image in Figure 2 shows the Hasse diagram of $\mathcal{L}_3^S$.

The k^{th} *Tamari lattice* is often defined on the set of binary plane trees with k vertices. However, there are several equivalent definitions that allow one to define isomorphic lattices on other sets of objects counted by the k^{th} Catalan number. Because the Tamari lattices are so multifaceted,

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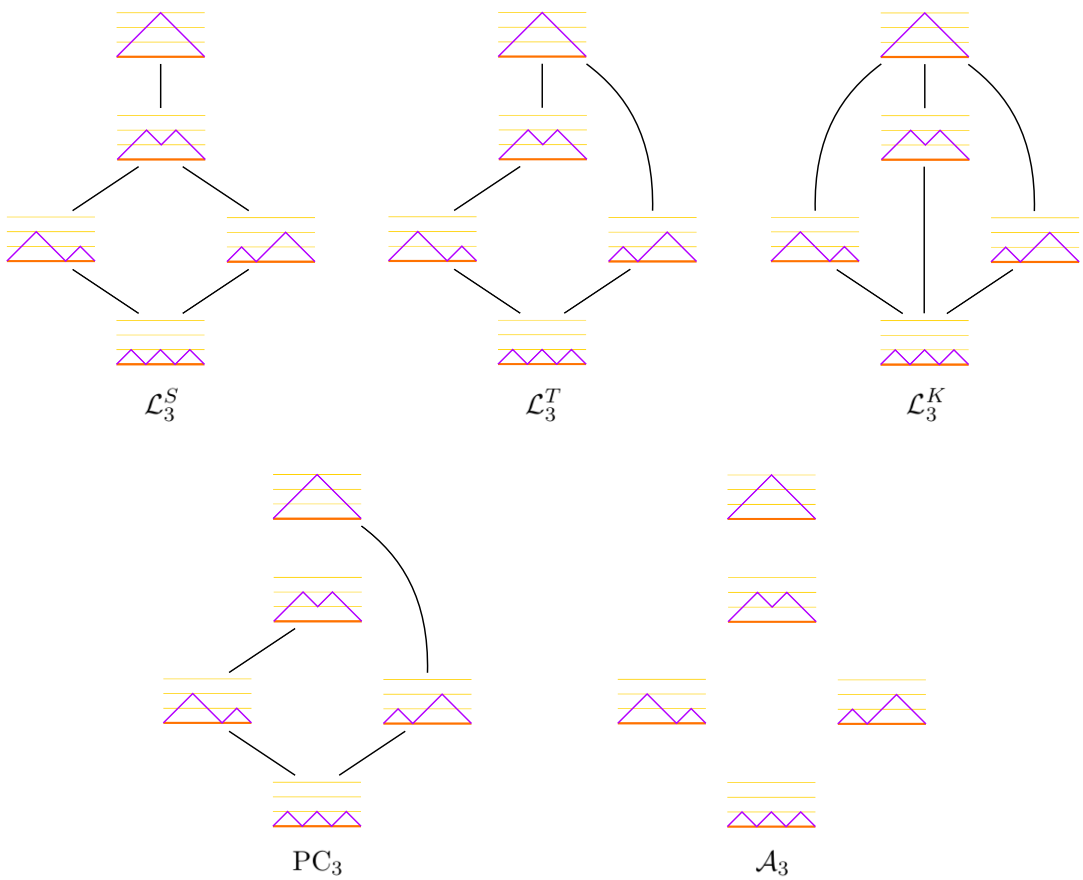


FIGURE 2. The Hasse diagrams of our Catalan posets for $k = 3$.

they have been extensively studied in combinatorics, group theory, theoretical computer science, algebraic geometry, and algebraic topology [4, 16, 20, 35, 37, 39, 42, 46, 49, 54]. In particular, Tamari lattices arise as the 1-skeletons of associahedra [46]. If $\leq_1$ and $\leq_2$ are two partial orders on the same set X, then we say the poset $(X, \leq_1)$ is an *extension* of the poset $(X, \leq_2)$ if $x \leq_2 y$ implies $x \leq_1 y$ for all $x, y \in X$. Bernardi and Bonichon [4] described how to define $\mathcal{L}_k^T$, the k^{th} Tamari lattice, so that its underlying set is $\mathbf{D}_k$; they proved that the Stanley lattice $\mathcal{L}_k^S$ is an extension of $\mathcal{L}_k^T$. We define $\mathcal{L}_k^T$ in Section 2.

In a now-classical paper, Kreweras [43] investigated the poset NC_k of all noncrossing partitions of the set $[k]$ ordered by refinement, showing, in particular, that this poset is a lattice. It is difficult to overstate the importance and ubiquity of noncrossing partitions and these lattices in mathematics [1, 4, 42, 43, 47, 51–53]. Using a bijection between Dyck paths and noncrossing partitions, Bernardi and Bonichon [4] defined an isomorphic copy of NC_k, denoted $\mathcal{L}_k^K$, so that its underlying set is $\mathbf{D}_k$. They also showed that $\mathcal{L}_k^T$ is an extension of $\mathcal{L}_k^K$. We call $\mathcal{L}_k^K$ the k^{th} *Kreweras lattice.*[1] We will find it more convenient to work with the noncrossing partition lattices instead of the Kreweras lattices, so we refer the interested reader to [4] for the definition of $\mathcal{L}_k^K$.

Bernardi and Bonichon used the name "Catalan lattices" to refer to $\mathcal{L}_k^S$, $\mathcal{L}_k^T$, and $\mathcal{L}_k^K$. Building off of earlier work of Bonichon [10], they gave unified bijections between intervals in these lattices

[1]We use the names "Kreweras lattice" and "noncrossing partition lattice" to distinguish the underlying sets, even though the lattices themselves are isomorphic.

and certain types of triangulations and realizers of triangulations. We find it appropriate to add two additional families of posets to this Catalan clan. The first is the family of *Pallo comb posets*, a relatively new family of posets introduced by Pallo in [50] as natural subposets of the Tamari lattices and studied afterward in [2, 20]. We let PC_k denote the k^{th} Pallo comb poset. These were defined on sets of binary trees in [20, 50] and on sets of triangulations in [2]; in Section 2, we define the Pallo comb posets on sets of Dyck paths. The second family of posets we add to the clan is the family of Catalan antichains. That is, we let $\mathcal{A}_k$ denote the antichain (poset with no order relations) defined on the set $\mathbf{D}_k$.

An *interval* in a poset P is a pair (x, y) of elements of P such that $x \leq y$. Let $\mathrm{Int}(P)$ be the set of all intervals of P. It is often interesting to count the intervals in combinatorial classes of posets, and the Catalan posets defined above are no exceptions. De Sainte-Catherine and Viennot [21] proved that

$$|\operatorname{Int}(\mathcal{L}_k^S)| = C_k C_{k+2} - C_{k+1}^2 = \frac{6}{(k+1)(k+2)^2(k+3)}\binom{2k}{k}\binom{2k+2}{k+1}. \tag{1}$$

Chapoton [16] proved that

$$|\operatorname{Int}(\mathcal{L}_k^T)| = \frac{2}{(3k+1)(3k+2)}\binom{4k+1}{k+1}. \tag{2}$$

In his initial investigation of the noncrossing partition lattices, Kreweras [43] proved that

$$|\operatorname{Int}(\mathcal{L}_k^K)| = |\operatorname{Int}(\mathrm{NC}_k)| = \frac{1}{2k+1}\binom{3k}{k}. \tag{3}$$

Aval and Chapoton [2] proved that

$$\sum_{k\geq 0} |\operatorname{Int}(\mathrm{PC}_k)| x^k = C(xC(x)), \tag{4}$$

where $C(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ is the generating function of the sequence of Catalan numbers. Of course, we also have

$$|\operatorname{Int}(\mathcal{A}_k)| = |\mathbf{D}_k| = C_k. \tag{5}$$

The formulas in (1), (2), (3), (4), (5) give rise to the OEIS sequences A005700, A000260, A001764, A127632, A000108, respectively [48].

Throughout this article, the word "permutation" refers to a permutation of a set of positive integers, written in one-line notation. Let S_n denote the set of permutations of the set $[n]$. The latter half of the present article's title refers to a special collection of permutations that arise in the study of West's stack-sorting map. This map, denoted by s, sends permutations of length n to permutations of length n. It is a slight variant of the stack-sorting algorithm that Knuth introduced in [41]. The map s was studied extensively in West's 1990 Ph.D. thesis [58] and has received a considerable amount of attention ever since [5–9, 11–15, 18, 19, 22–34, 36, 38, 55, 58, 59]. We give necessary background results concerning the stack-sorting map in Section 3, but the reader seeking additional historical motivation should consult one of the references [5, 8, 22–32]. There are multiple ways to define s, but the simplest is probably the following recursive definition. First, s sends the empty permutation to itself. If π is a permutation whose largest entry is n, then we can write $\pi = LnR$. We then define $s(\pi) = s(L)s(R)n$. For example,

$$s(35241) = s(3)\,s(241)\,5 = 3\,s(2)\,s(1)\,45 = 32145.$$

One of the central definitions concerning the stack-sorting map is that of the *fertility* of a permutation π; this is simply $|s^{-1}(\pi)|$, the number of preimages of π under s. Bousquet-Mélou called a permutation *sorted* if its fertility is positive. The following much more recent definition appeared first in [31].

Definition 1.1. We say a permutation is *uniquely sorted* if its fertility is 1. Let $\mathcal{U}_n$ denote the set of uniquely sorted permutations in S_n.

The following theorem from [31] characterizes uniquely sorted permutations. Recall that a *descent* of a permutation $\pi = \pi_1 \cdots \pi_n$ is an index $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. We let $\mathrm{Des}(\pi)$ denote the set of descents of the permutation π and let $\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$.

Theorem 1.1 ([31])**.** *A permutation of length n is uniquely sorted if and only if it is sorted and has exactly $\dfrac{n-1}{2}$ descents.*

Uniquely sorted permutations contain a huge amount of interesting hidden structure. The results in [31] hint that, in some loose sense, uniquely sorted permutations are to general permutations[2] what matchings are to general set partitions. For example, one immediate consequence of Theorem 1.1 is that there are no uniquely sorted permutations of even length (just as there are no matchings of a set of odd size). The authors of [31] defined a bijection between new combinatorial objects called "valid hook configurations" and certain weighted set partitions that Josuat-Vergès [40] studied in the context of free probability theory. They then showed that restricting this bijection to the set of valid hook configurations of uniquely sorted permutations induces a bijection between uniquely sorted permutations and those weighted set partitions that are matchings. This allowed them to prove that $|\mathcal{U}_{2k+1}| = A_{k+1}$, where $(A_m)_{m\geq 1}$ is OEIS sequence A180874 and is known as *Lassalle's sequence*. This fascinating new sequence first appeared in [45], where Lassalle proved a conjecture of Zeilberger by showing that the sequence is increasing. In fact, the bijection established in [31] produced three new combinatorial interpretations of Lassalle's sequence; the only combinatorial interpretation known prior involved the weighted matchings that Josuat-Vergès had examined. The article [31] also proves that the sequences $(A_{k+1}(\ell))_{\ell=1}^{2k+1}$ are symmetric, where $A_{k+1}(\ell)$ is the number of elements of $\mathcal{U}_{2k+1}$ with first entry ℓ.

The present article is meant to link uniquely sorted permutations that avoid certain patterns with intervals in the Catalan posets discussed above. Let $\mathcal{U}_n(\tau^{(1)}, \ldots, \tau^{(r)})$ denote the set of uniquely sorted permutations in S_n that avoid the patterns $\tau^{(1)}, \ldots, \tau^{(r)}$ (see the beginning of Section 3 for the definition of pattern avoidance). In Section 2, we define the Tamari lattices and Pallo comb posets. Section 3 reviews relevant background concerning the stack-sorting map and permutation patterns. Section 4 introduces new operators that act on permutations. We prove several properties of these operators that are used heavily in the remainder of the paper and in [26]. In Section 5, we find a bijection $\mathcal{U}_{2k+1}(312) \to \mathrm{Int}(\mathcal{L}_k^S)$, showing that 312-avoiding uniquely sorted permutations are counted by the numbers in (1). The proof that this map is surjective actually relies on a fun "energy argument" similar to the one used in the solution of the game "Conway's Soldiers." In Section 6, we find bijections $\mathcal{U}_{2k+1}(231) \to \mathcal{U}_{2k+1}(132)$ and $\mathcal{U}_{2k+1}(132) \to \mathrm{Int}(\mathcal{L}_k^T)$, showing that the permutations in $\mathcal{U}_{2k+1}(231)$ and the permutations in $\mathcal{U}_{2k+1}(132)$ are counted by the numbers in (2). In Section 7, we use generating trees to exhibit a bijection $\mathcal{U}_{2k+1}(312, 1342) \to \mathrm{Int}(\mathcal{L}_k^K)$, proving that the permutations in $\mathcal{U}_{2k+1}(312, 1342)$ are counted by the numbers in (3). In Section 8, we show that the permutations in $\mathcal{U}_{2k+1}(231, 4132)$ are in bijection with the intervals in $\mathrm{Int}(\mathrm{PC}_k)$.

[2] One could also substitute "general sorted permutations" for "general permutations" in this analogy.

In Section 9, we give bijections demonstrating that

$$|\mathcal{U}_{2k+1}(321)| = |\mathcal{U}_{2k+1}(132,231)| = |\mathcal{U}_{2k+1}(132,312)| = |\mathcal{U}_{2k+1}(231,312)| = C_k.$$

Thus, these sets of permutations are in bijection with intervals of the antichain $\mathcal{A}_k$. In fact, the bijection $\mathcal{U}_{2k+1}(312) \to \mathrm{Int}(\mathcal{L}_k^S)$ from Section 5 restricts to a bijection $\mathcal{U}_{2k+1}(312,231) \to \mathrm{Int}(\mathcal{A}_k)$. In Section 10, we quickly complete the enumeration of sets of the form $\mathcal{U}_{2k+1}(\tau^{(1)},\ldots,\tau^{(r)})$ when $\tau^{(1)},\ldots,\tau^{(r)} \in S_3$. We also formulate eighteen enumerative conjectures about sets of the form $\mathcal{U}_{2k+1}(\tau^{(1)},\tau^{(2)})$ with $\tau^{(1)} \in S_3$ and $\tau^{(2)} \in S_4$.

2. Tamari Lattices and Pallo Comb Posets

In this brief section, we define the Tamari lattices and Pallo comb posets. We will not actually need the definition of the Pallo comb posets in the rest of the article, but we include it here for the sake of completeness.

Definition 2.1. Given $\Lambda \in \mathbf{D}_k$, we can write $\Lambda = UD^{\gamma_1}UD^{\gamma_2}\cdots UD^{\gamma_k}$ for some nonnegative integers $\gamma_1,\ldots,\gamma_k$. Let $\mathrm{long}_j(\Lambda)$ be the smallest nonnegative integer t such that

$$\gamma_j + \gamma_{j+1} + \cdots + \gamma_{j+t} > t.$$

We call $\mathrm{long}_j(\Lambda)$ the *longevity* of the j^{th} up step of Λ. The *longevity sequence* of Λ is the tuple $(\mathrm{long}_1(\Lambda),\ldots,\mathrm{long}_k(\Lambda))$.

Geometrically, $\mathrm{long}_j(\Lambda)$ is the semilength of the longest Dyck path that we can obtain by starting where the j^{th} up step of Λ ends and following Λ. For instance, the longevity sequence of the Dyck path in Figure 1 is $(3,0,1,0,0)$. Theorem 2 in [49] and Theorem 1 in [50] characterize the Tamari lattices and Pallo comb posets (defined on sets of binary trees) in terms of "weight sequences" of binary trees. There is a bijection between Dyck paths and binary trees such that the weight sequence of the tree corresponding to $\Lambda \in \mathbf{D}_k$ is $(\mathrm{long}_k(\Lambda)+1,\ldots,\mathrm{long}_1(\Lambda)+1)$. We have used this correspondence to arrive at the following two definitions.

Definition 2.2. Given $\Lambda,\Lambda' \in \mathbf{D}_k$, we write $\Lambda \le_T \Lambda'$ if $\mathrm{long}_j(\Lambda) \le \mathrm{long}_j(\Lambda')$ for all $j \in [k]$. The k^{th} *Tamari lattice* is the poset $\mathcal{L}_k^T = (\mathbf{D}_k, \le_T)$.

Definition 2.3. Given $\Lambda,\Lambda' \in \mathbf{D}_k$, we write $\Lambda \le_{\text{Pallo}} \Lambda'$ if $\Lambda \le_T \Lambda'$ and if for every $j \in [k]$ such that $\mathrm{long}_j(\Lambda) < \mathrm{long}_j(\Lambda')$, we have $\mathrm{long}_\ell(\Lambda) \le j - \ell - 1$ for all $\ell \in [j-1]$. The k^{th} *Pallo comb poset* is $\mathrm{PC}_k = (\mathbf{D}_k, \le_{\text{Pallo}})$.

3. Stack-Sorting Background

Recall that S_n is the set of permutations of $[n]$ and that $\mathcal{U}_n$ is the set of uniquely sorted permutations in S_n (see Definition 1.1). The *normalization* of a permutation $\pi = \pi_1\cdots\pi_n$ is the permutation in S_n obtained from π by replacing the i^{th}-smallest entry of π by i for all $i \in [n]$. We say two permutations have the *same relative order* if their normalizations are equal. A permutation is called *normalized* if it is in S_n for some n. If $\sigma = \sigma_1\cdots\sigma_n$ and $\tau = \tau_1\cdots\tau_m$ are permutations, then we say σ *contains* the pattern τ if there are indices $i_1 < \cdots < i_m$ such that $\sigma_{i_1}\cdots\sigma_{i_m}$ has the same relative order as τ. Otherwise, we say σ *avoids* τ. Let $\mathrm{Av}(\tau^{(1)},\tau^{(2)},\ldots)$ be the set of normalized permutations that avoid the patterns $\tau^{(1)},\tau^{(2)},\ldots$ (this sequence of patterns could be finite or infinite). Let $\mathrm{Av}_n(\tau^{(1)},\tau^{(2)},\ldots) = \mathrm{Av}(\tau^{(1)},\tau^{(2)},\ldots)\cap S_n$ and $\mathcal{U}_n(\tau^{(1)},\tau^{(2)},\ldots) = \mathrm{Av}(\tau^{(1)},\tau^{(2)},\ldots)\cap\mathcal{U}_n$. Let $\mathcal{U}(\tau^{(1)},\tau^{(2)},\ldots)$ denote the set of all uniquely sorted permutations in $\mathrm{Av}(\tau^{(1)},\tau^{(2)},\ldots)$.

