

# CONSECUTIVE PATTERNS IN INVERSION SEQUENCES

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**ABSTRACT.** An inversion sequence of length  $n$  is an integer sequence  $e = e_1e_2\dots e_n$  such that  $0 \leq e_i < i$  for each  $i$ . Corteel–Martinez–Savage–Weselcouch and Mansour–Shattuck began the study of patterns in inversion sequences, focusing on the enumeration of those that avoid classical patterns of length 3. We initiate an analogous systematic study of *consecutive* patterns in inversion sequences, namely patterns whose entries are required to occur in adjacent positions. We enumerate inversion sequences that avoid consecutive patterns of length 3, and generalize some results to patterns of arbitrary length. Additionally, we study the notion of Wilf equivalence of consecutive patterns in inversion sequences, as well as generalizations of this notion analogous to those studied for permutation patterns. We classify patterns of length up to 4 according to the corresponding Wilf equivalence relations.

## 1. INTRODUCTION

Let  $\mathbf{I}_n$  denote the set of inversion sequences of length  $n$ , defined as integer sequences  $e = e_1e_2\dots e_n$  such that  $0 \leq e_i < i$  for each  $i$ . Let  $S_n$  denote the symmetric group of all permutations of  $[n] = \{1, 2, \dots, n\}$ . There is a natural bijection between  $S_n$  and  $\mathbf{I}_n$ , where each permutation  $\pi \in S_n$  is encoded by its inversion sequence

$$(1.1) \quad \Theta(\pi) = e = e_1e_2\dots e_n, \quad \text{with} \quad e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

Other closely-related bijections between  $S_n$  and  $\mathbf{I}_n$ , obtained by composing  $\Theta$  with trivial symmetries, appear in the literature as well. Corteel, Martinez, Savage and Weselcouch [8], and Mansour and Shattuck [17] initiated the study of classical patterns in inversion sequences, motivated by the potential of this correspondence to inform the study of patterns in permutations. Their systematic enumeration of classical patterns of length 3 turns out to be interesting in its own right, as it connects these patterns in inversion sequences to well-known sequences such as Bell numbers, Euler up/down numbers, Fibonacci numbers and Schröder numbers. Since the appearance of the work of Corteel *et al.* and Mansour and Shattuck, the interest in patterns in inversion sequences has been on the rise, and research in this area continues to proliferate [5, 6, 13, 14, 16, 18, 22].

Let us start with the basic definitions. We define a *pattern* to be a sequence  $p = p_1p_2\dots p_r$  with  $p_i \in \{0, 1, \dots, r-1\}$  for all  $i$ , where a value  $j > 0$  can appear in  $p$  only if  $j-1$  appears as well. Define the *reduction* of a word  $w = w_1w_2\dots w_k$  over the integers to be the word obtained by replacing all the occurrences of the  $i$ th smallest entry of  $w$  with  $i-1$  for all  $i$ . We say that an

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inversion sequence  $e$  contains the classical pattern  $p = p_1p_2 \dots p_r$  if there exists a subsequence  $e_{i_1}e_{i_2} \dots e_{i_r}$  of  $e$  whose reduction is  $p$ . Otherwise, we say that  $e$  avoids  $p$ .

**Example 1.1.** The inversion sequence  $e = 0021213$  avoids the classical pattern 210, but it contains the classical pattern 110. Indeed,  $e_3e_5e_6 = 221$  has reduction 110.

Note that, unlike patterns in permutations, patterns in inversion sequences may have repeated entries.

Corteel *et al.* [8] and Mansour and Shattuck [17] provide formulas for the number of inversion sequences of length  $n$  that avoid a given pattern  $p$  of length 3. In this paper, we initiate an analogous systematic study of consecutive patterns in inversion sequences.

We define consecutive patterns in inversion sequences in analogy to consecutive patterns in permutations, which were introduced by Elizalde and Noy [12]. We underline the entries of a consecutive pattern to distinguish it from the classical case.

**Definition 1.2.** An inversion sequence  $e$  contains the consecutive pattern  $p = \underline{p_1p_2 \dots p_r}$  if there is a consecutive subsequence  $e_i e_{i+1} \dots e_{i+r-1}$  of  $e$  whose reduction is  $p$ ; this subsequence is called an *occurrence* of  $p$  in position  $i$ . Otherwise, we say that  $e$  avoids  $p$ . Denote by  $\mathbf{I}_n(p)$  the set of inversion sequences of length  $n$  that avoid  $p$ .

**Example 1.3.** The inversion sequence  $e = 002031$  avoids the consecutive pattern 120, but it contains the consecutive pattern 010, since  $e_2e_3e_4 = 020$  is an occurrence of 010.

It is often useful to represent an inversion sequence  $e$  as an underdiagonal lattice path consisting of unit horizontal and vertical steps. Each entry  $e_i$  of  $e$  is represented by a horizontal step: a segment between the points  $(i-1, e_i)$  and  $(i, e_i)$ . The vertical steps are then inserted to make the path connected (see Figure 1).

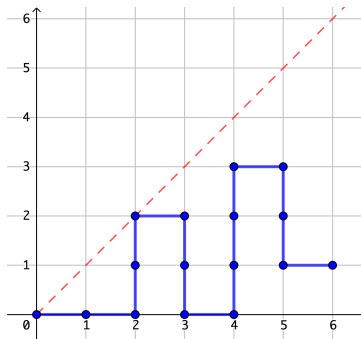


FIGURE 1. Representation of  $e = 002031 \in \mathbf{I}_6$  as a lattice path.

We are also interested in the following equivalence relations, defined in analogy to those for consecutive patterns in permutations (see [9, 10]).

**Definition 1.4.** Let  $p$  and  $p'$  be consecutive patterns in inversion sequences. We say that  $p$  and  $p'$  are

- *Wilf equivalent*, denoted by  $p \sim p'$ , if  $|\mathbf{I}_n(p)| = |\mathbf{I}_n(p')|$  for all  $n$ ;
- *strongly Wilf equivalent*, denoted by  $p \stackrel{s}{\sim} p'$ , if for each  $n$  and  $m$ , the number of inversion sequences in  $\mathbf{I}_n$  containing  $m$  occurrences of  $p$  is the same as for  $p'$ ;
- *super-strongly Wilf equivalent*, denoted by  $p \stackrel{ss}{\sim} p'$ , if the above condition holds for any set of prescribed positions for the  $m$  occurrences.

An equivalence of any one of these three types will be called a *generalized Wilf equivalence*. Henceforth, unless otherwise stated, patterns will refer to patterns in inversion sequences, and equivalence of patterns will refer to the equivalences in Definition 1.4.

## 2. SUMMARY OF RESULTS

In this paper we give recurrences to compute the numbers  $|\mathbf{I}_n(p)|$  of inversion sequences avoiding the consecutive pattern  $p$ , for  $p$  of length 3. Most of our recurrences use the refinement  $\mathbf{I}_{n,k} = \{e \in \mathbf{I}_n : e_n = k\}$  of  $\mathbf{I}_n$ , which allows us to define

$$(2.1) \quad \mathbf{I}_{n,k}(p) = \mathbf{I}_{n,k} \cap \mathbf{I}_n(p)$$

for a given pattern  $p$ . Note that  $\mathbf{I}_{n,k}(p) = \emptyset$  for  $k \geq n$  and  $\mathbf{I}_n(p) = \bigcup_{k=0}^{n-1} \mathbf{I}_{n,k}(p)$  for any pattern  $p$ .

**Example 2.1.** For the consecutive pattern  $p = \underline{001}$ , we have  $|\mathbf{I}_3(\underline{001})| = 4$ , as  $\mathbf{I}_{3,0}(\underline{001}) = \{000, 010\}$ ,  $\mathbf{I}_{3,1}(\underline{001}) = \{011\}$  and  $\mathbf{I}_{3,2}(\underline{001}) = \{012\}$ .

A summary of our results for consecutive patterns of length 3 is given in Table 1 for patterns with no repeated letters, and in Table 2 for patterns with repeated letters. In particular, we indicate which sequences match existing sequences in the On-Line Encyclopedia of Integer Sequences (OEIS) [20]. Here,  $\underline{1324}$  and  $\underline{1432}$  denote *vincular* (sometimes called *generalized*) permutation patterns, where the entries in underlined positions are required to be adjacent in an occurrence, as defined by Babson and Steingrímsson [2].

Pattern $p$	in the OEIS?	Recurrence for $ \mathbf{I}_{n,k}(p) $	$ \mathbf{I}_n(p) $ coincides with
<u>012</u>	A049774	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{l=1}^{k-1} \sum_{j=0}^{l-1} \sum_{i \geq j}  \mathbf{I}_{n-3,i}(p) $	$ S_n(\underline{321}) $
<u>021</u>	A071075	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - (n-2-k) \sum_{j=0}^{k-1}  \mathbf{I}_{n-2,j}(p) $	$ S_n(\underline{1324}) $
<u>102</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{j \geq 1} j  \mathbf{I}_{n-2,j}(p) $	
<u>120</u>	A200404	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{j > k} (n-2-j)  \mathbf{I}_{n-2,j}(p) $	$ S_n(\underline{1432}) $
<u>201</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - k \sum_{j > k}  \mathbf{I}_{n-2,j}(p) $	
<u>210</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{l=k+1}^{n-4} \sum_{j=l+1}^{n-3} \sum_{i \leq j}  \mathbf{I}_{n-3,i}(p) $	

TABLE 1. Consecutive patterns of length 3 with no repeated letters. Sequences A049774, A071075 and A200404 are known to count  $|S_n(\underline{321})|$ ,  $|S_n(\underline{1324})|$  and  $|S_n(\underline{1432})|$ , respectively, and exponential generating functions for A049774 and A071075 are known. All other results in the table are new.

Pattern $p$	in the OEIS?	Recurrence for $ \mathbf{I}_{n,k}(p) $
<u>000</u>	A052169	$ \mathbf{I}_n(p)  = \frac{(n+1)! - d_{n+1}}{n}$ , where $d_n$ is the number of derangements of $[n]$
<u>001</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{j < k}  \mathbf{I}_{n-2,j}(p) $
<u>010</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - (n-2-k)  \mathbf{I}_{n-2,k}(p) $
<u>011</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{j < k}  \mathbf{I}_{n-2,j}(p) $ if $k \neq n-1$ , and $ \mathbf{I}_{n,n-1}(p)  =  \mathbf{I}_{n-1}(p) $
<u>100</u> <sup>ss</sup> <u>110</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - \sum_{j > k}  \mathbf{I}_{n-2,j}(p) $
<u>101</u>	New	$ \mathbf{I}_{n,k}(p)  =  \mathbf{I}_{n-1}(p)  - k  \mathbf{I}_{n-2,k}(p) $

TABLE 2. Consecutive patterns of length 3 with repeated letters.

