On the asymptotic distinct prime partitions of integers

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Abstract

We discuss $Q(n)$, the number of ways a given integer n may be written as a sum of distinct primes, and study its asymptotic form $Q_{as}(n)$ valid in the limit $n \to \infty$. We obtain $Q_{as}(n)$ by Laplace inverting the fermionic partition function of primes, in number theory called the generating function of the distinct prime partitions, in the saddle-point approximation. We find that our result of $Q_{as}(n)$, which includes two higher-order corrections to the leading term in its exponent and a pre-exponential correction factor, approximates the exact $Q(n)$ far better than its simple leading-order exponential form given so far in the literature.

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I. INTRODUCTION

It has been shown [1] that the techniques of statistical mechanics may be used to obtain asymptotic forms for any type of partition of a positive integer n. The partition function of a gas in statistical mechanics contains information on the distribution of the total energy among the constituents and hence plays the same role as the generating function of the corresponding partitions in number theory. This relation was used in [1], where the number of partitions $P(n)$ studied in number theory was expressed in terms of the quantum density of states $\rho(E)$ given by the inverse Laplace transform of the partition function. Approximating the transform by the saddle-point method then yielded the asymptotic forms $P_{as}(n)$.

Recently, this method was applied to the partitions of integers into arbitrary primes in [2]. For a system whose single-particle levels are simply the primes p as an ordered set, its total energy is given as a sum of primes, and the corresponding density of states is related to the *number of arbitrary prime partitions* $P(n)$, assuming that the particles behave like *bosons*. The asymptotic form $P_{as}(n)$ obtained in [2] was found to approximate the exact $P(n)$ for large n far better than the asymptotic expressions given earlier in the literature [3, 4]. The same method was also applied more recently to distinct square partitions in [5].

In the present paper we study $Q(n)$, the number of ways a given integer n may be written as a sum of *distinct primes*. As only distinct primes are allowed in $Q(n)$, this corresponds to a system of *fermionic* particles with the primes p as single-particle levels. We again use the saddle-point method for Laplace inverting their partition function to obtain the asymptotic form $Q_{as}(n)$ valid in the limit $n \to \infty$. In numerical computations up to $n = 10^5$ we find that, like in [2], our result for $Q_{as}(n)$ approximates the exact $Q(n)$ far better than its simple leading-order exponential form given so far in the literature [6].

The plan of our paper is as follows. In Section II A, we establish the relation of $Q(n)$ to the partition function and the density of states $\rho^F(E)$. In Sec. II B we derive its asymptotic form using the saddle-point method, and in Sec. II C we give the explicit solution of the saddle-point equation leading to our final result for $Q_{as}(n)$. In Sec. III our asymptotic result is compared numerically with the exact function $Q(n)$ for the distinct prime partitions. We conclude the paper with a short summary in Sec. IV.

II. PARTITIONS INTO PRIMES

A. Fermionic partition function and its relation to $Q(n)$

Consider a large number N of fermions whose single-particle spectrum is given by the primes p. The total energy E of the system is given by

$$E = \sum_p n_p \, p \, . \tag{1}$$

(We use throughout dimensionless variables and take the particle mass m, the Planck constant $\hbar$ and the Boltzmann constant k to be unity: $m = \hbar = k = 1$.) Here and in the following, the sums $\sum_p$ run over all primes p, and n_p are the fermionic occupancies of the levels which must be zero or one, such that

$$\sum_p n_p = N \, , \qquad n_p = 0 \text{ or } 1 \, . \tag{2}$$

The number of possible energy partitions E with the restriction (2) shall be denoted by $Q_N(E)$, where the subscript N keeps track of the total number of particles. Although E is necessarily integer, we treat it as a continuous variable like in statistical mechanics. $Q_N(n)$ is the number of N-*restricted* fermionic partitions of n, i.e., the number of ways to write n as a *sum of* N *distinct primes*. In the limit $N \to \infty$, $Q_N(n)$ will tend towards the number of *unrestricted* but *distinct prime partitions* $Q(n)$ under consideration here.

For the purpose of this paper, we are only interested in the limit $N \to \infty$ of the fermionic partitions which then become unrestricted as stated above. The *quantum-statistical partition function* $Z^F(\beta)$ is in this limit given by

$$Z^F(\beta) = \prod_p [1 + e^{-\beta \, p}], \tag{3}$$

where $\beta = 1/kT$ is the inverse temperature and the product runs over all primes p. Taylor expanding the expontential in (3) and reordering the terms yields the alternative form of the partition function

$$Z^F(\beta) = \sum_{n=0}^{\infty} Q(n) \, e^{-n\beta} \, , \tag{4}$$

which in number theory is known as the *generating function* of the $Q(n)$. In the On-line Encyclopedia of Integer Sequences (OEIS) [7], the sequence of numbers $Q(n)$ is called the sequence A000586. Its first ten members are $Q(n) = 1, 0, 1, 1, 0, 2, 0, 2, 1, 1$ for $n = 0, ..., 9$, where $Q(0) = 1$ by definition. Note also that the $Q(n)$ are a subset of the (bosonic) prime partitions $P(n)$, called the sequence A000607 in [7].