The investigation of permutation patterns initiated with Knuth's introduction of a certain "stack-sorting algorithm" in [41]. West introduced the stack-sorting map s, which is a deterministic variant of Knuth's algorithm, in his dissertation [58]. Recall that the fertility of a permutation π is $|s^{-1}(\pi)|$. It follows from Knuth's analysis that $s^{-1}(123\cdots n) = \text{Av}_n(231)$ and that $|\text{Av}_n(231)| = C_n$, so the fertilities of increasing permutations are Catalan numbers. West also went to great lengths to calculate the fertilities of the permutations of the forms

$$23\cdots k1(k+1)\cdots n, \quad 12\cdots(k-2)k(k-1)(k+1)\cdots n, \quad \text{and} \quad k12\cdots(k-1)(k+1)\cdots n.$$

The very particular forms of these permutations indicates that computing fertilities of permutations is, a priori, a difficult task.

Bousquet-Mélou [12] provided an algorithm for determining whether or not a given permutation is sorted (i.e., has positive fertility). She then asked for a general method for computing the fertility of any given permutation. The present author accomplished this in much greater generality in [28–30,32] ([32] is joint with Kravitz) by introducing new combinatorial objects called "valid hook configurations." He and his coauthors have since developed and applied the theory of valid hook configurations in order to reprove several known results and establish many new results concerning the stack-sorting map [22, 23, 25, 26, 28–32].

Bousquet-Mélou's algorithm for determining if a permutation is sorted proceeds by describing how to compute the so-called "canonical tree" of a permutation, which is unique if it exists. A permutation is sorted if and only if it has a canonical tree. The current author [29] translated this algorithm into the language of valid hook configurations, defining the "canonical valid hook configuration" of a permutation. We are fortunate in this article that we do not need all of the definitions and main theorems concerning valid hook configurations. In order to work with uniquely sorted permutations, we will only need to define canonical valid hook configurations.

The *plot* of a permutation $\pi = \pi_1\cdots\pi_n$ is obtained by plotting the points (i, π_i) for all $i \in [n]$. A *hook* H of π is drawn by starting at a point (i, π_i) in the plot of π, moving vertically upward, and then moving to the right to connect with a different point (j, π_j). In order to do this, we must have $i < j$ and $\pi_i < \pi_j$. The point (i, π_i) is called the *southwest endpoint* of H, while (j, π_j) is called the *northeast endpoint* of H. We say a point (r, π_r) *lies strictly below* H if $i < r < j$ and $\pi_r < \pi_j$. We say (r, π_r) *lies weakly below* H if it lies strictly below H or if $r = j$. The left image in Figure 3 shows the plot of a permutation along with a single hook.

Canonical Valid Hook Configuration Construction

Recall that a descent of $\pi = \pi_1\cdots\pi_n$ is an index $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Let $d_1 < \cdots < d_k$ be the descents of π. The *canonical valid hook configuration* of π is the tuple $\mathcal{H} = (H_1, \ldots, H_k)$ of hooks of π defined as follows. First, the southwest endpoint of the hook H_i is (d_i, π_{d_i}). We let $\mathfrak{N}_i$ denote the northeast endpoint of H_i. We determine these northeast endpoints in the order $\mathfrak{N}_k, \mathfrak{N}_{k-1}, \ldots, \mathfrak{N}_1$. First, $\mathfrak{N}_k$ is the leftmost point lying above and to the right of (d_k, π_{d_k}). Next, $\mathfrak{N}_{k-1}$ is the leftmost point lying above and to the right of $(d_{k-1}, \pi_{d_{k-1}})$ that does not lie weakly below H_k. In general, $\mathfrak{N}_\ell$ is the leftmost point lying above and to the right of (d_ℓ, π_{d_ℓ}) that does not lie weakly below any of the hooks $H_k, H_{k-1}, \ldots, H_{\ell+1}$. If there is any time during this process when the point $\mathfrak{N}_\ell$ does not exist, then π does not have a canonical valid hook configuration. See the right part of Figure 3 for an example of this construction. $\lozenge$

The following useful proposition is a consequence of the discussion of canonical valid hook configurations in [29], although it is essentially equivalent to Bousquet-Mélou's algorithm in [12]. In

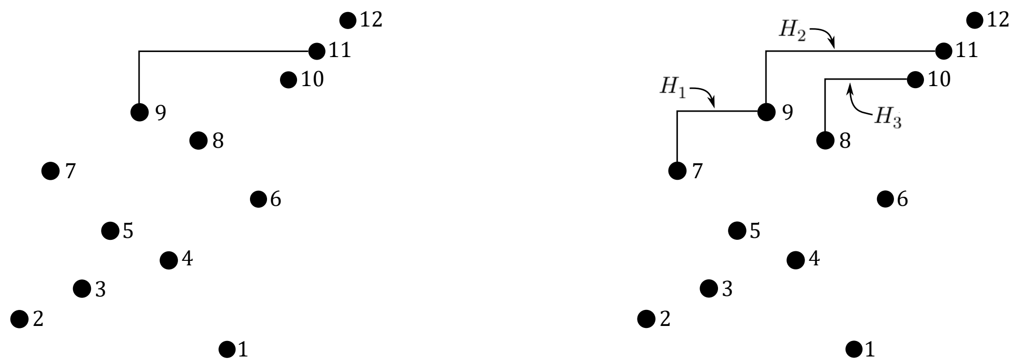


FIGURE 3. On the left is the plot of 2 7 3 5 9 4 8 1 6 10 11 12 along with one hook whose southwest endpoint is $(5, 9)$ and whose northeast endpoint is $(11, 11)$. The points lying strictly below this hook are $(6, 4), (7, 8), (8, 1), (9, 6), (10, 10)$. These five points and $(11, 11)$ are the points lying weakly below the hook. The right image shows the canonical valid hook configuration of 2 7 3 5 9 4 8 1 6 10 11 12.

combination with Theorem 1.1, this proposition allows us to determine whether or not a given permutation is uniquely sorted.

Proposition 3.1 ([29]). *A permutation is sorted if and only if it has a canonical valid hook configuration.*

We end this section by recording some lemmas regarding canonical valid hook configurations that will prove useful in subsequent sections. Let us say a point (i, π_i) in the plot of a permutation $\pi = \pi_1 \cdots \pi_n$ is a *descent top* of the plot of π if i is a descent of π. Similarly, say (i, π_i) is a *descent bottom* of the plot of π if $i - 1$ is a descent of π. The point (i, π_i) is called a *left-to-right maximum* of the plot of π if it is higher than all of the points to its left.

Lemma 3.1. *Let π be a uniquely sorted permutation of length $2k + 1$. Let $\mathfrak{N}_1, \ldots, \mathfrak{N}_k$ be the northeast endpoints of the hooks in the canonical valid hook configuration of π. Let $\mathrm{DB}(\pi)$ be the set of descent bottoms of the plot of π. The two k-element sets $\mathrm{DB}(\pi)$ and $\{\mathfrak{N}_1, \ldots, \mathfrak{N}_k\}$ form a partition of the set $\{(i, \pi_i) : 2 \leq i \leq 2k + 1\}$.*

Proof. Theorem 1.1 tells us that we do indeed have $|\mathrm{DB}(\pi)| = \mathrm{des}(\pi) = k$. Let $d_1 < \cdots < d_k$ be the descents of π, and let $\mathcal{H} = (H_1, \ldots, H_k)$ be the canonical valid hook configuration of π. By convention, H_i has southwest endpoint (d_i, π_{d_i}) and northeast endpoint $\mathfrak{N}_i$. We must show that $\mathrm{DB}(\pi)$ and $\{\mathfrak{N}_1, \ldots, \mathfrak{N}_k\}$ are disjoint. If this were not the case, then we would have $(d_\ell + 1, \pi_{d_\ell + 1}) = \mathfrak{N}_m$ for some $\ell, m \in \{1, \ldots, k\}$. The southwest endpoint of H_m is (d_m, π_{d_m}), and this must lie below and to the left of $\mathfrak{N}_m$. Thus, $m < \ell$. Also, $\mathfrak{N}_m$ lies strictly below the hook H_ℓ. Referring to the canonical valid hook configuration construction to see how $\mathfrak{N}_m$ was defined, we find that this is impossible. □

Lemma 3.2. *Let $\pi \in \mathcal{U}_{2k+1}(312)$, and let $\mathcal{H} = (H_1, \ldots, H_k)$ be the canonical valid hook configuration of π. Let $\mathfrak{N}_i$ denote the northeast endpoint of H_i. The left-to-right maxima of the plot of π are precisely the points $(1, \pi_1), \mathfrak{N}_1, \ldots, \mathfrak{N}_k$.*

Proof. Let $\mathrm{DB}(\pi)$ be the set of descent bottoms of the plot of π. Because π avoids 312, every point in the plot of π that is not in $\mathrm{DB}(\pi)$ must be a left-to-right maximum of the plot of π. On the other hand, none of the points in $\mathrm{DB}(\pi)$ are left-to-right maxima of the plot of π. The desired result now follows from Lemma 3.1. □

4. Permutation Operations

In this section, we establish several definitions and conventions regarding permutations. It will often be convenient to associate permutations with their plots. From this viewpoint, a permutation is essentially just an arrangements of points in the plane such that no two distinct points lie on a single vertical or horizontal line. When viewing permutations in this way, we do not distinguish between two permutations that have the same relative order. In other words, the plots that we draw are really meant to represent equivalence classes of permutations, where two permutations are equivalent if they have the same relative order. For example, if $\lambda = 21$ and $\mu = 12$, then $\begin{array}{cc} & \boxed{\lambda} \\ \boxed{\mu} & \end{array}$ represents (the equivalence class of) 1243.

Given a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$, we let $\text{rev}(\pi) = \pi_n \cdots \pi_1$ be the reverse of π. Let π^{-1} be the inverse of π in the group S_n; this is the permutation in S_n in which the entry i appears in the π_i^{th} position. Geometrically, we obtain the plot of $\text{rev}(\pi)$ by reflecting the plot of π through the line $x = (n+1)/2$. We obtain the plot of π^{-1} by reflecting the plot of π though the line $y = x$. Let $\text{rot}(\pi)$ (respectively, $\text{rot}^{-1}(\pi)$) be the permutation whose plot is obtained by rotating the plot of π counterclockwise (respectively, clockwise) by 90°. Equivalently, $\text{rot}(\pi) = \text{rev}(\pi^{-1})$.

The *sum* of two permutations μ and λ, denoted $\mu \oplus \lambda$, is the permutation obtained by placing the plot of λ above and to the right of the plot of μ. The *skew sum* of μ and λ, denoted $\mu \ominus \lambda$, is the permutation obtained by placing the plot of λ below and to the right of the plot of μ. In our geometric point of view, we have

$$\mu \oplus \lambda = \begin{array}{cc} & \boxed{\lambda} \\ \boxed{\mu} & \end{array} \quad \text{and} \quad \mu \ominus \lambda = \begin{array}{cc} \boxed{\mu} & \\ & \boxed{\lambda} \end{array}.$$

For each $i \in [n]$, we define four "sliding operators" on S_n. The first, denoted[3] swu_i, essentially takes the points in the plot of a permutation π that lie southwest of the point with height i and slides them up above all the points that are southeast of the point with height i. We illustrate this operator in Figure 4. To define this more precisely, let L_i (respectively, R_i) be the set of elements of $[i-1]$ that lie to the left (respectively, right) of i in π. If $\pi_j \geq i$, then the j^{th} entry of $\text{swu}_i(\pi)$ is π_j. If $\pi_j < \pi_i$, then either $\pi_j \in L_i$ or $\pi_j \in R_i$. If π_j is the m^{th}-smallest element of R_i, then the j^{th} entry of $\text{swu}_i(\pi)$ is m. If π_j is the m^{th}-largest element of L_i, then the j^{th} entry of $\text{swu}_i(\pi)$ is $i - m$.

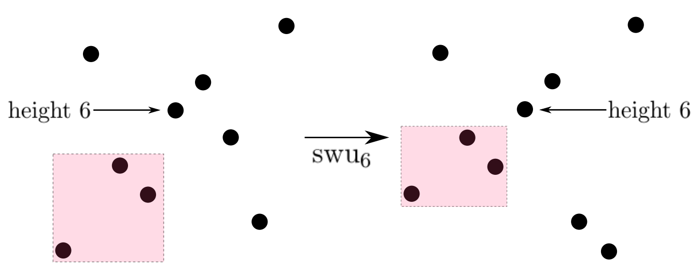


Figure 4. The operator swu_6 slides the points to the southwest of the point with height 6 (shaded in pink) up.

[3]The name of the operator stands for "southwest up."

The second operator we define is swd_i, which takes the points in the plot of π that lie southwest of the point with height i and slides them down below the points lying to the southeast of that point. We can define this operator formally by

$$\mathrm{swd}_i(\pi) = \mathrm{rev}(\mathrm{swu}_i(\mathrm{rev}(\pi))).$$

The third and fourth operators, swl_i and swr_i, are defined by

$$\mathrm{swl}_i(\pi) = \mathrm{rot}^{-1}(\mathrm{swu}_i(\mathrm{rot}(\pi))) \quad \text{and} \quad \mathrm{swr}_i(\pi) = \mathrm{rot}^{-1}(\mathrm{swd}_i(\mathrm{rot}(\pi))).$$

The operator swl_i takes the points to the southwest of the point in position i and slides them to the left; the operator swr_i slides them to the right. We illustrate swr_6 in Figure 5. We can also define these maps on arbitrary permutations by normalizing the permutations, applying the maps, and then "unnormalizing." For example, since $\mathrm{swu}_4(1243) = 2341$, we have $\mathrm{swu}_4(2496) = 4692$.

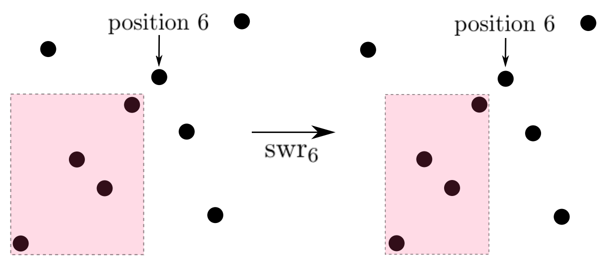


FIGURE 5. The operator swr_6 slides the points to the southwest of the point in position 6 (shaded in pink) to the right.

We define $\mathrm{swu} : S_n \to S_n$ by

$$\mathrm{swu} = \mathrm{swu}_1 \circ \mathrm{swu}_2 \circ \cdots \circ \mathrm{swu}_n\,.$$

An alternative recursive way of thinking of this map, which we illustrate in Figure 6, is as follows. Let us write $\pi = LnR$. We have

$$\mathrm{swu}(\pi) = (\mathrm{swu}(L) \oplus 1) \ominus \mathrm{swu}(R).$$

This recursive definition requires us to define swu on arbitrary permutations, which we can do by normalizating, applying swu, and then unnormalizing. The reader should imagine this sliding operator as acting on a collection of points in the plane instead of a string of numbers. Similarly, let

$$\mathrm{swd} = \mathrm{swd}_1 \circ \mathrm{swd}_2 \circ \cdots \circ \mathrm{swd}_n, \quad \mathrm{swl} = \mathrm{swl}_1 \circ \mathrm{swl}_2 \circ \cdots \circ \mathrm{swl}_n, \quad \mathrm{swr} = \mathrm{swr}_1 \circ \mathrm{swr}_2 \circ \cdots \circ \mathrm{swr}_n\,.$$

As before, we can also define these maps for arbitrary permutations. By a slight abuse of notation, we use the symbols swu, swd, swl, swr to denote the maps defined on all permutations of all lengths (or alternatively, on their equivalence classes).

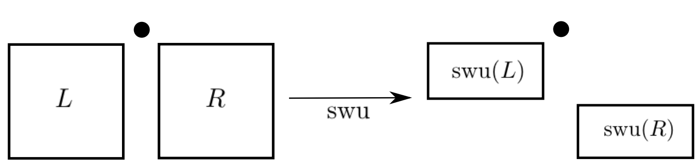


FIGURE 6. A recursive definition of the map swu.

Lemma 4.1. *The maps*

$$\mathrm{swu}:\mathrm{Av}(231)\to\mathrm{Av}(132)\quad \textit{and}\quad \mathrm{swd}:\mathrm{Av}(132)\to\mathrm{Av}(231)$$

are inverse bijections. The maps

$$\mathrm{swl}:\mathrm{Av}(132)\to\mathrm{Av}(312)\quad \textit{and}\quad \mathrm{swr}:\mathrm{Av}(312)\to\mathrm{Av}(132)$$

are inverse bijections.

Proof. We first prove by induction on n that

$$\mathrm{swu}:\mathrm{Av}_n(231)\to\mathrm{Av}_n(132)\quad \text{and}\quad \mathrm{swd}:\mathrm{Av}_n(132)\to\mathrm{Av}_n(231)$$

are inverse bijections. This is clear if $n=0$ or $n=1$, so assume $n\geq 2$. Choose $\pi\in\mathrm{Av}_n(231)$, and write $\pi=LnR$. Because π avoids 231, we have $\pi=L\oplus(1\ominus R)$. Furthermore, L and R avoid 231. The recursive definition of swu tells us that $\mathrm{swu}(\pi)=(\mathrm{swu}(L)\oplus 1)\ominus\mathrm{swu}(R)$. By induction, we find that $\mathrm{swu}(L)$ and $\mathrm{swu}(R)$ avoid 132, so $\mathrm{swu}(\pi)$ also avoids 132. Moreover, there is a recursive definition of swd analogous to the recursive definition of swu that yields

$$\mathrm{swd}(\mathrm{swu}(\pi))=\mathrm{swd}((\mathrm{swu}(L)\oplus 1)\ominus\mathrm{swu}(R))=\mathrm{swd}(\mathrm{swu}(L))\oplus(1\ominus\mathrm{swd}(\mathrm{swu}(R))).$$

By induction on n, this is just $L\oplus(1\ominus R)$, which is π. This shows that swd is a left inverse of swu, and a similar argument with the roles of swd and swu reversed shows that swd is also a right inverse of swu. The second statement now follows easily from the first if we use the fact that $\mathrm{swl}=\mathrm{rot}^{-1}\circ\mathrm{swu}\circ\mathrm{rot}$. □

Lemma 4.2. *The maps*

$$\mathrm{swu}:\mathrm{Av}(231,312)\to\mathrm{Av}(132,312)\quad \textit{and}\quad \mathrm{swd}:\mathrm{Av}(132,312)\to\mathrm{Av}(231,312)$$

are inverse bijections. The maps

$$\mathrm{swl}:\mathrm{Av}(132,231)\to\mathrm{Av}(231,312)\quad \textit{and}\quad \mathrm{swr}:\mathrm{Av}(231,312)\to\mathrm{Av}(132,231)$$

are inverse bijections.