Another main result of this paper is the classification of consecutive patterns of length 3 and 4 according to the generalized Wilf equivalence classes from Definition 1.4. The equivalence classes for length 3 are given by Tables 1 and 2. Specifically, the equivalence  $\underline{100} \stackrel{ss}{\sim} \underline{110}$  is the only generalized Wilf equivalence among consecutive patterns of length 3; that is, no two such patterns are even Wilf equivalent.

The next theorem describes the corresponding classes for patterns of length 4. The patterns  $p$  are listed in increasing order of the values of  $|\mathbf{I}_n(p)|$  for  $n \leq 10$ ; that is, from least avoided to most avoided in inversion sequences of length up to 10.

**Theorem 2.2.** *A complete list of the generalized Wilf equivalences between consecutive patterns of length 4 is as follows:*

- |   |  |
|---|--|
| <p>(i) <math>\underline{0102} \stackrel{ss}{\sim} \underline{0112}</math>.</p> <p>(ii) <math>\underline{0021} \stackrel{ss}{\sim} \underline{0121}</math>.</p> <p>(iii) <math>\underline{0100} \stackrel{ss}{\sim} \underline{0110}</math>.</p> <p>(iv) <math>\underline{1002} \stackrel{ss}{\sim} \underline{1012} \stackrel{ss}{\sim} \underline{1102}</math>.</p> <p>(v) <math>\underline{1200} \stackrel{ss}{\sim} \underline{1210} \stackrel{ss}{\sim} \underline{1220}</math>.</p> <p>(vi) <math>\underline{0211} \stackrel{ss}{\sim} \underline{0221}</math>.</p> <p>(vii) <math>\underline{1000} \stackrel{ss}{\sim} \underline{1110}</math>.</p> | <p>(viii) <math>\underline{1001} \stackrel{ss}{\sim} \underline{1011} \stackrel{ss}{\sim} \underline{1101}</math>.</p> <p>(ix) <math>\underline{2013} \stackrel{ss}{\sim} \underline{2103}</math>.</p> <p>(x) <math>\underline{2100} \stackrel{ss}{\sim} \underline{2210}</math>.</p> <p>(xi) <math>\underline{2001} \stackrel{ss}{\sim} \underline{2011} \stackrel{ss}{\sim} \underline{2101} \stackrel{ss}{\sim} \underline{2201}</math>.</p> <p>(xii) <math>\underline{2012} \stackrel{ss}{\sim} \underline{2102}</math>.</p> <p>(xiii) <math>\underline{2010} \stackrel{ss}{\sim} \underline{2110} \stackrel{ss}{\sim} \underline{2120}</math>.</p> <p>(xiv) <math>\underline{3012} \stackrel{ss}{\sim} \underline{3102}</math>.</p> |
|---|--|

Again, all the Wilf equivalences among consecutive patterns of length 4 are super-strong Wilf equivalences. This leads us to speculate the following, in analogy to Nakamura's conjecture for consecutive patterns in permutations [19, Conjecture 5.6], which remains open.

**Conjecture 2.3.** *Two consecutive patterns of length  $k$  in inversion sequences are strongly Wilf equivalent if and only if they are Wilf equivalent.*

It is in fact possible that a stronger version of Conjecture 2.3 holds, namely that all three types of generalized Wilf equivalence for consecutive patterns in inversion sequences (see Definition 1.4) coincide. The results in this paper show that this is the case for consecutive patterns of length at most 4.

On the other hand, following Martinez and Savage [18], one can reframe the notion of patterns of length 3 to consider triples of binary relations among the entries in an occurrence. In an upcoming followup paper [1], we study the consecutive analogue of this framework. Perhaps surprisingly, in the setting of consecutive patterns of relations, the notions of Wilf equivalence and strong Wilf equivalence do not coincide in general.

The rest of the paper is organized as follows. In Section 3 we study consecutive patterns of length 3, proving the results summarized in Tables 1 and 2. Section 4 is devoted to the classification of consecutive patterns of length 4 into generalized Wilf equivalence classes, proving Theorem 2.2. Some of our results apply to consecutive patterns of arbitrary length, and will be given in Proposition 3.2, and Theorems 4.17 and 4.18.

### 3. CONSECUTIVE PATTERNS OF LENGTH 3

In this section we enumerate inversion sequences avoiding each given consecutive pattern of length 3. Tables 3 and 4 provide the first 8 terms of the sequence  $|\mathbf{I}_n(p)|$  for patterns of length 3 without and with repeated letters, respectively.

Pattern $p$	$ \mathbf{I}_n(p) $ for $1 \leq n \leq 8$
<u>012</u>	1, 2, 5, 17, 70, 349, 2017, 13358
<u>021</u>	1, 2, 6, 23, 107, 585, 3671, 25986
<u>102</u>	1, 2, 6, 22, 96, 492, 2902, 19350
<u>120</u>	1, 2, 6, 23, 107, 582, 3622, 25369
<u>201</u>	1, 2, 6, 24, 118, 684, 4548, 34036
<u>210</u>	1, 2, 6, 24, 118, 684, 4554, 34192

TABLE 3. First 8 terms of the sequence  $|\mathbf{I}_n(p)|$  for consecutive patterns  $p$  of length 3 with no repeated letters.

Pattern $p$	$ \mathbf{I}_n(p) $ for $1 \leq n \leq 8$
<u>000</u>	1, 2, 5, 19, 91, 531, 3641, 28673
<u>001</u>	1, 2, 4, 11, 42, 210, 1292, 9352
<u>010</u>	1, 2, 5, 17, 76, 417, 2701, 20199
<u>011</u>	1, 2, 5, 17, 75, 407, 2621, 19524
<u>100</u> <sup>ss</sup> <u>110</u>	1, 2, 6, 23, 109, 618, 4098, 31173
<u>101</u>	1, 2, 6, 23, 109, 619, 4113, 31352

TABLE 4. First 8 terms of the sequence  $|\mathbf{I}_n(p)|$  for consecutive patterns  $p$  of length 3 with repeated letters.

3.1. **The pattern 000.** The following result provides a method to compute  $|\mathbf{I}_n(\underline{000})|$  recursively.

**Proposition 3.1.** *The sequence  $|\mathbf{I}_n(\underline{000})|$  satisfies the recurrence*

$$|\mathbf{I}_n(\underline{000})| = (n-1)|\mathbf{I}_{n-1}(\underline{000})| + (n-2)|\mathbf{I}_{n-2}(\underline{000})|,$$

for  $n \geq 3$ , with initial terms  $|\mathbf{I}_1(\underline{000})| = 1$  and  $|\mathbf{I}_2(\underline{000})| = 2$ .

*Proof.* It is clear that  $|\mathbf{I}_n(\underline{000})| = |\mathbf{I}_n|$  for  $n < 3$ , so  $|\mathbf{I}_1(\underline{000})| = 1$  and  $|\mathbf{I}_2(\underline{000})| = 2$ . For  $n \geq 3$ , let us count the inversion sequences  $e \in \mathbf{I}_n(\underline{000})$  by considering two cases.

If  $e_{n-1} \neq e_n$ , then there are  $|\mathbf{I}_{n-1}(\underline{000})|$  many ways to choose the first  $n-1$  entries of  $e$  and  $n-1$  possibilities for  $e_n$ . Namely,  $e_n$  may be any element of  $\{0, 1, \dots, n-1\} \setminus \{e_{n-1}\}$ .

If  $e_{n-1} = e_n$ , then it must be that  $e_{n-2} \neq e_{n-1}$ . So there are  $|\mathbf{I}_{n-2}(\underline{000})|$  many ways to choose the first  $n-2$  entries of  $e$  and  $n-2$  possibilities for  $e_{n-1} = e_n$ , which can be any element in  $\{0, 1, \dots, n-2\} \setminus \{e_{n-2}\}$ . Since every  $e \in \mathbf{I}_n(\underline{000})$  falls into one of these two cases, the stated recurrence follows.  $\square$

The following result generalizes Proposition 3.1 to patterns of arbitrary length, providing a recurrence for the number of inversion sequences avoiding the consecutive pattern consisting of  $r$  zeros. We denote this pattern by  $\underline{0}^r$ .

**Proposition 3.2.** *Let  $n \geq r \geq 2$ . The sequence  $|\mathbf{I}_n(\underline{0}^r)|$  satisfies the recurrence*

$$|\mathbf{I}_n(\underline{0}^r)| = \sum_{j=1}^{r-1} (n-r) |\mathbf{I}_{n-j}(\underline{0}^r)|,$$

with initial conditions  $|\mathbf{I}_n(\underline{0}^r)| = n!$  for  $1 \leq n < r$ .

*Proof.* Let  $A_j$  denote the subset of  $\mathbf{I}_n(\underline{0}^r)$  consisting of sequences  $e$  for which  $j$  is the largest integer such that  $e_{n-j+1} = e_{n-j+2} = \dots = e_n$ . Given that  $\mathbf{I}_n(\underline{0}^r) = \bigsqcup_{j=1}^{r-1} A_j$  (which denotes a disjoint union), we see that  $|\mathbf{I}_n(\underline{0}^r)| = \sum_{j=1}^{r-1} |A_j|$ .

Fix  $1 \leq j \leq r-1$ . Let us construct a sequence  $e \in A_j$ . There are  $|\mathbf{I}_{n-j}(\underline{0}^r)|$  possibilities for the first  $n-j$  entries of  $e$ . Once these entries are determined, we have  $n-j$  choices for the remaining (repeated) entry  $e_{n-j+1} = e_{n-j+2} = \dots = e_n$ , which can be any element in  $\{0, 1, \dots, n-j\} \setminus \{e_{n-j}\}$ . Since every sequence in  $A_j$  may be constructed this way and every sequence constructed in this manner is in  $A_j$ , we deduce that  $|A_j| = (n-j) |\mathbf{I}_{n-j}(\underline{0}^r)|$ , which concludes the proof.  $\square$

A derangement is a permutation without fixed points. The next result relates inversion sequences avoiding the pattern  $\underline{000}$  to derangements.

**Corollary 3.3.** *Let  $d_n$  be the number of derangements of length  $n$ . For  $n \geq 1$ ,*

$$|\mathbf{I}_n(\underline{000})| = \frac{(n+1)! - d_{n+1}}{n}.$$

*Proof.* We proceed by induction on  $n$ . The cases  $n = 1, 2$  are straightforward to verify, so let  $n \geq 3$ . Suppose that  $|\mathbf{I}_j(\underline{000})| = ((j+1)! - d_{j+1})/j$  for  $1 \leq j < n$ . By Proposition 3.1,

$$|\mathbf{I}_n(\underline{000})| = (n-1) |\mathbf{I}_{n-1}(\underline{000})| + (n-2) |\mathbf{I}_{n-2}(\underline{000})|,$$

so it follows from the inductive hypothesis that

$$\begin{aligned} |\mathbf{I}_n(\underline{000})| &= (n-1) \frac{n! - d_n}{n-1} + (n-2) \frac{(n-1)! - d_{n-1}}{n-2} \\ &= n! - d_n + (n-1)! - d_{n-1} \\ &= \frac{(n+1)! - n(d_n + d_{n-1})}{n}. \end{aligned}$$

It is well-known that the number of derangements satisfies the recurrence  $d_n = (n-1)(d_{n-1} + d_{n-2})$ . This implies that  $|\mathbf{I}_n(\underline{000})| = ((n+1)! - d_{n+1})/n$ .  $\square$

The sequence  $((n+1)! - d_{n+1})/n$  is listed as A052169 in [20]. It would be interesting to find a direct combinatorial proof of Corollary 3.3, such as a bijection between  $\mathbf{I}_n(\underline{000}) \times [n]$  and the set of permutations in  $S_{n+1}$  with at least one fixed point.

**3.2. The patterns 100 and 110.** Next we prove that the patterns 100 and 110 are Wilf equivalent. We begin by finding a recurrence for  $|\mathbf{I}_{n,k}(\underline{110})|$ , as defined in Equation (2.1). The idea is to exploit the fact that if  $e = e_1 e_2 \dots e_n \in \mathbf{I}_n(\underline{110})$ , then  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{110})$ , as we implicitly did in the proof of Proposition 3.1. Throughout this section, we use the convention that

$$(3.1) \quad |\mathbf{I}_0(p)| = 1$$

for any pattern  $p$ . This value also serves as an initial term in most of the recurrences below, even when not explicitly stated. We also use the convention that  $|\mathbf{I}_{n,j}(p)| = 0$  if  $n < 0$ .

**Proposition 3.4.** *Let  $n \geq 1$  and  $0 \leq k < n$ . The sequence  $|\mathbf{I}_{n,k}(\underline{110})|$  satisfies the recurrence*

$$|\mathbf{I}_{n,k}(\underline{110})| = |\mathbf{I}_{n-1}(\underline{110})| - \sum_{j=k+1}^{n-3} |\mathbf{I}_{n-2,j}(\underline{110})|.$$

*Proof.* It is straightforward to verify that the recurrence holds for  $n = 1, 2$  (note that the sum on the right-hand side is empty in these cases). Assume  $n \geq 3$ . For  $j \leq n-3$ , let

$$A_{n-1,j} = \{e \in \mathbf{I}_{n-1,j}(\underline{110}) : e_{n-2} = j\}.$$

Note that  $e \in \mathbf{I}_{n,k}(\underline{110})$  if and only if  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{110}) \setminus \bigsqcup_{j=k+1}^{n-3} A_{n-1,j}$  and  $e_n = k$ . Hence,

$$|\mathbf{I}_{n,k}(\underline{110})| = |\mathbf{I}_{n-1}(\underline{110})| - \sum_{j=k+1}^{n-3} |A_{n-1,j}|.$$

It remains to determine  $|A_{n-1,j}|$  for  $k+1 \leq j \leq n-3$ . Since  $e \in A_{n-1,j}$  if and only if  $e_1 e_2 \dots e_{n-2} \in \mathbf{I}_{n-2,j}(\underline{110})$  and  $e_{n-1} = j$ , we deduce that  $|A_{n-1,j}| = |\mathbf{I}_{n-2,j}(\underline{110})|$ .  $\square$

It is clear by definition that  $|\mathbf{I}_n(p)| = \sum_{k=0}^{n-1} |\mathbf{I}_{n,k}(p)|$  for any pattern  $p$ , so Proposition 3.4 allows us to compute  $|\mathbf{I}_n(\underline{110})|$  recursively. We now show that the number of inversion sequences avoiding the pattern 100 satisfies the same recurrence.

**Proposition 3.5.** *Let  $n \geq 1$  and  $0 \leq k < n$ . The sequence  $|\mathbf{I}_{n,k}(\underline{100})|$  satisfies the recurrence*

$$|\mathbf{I}_{n,k}(\underline{100})| = |\mathbf{I}_{n-1}(\underline{100})| - \sum_{j=k+1}^{n-3} |\mathbf{I}_{n-2,j}(\underline{100})|.$$

*Proof.* The result clearly holds for  $n = 1, 2$ , so we assume  $n \geq 3$ . Let

$$A_{n-1,k} = \{e \in \mathbf{I}_{n-1,k}(\underline{100}) : e_{n-2} > e_{n-1}\}.$$

If  $e \in \mathbf{I}_{n,k}(\underline{100})$ , then  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{100})$ . Additionally, since  $e_{n-2} e_{n-1} e_n$  is not an occurrence of  $\underline{100}$ , it is not possible to have  $e_{n-2} > e_{n-1} = k$ . Thus,  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{100}) \setminus A_{n-1,k}$ .

Conversely, given  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{100}) \setminus A_{n-1,k}$ , setting  $e_n = k$  yields an inversion sequence  $e = e_1 e_2 \dots e_n \in \mathbf{I}_{n,k}(\underline{100})$ . From this bijective correspondence between  $\mathbf{I}_{n,k}(\underline{100})$  and  $\mathbf{I}_{n-1}(\underline{100}) \setminus A_{n-1,k}$ , it follows that

$$(3.2) \quad |\mathbf{I}_{n,k}(\underline{100})| = |\mathbf{I}_{n-1}(\underline{100})| - |A_{n-1,k}|.$$

Let us determine  $|A_{n-1,k}|$ . First suppose  $0 \leq k \leq n-4$ . Note that  $e_1 e_2 \dots e_{n-1} \in A_{n-1,k}$  if and only if  $e_1 e_2 \dots e_{n-2} \in \mathbf{I}_{n-2,j}(\underline{100})$  for some  $k+1 \leq j \leq n-3$  and  $e_{n-1} = k$ . Hence,

$$(3.3) \quad |A_{n-1,k}| = \sum_{j=k+1}^{n-3} |\mathbf{I}_{n-2,j}(\underline{100})|.$$

In fact, this formula also holds for  $k = n-3, n-2$ , since in these cases the sum on the right-hand side is empty, and the left-hand side is zero by definition. The result now follows from Equations (3.2) and (3.3).  $\square$

Using Equation (3.1), the next corollary is an immediate consequence of Propositions 3.4 and 3.5.

**Corollary 3.6.** *The patterns  $\underline{110}$  and  $\underline{100}$  are Wilf equivalent. Furthermore, if  $n \geq 1$  and  $0 \leq k < n$ , then  $|\mathbf{I}_{n,k}(\underline{110})| = |\mathbf{I}_{n,k}(\underline{100})|$ .*

Next we show that, in fact, the patterns  $\underline{110}$  and  $\underline{100}$  are super-strongly Wilf equivalent. We begin by introducing some notation. Given a consecutive pattern  $p = \underline{p_1 p_2 \dots p_r}$  and  $e \in \mathbf{I}_n$ , we define

$$\text{Em}(p, e) = \{i : e_i e_{i+1} \dots e_{i+r-1} \text{ is an occurrence of } p\}.$$

**Example 3.7.** Let  $p = 100$  and  $e = 0021100300 \in \mathbf{I}_{10}$ . Then  $\text{Em}(p, e) = \{3, 5, 8\}$ .

With this notation, two consecutive patterns  $p$  and  $p'$  are super-strongly Wilf equivalent if and only if, for all  $n$  and all  $T \subseteq [n]$ ,

$$|\{e \in \mathbf{I}_n : \text{Em}(p, e) = T\}| = |\{e \in \mathbf{I}_n : \text{Em}(p', e) = T\}|.$$

**Definition 3.8.** We say that a consecutive pattern  $p$  is *non-overlapping* if it is impossible for two occurrences of  $p$  in an inversion sequence to overlap in more than one entry. Equivalently,  $p = \underline{p_1 \dots p_r}$  is non-overlapping if, for all  $1 < i < k$ , the reductions of  $p_1 \dots p_i$  and  $p_{r-i+1} \dots p_r$  do not coincide.



**Example 3.9.** The pattern  $\underline{110}$  is non-overlapping, but this is not the case for the pattern  $\underline{1100}$ . Indeed,  $e = 0000221100 \in \mathbf{I}_{10}$  is an example of an inversion sequence with two occurrences of  $\underline{1100}$  which overlap in two entries, namely  $e_5e_6e_7e_8$  and  $e_7e_8e_9e_{10}$ .

**Lemma 3.10.** *Let  $n$  be a positive integer and  $S \subseteq [n]$ , then*

$$|\{e \in \mathbf{I}_n : \text{Em}(\underline{110}, e) \supseteq S\}| = |\{e \in \mathbf{I}_n : \text{Em}(\underline{100}, e) \supseteq S\}|.$$

*Proof.* Consider the map

$$\Phi_S : \{e \in \mathbf{I}_n : \text{Em}(\underline{110}, e) \supseteq S\} \rightarrow \{e' \in \mathbf{I}_n : \text{Em}(\underline{100}, e') \supseteq S\}$$

that replaces the occurrences of  $\underline{110}$  in  $e$  in positions  $S$  with occurrences of  $\underline{100}$ . More specifically, for  $e$  in the domain,  $\Phi_S(e) = e'$  is defined as

$$e'_j = \begin{cases} e_{j+1}, & \text{if } j-1 \in S, \\ e_j, & \text{otherwise,} \end{cases}$$

for  $1 \leq j \leq n$ . Note that  $e' \in \mathbf{I}_n$  because, if  $j-1 \in S$ , then  $e_{j-1}e_j e_{j+1}$  is an occurrence of  $\underline{110}$ , and so  $e'_j = e_{j+1} < e_j < j$ .