Proof. To prove the first statement, we must show that $\mathrm{swu}(\mathrm{Av}_n(231,312))=\mathrm{Av}_n(132,312)$ for all $n\geq 1$. This is clear if $n\leq 2$, so we may assume $n\geq 3$ and induct on n. Choose $\pi\in\mathrm{Av}_n(231,312)$. Because π avoids 231, we can write $\pi=L\oplus(1\ominus R)$ for some permutations $L,R\in\mathrm{Av}(231,312)$. Note that R is a decreasing permutation because π avoids 312. By induction, $\mathrm{swu}(L)\in\mathrm{Av}(132,312)$. Also, $\mathrm{swu}(R)=R$. The recursive definition of swu tells us that $\mathrm{swu}(\pi)=(\mathrm{swu}(L)\oplus 1)\ominus\mathrm{swu}(R)$. The permutation $\mathrm{swu}(\pi)$ certainly avoids 132. Since $\mathrm{swu}(L)$ avoids 312 and $\mathrm{swu}(R)=R$ is decreasing, $\mathrm{swu}(\pi)$ avoids 312. This proves that $\mathrm{swu}(\mathrm{Av}_n(231,312))\subseteq\mathrm{Av}_n(132,312)$ for all $n\geq 1$. A similar argument with swu replaced by swd proves the reverse containment. The second statement now follows from the first if we use the fact that $\mathrm{swl}=\mathrm{rot}^{-1}\circ\mathrm{rev}\circ\mathrm{swd}\circ\mathrm{rev}\circ\mathrm{rot}$. □

In the following lemma, recall that $\mathrm{Des}(\pi)$ denotes the set of descents of π.

Lemma 4.3. *For every permutation π, we have $\mathrm{Des}(\mathrm{swu}(\pi))=\mathrm{Des}(\pi)$. If $\pi\in\mathrm{Av}(312)$, then $\mathrm{des}(\mathrm{swr}(\pi))=\mathrm{des}(\pi^{-1})=\mathrm{des}(\pi)$. If $\pi\in\mathrm{Av}(132)$, then $\mathrm{des}(\mathrm{swl}(\pi))=\mathrm{des}(\pi^{-1})=\mathrm{des}(\pi)$.*

Proof. For each $i\in[n]$ and $\sigma\in S_n$, it is clear from the definition of swu_i that $\mathrm{Des}(\mathrm{swu}_i(\sigma))=\mathrm{Des}(\sigma)$. The first claim now follows from the fact that $\mathrm{swu}(\pi)=\mathrm{swu}_1\circ\cdots\circ\mathrm{swu}_n(\pi)$. Now assume $\pi\in\mathrm{Av}_n(312)$. The second claim is trivial if $n\leq 1$, so we may assume $n\geq 2$ and induct on n. Because π avoids 312, we can write $\pi=\lambda\oplus(\mu\ominus 1)$ for some $\lambda\in\mathrm{Av}(312)$ and $\mu\in\mathrm{Av}(312)$. We have $\pi^{-1}=\lambda^{-1}\oplus(1\ominus\mu^{-1})$, so we can use induction to see that $\mathrm{des}(\pi)=\mathrm{des}(\lambda)+\mathrm{des}(\mu)+1=$

$\text{des}(\lambda^{-1}) + \text{des}(\mu^{-1}) + 1 = \text{des}(\pi^{-1})$. Similarly, the recursive definition of swr (which is analogous to the recursive definition that we gave for swu) tells us that $\text{swr}(\pi) = \text{swr}(\mu) \ominus (\text{swr}(\lambda) \oplus 1)$. We know by induction that $\text{des}(\text{swr}(\mu)) = \text{des}(\mu)$ and $\text{des}(\text{swr}(\lambda)) = \text{des}(\lambda)$. We want to prove that $\text{des}(\text{swr}(\pi)) = \text{des}(\pi)$. If μ is empty, then

$$\text{des}(\text{swr}(\pi)) = \text{des}(\text{swr}(\lambda)) = \text{des}(\lambda) = \text{des}(\pi).$$

If μ is nonempty, then

$$\text{des}(\text{swr}(\pi)) = \text{des}(\text{swr}(\mu)) + \text{des}(\text{swr}(\lambda)) + 1 = \text{des}(\mu) + \text{des}(\lambda) + 1 = \text{des}(\pi).$$

The proof of the third claim is completely analogous to the proof of the second. □

Lemma 4.4. *If π is a sorted permutation, then* $\text{swu}(\pi)$ *and* $\text{swd}(\pi)$ *are sorted. If, in addition, π avoids* 132*, then* $\text{swl}(\pi)$ *is sorted.*

Proof. Assume $\sigma \in S_n$ is sorted, and let $\mathcal{H}$ be its canonical valid hook configuration, which is guaranteed to exist by Proposition 3.1. For $i \in [n]$, we claim that $\text{swu}_i(\sigma)$ is sorted. To see this, note that the plot of $\text{swu}_i(\sigma)$ is obtained from the plot of σ by sliding some points up and sliding other points down. During this process, let us simply keep the hooks in $\mathcal{H}$ attached to their southwest and northeast endpoints. This is illustrated in Figure 7. The resulting configuration of hooks is the canonical valid hook configuration of $\text{swu}_i(\sigma)$.[4] Crucially, we are using the fact, which follows from the canonical valid hook configuration construction, that no hooks in $\mathcal{H}$ pass below the point with height i in the plot of σ (no hook can ever pass below any point in the plot). A similar argument shows that $\text{swd}_i(\sigma)$ is sorted. As i and σ were arbitrary, we find that if π is sorted, then $\text{swu}(\pi) = \text{swu}_1 \circ \cdots \circ \text{swu}_n(\pi)$ and $\text{swd}(\pi) = \text{swd}_1 \circ \cdots \circ \text{swd}_n(\pi)$ are sorted.

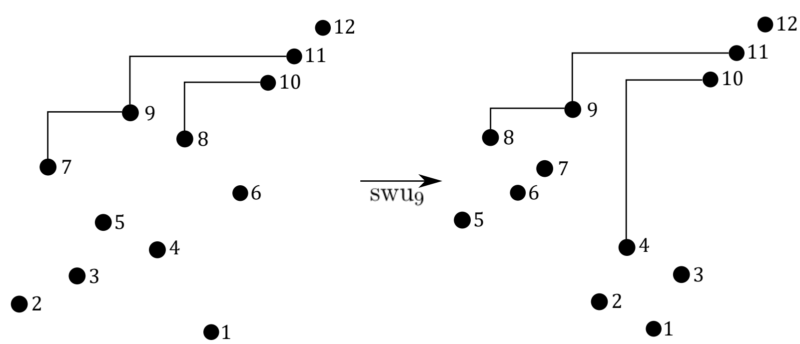


FIGURE 7. The canonical valid hook configuration of σ transforms into the canonical valid hook configuration of $\text{swu}_9(\sigma)$.

To help us prove the second statement, let us define $\text{def}(\sigma)$ to be the smallest nonnegative integer ℓ such that $\sigma \oplus (123 \cdots \ell)$ is sorted.[5] Thus, $\text{def}(\sigma) = 0$ if and only if σ is sorted. Roughly speaking, one can think of $\text{def}(\sigma)$ as the number of descent tops in the plot of σ that cannot find corresponding northeast endpoints for their hooks during the canonical valid hook configuration construction. From this interpretation, one can verify that for every permutation σ and every nonempty permutation τ, we have

(6) $\quad \text{def}(\sigma \oplus \tau) \leq \max\{0, \text{def}(\sigma) - 1\} + \text{def}(\tau) \quad \text{and} \quad \text{def}(\tau \ominus \sigma) = \text{def}(\tau) + \text{def}(\sigma) + 1.$

[4]It follows from the results in [27] that σ and $\text{swu}_i(\sigma)$ actually have the same fertility, but we do not need the full strength of that result in this article.

[5]The abbreviation def stands for "deficiency," not "Defant."

Furthermore, the inequality on the left is an equality when $\tau = 1$. That is,

$$\mathrm{def}(\sigma \oplus 1) = \max\{0, \mathrm{def}(\sigma) - 1\}. \tag{7}$$

We claim that if π avoids 132, then $\mathrm{def}(\mathrm{swl}(\pi)) \leq \mathrm{def}(\pi)$. In particular, this will prove that if π avoids 132 and is sorted, then $\mathrm{swl}(\pi)$ is sorted. The proof of this claim is by induction on the length n of the permutation π. We are done if $n \leq 2$ since $\mathrm{swl}(\pi) = \pi$ in that case, so we may assume that $n \geq 3$. Since π avoids 132, we can write $\pi = \mu \ominus (\lambda \oplus 1)$ for some permutations λ and μ that avoid 132. The recursive definition of swl tells us that $\mathrm{swl}(\pi) = \mathrm{swl}(\lambda) \oplus (\mathrm{swl}(\mu) \ominus 1)$. By induction,

$$\mathrm{def}(\mathrm{swl}(\lambda)) \leq \mathrm{def}(\lambda) \quad \text{and} \quad \mathrm{def}(\mathrm{swl}(\mu)) \leq \mathrm{def}(\mu). \tag{8}$$

If μ is nonempty, then we can apply (6), (7), and (8) to find that

$$\mathrm{def}(\mathrm{swl}(\pi)) = \mathrm{def}(\mathrm{swl}(\lambda) \oplus (\mathrm{swl}(\mu) \ominus 1)) \leq \max\{0, \mathrm{def}(\mathrm{swl}(\lambda)) - 1\} + \mathrm{def}(\mathrm{swl}(\mu) \ominus 1)$$
$$= \max\{0, \mathrm{def}(\mathrm{swl}(\lambda)) - 1\} + \mathrm{def}(\mathrm{swl}(\mu)) + \mathrm{def}(1) + 1 \leq \max\{0, \mathrm{def}(\lambda) - 1\} + \mathrm{def}(\mu) + \mathrm{def}(1) + 1$$
$$= \max\{0, \mathrm{def}(\lambda) - 1\} + \mathrm{def}(\mu) + 1 = \mathrm{def}(\lambda \oplus 1) + \mathrm{def}(\mu) + 1 = \mathrm{def}(\mu \ominus (\lambda \oplus 1)) = \mathrm{def}(\pi).$$

If μ is empty, then (7) and (8) imply that

$$\mathrm{def}(\mathrm{swl}(\pi)) = \mathrm{def}(\mathrm{swl}(\lambda) \oplus 1) = \max\{0, \mathrm{def}(\mathrm{swl}(\lambda)) - 1\}$$
$$\leq \max\{0, \mathrm{def}(\lambda) - 1\} = \mathrm{def}(\lambda \oplus 1) = \mathrm{def}(\pi). \qquad \square$$

5. Stanley Intervals and $\mathcal{U}_{2k+1}(312)$

Throughout this section, we assume $\pi = \pi_1 \cdots \pi_n \in \mathcal{U}_n(312)$ and let $n = 2k+1$. Let $\Lambda_i = D$ if $n - i \in \mathrm{Des}(\pi)$, and let $\Lambda_i = U$ otherwise. Let $\Lambda'_i = U$ if $i \in \mathrm{Des}(\mathrm{rot}(\pi))$, and let $\Lambda'_i = D$ otherwise. Form the words $\Lambda = \Lambda_1 \cdots \Lambda_{2k}$ and $\Lambda' = \Lambda'_1 \cdots \Lambda'_{2k}$ over the alphabet $\{U, D\}$, and let $\Lambda\!\!\Lambda_k(\pi) = (\Lambda, \Lambda')$. Figure 8 illustrates this procedure. Our goal is to show that $\Lambda\!\!\Lambda_k(\pi) \in \mathrm{Int}(\mathcal{L}_k^S)$ and that the resulting map

$$\Lambda\!\!\Lambda_k : \mathcal{U}_{2k+1}(312) \to \mathrm{Int}(\mathcal{L}_k^S)$$

is bijective.[6]

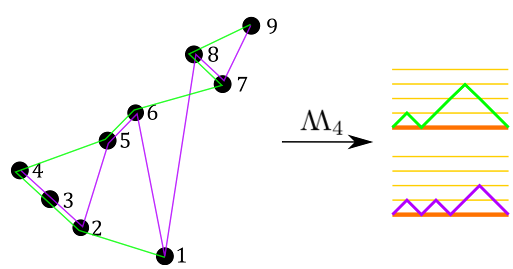
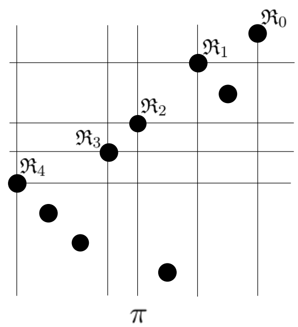


Figure 8. An example illustrating the definition of $\Lambda\!\!\Lambda_4$. Imagine taking the purple path drawn on the permutation and rotating it 180° to obtain the purple Dyck path on the bottom. Similarly, rotate the green path drawn on the permutation by 90° clockwise to obtain the *reverse* of the green Dyck path on the top.

Lemma 3.2 tells us that the left-to-right maxima of the plot of π are $(1, \pi_1), \mathfrak{N}_1, \ldots, \mathfrak{N}_k$, where $\mathfrak{N}_1, \ldots, \mathfrak{N}_k$ are the northeast endpoints of the hooks in the canonical valid hook configuration of π. It will be useful to keep in mind that $\pi_n = n$ because π is sorted. Let $\mathfrak{R}_0, \ldots, \mathfrak{R}_k$ be these left-to-right maxima, written in order from right to left (for example, $\mathfrak{R}_0 = (n, n)$ and $\mathfrak{R}_k = (1, \pi_1)$).

[6]We use the letter Λ to denote Dyck paths because it resembles an up step followed by a down step. The "double lambda" symbol $\Lambda\!\!\Lambda$ is meant to resemble two copies of Λ since the bijections output pairs of Dyck paths.

To improve readability, we also let $\mathfrak{P}(i) = (i, \pi_i)$. From π, we obtain a $k \times k$ matrix $M(\pi) = (m_{ij})$ as follows. First, let $m_{ij} = 0$ whenever $j \leq k - i$ (in other words, the matrix obtained by "turning $M(\pi)$ upside down" is upper-triangular). If $j > k - i$, then we let m_{ij} be the number of points in the plot of π that lie between $\mathfrak{R}_{k-j}$ and $\mathfrak{R}_{k-j+1}$ horizontally and lie between $\mathfrak{R}_i$ and $\mathfrak{R}_{i+1}$ vertically (we make the convention that all points "lie above $\mathfrak{R}_{k+1}$," even though $\mathfrak{R}_{k+1}$ is not actually a point that we have defined). Alternatively, we can imagine drawing vertical lines through the points $\mathfrak{R}_0, \ldots, \mathfrak{R}_k$ and horizontal lines through $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ to produces an array of cells as in Figure 9. The matrix $M(\pi)$ is now obtained by recording the number of points in each of these cells.



$$M(\pi) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

FIGURE 9. The array of cells and the corresponding matrix $M(\pi)$.

Remark 5.1. In the array of cells we have just described, the points appearing in each column must be decreasing in height from left to right because π avoids 312. Similarly, the points appearing in each row must be decreasing in height from left to right. This tells us that the permutation π is uniquely determined by the matrix $M(\pi)$. Indeed, the matrix tells us how many points to place in each cell, and the positions of all of the point relative to each other are then determined by the fact that the points within rows and columns are decreasing in height. ◊

We now return to the definition of $\Lambda\!\!\Lambda_k$. One can check that $\Lambda = UD^{\gamma_1}UD^{\gamma_2} \cdots UD^{\gamma_k}$ and $\Lambda' = UD^{\gamma_1'}UD^{\gamma_2'} \cdots UD^{\gamma_k'}$, where γ_i is the sum of the entries in column $k - i + 1$ of $M(\pi)$ and γ_i' is the sum of the entries in row i of $M(\pi)$. Because every nonzero entry in one of the first i rows of $M(\pi)$ is also in one of the last i columns of $M(\pi)$, we have

$$\gamma_1 + \cdots \gamma_i \geq \gamma_1' + \cdots \gamma_i' \quad \text{for all } i \in \{1, \ldots, k\}. \tag{9}$$

Lemma 5.1. *Preserving the notation from above, we have $\Lambda\!\!\Lambda_k(\pi) \in \mathrm{Int}(\mathcal{L}_k^S)$.*

Proof. Let us check that Λ and Λ' are actually Dyck paths. Because π has k descents, exactly half of the letters in Λ are equal to U. We can use Lemma 4.3 and the fact that $\mathrm{rot}(\pi) = \mathrm{rev}(\pi^{-1})$ to see that $\mathrm{des}(\mathrm{rot}(\pi)) = \mathrm{des}(\mathrm{rev}(\pi^{-1})) = 2k - \mathrm{des}(\pi^{-1}) = 2k - \mathrm{des}(\pi) = k$. Therefore, exactly half of the letters in Λ' are equal to U.

Recall that $n = 2k + 1$. Choose $p \in \{1, \ldots, 2k\}$, and let u be the number of appearances of the letter U in $\Lambda_1 \cdots \Lambda_p$. For $i \in \{1, \ldots, 2k\}$, we have $\Lambda_i = D$ if and only if $\mathfrak{P}(n - i + 1)$ is in $\mathrm{DB}(\pi)$, the set of descent bottoms in the plot of π. Because $\mathrm{DB}(\pi)$ and $\{\mathfrak{N}_1, \ldots, \mathfrak{N}_k\}$ form a partition of $\{\mathfrak{P}(r) : 2 \leq r \leq n\}$ by Lemma 3.1, we have $\Lambda_i = U$ if and only if $\mathfrak{P}(n - i + 1) \in \{\mathfrak{N}_1, \ldots, \mathfrak{N}_k\}$. This means that u is the number of northeast endpoints (in the canonical valid hook configuration of π) that lie in the set $\{\mathfrak{P}(n - p + 1), \ldots, \mathfrak{P}(n)\}$. Also, $p - u$ is the number of appearances of D in $\Lambda_1 \cdots \Lambda_p$, which is $|\,\mathrm{Des}(\pi) \cap \{n - p, \ldots, n - 1\}|$. This is the number of southwest endpoints that lie in the set $\{\mathfrak{P}(n - p), \ldots, \mathfrak{P}(n - 1)\}$. Each of these southwest endpoints must belong to a hook whose northeast endpoint is in $\{\mathfrak{P}(n-p+1),\ldots,\mathfrak{P}(n)\}$, so $p-u\leq u$. As p was arbitrary, this proves that Λ is a Dyck path.