To show that  $\Phi_S$  is a bijection, let us describe its inverse. For  $e' \in \mathbf{I}_n$  with  $\text{Em}(\underline{100}, e') \supseteq S$ , define  $\Psi_S(e') = e$  by

$$e_j = \begin{cases} e'_{j-1}, & \text{if } j-1 \in S, \\ e'_j, & \text{otherwise,} \end{cases}$$

for  $1 \leq j \leq n$ . Note that  $e \in \mathbf{I}_n$  because, if  $j-1 \in S$ , then  $e'_{j-1}e'_j e'_{j+1}$  is an occurrence of  $\underline{100}$ , and so  $e_j = e'_{j-1} < j-1 < j$ . Since  $\underline{110}$  and  $\underline{100}$  are non-overlapping, it is clear by construction that  $\Phi_S$  and  $\Psi_S$  are inverses of each other.  $\square$

The following lemma provides the final argument required to show that  $\underline{110} \stackrel{ss}{\sim} \underline{100}$ .

**Lemma 3.11.** *Let  $p$  and  $p'$  be two consecutive patterns such that*

$$|\{e \in \mathbf{I}_n : \text{Em}(p, e) \supseteq S\}| = |\{e \in \mathbf{I}_n : \text{Em}(p', e) \supseteq S\}|$$

*for all positive integers  $n$  and all  $S \subseteq [n]$ . Then  $p \stackrel{ss}{\sim} p'$ .*

*Proof.* Given  $S \subseteq [n]$ , let

$$f_=(S) = |\{e \in \mathbf{I}_n : \text{Em}(p, e) = S\}| \quad \text{and} \quad f_{\geq}(S) = |\{e \in \mathbf{I}_n : \text{Em}(p, e) \supseteq S\}|.$$

For any  $T \subseteq [n]$ , it is clear that

$$f_{\geq}(T) = \sum_{S \supseteq T} f_=(S),$$

so the Principle of Inclusion-Exclusion (see [21, Theorem 2.1.1]) implies that

$$(3.4) \quad f_=(T) = \sum_{S \supseteq T} (-1)^{|S \setminus T|} f_{\geq}(S).$$

Similarly, letting

$$f'_{\leq}(S) = |\{e \in \mathbf{I}_n : \text{Em}(p', e) = S\}| \quad \text{and} \quad f'_{\geq}(S) = |\{e \in \mathbf{I}_n : \text{Em}(p', e) \supseteq S\}|,$$

we have

$$(3.5) \quad f'_{\leq}(T) = \sum_{S \supseteq T} (-1)^{|S \setminus T|} f'_{\geq}(S).$$

By assumption,  $f_{\geq}(S) = f'_{\geq}(S)$  for all  $S \subseteq [n]$ , so Equations (3.4) and (3.5) imply that  $f_{\leq}(T) = f'_{\leq}(T)$  for all  $T \subseteq [n]$ , that is,

$$|\{e \in \mathbf{I}_n : \text{Em}(p, e) = T\}| = |\{e \in \mathbf{I}_n : \text{Em}(p', e) = T\}|.$$

This proves that  $p \stackrel{ss}{\sim} p'$ . □

The next proposition is an immediate consequence of Lemmas 3.10 and 3.11. A generalization to patterns of arbitrary length will be given in Theorem 4.12.

**Proposition 3.12.** *The patterns  $\underline{110}$  and  $\underline{100}$  are super-strongly Wilf equivalent.*

**3.3. Other patterns of length 3.** Applying the same technique used in the proof of Propositions 3.4 and 3.5, we have obtained recurrences for  $|\mathbf{I}_{n,k}(p)|$  for all consecutive patterns  $p$  of length 3. In each case, we consider an inversion sequence  $e = e_1 e_2 \dots e_n \in \mathbf{I}_{n,k}(p)$  and analyze the possibilities for  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(p)$ . We then determine that  $e_1 e_2 \dots e_{n-1}$  may be any sequence in  $\mathbf{I}_{n-1}(p)$ , with the exception of certain  $e_1 e_2 \dots e_{n-2} \in \mathbf{I}_{n-2,j}(p)$ , which are not allowed for some values of  $j$ . For this reason we obtain recurrences for  $|\mathbf{I}_{n,k}(p)|$  in terms of  $|\mathbf{I}_{n-1}(p)|$  and  $|\mathbf{I}_{n-2,j}(p)|$ , with  $j$  in a certain range.

We now list recurrences obtained for consecutive patterns of length 3 with no repeated letters.

**Proposition 3.13.** *Let  $n \geq 1$  and  $0 \leq k < n$ . Then the following recurrences hold:*

(a) For  $k < n - 1$ ,

$$|\mathbf{I}_{n,k}(\underline{021})| = |\mathbf{I}_{n-1}(\underline{021})| - (n - 2 - k) \sum_{j=0}^{k-1} |\mathbf{I}_{n-2,j}(\underline{021})|,$$

$$\text{and } |\mathbf{I}_{n,n-1}(\underline{021})| = |\mathbf{I}_{n-1}(\underline{021})|.$$

$$(b) |\mathbf{I}_{n,k}(\underline{102})| = |\mathbf{I}_{n-1}(\underline{102})| - \sum_{j \geq 1} j |\mathbf{I}_{n-2,j}(\underline{102})|.$$

$$(c) |\mathbf{I}_{n,k}(\underline{120})| = |\mathbf{I}_{n-1}(\underline{120})| - \sum_{j > k} (n - 2 - j) |\mathbf{I}_{n-2,j}(\underline{120})|.$$

$$(d) |\mathbf{I}_{n,k}(\underline{201})| = |\mathbf{I}_{n-1}(\underline{201})| - k \sum_{j > k} |\mathbf{I}_{n-2,j}(\underline{201})|.$$

Next, we describe the recurrence obtained for the pattern  $\underline{012}$ , which contains terms of the form  $|\mathbf{I}_{n-3,j}(\underline{012})|$ . In this case, when we consider an inversion sequence  $e = e_1 e_2 \dots e_n \in \mathbf{I}_{n,k}(\underline{012})$  and analyze the possibilities for  $e_1 e_2 \dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{012})$ , we determine that  $e_1 e_2 \dots e_{n-1}$  may be

any sequence in  $\mathbf{I}_{n-1}(\underline{012})$ , with the exception of certain  $e_1e_2\dots e_{n-3} \in \mathbf{I}_{n-3,j}(\underline{012})$ , which are not allowed for some values of  $j$ . Indeed, note that  $e = e_1e_2\dots e_n \in \mathbf{I}_{n,k}(\underline{012})$  if and only if:

$$e_1e_2\dots e_{n-1} \in \mathbf{I}_{n-1}(\underline{012}), \quad e_n = k \quad \text{and} \quad e_{n-2}, e_{n-1} \text{ do not satisfy } e_{n-2} < e_{n-1} < k.$$

Now  $e_1e_2\dots e_{n-1}$  belongs to  $\mathbf{I}_{n-1}(\underline{012})$  and satisfies  $e_{n-2} < e_{n-1} < k$  if and only if  $e_1e_2\dots e_{n-3} \in \mathbf{I}_{n-3,i}(\underline{012})$  for some  $i \geq e_{n-2}$ , and  $e_{n-2} < e_{n-1} < k$ . Hence, making the substitutions  $j = e_{n-2}$  and  $l = e_{n-1}$ , we deduce the following result.

**Proposition 3.14.** *Let  $n \geq 4$  and  $0 \leq k < n$ . Then the sequence  $|\mathbf{I}_{n,k}(\underline{012})|$  satisfies the recurrence*

$$|\mathbf{I}_{n,k}(\underline{012})| = |\mathbf{I}_{n-1}(\underline{012})| - \sum_{l=1}^{k-1} \sum_{j=0}^{l-1} \sum_{i \geq j} |\mathbf{I}_{n-3,i}(\underline{012})|,$$

with initial conditions  $|\mathbf{I}_{1,0}(\underline{012})| = |\mathbf{I}_{2,0}(\underline{012})| = |\mathbf{I}_{2,1}(\underline{012})| = |\mathbf{I}_{3,2}(\underline{012})| = 1$  and  $|\mathbf{I}_{3,0}(\underline{012})| = |\mathbf{I}_{3,1}(\underline{012})| = 2$ .

With some algebraic manipulations, the triple sum in Proposition 3.14 may be simplified to obtain the following recurrence, which can be computed more efficiently.

**Corollary 3.15.** *Let  $n \geq 4$  and  $0 \leq k < n$ . Then*

$$|\mathbf{I}_{n,k}(\underline{012})| = |\mathbf{I}_{n-1}(\underline{012})| - \sum_{i=0}^{k-3} (i+1) \left( k-1 - \frac{i}{2} \right) |\mathbf{I}_{n-3,i}(\underline{012})| - \frac{k(k-1)}{2} \sum_{i=k-2}^{n-4} |\mathbf{I}_{n-3,i}(\underline{012})|,$$

with the same initial conditions as in Proposition 3.14.

The following is the recurrence obtained for the pattern  $\underline{210}$  using a similar argument.

**Proposition 3.16.** *Let  $n \geq 1$  and  $0 \leq k < n$ . Then the sequence  $|\mathbf{I}_{n,k}(\underline{210})|$  satisfies the recurrence*

$$|\mathbf{I}_{n,k}(\underline{210})| = |\mathbf{I}_{n-1}(\underline{210})| - \sum_{l=k+1}^{n-4} \sum_{j=l+1}^{n-3} \sum_{i \leq j} |\mathbf{I}_{n-3,i}(\underline{210})|.$$

As in Corollary 3.15, we may simplify the triple sum in Proposition 3.16 to obtain the following recurrence.

**Corollary 3.17.** *Let  $n \geq 1$  and  $0 \leq k < n$ . If  $1 \leq n \leq 4$  or  $0 \leq n-5 < k$ , then  $|\mathbf{I}_{n,k}(\underline{210})| = |\mathbf{I}_{n-1}(\underline{210})|$ . Otherwise (i.e.,  $0 \leq k \leq n-5$ ),*

$$|\mathbf{I}_{n,k}(\underline{210})| = |\mathbf{I}_{n-1}(\underline{210})| - \frac{1}{2} \left[ (n-k-4)(n-k-3) \sum_{i=0}^{k+2} |\mathbf{I}_{n-3,i}(\underline{210})| + \sum_{i=k+3}^{n-4} (n-i-2)(n+i-2k-5) |\mathbf{I}_{n-3,i}(\underline{210})| \right].$$

Similarly, it is possible to write recurrences for all the remaining consecutive patterns of length 3 with repeated letters. We list them next.