Let us keep p and u as in the preceding paragraph. Let u' be the number of appearances of U in $\Lambda_1'\cdots\Lambda_p'$. We will show that $u'\geq u$. As p was arbitrary, this will prove that Λ' is a Dyck path and that $\Lambda \leq_S \leq \Lambda'$. We can write

$$\Lambda_1\cdots\Lambda_p = UD^{\gamma_1}UD^{\gamma_2}\cdots UD^{\gamma_{u-1}}UD^{\delta} \quad \text{and} \quad \Lambda_1'\cdots\Lambda_p' = UD^{\gamma_1'}UD^{\gamma_2'}\cdots UD^{\gamma_{u'-1}'}UD^{\delta'}$$

for some $\delta\in\{0,\ldots,\gamma_u\}$ and $\delta'\in\{0,\ldots,\gamma_{u'}'\}$. For the sake of finding a contradiction, assume $u\geq u'+1$. Since $\Lambda_1\cdots\Lambda_p$ and $\Lambda_1'\cdots\Lambda_p'$ have the same length, the former must have fewer copies of D than the latter has. In symbols, this says that $\gamma_1+\cdots\gamma_{u-1}+\delta<\gamma_1'+\cdots\gamma_{u-1}'+\delta'$. Using (9) and the assumption that $u\geq u'+1$, we obtain our desired contradiction from the chain of inequalities

$$\gamma_1+\cdots\gamma_{u-1}+\delta\geq\gamma_1+\cdots\gamma_{u-1}\geq\gamma_1+\cdots\gamma_{u'}\geq\gamma_1'+\cdots\gamma_{u'}'\geq\gamma_1'+\cdots\gamma_{u'-1}'+\delta'. \qquad \square$$

We need one additional technical lemma before we can prove the invertibility of $\Lambda\!\!\Lambda_k$. Given a $k\times k$ matrix $M=(m_{ij})$ and indices $r,r',c,c'\in\{1,\ldots,k\}$, consider the matrix obtained by deleting all rows of M except rows r and r' and deleting all columns of M except columns c and c'. We say this new matrix is a *lower* 2×2 *submatrix* of M if $k+1-c\leq r<r'$ and $c<c'$. Define the *energy* of M to be $e(M)=\sum_{i=1}^k\sum_{j=1}^k 2^{i-j}m_{ij}$.

Lemma 5.2. *Let $a_1,\ldots,a_k,b_1,\ldots,b_k$ be nonnegative integers such that $a_1+\cdots+a_k=b_1+\cdots+b_k$ and $a_{k-i+1}+\cdots+a_k\leq b_{k-i+1}+\cdots+b_k$ for all $i\in\{1,\ldots,k\}$. There exists a $k\times k$ matrix $M=(m_{ij})$ with nonnegative integer entries such that*

(i) $m_{ij}=0$ whenever $j\leq k-i$;
(ii) the sum of the entries in column i of M is b_i for every $i\in\{1,\ldots,k\}$;
(iii) the sum of the entries in row i of M is a_{k-i+1} for every $i\in\{1,\ldots,k\}$;
(iv) in every lower 2×2 submatrix of M, either the bottom left entry or the top right entry is 0.

Proof. Let us first prove that there is a $k\times k$ matrix M satisfying properties (i)–(iii). Let $R=a_1+\cdots+a_k=b_1+\cdots+b_k$. We induct on both k and R, observing that the proof is trivial if $k=1$ or $R=0$. Assume $k\geq 2$ and $R\geq 1$. Let us first consider the case in which $b_k=0$. Since $a_k\leq b_k$, we have $a_k=0$ as well. Notice that $a_1+\cdots+a_{k-1}=b_1+\cdots+b_{k-1}$ and $a_{(k-1)-i+1}+\cdots+a_{k-1}\leq b_{(k-1)-i+1}+\cdots+b_{k-1}$ for all $i\in\{1,\ldots,k-1\}$. Using the induction hypothesis (inducting on k), we find that there is a matrix $M'=(m_{ij}')$ such that the properties (i)–(iii) are satisfied when we replace k by $k-1$ and replace M by M'. Now let $m_{ij}=0$ when $i=1$ or $j=k$, and let $m_{ij}=m_{(i-1)j}'$ when $i\geq 2$ and $j\leq k-1$. The matrix $M=(m_{ij})$ satisfies properties (i)–(iii).

We now consider the case in which $b_k\geq 1$. Let ℓ be the smallest positive integer such that $a_{k-\ell+1}\geq 1$. Let $b_i'=b_i$ for $i\neq k$, and let $b_k'=b_k-1$. Let $a_i'=a_i$ for $i\neq k-\ell+1$, and let $a_{k-\ell+1}'=a_{k-\ell+1}-1$. Note that $a_1'+\cdots+a_k'=b_1'+\cdots+b_k'=R-1$. If $i<\ell$, then $a_{k-i+1}'+\cdots+a_k'=a_{k-i+1}+\cdots+a_k=0\leq b_{k-i+1}'+\cdots+b_k'$. If $i\geq\ell$, then $a_{k-i+1}'+\cdots+a_k'=a_{k-i+1}+\cdots+a_k-1\leq b_{k-i+1}+\cdots+b_k-1=b_{k-i+1}'+\cdots+b_k'$. This shows that the hypotheses of the lemma are satisfied by $a_1',\ldots,a_k',b_1',\ldots,b_k'$. By induction on R, we see that there is a matrix $M'=(m_{ij}')$ such that properties (i)–(iii) are satisfied when we replace $a_1,\ldots,a_k,b_1,\ldots,b_k$ by $a_1',\ldots,a_k',b_1',\ldots,b_k'$ and replace M by M'. Let $m_{ij}=m_{ij}'$ when $(i,j)\neq(k-\ell+1,k)$, and let $m_{(k-\ell+1)k}=m_{(k-\ell+1)k}'+1$. The matrix $M=(m_{ij})$ satisfies properties (i)–(iii).

If M does not satisfy property (iv), then we can define a "move" on M as follows. Choose $r, r', c, c' \in \{1, \ldots, k\}$ with $k+1-c \leq r < r'$ and $c < c'$ such that $m_{r'c}$ and $m_{rc'}$ are positive. Now replace the entries $m_{rc}, m_{rc'}, m_{r'c}, m_{r'c'}$ with the entries $m_{rc}+1, m_{rc'}-1, m_{r'c}-1, m_{r'c'}+1$, respectively. Performing a move produces a new matrix $\widetilde{M}$ that still satisfies properties (i)–(iii). Considering the energies of these matrices, which we defined above, we find that

$$e(M)-e(\widetilde{M}) = -2^{r-c}+2^{r-c'}+2^{r'-c}-2^{r'-c'} \geq -2^{(r'-1)-c}+2^{r-c'}+2^{r'-c}-2^{r'-(c+1)} = 2^{r-c'} \geq 2^{1-k}.$$

This shows that after applying a finite sequence of moves, we will eventually obtain a matrix that satisfies all of the properties (i)–(iv). □

Theorem 5.1. *For each nonnegative integer k, the map $\Lambda\!\!\Lambda_k : \mathcal{U}_{2k+1}(312) \to \operatorname{Int}(\mathcal{L}_k^S)$ is a bijection.*

Proof. We first prove surjectivity. Fix $(\Lambda, \Lambda') \in \operatorname{Int}(\mathcal{L}_k^S)$, and write $\Lambda = UD^{\gamma_1}UD^{\gamma_2}\cdots UD^{\gamma_k}$ and $\Lambda' = UD^{\gamma_1'}UD^{\gamma_2'}\cdots UD^{\gamma_k'}$. Put $a_i = \gamma'_{k-i+1}$ and $b_i = \gamma_{k-i+1}$. The fact that Λ and Λ' are Dyck paths guarantees that $a_1 + \cdots + a_k = b_1 + \cdots + b_k = k$. The fact that $\Lambda \leq_S \Lambda'$ tells us that $a_{k-i+1} + \cdots + a_k \leq b_{k-i+1} + \cdots + b_k$ for all $i \in \{1, \ldots, k\}$. Appealing to Lemma 5.2, we obtain a matrix $M = (m_{ij})$ satisfying the properties (i)–(iv) listed in the statement of that Lemma. It follows from Remark 5.1 that we can use such a matrix to obtain a permutation $\pi \in S_n$ (where $n = 2k+1$) with $M(\pi) = M$ and with the property that $\Lambda_i = D$ if and only if $n-i \in \operatorname{Des}(\pi)$ and $\Lambda_i' = U$ if and only if $i \in \operatorname{Des}(\operatorname{rot}(\pi))$. Because D appears exactly k times in Λ, the permutation π has exactly k descents. The construction of π described in Remark 5.1 along with property (iv) from Lemma 5.2 guarantee that π avoids 312. To see that π is uniquely sorted, it suffices by Theorem 1.1 and Proposition 3.1 to see that it has a canonical valid hook configuration. This follows from the fact that every prefix of Λ contains at least as many copies of U as copies of D. Indeed, if $d_1 < \ldots < d_k$ are the descents of π, then this property of Λ guarantees that the plot of π has at least ℓ left-to-right maxima to the right of $\mathfrak{P}(d_{k-\ell+1})$ for every $\ell \in \{1, \ldots, k\}$. This means that it is always possible to find a northeast endpoint for the hook $H_{k-\ell+1}$ when we construct the canonical valid hook configuration of π. Consequently, $\pi \in \mathcal{U}_{2k+1}(312)$. Properties (ii) and (iii) from Lemma 5.2 ensure that $\Lambda\!\!\Lambda_k(\pi) = (\Lambda, \Lambda')$.

To prove injectivity, let us assume by way of contradiction that there are distinct $\pi, \pi' \in \mathcal{U}_{2k+1}(312)$ with $\Lambda\!\!\Lambda_k(\pi) = \Lambda\!\!\Lambda_k(\pi') = (\Lambda, \Lambda')$, where (Λ, Λ') is as above. According to Remark 5.1, the matrices $M(\pi) = (m_{ij})$ and $M(\pi') = (m'_{ij})$ uniquely determine π and π', respectively. Therefore, these matrices are distinct. However, both of these matrices satisfy properties (i)–(iv) from Lemma 5.2, where $a_i = \gamma'_{k-i+1}$ and $b_i = \gamma_{k-i+1}$. Because they are distinct, we can find a pair (i_0, j_0) with $m_{i_0j_0} \neq m'_{i_0j_0}$. We may assume that j_0 was chosen maximally, which means $m_{ij} = m'_{ij}$ whenever $j > j_0$. We may assume that i_0 was chosen maximally after j_0 was chosen, meaning $m_{ij_0} = m'_{ij_0}$ whenever $i > i_0$. We may assume without loss of generality that $m_{i_0j_0} > m'_{i_0j_0}$. Because $M(\pi)$ and $M(\pi')$ satisfy property (ii), their j_0^{th} columns have the same sum. This means that there exists $i_1 \neq i_0$ with $m_{i_1j_0} < m'_{i_1j_0}$. In particular, $m'_{i_1j_0}$ is positive. The maximality of i_0 guarantees that $i_1 < i_0$. Because $M(\pi)$ and $M(\pi')$ satisfy property (iii), their i_1^{th} rows have the same sum. This means that there exists $j_1 \neq j_0$ with $m_{i_1j_1} > m'_{i_1j_0}$. The maximality of j_0 guarantees that $j_1 < j_0$. Since $M(\pi)$ satisfies property (i) and $m_{i_1j_1} > 0$, we must have $k+1-j_1 \leq i_1$. Now, the j_1^{th} columns of $M(\pi)$ and $M(\pi')$ have the same sum, so there exists $i_2 \neq i_1$ such that $m_{i_2j_1} < m'_{i_2j_1}$. If $i_2 > i_1$, then $m'_{i_2j_1}$ and $m'_{i_1j_0}$ are positive numbers that form the bottom left and top right entries in a lower 2×2 submatrix of $M(\pi)$. This is impossible since $M(\pi)$ satisfies property (iv), so we must have $i_2 < i_1$. Continuing in this fashion, we find decreasing sequences of positive integers $i_0 > i_1 > i_2 > \cdots$ and $j_0 > j_1 > j_2 > \cdots$. This is our desired contradiction. □

Combining Theorem 5.1 with equation (1) yields the following corollary.

Corollary 5.1. *For each nonnegative integer k,*

$$|\mathcal{U}_{2k+1}(312)| = C_k C_{k+2} - C_{k+1}^2 = \frac{6}{(k+1)(k+2)^2(k+3)}\binom{2k}{k}\binom{2k+2}{k+1}.$$

6. Tamari Intervals, $\mathcal{U}_{2k+1}(132)$, and $\mathcal{U}_{2k+1}(231)$

In Section 4, we introduced sliding operators $\mathrm{swu}, \mathrm{swd}, \mathrm{swl}, \mathrm{swr}$. In the previous section, we found bijections $\Lambda_k : \mathcal{U}_{2k+1}(312) \to \mathrm{Int}(\mathcal{L}_k^S)$, where $\mathcal{L}_k^S$ is the k^{th} Stanley lattice. Recall that $\mathcal{L}_k^T$ is the k^{th} Tamari lattice, which we defined in Section 2. The purpose of the current section is to show that for each nonnegative integer k, the maps $\mathrm{swu} : \mathcal{U}_{2k+1}(231) \to \mathcal{U}_{2k+1}(132)$ and $\Lambda_k \circ \mathrm{swl} : \mathcal{U}_{2k+1}(132) \to \mathrm{Int}(\mathcal{L}_k^T)$ are bijections.[7] We have actually already done all of the heavy lifting needed to establish the first of these bijections.

Theorem 6.1. *For each nonnegative integer k, the maps* $\mathrm{swu} : \mathcal{U}_{2k+1}(231) \to \mathcal{U}_{2k+1}(132)$ *and* $\mathrm{swd} : \mathcal{U}_{2k+1}(132) \to \mathcal{U}_{2k+1}(231)$ *are inverse bijections.*

Proof. Lemma 4.1 tells us that $\mathrm{swu} : \mathrm{Av}(231) \to \mathrm{Av}(132)$ and $\mathrm{swd} : \mathrm{Av}(132) \to \mathrm{Av}(231)$ are inverse bijections. These maps also preserve lengths of permutations, so it suffices to show that they map uniquely sorted permutations to uniquely sorted permutations. If $\pi \in \mathcal{U}_{2k+1}(231)$, then we know from Theorem 1.1 and Lemma 4.3 that $\mathrm{des}(\mathrm{swu}(\pi)) = \mathrm{des}(\pi) = k$. Lemma 4.4 tells us that $\mathrm{swu}(\pi)$ is sorted, so it follows from Theorem 1.1 that $\mathrm{swu}(\pi)$ is uniquely sorted. This shows that $\mathrm{swu}(\mathcal{U}_{2k+1}(231)) \subseteq \mathcal{U}_{2k+1}(132)$, and a similar argument proves the reverse containment. $\square$

We now proceed to establish our bijections between 132-avoiding uniquely sorted permutations and intervals in Tamari lattices. This essentially amounts to proving that if $\pi \in \mathcal{U}_{2k+1}(312)$, then $\mathrm{swr}(\pi)$ is sorted if and only if $\Lambda_k(\pi) \in \mathrm{Int}(\mathcal{L}_k^T)$. We do this in the following two propositions.

Proposition 6.1. *If $\pi \in \mathcal{U}_{2k+1}(312)$ is such that $\Lambda_k(\pi) \in \mathrm{Int}(\mathcal{L}_k^T)$, then* $\mathrm{swr}(\pi)$ *is sorted.*

Proof. We prove the contrapositive. Assume $\mathrm{swr}(\pi)$ is not sorted. Let $n = 2k+1$. Since π is sorted and $\mathrm{swr}(\pi) = \mathrm{swr}_1 \circ \cdots \circ \mathrm{swr}_n(\pi)$ by definition, there exists $i \in [n]$ such that the permutation $\pi' := \mathrm{swr}_{i+1} \circ \cdots \circ \mathrm{swr}_n(\pi)$ is sorted while $\pi'' := \mathrm{swr}_i \circ \cdots \circ \mathrm{swr}_n(\pi) = \mathrm{swr}_i(\pi')$ is not. Write $\pi' = \pi'_1 \cdots \pi'_n$ and $\pi'' = \pi''_1 \cdots \pi''_n$. Let $a = \pi'_i$, and let $\ell \in [n]$ be such that $\pi_\ell = a$. Because π avoids 312, its plot has the shape shown in Figure 10. Using the definitions of the maps $\mathrm{swr}_i, \ldots, \mathrm{swr}_n$, we find that the shapes of π' and π'' are as shown in Figure 10. The boxes in these diagrams are meant to represent places where there *could* be points, but boxes could be empty.

Because π' is sorted, it has a canonical valid hook configuration $\mathcal{H}'$ by Proposition 3.1. Let $Q(\lambda)$ (respectively, $Q(\sigma')$) be the number of hooks in $\mathcal{H}'$ with northeast endpoints in λ (respectively, σ') whose southwest endpoints are not in λ (respectively, σ'). Recall the definition of the deficiency statistic def from the proof of Lemma 4.4. Again, one can think of $\mathrm{def}(\tau)$ as the number of descent tops in the plot of τ that cannot find northeast endpoints within τ for their hooks. Every southwest endpoint counted by $\mathrm{def}(\lambda)$ or $\mathrm{def}(\mu')$ belongs to a hook whose northeast endpoint is counted by either $Q(\lambda)$ or $Q(\sigma')$. Therefore, $Q(\sigma') + Q(\lambda) \geq \mathrm{def}(\lambda) + \mathrm{def}(\mu')$. Hence, $Q(\lambda) \geq \mathrm{def}(\lambda) + \mathrm{def}(\mu') - Q(\sigma')$.

[7] It is not clear at this point that the composition $\Lambda_k \circ \mathrm{swl} : \mathcal{U}_{2k+1}(132) \to \mathrm{Int}(\mathcal{L}_k^T)$ is even well-defined, but we will see that it is.