**Proposition 3.18.** *Let  $n \geq 1$  and  $0 \leq k < n$ . Then the following recurrences hold:*

- (a)  $|\mathbf{I}_{n,k}(\underline{001})| = |\mathbf{I}_{n-1}(\underline{001})| - \sum_{j < k} |\mathbf{I}_{n-2,j}(\underline{001})|.$
- (b)  $|\mathbf{I}_{n,k}(\underline{010})| = |\mathbf{I}_{n-1}(\underline{010})| - (n-2-k) |\mathbf{I}_{n-2,k}(\underline{010})|.$
- (c) For  $k < n-1$ ,

$$|\mathbf{I}_{n,k}(\underline{011})| = |\mathbf{I}_{n-1}(\underline{011})| - \sum_{j < k} |\mathbf{I}_{n-2,j}(\underline{011})|,$$

and  $|\mathbf{I}_{n,n-1}(\underline{011})| = |\mathbf{I}_{n-1}(\underline{011})|.$

- (d)  $|\mathbf{I}_{n,k}(\underline{101})| = |\mathbf{I}_{n-1}(\underline{101})| - k |\mathbf{I}_{n-2,k}(\underline{101})|.$

**3.4. From patterns in inversion sequences to patterns in permutations.** In this subsection we discuss some correspondences between consecutive patterns of length 3 in inversion sequences and permutation patterns. We exploit the bijection  $\Theta : S_n \rightarrow \mathbf{I}_n$  defined by (1.1). For  $\pi \in S_n$  and  $1 \leq i \leq n$ , let

$$E_i(\pi) = \{j \in [n] : j < i \text{ and } \pi_j > \pi_i\}.$$

Note that the entries of  $e = \Theta(\pi)$  are given by  $e_i = |E_i|$ .

**Proposition 3.19.** *Let  $\pi \in S_n$ , and let  $e = \Theta(\pi)$  be its corresponding inversion sequence. Then  $\pi$  avoids  $\underline{321}$  if and only if  $e$  avoids  $\underline{012}$ . Hence,  $|\mathbf{I}_n(\underline{012})| = |S_n(\underline{321})|$  for all  $n$ .*

*Proof.* Suppose  $\pi_i > \pi_{i+1}$ . Then every  $j < i$  such that  $\pi_j > \pi_i$  satisfies  $\pi_j > \pi_{i+1}$ . It follows that  $E_i(\pi) \subseteq E_{i+1}(\pi)$ . Since  $i+1 \in E_{i+1}(\pi) \setminus E_i(\pi)$ , this inclusion is strict, and so  $e_i < e_{i+1}$ .

Conversely, suppose  $e_i < e_{i+1}$ . If  $\pi_i < \pi_{i+1}$ , then  $i \notin E_{i+1}(\pi)$ , so there must exist some  $j \in E_{i+1}(\pi) \setminus E_i(\pi)$  with  $j < i$ . But then  $\pi_{i+1} < \pi_j < \pi_i$ , which contradicts  $\pi_i < \pi_{i+1}$ . It follows that  $\pi_i > \pi_{i+1}$ .

We deduce that

$$(3.6) \quad \pi_i > \pi_{i+1} \quad \text{if and only if} \quad e_i < e_{i+1}.$$

Thus,  $\pi$  contains  $\underline{321}$  if and only if  $e$  contains  $\underline{012}$ . Equivalently,  $\pi$  avoids  $\underline{321}$  if and only if  $e$  avoids  $\underline{012}$ . Consequently, the map  $\Theta$  induces a bijection between  $S_n(\underline{321})$  and  $\mathbf{I}_n(\underline{012})$ .  $\square$

The exponential generating function for permutations avoiding  $\underline{321}$  is known, see [7, 12] and sequence A049774 in [20]. It follows from Proposition 3.19 and [12, Theorem 4.1] that

$$\sum_{n \geq 0} |\mathbf{I}_n(\underline{012})| \frac{z^n}{n!} = \frac{\sqrt{3}}{2} \frac{e^{z/2}}{\cos(\sqrt{3}z/2 + \pi/6)}.$$

The recurrence in Proposition 3.14 appears to be new.

By (3.6),  $\pi$  contains  $\underline{(r+1)r \dots 1}$  if and only if  $e$  contains  $\underline{01 \dots r}$ . Thus, Proposition 3.19 is generalized as follows.

**Proposition 3.20.** *Let  $\pi \in S_n$ , and let  $e = \Theta(\pi)$  be its corresponding inversion sequence. Then  $\pi$  avoids  $\underline{(r+1)r\dots 1}$  if and only if  $e$  avoids  $\underline{01\dots r}$ . Hence,  $|\mathbf{I}_n(\underline{01\dots r})| = |S_n(\underline{(r+1)r\dots 1})|$  for all  $n$ .*

The next result relates  $\underline{120}$  and  $\underline{021}$  to vincular patterns in permutations.

**Proposition 3.21.** *Let  $\pi \in S_n$ , and let  $e = \Theta(\pi)$  be its corresponding inversion sequence. Then:*

- (a)  $\pi$  avoids  $\underline{3214}$  if and only if  $e$  avoids  $\underline{120}$ ;
- (b)  $\pi$  avoids  $\underline{2413}$  if and only if  $e$  avoids  $\underline{021}$ .

In particular,  $|\mathbf{I}_n(\underline{120})| = |S_n(\underline{3214})|$  and  $|\mathbf{I}_n(\underline{021})| = |S_n(\underline{2413})|$  for all  $n$ .

*Proof.* The proof of both results is quite similar, so we only prove (a). It suffices to show that  $\pi$  contains  $\underline{3214}$  if and only if  $e$  contains  $\underline{120}$ .

Suppose that  $\pi$  contains an occurrence  $\pi_j\pi_i\pi_{i+1}\pi_{i+2}$  of  $\underline{3214}$ , where  $j < i$  and  $\pi_{i+1} < \pi_i < \pi_j < \pi_{i+2}$ . As in the proof of Proposition 3.19, we have  $e_i < e_{i+1}$ . If  $l \in E_{i+2}(\pi)$ , then  $l < i+2$  and  $\pi_l > \pi_{i+2}$ . Since  $\pi_i, \pi_{i+1} < \pi_{i+2}$ , it must be that  $l < i$ , and so  $l \in E_i(\pi)$ . We deduce that  $E_{i+2}(\pi) \subseteq E_i(\pi)$ . In fact, since  $j \in E_i(\pi) \setminus E_{i+2}(\pi)$ , the inclusion is strict, so  $e_{i+2} < e_i$ . It follows that  $e_i e_{i+1} e_{i+2}$  is an occurrence of  $\underline{120}$ .

Conversely, suppose that  $e$  contains an occurrence  $e_i e_{i+1} e_{i+2}$  of  $\underline{120}$ , where  $e_{i+2} < e_i < e_{i+1}$ . As in the proof of Proposition 3.19, we have  $\pi_i > \pi_{i+1}$ . If  $\pi_i > \pi_{i+2}$ , then  $E_i(\pi) \subseteq E_{i+2}(\pi)$ , contradicting that  $e_i > e_{i+2}$ . It follows that  $\pi_{i+2} > \pi_i$ , and so  $\pi_i \pi_{i+1} \pi_{i+2}$  is an occurrence of  $\underline{213}$ . Finally, since  $e_i > e_{i+2}$ , there exists  $j \in E_i(\pi) \setminus E_{i+2}(\pi)$ . In particular,  $j < i$  and  $\pi_i < \pi_j < \pi_{i+2}$ , and thus  $\pi_j \pi_i \pi_{i+1} \pi_{i+2}$  is an occurrence of  $\underline{3214}$ .  $\square$

Permutations avoiding the vincular pattern  $\underline{1432}$  were studied by Baxter and Pudwell [3]. The sequence  $|S_n(\underline{1432})|$  appears as A200404 in [20], but no enumerative results seem to be known. The next corollary shows that  $|\mathbf{I}_n(\underline{120})| = |S_n(\underline{1432})|$ , and so we can use the recurrence in Proposition 3.13(c) to compute these numbers. Given a permutation  $\pi = \pi_1 \pi_2 \dots \pi_n \in S_n$ , its reverse-complement is the permutation  $\pi^{RC}$ , where  $(\pi^{RC})_i = n+1 - \pi_{n+1-i}$  for  $1 \leq i \leq n$ .

**Corollary 3.22.** *There is a bijection between  $\mathbf{I}_n(\underline{120})$  and  $S_n(\underline{1432})$ .*

*Proof.* The map  $\pi \rightarrow \pi^{RC}$  induces a bijection between  $S_n(\underline{3214})$  and  $S_n(\underline{1432})$ . Indeed, for  $1 \leq j < i \leq n$ , we have  $\pi_{i+1} < \pi_i < \pi_j < \pi_{i+2}$  if and only if  $n+1 - \pi_{i+2} < n+1 - \pi_j < n+1 - \pi_i < n+1 - \pi_{i+1}$ , which is equivalent to  $\pi_{n-i-1}^{RC} < \pi_{n-j+1}^{RC} < \pi_{n-i+1}^{RC} < \pi_{n-i}^{RC}$ . Thus,  $\pi$  contains  $\underline{3214}$  if and only if  $\pi^{RC}$  contains  $\underline{1432}$ . The result now follows from Proposition 3.21(a).  $\square$

Next we show that the sequence counting  $\underline{021}$ -avoiding inversion sequences coincides with that counting  $\underline{1324}$ -avoiding permutations, which is given by A071075 in [20].