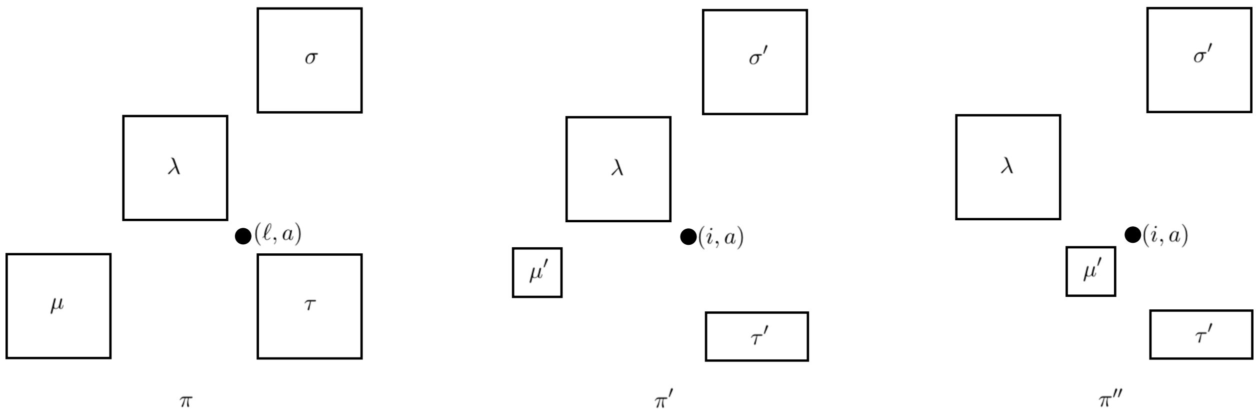


FIGURE 10. The shapes of the plots of π, π', π''.

When we try to construct the canonical valid hook configuration of π'', we must fail at some point because π'' is not sorted. The plots of π' and π'' are the same to the right of the point (i, a), so this failure must occur when we try to find the northeast endpoint of a hook whose southwest endpoint is in either λ or μ'. All choices for these northeast endpoints are either (i, a) or are in σ', and the choices in σ' contain all of the points in σ' that are counted by $Q(\sigma')$. It follows that $Q(\sigma') + 1 < \text{def}(\lambda) + \text{def}(\mu')$. Using the last line from the preceding paragraph, we find that $Q(\lambda) \geq \text{def}(\lambda) + \text{def}(\mu') - Q(\sigma') > 1$. Therefore, $Q(\lambda) \geq 2$. It follows that in the plot of π', there are at least two points in λ that are northeast endpoints of hooks in $\mathcal{H}'$ whose southwest endpoints are not in λ. These points (after they have been slid horizontally) are still northeast endpoints of hooks in the canonical valid hook configuration $\mathcal{H}$ of π. Indeed, this is a consequence of Lemma 3.1 because π is uniquely sorted and these points are left-to-right maxima of the plot of π. In the plot of π, the hooks with these two points as northeast endpoints must have southwest endpoints that are not in λ.

Every time we mention hooks, southwest endpoints, or northeast endpoints in the remainder of the proof, we refer to those of the canonical valid hook configuration $\mathcal{H}$ of π. Note that $\pi_n = n$ because π is sorted. Lemma 3.2 tells us that the left-to-right maxima of the plot of π are $(1, \pi_1), \mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_k$, where $\mathfrak{N}_1, \ldots, \mathfrak{N}_k$ are the northeast endpoints. Let $\mathfrak{R}_0, \ldots, \mathfrak{R}_k$ be these left-to-right maxima, written in order from right to left (so $\mathfrak{R}_0 = (n, n)$ and $\mathfrak{R}_k = (1, \pi_1)$). Let $\Lambda_k(\pi) = (\Lambda, \Lambda')$, where $\Lambda = UD^{\gamma_1}UD^{\gamma_2} \cdots UD^{\gamma_k}$ and $\Lambda' = UD^{\gamma'_1}UD^{\gamma'_2} \cdots UD^{\gamma'_k}$. Because π avoids 312, the points lying horizontally between $\mathfrak{R}_{i-1}$ and $\mathfrak{R}_i$ are decreasing in height from left to right for every $i \in [k]$. Similarly, the points lying vertically between $\mathfrak{R}_i$ and $\mathfrak{R}_{i+1}$ are decreasing in height from left to right for every $i \in [k]$. This implies that γ_i is the number of points lying horizontally between $\mathfrak{R}_{i-1}$ and $\mathfrak{R}_i$, while γ'_i is the number of points lying vertically between $\mathfrak{R}_i$ and $\mathfrak{R}_{i+1}$ (we make the convention that all points "lie above $\mathfrak{R}_{k+1}$," even though $\mathfrak{R}_{k+1}$ is not actually a point that we have defined).

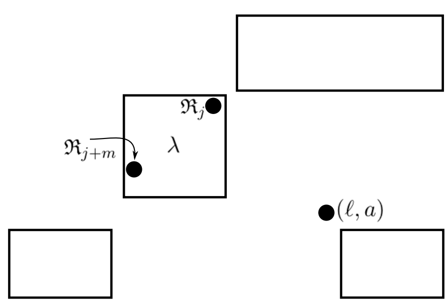
We saw above that there are (at least) two points in λ that are northeast endpoints of hooks whose southwest endpoints are not in λ. Their southwest endpoints must be in μ. Since these points are northeast endpoints, they are $\mathfrak{R}_{j-1}$ and $\mathfrak{R}_{j+m-1}$ for some $j \in [k]$ and $m \geq 1$. We may assume that we have chosen these points as far left as possible. In particular, $\mathfrak{R}_{j+m-1}$ is the leftmost point in λ.

For $r \in \{j, \ldots, j+m-1\}$, let $\zeta^{(r)}$ be the permutation whose plot is the portion of λ obtained by deleting everything to the left of $\mathfrak{R}_r$ and everything equal to or to the right of $\mathfrak{R}_{j-1}$. Because none of the hooks with southwest endpoints in λ have $\mathfrak{R}_{j-1}$ as their northeast endpoints, all of the southwest endpoints in $\zeta^{(r)}$ belong to hooks whose northeast endpoints are in $\zeta^{(r)}$. When $r = j$, this implies that there are no points horizontally between $\mathfrak{R}_{j-1}$ and $\mathfrak{R}_j$. Thus, $\gamma_j = 0$. When $r = j+1$, this implies that there is at most one point other than $\mathfrak{R}_j$ that lies horizontally between $\mathfrak{R}_{j-1}$ and $\mathfrak{R}_{j+1}$. Thus, $\gamma_j + \gamma_{j+1} \leq 1$. Continuing in this way, we find that $\gamma_j + \cdots + \gamma_{j+v} \leq v$ for all $v \in \{0, \ldots, m-1\}$. Referring to Definition 2.1, we find that $\text{long}_j(\Lambda) \geq m$.

Now recall that we chose the point $\mathfrak{R}_{j-1}$ as far left as possible subject to the conditions that it is not the leftmost point in λ and that it is the northeast endpoint of a hook whose southwest endpoint is in μ. When the northeast endpoint of this hook was determined in the canonical valid hook configuration construction, we did not choose any of the points $\mathfrak{R}_j, \ldots, \mathfrak{R}_{j+m-2}$. This means that we *could not* have chosen any of these points, so they must have already been northeast endpoints of other hooks. These other hooks must all have their southwest endpoints in λ (since we chose $\mathfrak{R}_{j-1}$ as far left as possible). These southwest endpoints are descent tops, and the corresponding descent bottoms are not left-to-right maxima. This tells us that there are at least $m-1$ points other than $\mathfrak{R}_{j+1}, \ldots, \mathfrak{R}_{j+m-2}$ that lie horizontally between $\mathfrak{R}_j$ and $\mathfrak{R}_{j+m-1}$. All of these points are in λ, so they must lie above $\mathfrak{R}_{j+m}$ and below $\mathfrak{R}_{j+1}$. This forces $\gamma'_{j+1} + \cdots + \gamma'_{j+m-1} \geq m-1$. However, (ℓ, a) is another point that lies above $\mathfrak{R}_{j+m}$ and below $\mathfrak{R}_{j+1}$, so we actually have $\gamma'_{j+1} + \cdots + \gamma'_{j+m-1} > m-1$. Since $\gamma'_j \geq 0$, this means that $\gamma'_j + \cdots + \gamma'_{j+m-1} > m-1$. According to Definition 2.1, $\text{long}_j(\Lambda') \leq m-1$. We have seen that $\text{long}_j(\Lambda) \geq m$, so it is immediate from the definition of the Tamari lattice $\mathcal{L}_k^T$ that $(\Lambda, \Lambda') \notin \text{Int}(\mathcal{L}_k^T)$. □

Proposition 6.2. *If $\pi \in \mathcal{U}_{2k+1}(312)$ is such that* $\text{swr}(\pi)$ *is sorted, then* $\mathfrak{M}_k(\pi) \in \text{Int}(\mathcal{L}_k^T)$.

Proof. Let $\mathfrak{M}_k(\pi) = (\Lambda, \Lambda')$, where $\Lambda = UD^{\gamma_1}UD^{\gamma_2}\cdots UD^{\gamma_k}$ and $\Lambda' = UD^{\gamma'_1}UD^{\gamma'_2}\cdots UD^{\gamma'_k}$. We are going to prove the contrapositive of the proposition, so assume $(\Lambda, \Lambda') \notin \text{Int}(\mathcal{L}_k^T)$. This means that there exists $j \in [k]$ such that $\text{long}_j(\Lambda) > \text{long}_j(\Lambda')$. As in the proof of the previous proposition, we let $\mathfrak{R}_0, \ldots, \mathfrak{R}_k$ be the left-to-right maxima of the plot of π listed in order from right to left. Let (ℓ, a) be the highest point in the plot of π that appears below and to the right of $\mathfrak{R}_j$. Let $\mathfrak{R}_{j+m}$ be the leftmost point that is higher than (ℓ, a). Using the assumption that π avoids 312, we find that the plot of π has the following shape:



The image is meant to indicate that $\mathfrak{R}_j$ is the highest and the rightmost point in λ and that $\mathfrak{R}_{j+m}$ is the leftmost point in λ. The boxes in this diagram represent places where there could be points, but boxes could be empty.

Let $t = \text{long}_j(\Lambda)$ and $t' = \text{long}_j(\Lambda')$. Since $t > t' \geq 0$, it follows from Definition 2.1 that $\gamma_j = 0$ and

$$\gamma_j + \cdots + \gamma_{j+t'} \leq t' < \gamma'_j + \cdots + \gamma'_{j+t'}. \tag{10}$$

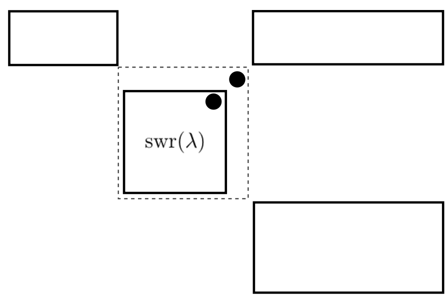
Now, $\gamma_j + \cdots + \gamma_{j+t'}$ is the number of points in the plot of π other than $\mathfrak{R}_j, \ldots, \mathfrak{R}_{j+t'-1}$ lying horizontally between $\mathfrak{R}_{j-1}$ and $\mathfrak{R}_{j+t'}$. Since $\gamma_j = 0$, this is actually the same as the number of points in the plot of π other than $\mathfrak{R}_{j+1}, \ldots, \mathfrak{R}_{j+t'-1}$ lying horizontally between $\mathfrak{R}_j$ and $\mathfrak{R}_{j+t'}$. Similarly, $\gamma'_j + \cdots + \gamma'_{j+t'}$ is the number of points other than $\mathfrak{R}_{j+1}, \ldots, \mathfrak{R}_{j+t'}$ lying vertically between $\mathfrak{R}_j$ and $\mathfrak{R}_{j+t'+1}$. If $t' \leq m-1$, then all of these points counted by $\gamma'_j + \cdots + \gamma'_{j+t'}$ are in λ. In fact, they all lie horizontally between $\mathfrak{R}_j$ and $\mathfrak{R}_{j+t'}$, so they are among the points counted by $\gamma_j + \cdots + \gamma_{j+t'}$. This contradicts (10), so we must have $t' \geq m$. This means that $t \geq m+1$, so it follows from Definition 2.1 that $\gamma_j + \cdots + \gamma_{j+m} \leq m$. The points in the plot of π other than $\mathfrak{R}_{j+1}, \ldots, \mathfrak{R}_{j+m-1}$ that lie horizontally between $\mathfrak{R}_j$ and $\mathfrak{R}_{j+m}$ are all in λ. Letting $|\lambda|$ denote the number of points in λ, we find that

$$|\lambda| = \gamma_j + \cdots + \gamma_{j+m} + m + 1 \leq 2m + 1. \tag{11}$$

The $m+1$ points $\mathfrak{R}_j, \ldots, \mathfrak{R}_{j+m}$, which lie in λ, are not descent bottoms in the plot of λ, so it follows from (11) that $\operatorname{des}(\lambda) \leq |\lambda| - (m+1) \leq (|\lambda|-1)/2$. We know that λ avoids 312 because π does, so we can use Lemma 4.3 to see that

$$\operatorname{des}(\operatorname{swr}(\lambda)) \leq (|\lambda|-1)/2. \tag{12}$$

We now check that $\operatorname{swr}(\pi)$ has the following shape:



Because $\pi \in \mathcal{U}_{2k+1}(312)$, we know from Theorem 1.1 and Lemma 4.3 that $\operatorname{des}(\operatorname{swr}(\pi)) = \operatorname{des}(\pi) = k$. Our goal is to show that $\operatorname{swr}(\pi)$ is not sorted, so suppose by way of contradiction that it is sorted. Theorem 1.1 tells us that $\operatorname{swr}(\pi)$ is uniquely sorted. Let $\mathfrak{N}'_1, \ldots, \mathfrak{N}'_k$ be the northeast endpoints of the hooks in the canonical valid hook configuration of $\operatorname{swr}(\pi)$. Let $\operatorname{DB}(\operatorname{swr}(\pi))$ be the set of descent bottoms in the plot of $\operatorname{swr}(\pi)$. Let $\mathfrak{Q} = (1, \operatorname{swr}(\pi)_1)$ be the leftmost point in the plot of $\operatorname{swr}(\pi)$. According to Lemma 3.1, the sets $\operatorname{DB}(\operatorname{swr}(\pi))$, $\{\mathfrak{N}'_1, \ldots, \mathfrak{N}'_k\}$, and $\{\mathfrak{Q}\}$ form a partition of the set of points in the plot of $\operatorname{swr}(\pi)$.

Now refer back to the above image of the plot of $\operatorname{swr}(\pi)$. Let $\mathcal{E}$ be the set of points lying in the dotted region. It is possible that there is a point in this plot lying above and to the left of all of the points in $\mathcal{E}$. It is also possible that there is no such point. In either case,

$$|(\operatorname{DB}(\operatorname{swr}(\pi)) \cup \{\mathfrak{Q}\}) \cap \mathcal{E}| \leq \operatorname{des}(\operatorname{swr}(\lambda)) + 1. \tag{13}$$

Now note that there are no points in the plot of $\operatorname{swr}(\pi)$ lying below and to the left of the dotted region. This means that every element of $\{\mathfrak{N}'_1, \ldots, \mathfrak{N}'_k\} \cap \mathcal{E}$ is the northeast endpoint of a hook (in the canonical valid hook configuration of $\operatorname{swr}(\pi)$) whose southwest endpoint is also in $\mathcal{E}$. Hence, $|\{\mathfrak{N}'_1, \ldots, \mathfrak{N}'_k\} \cap \mathcal{E}|$ is at most the number of southwest endpoints that lie in $\mathcal{E}$. Now recall from the canonical valid hook configuration construction that the southwest endpoints of hooks are precisely the descent tops in the plot. It follows that the number of southwest endpoints in $\mathcal{E}$ is the number of descent tops in $\mathcal{E}$, which is $\operatorname{des}(\operatorname{swr}(\lambda))$. Combining this observation with (12) and (13) yields

$$|\mathcal{E}| = |(\operatorname{DB}(\operatorname{swr}(\pi)) \cup \{\mathfrak{Q}\}) \cap \mathcal{E}| + |\{\mathfrak{N}'_q, \ldots, \mathfrak{N}'_k\} \cap \mathcal{E}| \leq \operatorname{des}(\operatorname{swr}(\lambda)) + 1 + \operatorname{des}(\operatorname{swr}(\lambda)) \leq |\lambda|.$$

This is our desired contradiction because $|\mathcal{E}| = |\lambda| + 1$. $\square$

Theorem 6.2. *For each nonnegative integer k, the map $\Lambda_k \circ \mathrm{swl} : \mathcal{U}_{2k+1}(132) \to \mathrm{Int}(\mathcal{L}_k^T)$ is a bijection.*

Proof. First, recall from Lemma 4.1 that $\mathrm{swl} : \mathrm{Av}(132) \to \mathrm{Av}(312)$ and $\mathrm{swr} : \mathrm{Av}(312) \to \mathrm{Av}(132)$ are inverse bijections. If $\pi \in \mathcal{U}_{2k+1}(132)$, then we know from Theorem 1.1 that π is sorted and has k descents. Lemmas 4.3 and 4.4 guarantee that $\mathrm{swl}(\pi)$ is sorted and has k descents, so it follows from Theorem 1.1 that $\mathrm{swl}(\pi) \in \mathcal{U}_{2k+1}(312)$. This means that it actually makes sense to apply Λ_k to $\mathrm{swl}(\pi)$. Since $\mathrm{swr}(\mathrm{swl}(\pi)) = \pi$ is sorted, Proposition 6.2 tells us that $\Lambda_k(\mathrm{swl}(\pi)) \in \mathrm{Int}(\mathcal{L}_k^T)$. Hence, $\Lambda_k \circ \mathrm{swl}$ does indeed map $\mathcal{U}_{2k+1}$ into $\mathrm{Int}(\mathcal{L}_k^T)$. The injectivity of the map $\Lambda_k \circ \mathrm{swl} : \mathcal{U}_{2k+1}(132) \to \mathrm{Int}(\mathcal{L}_k^T)$ follows from the injectivity of Λ_k and the injectivity of swl on $\mathcal{U}_{2k+1}(132)$. To prove surjectivity, choose $(\Lambda, \Lambda') \in \mathrm{Int}(\mathcal{L}_k^T)$. Let $\sigma = \Lambda_k^{-1}(\Lambda, \Lambda')$. We know by the definition of Λ_k that $\sigma \in \mathcal{U}_{2k+1}(312)$, so σ has k descents. According to Lemma 4.3, $\mathrm{swr}(\sigma)$ has k descents. Since $\Lambda_k(\sigma) \in \mathrm{Int}(\mathcal{L}_k^T)$, it follows from Proposition 6.1 that $\mathrm{swr}(\sigma)$ is sorted. By Theorem 1.1, $\mathrm{swr}(\sigma) \in \mathcal{U}_{2k+1}(132)$. This proves surjectivity since $\Lambda_k \circ \mathrm{swl}(\mathrm{swr}(\sigma)) = \Lambda_k(\sigma) = (\Lambda, \Lambda')$. □

Combining Theorem 6.1, Theorem 6.2, and equation (2) yields the following corollary.

Corollary 6.1. *For each nonnegative integer k,*

$$|\mathcal{U}_{2k+1}(132)| = |\mathcal{U}_{2k+1}(231)| = \frac{2}{(3k+1)(3k+2)}\binom{4k+1}{k+1}.$$

7. Noncrossing Partition Intervals and $\mathcal{U}_{2k+1}(312, 1342)$

As mentioned in the introduction, the Kreweras lattices $\mathcal{L}_k^K$ are isomorphic to the noncrossing partition lattices NC_k and are sublattices of the Tamari lattices $\mathcal{L}_k^T$. We want to find a bijection between uniquely sorted permutations avoiding 312 and 1342 and intervals in Kreweras (equivalently, noncrossing partition) lattices. Since $\mathcal{U}_{2k+1}(312, 1342) \subseteq \mathcal{U}_{2k+1}(312)$ and $\mathrm{Int}(\mathcal{L}_k^K) \subseteq \mathrm{Int}(\mathcal{L}_k^S)$, one might hope that the map Λ_k from Section 5 would yield our desired bijection. More precisely, it would be nice if we had $\Lambda_k(\mathcal{U}_{2k+1}(312, 1342)) = \mathrm{Int}(\mathcal{L}_k^K)$. This, however, is not the case; we must define a completely different map. We find it convenient to only work with the noncrossing partition lattices in this section. Thus, our goal is to prove the following theorem.

Theorem 7.1. *For each nonnegative integer k, there is a bijection*

$$\Upsilon_k : \mathcal{U}_{2k+1}(312, 1342) \to \mathrm{Int}(\mathrm{NC}_k).$$

The proof of Theorem 7.1 in the specific case $k = 0$ is trivial (we make the convention that $\mathrm{NC}_0 = \{\emptyset\}$), so we will assume throughout this section that $k \geq 1$.

To prove this theorem, we make use of *generating trees*, an enumerative tool that was introduced in [17] and studied heavily afterward [3, 56, 57]. To describe a generating tree of a class of combinatorial objects, we first specify a scheme by which each object of size n can be uniquely generated from an object of size $n-1$. We then label each object with the number of objects it generates. The generating tree consists of an "axiom" that specifies the labels of the objects of size 1 along with a "rule" that describes the labels of the objects generated by each object with a given label. For example, in the generating tree

$$\text{Axiom: } (2) \qquad \text{Rule: } (1) \rightsquigarrow (2), \quad (2) \rightsquigarrow (1)(2),$$

the axiom (2) tells us that we begin with a single object of size 1 that has label 2. The rule $(1) \rightsquigarrow (2)$, $(2) \rightsquigarrow (1)(2)$ tells us that each object of size $n-1$ with label 1 generates a single object of size n with label 2, whereas each object of size $n-1$ with label 2 generates one object of size n with label 1 and one object of size n with label 2. This example generating tree describes objects counted by the Fibonacci numbers.

We are going to describe a generating tree for the class of intervals in noncrossing partition lattices and a generating tree for the class[8] $\mathcal{U}(312, 1342)$. We will find that there is a natural isomorphism between these two generating trees. This isomorphism yields the desired bijections Υ_k.

Remark 7.1. It is actually possible to give a short description of the bijection Υ_k that does not rely on generating trees. We do this in the following paragraph. However, it is not at all obvious from the definition we are about to give that this map is indeed a bijection from $\mathcal{U}_{2k+1}(312, 1342)$ to $\mathrm{Int}(\mathrm{NC}_k)$. The current author was able to prove this directly, but the proof ended up being very long and tedious. For this reason, we will content ourselves with merely defining the map. We also omit the proof that this map is indeed the same as the map Υ_k that we will obtain later via generating tees, although this fact can be proven by tracing carefully through the relevant definitions. In order to avoid potential confusion arising from the fact that we have given different definitions of these maps and have not proven them to be equivalent, we use the symbol Υ'_k for the map defined in the next paragraph.

Suppose we are given $\pi \in \mathcal{U}_{2k+1}(312, 1342)$. Because π is sorted, we know from Proposition 3.1 that it has a canonical valid hook configuration $\mathcal{H}$. Let $\mathfrak{W}_1, \ldots, \mathfrak{W}_k$ be the northeast endpoints of the hooks in $\mathcal{H}$ listed in increasing order of height. Let $\mathfrak{U}_\ell$ be the southwest endpoint of the hook whose northeast endpoint is $\mathfrak{W}_\ell$. The *partner* of $\mathfrak{W}_\ell$, which we denote by $\mathfrak{V}_\ell$, is the point immediately to the right[9] of $\mathfrak{U}_\ell$ in the plot of π. Let ρ be the partition of $[k]$ obtained as follows. Place numbers $\ell, m \in [k]$ in the same block of ρ if $\mathfrak{V}_\ell$ appears immediately above and immediately to the left of $\mathfrak{V}_m$ in the plot of π. Then, close all of these blocks by transitivity. Let κ be the partition of $[k]$ obtained as follows. Place numbers $\ell, m \in [k]$ in the same block of κ if they are in the same block of ρ or if $\mathfrak{W}_\ell$ appears immediately above and immediately to the left of $\mathfrak{V}_m$ in the plot of π. Then, close all of these blocks by transitivity. Let $\Upsilon'_k(\pi) = (\rho, \kappa)$. Figure 11 shows an example application of each of the maps $\Upsilon'_1, \Upsilon'_2, \Upsilon'_3, \Upsilon'_4$ (which are secretly the same as the maps $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ defined later). At this point in time, the reader should ignore the horizontal maps, the green arrows, and the green shading in Figure 11. $\lozenge$

We now proceed to describe the generating tree for the combinatorial class of intervals in noncrossing partition lattices. Let us say an interval $(\rho, \kappa) \in \mathrm{Int}(\mathrm{NC}_k)$ *generates* an interval $(\widetilde{\rho}, \widetilde{\kappa}) \in \mathrm{Int}(\mathrm{NC}_{k+1})$ if ρ and κ are the partitions obtained by removing the number $k+1$ from its blocks in $\widetilde{\rho}$ and $\widetilde{\kappa}$, respectively. Each interval in $\mathrm{Int}(\mathrm{NC}_k)$ will generate multiple intervals in $\mathrm{Int}(\mathrm{NC}_{k+1})$; to understand the possibilities here, we introduce operations $\mathfrak{u}, \mathfrak{v}_1, \ldots, \mathfrak{v}_k, \mathfrak{w}_1, \ldots, \mathfrak{w}_k$. Given $(\rho, \kappa) \in \mathrm{Int}(\mathrm{NC}_k)$, let $\mathfrak{u}(\rho, \kappa)$ be the interval whose first and second partitions are obtained by appending the singleton block $\{k+1\}$ to ρ and κ, respectively. For $r \in \{1, \ldots, k\}$, let $\mathfrak{v}_r(\rho, \kappa)$ be the interval whose first and second partitions are obtained by adding the number $k+1$ to the blocks containing r in ρ and κ, respectively. Let $\mathfrak{w}_r(\rho, \kappa)$ be the interval whose first partition is obtained by appending the singleton block $\{k+1\}$ to ρ and whose second partition is obtained by adding the number $k+1$ to the block in κ that contains r. We will always say that $\mathfrak{u}$ is an *allowable operation* of (ρ, κ). We say $\mathfrak{v}_r$ is an allowable operation of (ρ, κ) if $\mathfrak{v}_r(\rho, \kappa) \in \mathrm{Int}(\mathrm{NC}_{k+1})$ and r is minimal in its block in ρ. We say $\mathfrak{w}_r$ is an allowable operation of (ρ, κ) if $\mathfrak{w}_r(\rho, \kappa) \in \mathrm{Int}(\mathrm{NC}_{k+1})$ and r is

[8]Every combinatorial class has a "size function." The "size" of a permutation of length $2k+1$ in this class is k.

[9]We say a point $\mathfrak{Y}$ is *immediately to the right of* a point $\mathfrak{X}$ if $\mathfrak{Y}$ is the leftmost point to the right of $\mathfrak{X}$. The phrases "immediately to the left," "immediately above," and "immediately below" are defined similarly.

maximal in its block in κ. As mentioned before, every interval $(\widetilde{\rho},\widetilde{\kappa}) \in \mathrm{Int}(\mathrm{NC}_{k+1})$ is generated by a unique $(\rho,\kappa) \in \mathrm{Int}(\mathrm{NC}_k)$. Furthermore, there is a unique allowable operation of (ρ,κ) that sends (ρ,κ) to $(\widetilde{\rho},\widetilde{\kappa})$. Consequently, the number of intervals generated by (ρ,κ) is precisely the number of allowable operations of (ρ,κ). The bottom of Figure 11 depicts applications of some allowable operations.

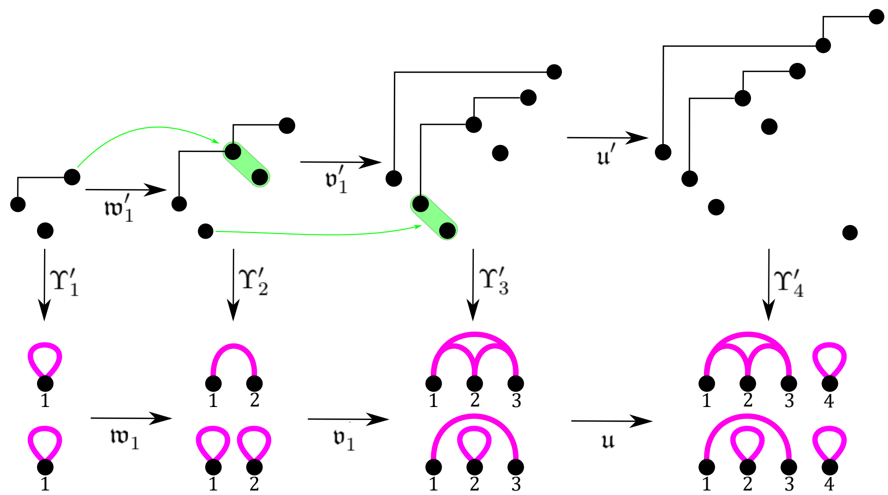


FIGURE 11. Examples illustrating the various maps and operations defined in this section. We find it convenient to draw the plots of uniquely sorted permutations with their canonical valid hook configurations. The green arrows indicate the points that split into two when we apply $\mathfrak{w}_1'$ and $\mathfrak{v}_1'$.

Consider the following ordering on these operations:

$$\mathfrak{v}_1 \prec \mathfrak{w}_1 \prec \mathfrak{v}_2 \prec \mathfrak{w}_2 \prec \cdots \prec \mathfrak{v}_k \prec \mathfrak{w}_k \prec \mathfrak{u}.$$

Let $(\rho,\kappa) \in \mathrm{Int}(\mathrm{NC}_k)$. For each operation $\mathfrak{o}$, let $F(\mathfrak{o})$ be the set of allowable operations of (ρ,κ) preceding $\mathfrak{o}$ in the ordering $\prec$. If $\mathfrak{v}_r$ is an allowable operation of (ρ,κ), then it is not difficult to check that the allowable operations of the interval $\mathfrak{v}_r(\rho,\kappa) \in \mathrm{Int}(\mathrm{NC}_{k+1})$ are precisely the operations in $F(\mathfrak{v}_r) \cup \{\mathfrak{v}_r, \mathfrak{w}_{k+1}, \mathfrak{u}\}$. If $\mathfrak{w}_r$ is an allowable operation of (ρ,κ), then it is not difficult to check that the allowable operations of the interval $\mathfrak{w}_r(\rho,\kappa) \in \mathrm{Int}(\mathrm{NC}_{k+1})$ are precisely the operations in $F(\mathfrak{w}_r) \cup \{\mathfrak{v}_{k+1}, \mathfrak{w}_{k+1}, \mathfrak{u}\}$. Finally, the allowable operations of $\mathfrak{u}(\rho,\kappa)$ are the elements of $F(\mathfrak{u}) \cup \{\mathfrak{v}_{k+1}, \mathfrak{w}_{k+1}, \mathfrak{u}\}$. Now recall that the label of (ρ,κ) in our generating tree is the number of allowable operations of (ρ,κ). This is precisely $|F(\mathfrak{u})|+1$. The above discussion tells us that the labels of the intervals generated by (ρ,κ) are $3,4,5,\ldots,|F(\mathfrak{u})|+3$. In summary, a generating tree of the class of intervals in noncrossing partition lattices is

$$\text{Axiom: } (3) \qquad \text{Rule: } (m) \rightsquigarrow (3)(4)\cdots(m+2) \quad \text{for every } m \in \mathbb{N}. \tag{14}$$

Let us remark that it is proven in [3] that the generating tree in (14) describes objects counted by the 3-Catalan numbers $\frac{1}{2k+1}\binom{3k}{k}$. Thus, we have actually reproven equation (3).

We now want to describe a generating tree for the combinatorial class $\mathcal{U}(312,1342)$. We associate such permutations with their canonical valid hook configurations. Suppose $\pi = \pi_1 \cdots \pi_{2k+1} \in \mathcal{U}_{2k+1}(312,1342)$, and let $\mathfrak{P}(i)$ be the point (i,π_i) in the plot of π. Given a point $\mathfrak{X}$ in the plot π, let $\mathrm{split}_{\mathfrak{X}}(\pi)$ be the permutation whose plot is obtained by inserting a new point immediately below and immediately to the right of $\mathfrak{X}$ and then normalizing. In other words, we "split" the point $\mathfrak{X}$ into two points in such a way that one of the new points is below and to the right of the other. Using Lemma 3.1, one can show that there is a chain of hooks connecting the point $\mathfrak{P}(1)$ to the point $\mathfrak{P}(2k+1)$. We call this chain of hooks (including the endpoints of the hooks in the chain) the *skyline* of π. For example, if $\pi = 432657819$ is the permutation in the top right of Figure 11, then the skyline of π contains the points $\mathfrak{P}(1), \mathfrak{P}(7), \mathfrak{P}(9)$.

Theorem 1.1 tells us that π has exactly k descents, say $d_1 < \cdots < d_k$. Let $d_{k+1} = 2k+1$. Let $\mathfrak{u}'(\pi) = (\pi \ominus 1) \oplus 1$ (recall the notation from Section 4). For $r \in \{1, \ldots, k\}$, let $\mathfrak{v}'_r(\pi) = \text{split}_{\mathfrak{P}(d_r+1)}(\pi) \oplus 1$ and $\mathfrak{w}'_r(\pi) = \text{split}_{\mathfrak{P}(d_{r+1})}(\pi) \oplus 1$. We will always say that $\mathfrak{u}'$ is an *allowable operation* of π. We say $\mathfrak{v}'_r$ is an allowable operation of π if $\mathfrak{P}(d_r)$ is in the skyline of π. We say $\mathfrak{w}'_r$ is an allowable operation of π if $\mathfrak{P}(d_{r+1})$ is in the skyline of π and is not immediately above $\mathfrak{P}(d_{r+1}+1)$ in the plot of π. The top of Figure 11 illustrates the application of some of these allowable operations. Recall that in a uniquely sorted permutation, the *partner* of the northeast endpoint of a hook H in the canonical valid hook configuration is the point immediately to the right of the southwest endpoint of H. Let us say a permutation $\pi \in \mathcal{U}_{2k+1}(312, 1342)$ *generates* a permutation $\widetilde{\pi} \in \mathcal{U}_{2k+3}(312, 1342)$ if the plot of π is obtained by removing the highest point in the plot of $\widetilde{\pi}$ (which is also the rightmost point and is also a northeast endpoint of a hook in the canonical valid hook configuration of $\widetilde{\pi}$) and the partner of that point from the plot of $\widetilde{\pi}$ (and then normalizing).

We have defined the rule by which a permutation in $\mathcal{U}_{2k+1}(312, 1342)$ generates permutations in $\mathcal{U}_{2k+3}(312, 1342)$. In order to better understand this rule, we prove the following lemmas. The reader may find it helpful to refer to the top of Figure 11 while reading the proofs that follow.

Lemma 7.1. *Every permutation $\widetilde{\pi} \in \mathcal{U}_{2k+3}(312, 1342)$ is generated by a unique permutation $\pi \in \mathcal{U}_{2k+1}(312, 1342)$. Furthermore, there is an allowable operation of π that sends π to $\widetilde{\pi}$.*

Proof. Let π be the permutation that generates $\widetilde{\pi}$ (it is clearly unique). We first want to show that $\pi \in \mathcal{U}_{2k+1}(312, 1342)$. Let $\mathfrak{P}(i)$ and $\widetilde{\mathfrak{P}}(i)$ denote the points (i, π_i) and $(i, \widetilde{\pi}_i)$ in the plots of π and $\widetilde{\pi}$, respectively. Since $\widetilde{\pi}$ is sorted, it has a canonical valid hook configuration $\widetilde{\mathcal{H}}$. The point $\widetilde{\mathfrak{P}}(2k+3)$ is the northeast endpoint of a hook H in $\widetilde{\mathcal{H}}$. Let $\widetilde{\mathfrak{P}}(\ell-1)$ be the southwest endpoint of H so that $\widetilde{\mathfrak{P}}(\ell)$ is the partner of $\widetilde{\mathfrak{P}}(2k+3)$. The plot of π is obtained from the plot of $\widetilde{\pi}$ by removing $\widetilde{\mathfrak{P}}(2k+3)$ and $\widetilde{\mathfrak{P}}(\ell)$ and normalizing. Note that π avoids 312 and 1342 because $\widetilde{\pi}$ does. We know that $\widetilde{\mathfrak{P}}(2k+3)$ is not a descent bottom of the plot of $\widetilde{\pi}$ and that $\widetilde{\mathfrak{P}}(\ell)$ is a descent bottom. It follows that $\text{des}(\pi) = \text{des}(\widetilde{\pi}) - 1$ (we are using the fact that π avoids 312). Theorem 1.1 tells us that $\text{des}(\widetilde{\pi}) = k+1$, so $\text{des}(\pi) = k$. That same theorem now tells us that in order to prove $\pi \in \mathcal{U}_{2k+1}(312, 1342)$, it suffices to prove that π is sorted. By Proposition 3.1, we need to show that π has a canonical valid hook configuration.