**Corollary 3.23.** *For all  $n$ ,  $|\mathbf{I}_n(\underline{021})| = |S_n(\underline{1324})|$ .*

*Proof.* As in proof of Corollary 3.22, the map  $\pi \rightarrow \pi^{RC}$  induces a bijection between  $S_n(\underline{2413})$  and  $S_n(\underline{2413})$ . Similarly, by applying the reversal operation  $\pi_1\pi_2\dots\pi_n \rightarrow \pi_n\dots\pi_2\pi_1$ , we see that  $|S_n(\underline{132})| = |S_n(\underline{231})|$ . It follows from [11, Proposition 3.1] or [15, Corollary 15] that  $|S_n(\underline{1324})| = |S_n(\underline{2314})|$ . Finally, a result of Baxter and Shattuck [4] states that  $|S_n(\underline{2413})| = |S_n(\underline{2314})|$ . Combining these with Proposition 3.21(b), we obtain

$$|\mathbf{I}_n(\underline{021})| = |S_n(\underline{2413})| = |S_n(\underline{2413})| = |S_n(\underline{2314})| = |S_n(\underline{1324})|. \quad \square$$

The exponential generating function for permutations avoiding  $\underline{1324}$  can be deduced by combining [11, Proposition 3.1] and [12, Theorem 4.1]. It follows from Corollary 3.23 that

$$\sum_{n \geq 0} |\mathbf{I}_n(\underline{021})| \frac{z^n}{n!} = \exp \left( \int_0^z \frac{dt}{1 - \int_0^t e^{-u^2/2} du} \right).$$

A functional equation satisfied by the ordinary generating function for  $\underline{1324}$ -avoiding permutations is given by Baxter and Shattuck [4]. The recurrence in Proposition 3.13(a) appears to be new.

#### 4. CONSECUTIVE PATTERNS OF LENGTH 4

In this section we study consecutive patterns of length 4 in inversion sequences. Specifically, we determine the Wilf equivalence classes for consecutive patterns of length 4. This classification is summarized in Theorem 2.2. There are a total of 75 consecutive patterns of length 4, which fall into 55 Wilf equivalence classes. This is also the number of both strongly and super-strongly Wilf equivalence classes because, as mentioned before, all the equivalences among consecutive patterns of length 4 are super-strong equivalences. Of these 55 classes, 1 contains four patterns, 4 contain three patterns, 9 contain two patterns, and 41 contain one pattern.

The only consecutive pattern  $p$  of length 4 for which the sequence  $|\mathbf{I}_n(p)|$  appears in the OEIS [20] is  $p = \underline{0123}$ . We have that  $|\mathbf{I}_n(\underline{0123})| = |S_n(\underline{1234})|$ , which is sequence A117158 in [20]. By Proposition 3.20, the map  $\Theta$  gives a bijection between  $\mathbf{I}_n(\underline{0123})$  and  $S_n(\underline{4321})$ . The exponential generating function for this sequence is known [7, 12]. It follows from [12, Theorem 4.3] that

$$\sum_{n \geq 0} |\mathbf{I}_n(\underline{0123})| \frac{z^n}{n!} = \frac{2}{\cos z - \sin z + e^{-z}}.$$

**4.1. Equivalences proved via bijections.** In this subsection we use bijections to prove equivalences (ii)–(vi), (viii), (ix), (xi), (xii) and (xiv) in Theorem 2.2. All the equivalences established in this subsection are in fact stronger than super-strong Wilf equivalence: we show not only that  $p \stackrel{ss}{\sim} p'$ , but rather that

$$(4.1) \quad \left| \{e \in \mathbf{I}_n : \text{Em}(p, e) = S, \text{Em}(p', e) = T\} \right| = \left| \{e \in \mathbf{I}_n : \text{Em}(p, e) = T, \text{Em}(p', e) = S\} \right|$$

for all  $n$  and all  $S, T \subseteq [n]$ . In other words, the joint distribution of the positions of the occurrences of  $p$  and  $p'$  is symmetric.

This is proved by exhibiting a bijection from  $\mathbf{I}_n$  to itself that changes all occurrences of  $p$  into occurrences of  $p'$ , and vice versa. The following notion will help us construct such a bijection.

**Definition 4.1.** Two consecutive patterns  $p$  and  $p'$  are *mutually non-overlapping* if it is impossible for an occurrence of  $p$  and an occurrence of  $p'$  in an inversion sequence to overlap in more than one entry. Equivalently,  $p$  and  $p'$  are mutually non-overlapping if, for every  $i > 1$ , the reduction of the first  $i$  entries of either of the patterns does not coincide with the reduction of the last  $i$  entries of the other pattern.

**Example 4.2.** The patterns 110 and 010 are mutually non-overlapping. However, the same cannot be said about the patterns 110 and 100. For example, the inversion sequence  $e = 0002211 \in \mathbf{I}_7$  has occurrences of 110 and 100 overlapping in two entries,  $e_5$  and  $e_6$ .

Next we prove equivalence (ii) in Theorem 2.2.

**Proposition 4.3.** *The patterns  $p = \underline{0021}$  and  $p' = \underline{0121}$  satisfy Equation (4.1). In particular,  $\underline{0021} \stackrel{ss}{\sim} \underline{0121}$ .*

*Proof.* We define a map  $\phi : \mathbf{I}_n \rightarrow \mathbf{I}_n$  that changes every occurrence of 0021 into an occurrence of 0121, and vice versa. Specifically, let  $\phi(e) = e'$ , where

$$e'_j = \begin{cases} e_{j+2}, & \text{if } e_{j-1}e_j e_{j+1}e_{j+2} \text{ is an occurrence of } \underline{0021}, \\ e_{j-1}, & \text{if } e_{j-1}e_j e_{j+1}e_{j+2} \text{ is an occurrence of } \underline{0121}, \\ e_j, & \text{otherwise.} \end{cases}$$

For instance, if  $e = 00032454$ , then  $\phi(e) = e' = 00232254$ .

First we show that  $e'$  is an inversion sequence by checking that  $e'_j < j$  for all  $j$ . Indeed, if  $e_{j-1}e_j e_{j+1}e_{j+2}$  is an occurrence of 0121, then  $e'_j = e_{j-1} < j-1 < j$ . Additionally, if  $e_{j-1}e_j e_{j+1}e_{j+2}$  is an occurrence of 0021, then  $e'_j = e_{j+2} < e_{j+1} < j+1$ , so  $e'_j < j$ .

Since 0021 and 0121 are non-overlapping and mutually non-overlapping, and  $\phi$  does not change the first or the last entry of any occurrence of these patterns, it is clear that  $\phi^{-1} = \phi$ , so  $\phi$  is a bijection. Furthermore, the positions of the occurrences of 0021 (respectively, 0121) in  $e \in \mathbf{I}_n$  are precisely the positions of the occurrences of 0121 (respectively, 0021) in  $\phi(e)$ . It follows that Equation (4.1) holds and that  $\underline{0021} \stackrel{ss}{\sim} \underline{0121}$ .  $\square$

The idea in the proof of Proposition 4.3 can be exploited to prove a more general result. In order to do this, we need to introduce some terminology.

**Definition 4.4.** Let  $p = \underline{p_1 p_2 \dots p_r}$  and  $p' = \underline{p'_1 p'_2 \dots p'_r}$  be two consecutive patterns such that  $p_1 = p'_1$ ,  $p_r = p'_r$ , and  $\max_i \{p_i\} = \max_i \{p'_i\} = d$  (that is,  $p$  and  $p'$  have the same number  $d+1$  of distinct entries). Suppose that  $e \in \mathbf{I}_n$  has an occurrence of  $p$  in position  $i$ , and let  $f : \{0, 1, \dots, d\} \rightarrow \{e_i, e_{i+1}, \dots, e_{i+r-1}\}$  (where the right-hand side is viewed as a set, disregarding repetitions) be the order-preserving bijection such that  $e_i e_{i+1} \dots e_{i+r-1} = f(p_1) f(p_2) \dots f(p_r)$ .

Define a sequence  $e' = e'_1 e'_2 \dots e'_n$  by setting

$$e'_i e'_{i+1} \dots e'_{i+r-1} = f(p'_1) f(p'_2) \dots f(p'_r),$$

and  $e'_j = e_j$  for  $j < i$  or  $j > i + r - 1$ . This operation is called a *change* of the occurrence  $e_i e_{i+1} \dots e_{i+r-1}$  of  $p$  into an occurrence of  $p'$ . If, additionally,  $e' \in \mathbf{I}_n$  (that is,  $e'_j < j$  for all  $j$ ), then we say that this change is *valid*.

**Example 4.5.** Consider  $e = 00134015331 \in \mathbf{I}_{11}$ . It is not valid to change the occurrence  $e_2 e_3 e_4 e_5 e_6$  of 01230 into an occurrence of 03210. Indeed, such a change would yield  $e' = 00431015331$ , which is not an inversion sequence. On the other hand, it is valid to change the occurrence  $e_7 e_8 e_9 e_{10} e_{11}$  of 02110 into an occurrence of 02100, and this change yields  $e' = 00134015311 \in \mathbf{I}_{11}$ .

**Definition 4.6.** Let  $p = p_1 p_2 \dots p_r$  and  $p' = p'_1 p'_2 \dots p'_r$  be two consecutive patterns such that

$$p_1 = p'_1, \quad p_r = p'_r, \quad \text{and} \quad \max_i \{p_i\} = \max_i \{p'_i\}.$$

We say that  $p$  is *changeable* for  $p'$  if, for all  $1 \leq i \leq r$ ,

$$p'_i \leq \max(\{p_j : 1 \leq j \leq i\} \cup \{p_j - j + i : i < j \leq r\}).$$

If  $p$  is changeable for  $p'$  and  $p'$  is changeable for  $p$ , then we say that  $p$  and  $p'$  are *interchangeable*.

**Example 4.7.** Consider the consecutive patterns  $p = 01230$  and  $p' = 03210$ . Note that

$$p'_2 = 3 > 1 = \max(\{p_j : 1 \leq j \leq 2\} \cup \{p_j - j + 2 : 2 < j \leq 5\}).$$

Hence, 01230 is not changeable for 03210.

On the other hand, the patterns  $p = 0021$  and  $p' = 0121$  are interchangeable. Indeed, note that in this case  $p'_2 = 1 = \max(\{p_j : 1 \leq j \leq 2\} \cup \{p_j - j + 2 : 2 < j \leq 4\})$  and  $p'_i = p_i$  for  $i \neq 2$ , so  $p$  is changeable for  $p'$ . Similarly,  $p'$  is changeable for  $p$  because  $p_i \leq p'_i$  for all  $i$ .

The next lemma relates Definitions 4.4 and 4.6, showing why the concept of changeability is useful.

**Lemma 4.8.** *Let  $p = p_1 p_2 \dots p_r$  and  $p' = p'_1 p'_2 \dots p'_r$  be two consecutive patterns. Then  $p$  is changeable for  $p'$  if and only if, for every inversion sequence  $e \in \mathbf{I}_n$ , it is valid to change any given occurrence of  $p$  in  $e$  into an occurrence of  $p'$ .*

*Proof.* First we prove the forward direction. Suppose that  $p$  is changeable for  $p'$ . Let  $e \in \mathbf{I}_n$  be an inversion sequence with an occurrence of  $p$  in position  $i$ . Let  $e'$  be the sequence obtained by changing the occurrence  $e_i e_{i+1} \dots e_{i+r-1}$  of  $p$  into an occurrence of  $p'$ . To prove that this change is valid, it suffices to show that  $e'_{i+t-1} < i+t-1$  for all  $t$  such that  $p'_t \neq p_t$ .