Suppose first that ℓ is not a descent of $\widetilde{\pi}$. Since $\widetilde{\pi}_{\ell-1} > \widetilde{\pi}_\ell$, it follows from the fact that $\widetilde{\pi}$ avoids 312 that $\widetilde{\pi}_{\ell+1} > \widetilde{\pi}_{\ell-1}$. Referring to the canonical valid hook configuration construction, we find that the hook with southwest endpoint $\widetilde{\mathfrak{P}}(\ell-1)$ must have $\widetilde{\mathfrak{P}}(\ell+1)$ as its northeast endpoint. This hook is H, and its northeast endpoint is $\widetilde{\mathfrak{P}}(2k+3)$. This shows that $\ell = 2k+2$. The canonical valid hook configuration of π is now obtained by removing the points $\widetilde{\mathfrak{P}}(2k+3)$ and $\widetilde{\mathfrak{P}}(\ell) = \widetilde{\mathfrak{P}}(2k+2)$ along with the hook H and then normalizing. Note also that the assumption that $\widetilde{\pi}$ avoids 312 and 1342 forces us to have either $\widetilde{\pi}_\ell = 1$ or $\widetilde{\pi}_\ell = \widetilde{\pi}_{\ell-1} - 1$. In the first case, $\mathfrak{u}'(\pi) = \widetilde{\pi}$. In the second case, $\mathfrak{w}'_k(\pi) = \widetilde{\pi}$.

Next, assume ℓ is a descent of $\widetilde{\pi}$. In this case, $\widetilde{\mathfrak{P}}(\ell)$ is a descent top of the plot of $\widetilde{\pi}$, so it is the southwest endpoint of a hook H' in $\widetilde{\mathcal{H}}$. Let us draw a new hook H'' whose southwest endpoint is $\widetilde{\mathfrak{P}}(\ell-1)$ (which is the southwest endpoint of H) and whose northeast endpoint is the northeast endpoint of H'. If we now remove the points $\widetilde{\mathfrak{P}}(2k+3)$ and $\widetilde{\mathfrak{P}}(\ell)$ along with the hooks H and H' (but keep the hook H'') and then normalize, we obtain the canonical valid hook configuration of π. To see an example of this, consider the permutations $\pi = 21435$ and $\widetilde{\pi} = 3215467$, which appear in the top of Figure 11 ($\ell = 2$ in this example). This completes the proof that π is sorted. Notice also that the point $\widetilde{\mathfrak{P}}(\ell-1)$ is in the skyline of $\widetilde{\pi}$. It follows from this procedure that the point $\mathfrak{P}(\ell-1)$ is in the skyline of π.

We still need to show that there is an allowable operation of π that sends π to $\widetilde{\pi}$ when ℓ is a descent of $\widetilde{\pi}$. In this case, $\ell-1$ is a descent of π. As before, we let $d_1 < \cdots < d_k$ be the descents of π. Let r be such that $\ell - 1 = d_r$. As mentioned at the end of the last paragraph, $\mathfrak{P}(d_r) = \mathfrak{P}(\ell-1)$ is in the skyline of π. We have two cases to consider. First, assume $\widetilde{\mathfrak{P}}(\ell)$ lies immediately above $\widetilde{\mathfrak{P}}(\ell+1)$ in the plot of $\widetilde{\pi}$. In this case, we find that $\mathfrak{v}'_r$ is an allowable operation of π that sends π to $\widetilde{\pi}$. For the second case, assume $\widetilde{\mathfrak{P}}(\ell)$ does not lie immediately above $\widetilde{\mathfrak{P}}(\ell+1)$ in the plot of $\widetilde{\pi}$. This means that $\mathfrak{P}(d_r)$ does not lie immediately above $\mathfrak{P}(d_r+1)$ in the plot of π. Using the fact that $\widetilde{\pi}$ avoids 312 and 1342, we also find that $\widetilde{\mathfrak{P}}(\ell-1)$ is immediately above and to the left of $\widetilde{\mathfrak{P}}(\ell)$ in the plot of $\widetilde{\pi}$. Therefore, $\mathfrak{w}'_{r-1}$ is an allowable operation of π that sends π to $\widetilde{\pi}$. We should check that $r \geq 2$ in this case. Indeed, it follows from Lemma 3.1 that $d_1 = 1$. If we had $r = 1$, then the fact that $\mathfrak{P}(d_r)$ does not lie immediately above $\mathfrak{P}(d_r+1)$ in the plot of π would force π to contain a 312 pattern. □

Lemma 7.2. *If $\mathfrak{o}'$ is an allowable operation of a permutation $\pi \in \mathcal{U}_{2k+1}(312, 1342)$, then the permutation $\widetilde{\pi} = \mathfrak{o}'(\pi)$ is in $\mathcal{U}_{2k+3}(312, 1342)$. Furthermore, $\mathfrak{o}'$ is the only allowable operation of π that sends π to $\widetilde{\pi}$.*

Proof. Let $\mathcal{H}$ be the canonical valid hook configuration of π. Let $d_1 < \cdots < d_k$ be the descents of π. Let $\mathfrak{P}(i)$ and $\widetilde{\mathfrak{P}}(i)$ denote the points (i, π_i) and $(i, \widetilde{\pi}_i)$. Since π has k descents and avoids 312 and 1342, it is easy to check that $\widetilde{\pi}$ has $k+1$ descents and avoids those same patterns. According to Theorem 1.1, we need to show that $\widetilde{\pi}$ is sorted. By Proposition 3.1, this amounts to showing that $\widetilde{\pi}$ has a canonical valid hook configuration $\widetilde{\mathcal{H}}$. To do this, we simply reverse the process described in the proof of Lemma 7.1 that allowed us to obtain the canonical valid hook configuration of π from that of $\widetilde{\pi}$. More precisely, if $\mathfrak{o}'$ is $\mathfrak{u}'$ or $\mathfrak{w}'_k$, then we keep all of the hooks from $\mathcal{H}$ the same (modulo normalization of the plot) and attach a new hook with southwest endpoint $\widetilde{\mathfrak{P}}(2k+1)$ and northeast endpoint $\widetilde{\mathfrak{P}}(2k+3)$. Otherwise, there exists $r \in \{1, \ldots, k\}$ such that $\mathfrak{o}'$ is either $\mathfrak{v}'_r$ or $\mathfrak{w}'_{r-1}$. Let $\ell - 1 = d_r$, and note that d_r is a descent of both π and $\widetilde{\pi}$. Also, ℓ is a descent of $\widetilde{\pi}$. We obtain a configuration of hooks of $\widetilde{\pi}$ by keeping all of the hooks in $\mathcal{H}$ unchanged (modulo normalization). Let H'' be the hook in this configuration with southwest endpoint $\widetilde{\mathfrak{P}}(d_r) = \widetilde{\mathfrak{P}}(\ell-1)$. Now form a new hook H of $\widetilde{\pi}$ with southwest endpoint $\widetilde{\mathfrak{P}}(\ell-1)$ and northeast endpoint $\widetilde{\mathfrak{P}}(2k+3)$. Form another new hook H' of $\widetilde{\pi}$ whose southwest endpoint is $\widetilde{\mathfrak{P}}(\ell)$ and whose northeast endpoint is the northeast endpoint of H''. Removing the hook H'', we obtain the canonical valid hook configuration $\widetilde{\mathcal{H}}$ of $\widetilde{\pi}$.

We need to show that there is at most one allowable operation of π that sends π to $\widetilde{\pi}$. If $\widetilde{\pi}_{2k+2} = 1$, then the only such operation is $\mathfrak{u}'$. Now assume $\widetilde{\pi}_{2k+2} \neq 1$. Let $\widetilde{\mathfrak{P}}(d_r)$ be the southwest endpoint of the hook in $\widetilde{\mathcal{H}}$ whose northeast endpoint is $\widetilde{\mathfrak{P}}(2k+3)$. Analyzing the canonical valid hook configuration construction, we find that $\widetilde{\mathcal{H}}$ must have been formed via the construction described in the preceding paragraph. In other words, the only possibilities for allowable operations that send π to $\widetilde{\pi}$ are $\mathfrak{v}'_r$ and $\mathfrak{w}'_{r-1}$. If $\mathfrak{P}(d_r)$ is immediately above $\mathfrak{P}(d_r+1)$ in the plot of π, then the only possibility is $\mathfrak{v}'_r$ because $\mathfrak{w}'_{r-1}$ is not an allowable operation of π. If $\mathfrak{P}(d_r)$ is not immediately above $\mathfrak{P}(d_r+1)$ in the plot of π, then the permutations $\mathfrak{v}'_r(\pi)$ and $\mathfrak{w}'_{r-1}(\pi)$ are distinct. In this case, there is again at most one possibility for an allowable operation of π that sends π to $\widetilde{\pi}$. □

We are now in a position to describe our generating tree. Lemmas 7.1 and 7.2 tell us that every permutation in $\mathcal{U}_{2k+3}(312, 1342)$ is generated by a unique permutation in $\mathcal{U}_{2k+1}(312, 1342)$ and that the number of permutations in $\mathcal{U}_{2k+3}(312, 1342)$ that a permutation $\pi \in \mathcal{U}_{2k+1}(312, 1342)$ generates is the number of allowable operations of π. Consider the following ordering:

$$\mathfrak{v}'_1 \prec \mathfrak{w}'_1 \prec \mathfrak{v}'_2 \prec \mathfrak{w}'_2 \prec \cdots \prec \mathfrak{v}'_k \prec \mathfrak{w}'_k \prec \mathfrak{u}'.$$

Choose $\pi \in \mathcal{U}_{2k+1}(312, 1342)$. For each operation $\mathfrak{o}'$, let $F'(\mathfrak{o}')$ be the set of allowable operations of π preceding $\mathfrak{o}'$ in the ordering $\prec$. In the proof of Lemma 7.2, we described the procedure that produces the canonical valid hook configuration of $\widetilde{\pi}$ from that of π when $\widetilde{\pi}$ is a permutation that π generates. If $\mathfrak{v}'_r$ is an allowable operation of π and $\widetilde{\pi} = \mathfrak{v}'_r(\pi)$, then we can trace through this procedure to see that the allowable operations of $\widetilde{\pi}$ are precisely the operations in $F'(\mathfrak{v}'_r) \cup \{\mathfrak{v}'_r, \mathfrak{w}'_{k+1}, \mathfrak{u}'\}$. If $\mathfrak{w}'_r$ is an allowable operation of π and $\widetilde{\pi} = \mathfrak{w}'_r(\pi)$, then we can trace through this procedure to see that the allowable operations of $\widetilde{\pi}$ are precisely the operations in $F'(\mathfrak{w}'_r) \cup \{\mathfrak{v}'_{k+1}, \mathfrak{w}'_{k+1}, \mathfrak{u}'\}$. Finally, the allowable operations of $\mathfrak{u}'(\pi)$ are the elements of $F'(\mathfrak{u}') \cup \{\mathfrak{v}'_{k+1}, \mathfrak{w}'_{k+1}, \mathfrak{u}'\}$. The number of allowable operations of π is $|F'(\mathfrak{u}')| + 1$. The above discussion tells us that the labels of the permutations generated by π are $3, 4, 5, \ldots, |F'(\mathfrak{u}')| + 3$. In summary, a generating tree of the combinatorial class $\mathcal{U}(312, 1342)$ is

$$\text{Axiom: } (3) \qquad \text{Rule: } (m) \rightsquigarrow (3)(4)\cdots(m+2) \quad \text{for every } m \in \mathbb{N}. \tag{15}$$

Of course, (14) and (15) are identical. Thus, there is a natural isomorphism[10] between the generating trees of intervals in noncrossing partition lattices and uniquely sorted permutations avoiding 312 and 1342. In fact, this isomorphism is unique. Finally, we obtain the bijections $\Upsilon_k : \mathcal{U}_{2k+1}(312, 1342) \to \text{Int}(\text{NC}_k)$ from this isomorphism of generating trees in the obvious fashion, proving Theorem 7.1. Using the equation (3), we obtain the following corollary.

Corollary 7.1. *For each nonnegative integer k,*

$$|\mathcal{U}_{2k+1}(312, 1342)| = \frac{1}{2k+1}\binom{3k}{k}.$$

8. Pallo Comb Intervals and $\mathcal{U}_{2k+1}(231, 4132)$

Aval and Chapoton showed how to decompose the intervals in Pallo comb posets in order to obtain the identity (4). In this section, we show how to decompose permutations in $\mathcal{U}_{2k+1}(231, 4132)$ in order to obtain a similar identity that proves these permutations are in bijection with Pallo comb intervals. More precisely, we have the following theorem.

Theorem 8.1. *We have*

$$\sum_{k\geq 0} |\mathcal{U}_{2k+1}(231, 4132)| x^k = C(xC(x)),$$

where $C(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ *is the generating function of the sequence of Catalan numbers.*

We know that the Pallo comb poset PC_k is a subposet of the Tamari lattice $\mathcal{L}_k^T$. If we combine Theorems 6.1 and 6.2, we find a bijection $\Lambda_k \circ \text{swl} \circ \text{swu}$ from $\mathcal{U}_{2k+1}(231, 4132)$ to a subset of $\text{Int}(\mathcal{L}_k^T)$. One might hope to prove Theorem 8.1 by showing that this subset is precisely $\text{Int}(\text{PC}_k)$ and then invoking (4). Unfortunately, this is not the case when $k = 3$ (and probably also when $k \geq 4$). Before we can prove Theorem 8.1, we need the following lemma.

[10] We haven't formally defined "isomorphisms" of generating trees, but we expect the notion will be apparent.

Lemma 8.1. *For each nonnegative integer k, $|\mathcal{U}_{2k+1}(132,231)| = C_k$.*

Proof.[11] It is known that $|\operatorname{Av}_{2k+1}(132,231)| = 2^{2k}$. One way to prove this is to first observe that a permutation avoids 132 and 231 if and only if it can be written as a decreasing sequence followed by an increasing sequence. Given $\pi \in \operatorname{Av}_{2k+1}(132,231)$, we can write $\pi = L1R$, where L is decreasing and R is increasing. Let $w_\ell = U$ if $2k+2-\ell$ is an entry in R, and let $w_\ell = D$ if $2k+2-\ell$ is an entry in L. We obtain a word $w = w_1 \cdots w_{2k} \in \{U,D\}^{2k}$. The map $\pi \mapsto w$ is a bijection between $\operatorname{Av}_{2k+1}(132,231)$ and $\{U,D\}^{2k}$. The permutation π has exactly k descents if and only if the letter D appears exactly k times in the corresponding word w. Furthermore, π has a canonical valid hook configuration (meaning it is sorted) if and only if every prefix of w contains at least as many occurrences of the letter U as occurrences of D. Using Theorem 1.1, we see that π is uniquely sorted if and only if w is a Dyck path. □

Proof of Theorem 8.1. Let $B(x) = \sum_{k\geq 0} |\mathcal{U}_{2k+1}(231,4132)|x^k$ and $\widetilde{B}(x) = \sum_{n\geq 1} |\mathcal{U}_n(231,4132)|x^n$. Since there are no uniquely sorted permutations of even length, we have $\widetilde{B}(x) = xB(x^2)$. Therefore, our goal is to show that $\widetilde{B}(x) = E(x)$, where $E(x) = xC(x^2C(x^2))$. Using the standard Catalan functional equation $C(x) = 1 + xC(x)^2$, we find that $E(x) = x + xC(x^2)E(x)^2$. This last equation and the condition $E(x) = x + O(x^2)$ uniquely determine the power series $E(x)$. Since $\widetilde{B}(x) = x + O(x^2)$, we are left to prove that

$$\widetilde{B}(x) = x + xC(x^2)\widetilde{B}(x)^2. \tag{16}$$

The term x in (16) represents the permutation 1. Now suppose $\pi \in \mathcal{U}_n(231,4132)$, where $n = 2k+1 \geq 3$. Proposition 3.1 tells us that π has a canonical valid hook configuration $\mathcal{H}$, and Lemma 3.1 tells us that the point $(2k+1, 2k+1)$ is the northeast endpoint of a hook H in $\mathcal{H}$. Let (i, π_i) be the southwest endpoint of H. We will say π is *nice* if $i = 1$. Let us first consider the case in which π is nice.

Because π avoids 231, we can write $\pi = \pi_1\lambda\mu(2k+1)$, where $\lambda \in S_{\pi_1 - 1}$ and μ is a permutation of $\{\pi_1+1, \ldots, 2k\}$. Because π avoids 231 and 4132, λ avoids 132 and 231. As mentioned in the proof of Lemma 8.1, λ is a decreasing sequence followed by an increasing sequence. Let m be the largest integer such that the subpermutation τ of λ formed by the entries $1, 2, \ldots, 2m+1$ is in $\mathcal{U}_{2m+1}(132,231)$. We can write $\lambda = L\tau R$, where $\tau \in \mathcal{U}_{2m+1}(132,231)$, L is decreasing, and R is increasing. We claim that $\pi' = \pi_1 LR\mu$ is a uniquely sorted permutation that avoids 231 and 4132. It is easy to check that π' is a permutation of length $2k-2m-1$ that avoids 231 and 4132 and has exactly $k-m-1$ descents; we need to show that it has a canonical valid hook configuration $\mathcal{H}'$. We obtain $\mathcal{H}'$ from the canonical valid hook configuration $\mathcal{H} = (H_1, \ldots, H_k)$ of π as follows. Let ℓ be the length of L. For all $r \in \{1, \ldots, \ell+m+1\}$, the southwest endpoint of H_r is (r, π_r). For $r \in \{1, \ldots, \ell\}$, let H_r' be the hook of π' with southwest endpoint $(r, \pi_r) = (r, \pi_r')$ whose northeast endpoint has the same height as the northeast endpoint of H_{r+1}. For $r \in \{\ell+m+2, \ldots, k\}$, let H_r' be the hook of π' whose southwest and northeast endpoints have the same heights as the southwest and northeast endpoints of H_r, respectively. The canonical valid hook configuration of π' is $\mathcal{H}' = (H_1', \ldots, H_\ell', H_{\ell+m+2}', \ldots, H_k')$. See Figure 12 for an example of this construction.

One can use Lemma 3.1 and the fact that π is uniquely sorted to see that the last entry in L is $2m+2$, the smallest entry in π'. Thus, if we let π'' be the normalization of π', then we have obtained from π the pair $(\pi'', \tau) \in \mathcal{U}_{2k-2m-1}(231,4132) \times \mathcal{U}_{2m+1}(132,231)$. We can reverse this procedure. If we are given (π'', τ), then we can increase all of the entries in π'' by $2m+1$ to form

[11] One could alternatively prove this lemma by showing that $\Lambda_k \circ \operatorname{swl} : \mathcal{U}_{2k+1}(132,231) \to \operatorname{Int}(\mathcal{A}_k)$ is a bijection.

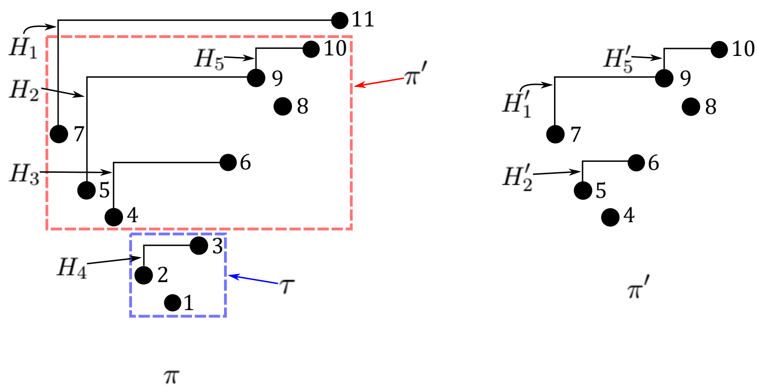


FIGURE 12. Decomposing the nice permutation π into pieces. In this example, $k = 5$, $\ell = 2$, and $m = 1$.

π'. We then insert τ after the smallest entry in π' and append the entry $2k+1$ to the end to form the permutation π. Lemma 8.1 tells us that $\sum_{n\geq 1} |\mathcal{U}_n(132, 231)|x^n = xC(x^2)$, so it follows that the generating function that counts nice permutations in $\mathcal{U}(231, 4132)$ is $x^2C(x^2)\widetilde{B}(x)$.

We now consider the general case in which π is not necessarily nice. Let $\sigma = \pi_1 \cdots \pi_{i-1}$, and let σ' be the normalization of $\pi_i \cdots \pi_{2k+1}$. Let σ'' be the normalization of $\sigma\pi_i$. Because π avoids 231, $\pi = \sigma \oplus \sigma'$ (so $\sigma'' = \sigma i$). The permutation σ'' is in $\mathcal{U}_i(231, 4132)$. The permutation σ' is a nice permutation in $\mathcal{U}(231, 4132)$. If we were given the permutation $\sigma'' \in \mathcal{U}(231, 4132)$ and the nice permutation $\sigma' \in \mathcal{U}(231, 4132)$, then we could easily reobtain π by first deleting the last entry of σ'' to form σ and then writing $\pi = \sigma \oplus \sigma'$. It follows that $\widetilde{B}(x) - x = \frac{1}{x}(x^2C(x^2)\widetilde{B}(x))\widetilde{B}(x) = xC(x^2)\widetilde{B}(x)^2$, which is (16) (the $\frac{1}{x}$ comes from the fact that $\pi_1 \cdots \pi_i$ and $\pi_i \cdots \pi_{2k+1}$ overlap in the entry π_i). $\square$

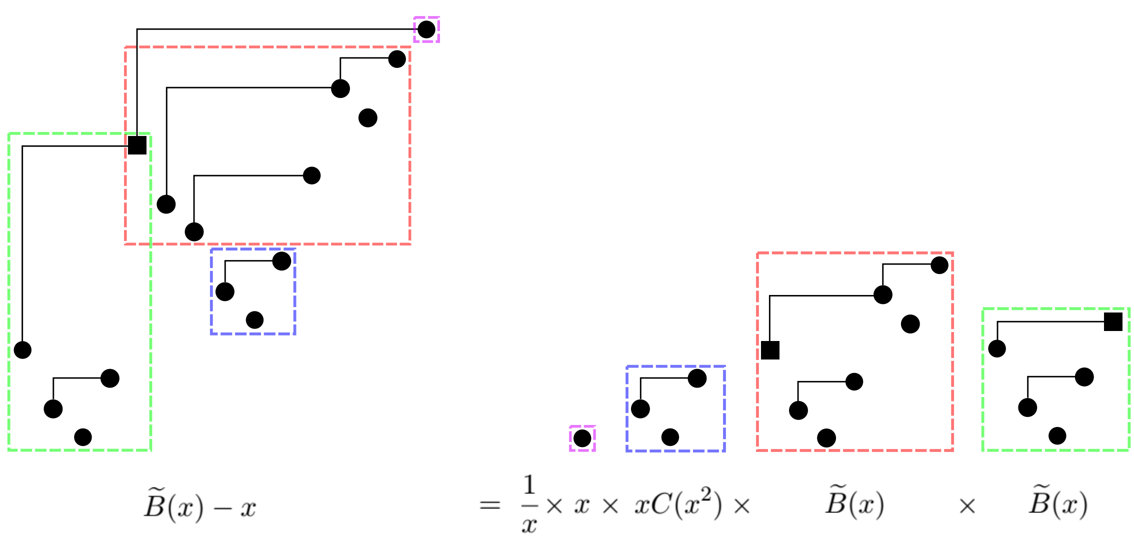


FIGURE 13. An illustration of the equation $\widetilde{B}(x) - x = xC(x^2)\widetilde{B}(x)^2$ from the proof of Theorem 8.1. The factor $\frac{1}{x}$ comes from the fact that the point marked with the square appears twice on the right-hand side.

9. Catalan Antichain Intervals

In this section, we prove that

$$|\mathcal{U}_{2k+1}(321)| = |\mathcal{U}_{2k+1}(231, 312)| = |\mathcal{U}_{2k+1}(132, 231)| = |\mathcal{U}_{2k+1}(132, 312)| = C_k.$$

These results fit into the theme of this article if we interpret C_k as the number of intervals in the antichain $\mathcal{A}_k$.

Theorem 9.1. *For each nonnegative integer k, we have $|\mathcal{U}_{2k+1}(321)| = C_k$.*

Proof. A *parking function of length k* is a tuple $(a_1, \ldots, a_k)$ of positive integers such that $a_i \leq i$ for all $i \in [k]$. We say this parking function is *nondecreasing* if $a_1 \leq \cdots \leq a_k$. It is well known that the number of nondecreasing parking functions of length k is C_k. Given $\pi = \pi_1 \cdots \pi_{2k+1} \in \mathcal{U}_{2k+1}(321)$, put $a_i = \pi_{2i} - i + 1$. We claim that $(a_1, \ldots, a_k)$ is a nondecreasing parking function.

Note that $(2, \pi_2)$ is not the northeast endpoint of a hook in the canonical valid hook configuration of π. Lemma 3.1 tells us that $(2, \pi_2)$ is a descent bottom in the plot of π, so 1 is a descent of π. Since π avoids 321, no two descents of π are consecutive integers. We know by Theorem 1.1 that π has k descents, so these descents must be $1, 3, 5, \ldots, 2k-1$. Choose $i \in [k]$. Since π avoids 321 and $\pi_{2i-1} > \pi_{2i}$, all of the elements of $[\pi_{2i}-1]$ appear to the left of π_{2i} in π. Because π_{2i-1} is an additional entry that appears to the left of π_{2i} in π, we must have $2i-1 \geq \pi_{2i}$. It follows that $a_i = \pi_{2i}-i+1 \leq i$. As i was arbitrary, $(a_1, \ldots, a_k)$ is a parking function. If $i \in [k-1]$, then $\pi_{2i+2} \geq \pi_{2i} + 1$ since π avoids 321 and $\pi_{2i-1} > \pi_{2i}$. This means that $a_{i+1} = \pi_{2i+2} - (i+1) + 1 \geq \pi_{2i} - i + 1 = a_i$. As i was arbitrary, $(a_1, \ldots, a_k)$ is nondecreasing.

Given the nondecreasing parking function $(a_1, \ldots, a_k)$, we can reobtain the permutation π. Indeed, the values of $\pi_2, \pi_4, \ldots, \pi_{2k}$ are determined by the definition $a_i = \pi_{2i}-i+1$. The other entries of π are determined by the fact that $\pi_1 < \pi_3 < \cdots < \pi_{2k+1}$, which is an easy consequence of the fact that π is a 321-avoiding sorted permutation whose descents are $1, 3, 5, \ldots, 2k-1$. Thus, π_{2i-1} must be the i^{th}-smallest element of $[2k+1] \setminus \{\pi_2, \pi_4, \ldots, \pi_{2k}\}$. We want to check that the permutation π obtained in this way is indeed in $\mathcal{U}_{2k+1}(321)$. One can easily check that this permutation avoids 321 and has k descents. We must show that it has a canonical valid hook configuration $\mathcal{H} = (H_1, \ldots, H_k)$. This is easy; H_i is simply the hook with southwest endpoint $(2i-1, \pi_{2i-1})$ and northeast endpoint $(2i+1, \pi_{2i+1})$. □

In the following theorems, recall the bijection $\Lambda\!\!\Lambda_k : \mathcal{U}_{2k+1}(312) \to \text{Int}(\mathcal{L}_k^S)$ from Theorem 5.1.

Theorem 9.2. *For each nonnegative integer k, the restriction of $\Lambda\!\!\Lambda_k$ to $\mathcal{U}_{2k+1}(231, 312)$ is a bijection from $\mathcal{U}_{2k+1}(231, 312)$ to $\text{Int}(\mathcal{A}_k)$. Hence, $|\mathcal{U}_{2k+1}(231, 312)| = C_k$.*

Proof. A permutation is called *layered* if can be written as $\text{Dec}_{a_1} \oplus \text{Dec}_{a_2} \oplus \cdots \oplus \text{Dec}_{a_m}$ for some positive integers $a_1, \ldots, a_m$, where $\text{Dec}_a = a(a-1)\cdots 1$ is the decreasing permutation in S_a. It is a standard fact among permutation pattern enthusiasts that a permutation $\pi \in S_n$ is layered if and only if it avoids 231 and 312. It is straightforward to check that a permutation $\pi \in \mathcal{U}_{2k+1}(312)$ is layered if and only if $\Lambda\!\!\Lambda_k(\pi) \in \text{Int}(\mathcal{A}_k)$ (meaning $\Lambda\!\!\Lambda_k(\pi) = (\Lambda, \Lambda)$ for some $\Lambda \in \mathbf{D}_k$). □

Theorem 9.3. *For each nonnegative integer k, the restriction of $\Lambda\!\!\Lambda_k \circ \text{swl}$ to $\mathcal{U}_{2k+1}(132, 231)$ is a bijection from $\mathcal{U}_{2k+1}(132, 231)$ to $\text{Int}(\mathcal{A}_k)$. Hence, $|\mathcal{U}_{2k+1}(132, 231)| = C_k$.*

Proof. We already know from Lemma 8.1 that $|\mathcal{U}_{2k+1}(132, 231)| = C_k$. Now suppose $\pi \in \mathcal{U}_{2k+1}(132, 231)$. We know by Theorem 1.1 that π is sorted and has k descents. Lemmas 4.3 and

4.4 tell us that $\mathrm{swl}(\pi)$ is sorted and has k descents, so it follows from Theorem 1.1 that $\mathrm{swl}(\pi)$ is uniquely sorted. We know from Lemma 4.2 that $\mathrm{swl}(\pi) \in \mathrm{Av}(231, 312)$. Lemma 4.2 also tells us that swl is injective on $\mathcal{U}_{2k+1}(132, 231)$. We have proven that $\mathrm{swl} : \mathcal{U}_{2k+1}(132, 231) \to \mathcal{U}_{2k+1}(231, 312)$ is an injection. It must also be surjective because $|\mathcal{U}_{2k+1}(132, 231)| = |\mathcal{U}_{2k+1}(231, 312)| = C_k$ by Lemma 8.1 and Theorem 9.2. The proof of the theorem now follows from Theorem 9.2. □

Theorem 9.4. *For each nonnegative integer k, the restriction of $\Lambda_k \circ \mathrm{swd}$ to $\mathcal{U}_{2k+1}(132, 312)$ is a bijection from $\mathcal{U}_{2k+1}(132, 312)$ to $\mathrm{Int}(\mathcal{A}_k)$. Hence, $|\mathcal{U}_{2k+1}(132, 312)| = C_k$.*

Proof. Theorem 6.1 and Lemma 4.2 tell us that $\mathrm{swd} : \mathcal{U}_{2k+1}(132) \to \mathcal{U}_{2k+1}(231)$ and $\mathrm{swd} : \mathrm{Av}(132, 312) \to \mathrm{Av}(231, 312)$ are bijections. It follows that $\mathrm{swd} : \mathcal{U}_{2k+1}(132, 312) \to \mathcal{U}_{2k+1}(231, 312)$ is a bijection, so the proof of the theorem follows from Theorem 9.2. □

10. Concluding Remarks

One of our primary focuses in this paper has been the enumeration of sets of the form $\mathcal{U}_{2k+1}(\tau^{(1)}, \ldots, \tau^{(r)})$. We can actually complete this enumeration for all possible cases in which the patterns $\tau^{(1)}, \ldots, \tau^{(r)}$ are of length 3. It is easy to check that $\mathcal{U}_{2k+1}(123)$ and $\mathcal{U}_{2k+1}(213)$ are empty when $k \geq 2$, so we only need to focus on the cases in which $\{\tau^{(1)}, \ldots, \tau^{(r)}\} \subseteq \{132, 231, 312, 321\}$. When $r = 0$ (so we consider the set $\mathcal{U}_{2k+1}$), the enumeration is completed in [31] and is given by Lassalle's sequence (as discussed in the introduction). The cases in which $r = 1$ are handled in Corollary 5.1, Corollary 6.1, and Theorem 9.1. Three of the six cases in which $r = 2$ are handled in Theorems 9.3, 9.2, and 9.4. In the following theorem, we finish the other three cases in which $r = 2$ along with all of the cases in which $r = 3$ or $r = 4$.

Theorem 10.1. *For each nonnegative integer k, we have*

$$|\mathcal{U}_{2k+1}(231, 321)| = |\mathcal{U}_{2k+1}(312, 321)| = |\mathcal{U}_{2k+1}(231, 312, 321)| = |\mathcal{U}_{2k+1}(132, 231, 312)| = 1.$$

For each $k \geq 2$, we have $\mathcal{U}_{2k+1}(132, 321) = \emptyset$.

Proof. We may assume $k \geq 2$. The proof of Theorem 9.1 shows that if $\pi = \pi_1 \cdots \pi_{2k+1} \in \mathcal{U}_{2k+1}(321)$, then $\pi_1 < \pi_3 < \cdots < \pi_{2k+1}$, and the descents of π are $1, 3, 5, \ldots, 2k-1$. It easily follows that

$$\mathcal{U}_{2k+1}(231, 321) = \mathcal{U}_{2k+1}(312, 321) = \mathcal{U}_{2k+1}(231, 312, 321) = \{214365 \cdots (2k)(2k-1)(2k+1)\}$$

and that $\mathcal{U}_{2k+1}(132, 321) = \emptyset$. Every element of $\mathrm{Av}(132, 231, 312)$ is of the form $L \oplus R$, where L is a decreasing permutation and R is an increasing permutation. A uniquely sorted permutation of length $2k+1$ must have k descents, so

$$\mathcal{U}_{2k+1}(132, 231, 312) = \{(k+1)k \cdots 1(k+2)(k+3) \cdots (2k+1)\}.$$ □

Theorem 10.1 implies that

$$\mathcal{U}_{2k+1}(132, 231, 321) = \mathcal{U}_{2k+1}(132, 312, 321) = \mathcal{U}_{2k+1}(132, 231, 312, 321) = \emptyset,$$

so we have completed the enumeration of $\mathcal{U}_{2k+1}(\tau^{(1)}, \ldots, \tau^{(r)})$ when $\tau^{(1)}, \ldots, \tau^{(r)}$ are of length 3. Since we enumerated $\mathcal{U}_{2k+1}(312, 1342)$ in Corollary 7.1 and enumerated $\mathcal{U}_{2k+1}(231, 4132)$ in Theorem 8.1, it is natural to look at other sets of the form $\mathcal{U}_{2k+1}(\tau^{(1)}, \tau^{(2)})$ with $\tau^{(1)} \in S_3$ and $\tau^{(2)} \in S_4$. To this end, we have eighteen conjectures. Each row of the following table represents the conjecture that the class of (normalized) uniquely sorted permutations (of odd length) avoiding the given patterns is counted by the corresponding OEIS sequence.

<table>
<tr><th>Patterns</th><th>OEIS Sequence</th></tr>
<tr><td>312, 1432
312, 2431
312, 3421
132, 3412
231, 1423</td><td>A001764</td></tr>
<tr><td>312, 1243</td><td>A122368</td></tr>
</table>

<table>
<tr><th>Patterns</th><th>OEIS Sequence</th></tr>
<tr><td>132, 3421
132, 4312
231, 1243</td><td>A001700</td></tr>
<tr><td>132, 2341
132, 4123</td><td>A109081</td></tr>
<tr><td>312, 2341</td><td>A006605</td></tr>
</table>

Patterns	OEIS Sequence
312, 3241	A279569
312, 4321	A063020
132, 4231	A071725
231, 1432	A001003
231, 4312	A127632
231, 4321	A056010

Table 1. Conjectural OEIS sequences enumerating sets of the form $\mathcal{U}_{2k+1}(\tau^{(1)}, \tau^{(2)})$.

Note that the OEIS sequences A001764 and A127632 give the numbers appearing in (3) and (4), respectively. A couple of especially well-known sequences appearing in Table 1 are A001700, which consists of the binomial coefficients $\binom{2k-1}{k}$, and A001003, which consists of the little Schröder numbers. We have also calculated the first few values of $|\mathcal{U}_{2k+1}(231, 4123)|$; beginning at $k = 0$, they are $1, 1, 3, 10, 36, 138, 553, 2288, 9699, 41908$. This sequence appears to be new, so we have added it as sequence A307346 in the OEIS. We have also computed the first few terms in each of the 24 sequences $(|\mathcal{U}_{2k+1}(\tau)|)_{k\geq 0}$ for $\tau \in S_4$; these sequences appear to be new.

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