Let  $t$  be such that  $p'_t \neq p_t$ , and let  $M_t = \max(\{p_j : 1 \leq j \leq t\} \cup \{p_j - j + t : t < j \leq r\})$ . By assumption,  $p'_t \leq M_t$ . If  $M_t = p_j$  for some  $1 \leq j \leq t$ , then  $p'_t \leq p_j$  implies that  $e'_{i+t-1} \leq e_{i+j-1} < i+j-1 \leq i+t-1$ . On the other hand, if  $M_t = p_j - j + t$  for some  $t < j \leq r$ , then  $p'_t \leq p_j - j + t$  implies that  $e'_{i+t-1} \leq e_{i+j-1} - j + t < (i+j-1) - j + t = i+t-1$ .



To prove the converse, suppose now that  $p$  is not changeable for  $p'$ . We construct an inversion sequence  $e$  with an occurrence of  $p$  whose change into an occurrence of  $p'$  is not valid. Let  $t$  be such that

$$(4.2) \quad p'_t > \max(\{p_j : 1 \leq j \leq t\} \cup \{p_j - j + t : t < j \leq r\}).$$

Let  $s$  be the smallest integer such that  $c = 0^s p$  is an inversion sequence and  $s + t - p'_t$  is nonnegative. Equivalently,

$$s = \max(\{p_j - j + 1 : 1 \leq j \leq r\} \cup \{p'_t - t\}).$$

We write  $c = c_1 c_2 \dots c_{s+r}$ , where  $c_i = 0$  for  $1 \leq i \leq s$  and  $c_{s+j} = p_j$  for  $1 \leq j \leq r$ . Now, define  $e = e_1 e_2 \dots e_{s+r}$  by setting

$$e_i = \begin{cases} c_i + s + t - p'_t, & \text{if } c_i \geq p'_t, \\ c_i, & \text{otherwise.} \end{cases}$$

Next we show that  $e$  is an inversion sequence. If  $i$  is such that  $e_i = c_i$ , then  $e_i < i$ , because  $c_i$  is an inversion sequence. If  $i$  is such that  $e_i \neq c_i$ , then  $i = s + j$  for some  $1 \leq j \leq r$ , and  $p'_t \leq c_i = c_{s+j} = p_j$ . By inequality (4.2),  $p'_t \leq p_j$  implies that  $t < j \leq r$ , and so  $p'_t > p_j - j + t$ . Hence,

$$e_i = c_i + s + t - p'_t = p_j + s + t - p'_t < p_j + s + t - (p_j - j + t) = s + j = i.$$

We have shown that  $e_i < i$  in any case, and so  $e \in \mathbf{I}_{s+r}$ .

Since  $c$  has an occurrence of  $p$  in position  $s + 1$ , so does  $e$ . Indeed, by our definition of  $s$ , we know that  $s + t - p'_t \geq 0$ , so adding  $s + t - p'_t$  to the values  $c_i$  for which  $c_i \geq p'_t$  does not alter their relative order. Let  $e'$  be the sequence obtained by changing the occurrence  $e_{s+1} e_{s+2} \dots e_{s+r}$  of  $p$  into an occurrence of  $p'$ . Then  $e'_{s+t} = p'_t + s + t - p'_t = s + t$ . This implies that  $e' \notin \mathbf{I}_{s+r}$ , so the change is not valid.  $\square$

**Example 4.9.** We showed in Example 4.7 that 01230 is not changeable for 03210. By Lemma 4.8, this is equivalent to the existence of some inversion sequence with an occurrence of 01230 whose change into  $p'$  is not valid. Example 4.5 gives such an inversion sequence.

On the other hand, we determined in Example 4.7 that 0021 and 0121 are interchangeable. By Lemma 4.8, this is equivalent to the fact that, in every  $e \in \mathbf{I}_n$ , it is valid to change any occurrence of 0021 (respectively, 0121) in  $e$  into an occurrence of 0121 (respectively, 0021). This property was shown in the proof of Proposition 4.3.

**Theorem 4.10.** *Let  $p$  and  $p'$  be non-overlapping, mutually non-overlapping and interchangeable consecutive patterns. Then  $p$  and  $p'$  satisfy Equation (4.1). In particular,  $p \stackrel{ss}{\sim} p'$ .*

*Proof.* Mimicking the proof of Proposition 4.3, we define a map  $\phi : \mathbf{I}_n \rightarrow \mathbf{I}_n$  that changes occurrences of  $p$  into occurrences of  $p'$  and vice versa. Since  $p$  and  $p'$  are interchangeable, Lemma 4.8 implies that it is possible to make these changes. Moreover, since  $p$  and  $p'$  are non-overlapping and mutually non-overlapping, all of these changes can be made simultaneously, and therefore  $\phi$  is well-defined and  $\phi^{-1} = \phi$ . Hence,  $\phi$  is a bijection and the result follows.  $\square$

We now list equivalences between consecutive patterns of length 4 that follow as particular instances of Theorem 4.10. In each case, it can be easily verified that the patterns in question are non-overlapping, mutually non-overlapping and interchangeable. We use the same labeling from Theorem 2.2.

**Corollary 4.11.** *The following equivalences hold:*

- |   |   |
|---|---|
| (iii) $\underline{0100} \overset{ss}{\sim} \underline{0110}$ .                                      | (ix) $\underline{2013} \overset{ss}{\sim} \underline{2103}$ .   |
| (iv) $\underline{1002} \overset{ss}{\sim} \underline{1012} \overset{ss}{\sim} \underline{1102}$ .   | (xi) $\underline{2001} \overset{ss}{\sim} \underline{2011} \overset{ss}{\sim} \underline{2101} \overset{ss}{\sim} \underline{2201}$ . |
| (v) $\underline{1200} \overset{ss}{\sim} \underline{1210} \overset{ss}{\sim} \underline{1220}$ .    | (xii) $\underline{2012} \overset{ss}{\sim} \underline{2102}$ .  |
| (vi) $\underline{0211} \overset{ss}{\sim} \underline{0221}$ .                                       | (xiv) $\underline{3012} \overset{ss}{\sim} \underline{3102}$ .  |
| (viii) $\underline{1001} \overset{ss}{\sim} \underline{1011} \overset{ss}{\sim} \underline{1101}$ . |   |

**4.2. Equivalences between non-overlapping patterns proved via inclusion-exclusion.** The equivalences between consecutive patterns of length 4 in this subsection follow from the next result, which has weaker hypotheses than Theorem 4.10. It relaxes the condition of  $p$  and  $p'$  being mutually non-overlapping, at the expense of not proving Equation (4.1) and not producing a direct bijection changing all the occurrences of  $p$  into occurrences of  $p'$ .

**Theorem 4.12.** *Let  $p$  and  $p'$  be non-overlapping and interchangeable consecutive patterns. Then  $p \overset{ss}{\sim} p'$ .*

*Proof.* We mimic the proof of Lemma 3.10 and prove that

$$|\{e \in \mathbf{I}_n : \text{Em}(p, e) \supseteq S\}| = |\{e \in \mathbf{I}_n : \text{Em}(p', e) \supseteq S\}|$$

for all positive integers  $n$  and all  $S \subseteq [n]$ .

Fix  $n$  and  $S$ , and consider the map

$$\Phi_S : \{e \in \mathbf{I}_n : \text{Em}(p, e) \supseteq S\} \rightarrow \{e \in \mathbf{I}_n : \text{Em}(p', e) \supseteq S\}$$

that changes the occurrences of  $p$  in  $e$  in positions  $S$  into occurrences of  $p'$ . Given that  $p$  is non-overlapping and interchangeable for  $p'$ , Lemma 4.8 implies that  $\Phi_S$  is well-defined.

Similarly, we define the map  $\Psi_S : \{e \in \mathbf{I}_n : \text{Em}(p', e) \supseteq S\} \rightarrow \{e \in \mathbf{I}_n : \text{Em}(p, e) \supseteq S\}$  that changes the occurrences of  $p'$  in  $e$  in positions  $S$  into occurrences of  $p$ . Since  $p'$  is non-overlapping and interchangeable for  $p$ , we deduce again that  $\Psi_S$  is well-defined. It is clear that  $\Phi_S$  and  $\Psi_S$  are inverses of each other, which implies that  $\Phi_S$  is a bijection. The result now follows from Lemma 3.11.  $\square$

The next corollary is a consequence of Theorem 4.12. The equivalences are labeled as in Theorem 2.2.

**Corollary 4.13.** *The following equivalences hold:*

(vii)  $\underline{1000} \overset{ss}{\sim} \underline{1110}$ .

(x)  $\underline{2100} \overset{ss}{\sim} \underline{2210}$ .

*Proof.* It is straightforward to verify that  $\underline{1000}$  and  $\underline{1110}$  are non-overlapping and interchangeable, and that the same is true for  $\underline{2100}$  and  $\underline{2210}$ . The result follows from Theorem 4.12.  $\square$

Note that the patterns  $\underline{1000}$  and  $\underline{1110}$  (respectively,  $\underline{2100}$  and  $\underline{2210}$ ) may overlap with one another, so it is not possible to deduce Corollary 4.13 from Theorem 4.10.

**4.3. Equivalences between overlapping patterns proved via inclusion-exclusion.** All the equivalences proved so far have relied on the patterns being non-overlapping. In this subsection we adapt the proof of Theorem 4.12 to patterns that may have occurrences overlapping in more than one entry. We start by proving equivalence (i) from Theorem 2.2.

**Proposition 4.14.** *The patterns  $\underline{0102}$  and  $\underline{0112}$  are super-strongly Wilf equivalent.*

*Proof.* We will prove that, for all  $n$  and all  $S \subseteq [n]$ ,

$$(4.3) \quad |\{e \in \mathbf{I}_n : \text{Em}(\underline{0102}, e) \supseteq S\}| = |\{e \in \mathbf{I}_n : \text{Em}(\underline{0112}, e) \supseteq S\}|.$$

Note that there is no loss of generality in assuming that there are no consecutive elements in  $S$ , because no two occurrences of  $\underline{0102}$  (or  $\underline{0112}$ ) can overlap in three entries.

We can write  $S$  uniquely as a disjoint union of *blocks*, which we define as maximal subsets whose entries form an arithmetic sequence with difference 2. Explicitly,  $S = \bigsqcup_{j=1}^m B_j$ , where  $B_j = \{i_j, i_j + 2, i_j + 4, \dots, i_j + 2l_j\}$  and  $i_j + 2(l_j + 1) < i_{j+1}$  for  $1 \leq j \leq m - 1$ .

Define a map  $\Phi_S : \{e \in \mathbf{I}_n : \text{Em}(\underline{0102}, e) \supseteq S\} \rightarrow \{e \in \mathbf{I}_n : \text{Em}(\underline{0112}, e) \supseteq S\}$  by setting  $\Phi_S(e) = e'$ , where

$$e'_i = \begin{cases} e_{i-1}, & \text{if } i-2 \in S, \\ e_i, & \text{otherwise.} \end{cases}$$

In other words,  $\Phi_S$  changes occurrences of  $\underline{0102}$  in positions in  $S$  into occurrences of  $\underline{0112}$ . To see that  $\Phi_S$  is a bijection, we define its inverse map  $\Psi_S$  by setting  $\Psi_S(e) = e'$ , where

$$e'_i = \begin{cases} e_{i_j}, & \text{if } i-2 \in B_j, \\ e_i, & \text{otherwise.} \end{cases}$$

The map  $\Psi_S$  changes occurrences of  $\underline{0112}$  in positions in  $S$  into occurrences of  $\underline{0102}$ .

Since  $\Phi_S$  is a bijection, Equation (4.3) is proved, and the result follows from Lemma 3.11.  $\square$

A schematic diagram of the maps  $\Phi_S$  and  $\Psi_S$  from the above proof appears in Figure 2. An example of the computation of  $\Phi_S$  is given next.

**Example 4.15.** Let  $S = \{3, 5, 9\}$  and  $e = 0102040523262889 \in \mathbf{I}_{16}$ , which satisfy  $S \subseteq \text{Em}(\underline{0102}, e) = \{1, 3, 5, 9, 11\}$ . Changing the occurrences of  $\underline{0102}$  in positions  $S$  into occurrences of  $\underline{0112}$ , we obtain  $\Phi_S(e) = e' = 0102244523362889$ . Note that  $S \subseteq \text{Em}(\underline{0112}, e') = \{3, 5, 9, 13\}$ , and that  $\Psi_S(e') = e$ .

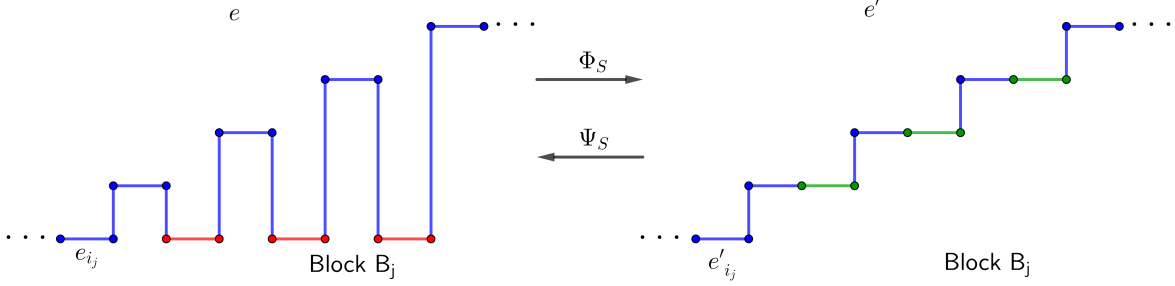


FIGURE 2. Schematic diagram of the behavior of  $\Phi_S$  and  $\Psi_S$  from Proposition 4.14 on occurrences within a block.

Next we prove equivalence (xiii) from Theorem 2.2.

**Proposition 4.16.** *The patterns  $\underline{2010}$ ,  $\underline{2110}$  and  $\underline{2120}$  are super-strongly Wilf equivalent.*

*Proof.* For any  $n$  and  $S \subseteq [n]$ , we construct a bijection

$$\Phi_S : \{e \in \mathbf{I}_n : \text{Em}(p, e) \supseteq S\} \rightarrow \{e \in \mathbf{I}_n : \text{Em}(p', e) \supseteq S\}$$

that changes occurrences of  $p = \underline{2010}$  in positions in  $S$  into occurrences of  $p' = \underline{2110}$  (likewise,  $p' = \underline{2120}$ ). The construction is very similar to the proof of Proposition 4.14, and so the details are omitted. Using Lemma 3.11, the result follows.  $\square$

*Proof of Theorem 2.2.* All the listed super-strong Wilf equivalences follow from Propositions 4.3, 4.14 and 4.16, and Corollaries 4.11 and 4.13. A brute force computation of the first ten terms of the sequences  $|\mathbf{I}_n(p)|$  for each consecutive pattern  $p$  of length 4 shows that there are no additional equivalences, and so the list gives a complete classification.  $\square$

We can generalize Proposition 4.14 to patterns of the form  $\underline{0^r 1 0^r 2 0^r \dots (s-1) 0^r s}$  and  $\underline{0^r 1 1^r 2 2^r \dots (s-1) (s-1)^r s}$ , with  $r \geq 1$  and  $s \geq 2$ , by proving a statement analogous to (4.3). Indeed, we generalize the notion of a block to be a maximal subset of  $S$  whose elements form an increasing sequence where consecutive differences are of the form  $a(r+1)$ , with  $1 \leq a \leq s-1$ . Again, we can write  $S$  uniquely as a disjoint union of blocks, say  $S = \bigsqcup_{j=1}^m B_j$ . We define a bijection  $\Phi_S$ , analogous to the one in the proof of Proposition 4.14, which changes occurrences of  $\underline{0^r 1 0^r 2 0^r \dots (s-1) 0^r s}$  in positions in  $S$  into occurrences of  $\underline{0^r 1 1^r 2 2^r \dots (s-1) (s-1)^r s}$ , as suggested by the schematic diagram in Figure 3. Specifically, we set  $\Phi_S(e) = e'$ , where

$$e'_i = \begin{cases} e_{i-b-1}, & \text{if } i - a(r+1) - b \in S \text{ for some } 1 \leq a \leq s-1 \text{ and } 0 \leq b \leq r-1, \\ e_i, & \text{otherwise.} \end{cases}$$

**Theorem 4.17.** *For every  $r \geq 1$  and  $s \geq 2$ , the patterns  $\underline{0^r 1 0^r 2 0^r \dots (s-1) 0^r s}$  and  $\underline{0^r 1 1^r 2 2^r \dots (s-1) (s-1)^r s}$  are super-strongly Wilf equivalent.*

Similarly, we can generalize Proposition 4.16 as follows.

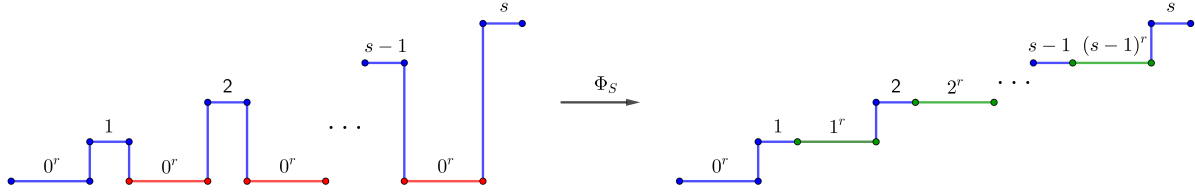


FIGURE 3. Schematic diagram showing how  $\Phi_S$  changes an occurrence of  $0^r 1 0^r 2 0^r \dots (s-1) 0^r s$  into an occurrence of  $0^r 1 1^r 2 2^r \dots (s-1) (s-1)^r s$ .

**Theorem 4.18.** *For every  $r \geq 1$  and  $s \geq 2$ , the patterns  $s 0^r (s-1) 0^r \dots 0^r 1$ ,  $s(s-1)^r s(s-2)^r s \dots s 0^r$  and  $s(s-1)^r (s-1)(s-2)^r (s-2) \dots 1 0^r$  are super-strongly Wilf equivalent.*

#### 4.4. A note regarding patterns extending 110 and 100.

**Definition 4.19.** Let  $r > s > 0$ . Suppose  $p = p_1 p_2 \dots p_r$  and  $p' = p'_1 p'_2 \dots p'_s$  are consecutive patterns of length  $r$  and  $s$ , respectively. We say  $p$  extends  $p'$  on the right (respectively, left) if the leftmost (respectively, rightmost)  $s$  entries of any occurrence of  $p$  in an inversion sequence form an occurrence of  $p'$ . If  $p$  extends  $p'$  either on the left or on the right, then we say  $p$  extends  $p'$ .

**Example 4.20.** The patterns of length 4 extending 110 are 1102, 1101, 2201, 1100 and 2210 on the right; and 2110, 1110, 1220, 0110 and 0221 on the left. The patterns of length 4 extending 100 are 1002, 1001, 2001, 1000 and 2110 on the right; and 2100, 1100, 1200, 0100 and 0211 on the left.

We proved in Proposition 3.12 that 100  $\overset{ss}{\sim}$  110, so it is natural to ask whether  $p \overset{ss}{\sim} p'$  for some patterns  $p$  and  $p'$  of length 4 extending 100 and 110, respectively. It follows from Theorem 2.2 that, in fact, there is a bijective correspondence between patterns of length 4 extending 100 and those extending 110, preserving super-strong Wilf equivalence classes.

**Corollary 4.21.** *Consider the bijection from patterns of length 4 extending 100 to patterns of length 4 extending 110 given by the following assignment:*

$$\begin{array}{ccccc} \underline{0100} \mapsto \underline{0110}, & \underline{1200} \mapsto \underline{1220}, & \underline{1000} \mapsto \underline{1110}, & \underline{1100} \mapsto \underline{1100}, & \underline{2001} \mapsto \underline{2201}, \\ \underline{1002} \mapsto \underline{1102}, & \underline{0211} \mapsto \underline{0221}, & \underline{1001} \mapsto \underline{1101}, & \underline{2100} \mapsto \underline{2210}, & \underline{2110} \mapsto \underline{2110}. \end{array}$$

Then, for each pair  $p \mapsto p'$ , we have that  $p \overset{ss}{\sim} p'$ .

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