From the partition function, we obtain the *many-body density of states* $\rho^F(E)$ by an inverse Laplace transform:

$$\rho^F(E) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\beta \, Z^F(\beta) \, e^{\beta E}. \tag{5}$$

Hereby β is taken as a complex variable and the integration above runs along the imaginary axis of the complex β plane. Later the Laplace inversion shall be taken in the saddle-point approximation.

It is important now to realize that $Q(n)$ is related to the density of states $\rho^F(E)$ in the following way. Taking directly the exact inverse Laplace transform of (4), we find

$$\rho^F(E) = \sum_{n=0}^{\infty} Q(n) \, \delta(E - n) \,, \tag{6}$$

where $\delta(E - n)$ is the Dirac delta function peaked at $E = n$. We see thus that $\rho^F(E)$ can also be understood as the *density of distinct prime partitions*. Like it was argued in [5] for the distinct square partitions, averaging $\rho^F(E)$ over a sufficiently large energy interval ΔE is asymptotically the same as averaging $Q(n)$ over a sufficiently large interval Δn:

$$\langle \, \rho^F(E) \, \rangle_{\Delta E} \;\sim\; \langle \, Q(n) \, \rangle_{\Delta n} \quad \text{for} \;\; E, n \to \infty \,. \tag{7}$$

Therefore determing the *asymptotic average part* $\rho^F_{as}(E)$ of the density of states valid in the limit $E \to \infty$, which can be obtained by the saddle-point approximation to its inverse Laplace transform (5), and equating $E = n$ will give the *average asymptotic form* $Q_{as}(n)$ of the distinct prime partitions.

B. Asymptotic partition function from saddle-point approximation

We first rewrite the inverse Laplace transform (5) by taking the natural log of Z^F into the exponent:

$$\rho^F(E) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\beta\, e^{\beta E + \ln Z^F(\beta)}\,. \tag{8}$$

We now evaluate this integral using the saddle-point method (also called the method of steepest descent). We define the exponent above as the canonical entropy function

$$S^F(E,\beta) = \beta E + \ln Z^F(\beta)\,. \tag{9}$$

Applying the saddle-point method to (8) requires to find a stationary point β_0 of the function $S^F(E,\beta)$ by solving the *saddle-point equation*

$$\left.\frac{\partial S^F(E,\beta)}{\partial\beta}\right|_{\beta_0} = E + \frac{\partial Z^F(E,\beta_0)/\partial\beta}{Z^F(E,\beta_0)} = 0\,. \tag{10}$$

If this equation has a solution β_0, which will be a function $\beta_0(E)$, one evaluates the successive partial derivatives of $S^F(E,\beta)$ at β_0:

$$S^{F(n)}(E,\beta_0) = \left.\frac{\partial^n S^F(E,\beta)}{\partial\beta^n}\right|_{\beta_0}\,. \tag{11}$$

The approximate result of the inverse Laplace transform then is given by

$$\rho^F_{as}(E) = \frac{e^{S^F(E,\beta_0)}}{\sqrt{2\pi S^{F(2)}(E,\beta_0)}}\left[1+\cdots\right], \tag{12}$$

where the dots indicate the so-called cumulants involving higher derivatives of the entropy, which become more important for large β (see, e.g., Ref. [8]). Since we are interested here in the limit $\beta \to 0$ relevant for the asymptotics of large E, we can neglect these cumulants.

Next, we take the natural log of the partition function given in Eq.(3)

$$\ln Z^F(\beta) = \sum_{p=2}^{\infty} \ln\left(1 + e^{-\beta p}\right) \tag{13}$$

and approximate it by the integral

$$\ln Z^F(\beta) \sim \int_2^{\infty} dx\, g_{av}(x)\ln\left(1+e^{-\beta x}\right), \tag{14}$$

where $g_{av}(x) = \frac{1}{\ln(x)}$ is the approximate density of primes, using the prime number theorem. If the density $g_{av}(x)$ were exact, then the integral would give the exact result (13).

The evaluation the integral in the limit $\beta \to 0$ follows closely the method outlined in [2]. Denoting $y = \beta x$, the integral becomes

$$\ln Z^F(\beta) \sim \frac{1}{\beta}\int_{2\beta}^{\infty} dy\, \frac{1}{\ln(\frac{y}{\beta})} \ln\left(1+e^{-y}\right) = -\frac{1}{\beta\ln\beta}\int_{2\beta}^{\infty} dy\, \frac{1}{1-\frac{\ln(y)}{\ln(\beta)}} \ln\left(1+e^{-y}\right). \tag{15}$$

In the limit $\beta \to 0$ we may write this integral as an asymptotic series

$$\ln Z^F(\beta) \sim -\frac{1}{\beta\ln\beta}\int_{2\beta}^{\infty} dy \left[1+\sum_{k=1}^{\infty}\left(\frac{\ln(y)}{\ln(\beta)}\right)^k\right] \ln\left(1+e^{-y}\right). \tag{16}$$

This is now a series in the expansion parameter $1/\ln(\beta)$ since each term is divided by the power $(\ln\beta)^k$. As we shall see later, in the leading saddle-point approximation $\ln(\beta_0) \approx \ln(E)$ and hence this is an asymptotic series in $1/\ln(E)$ as well. In the asymptotic limit we take the lower limit of the integral to be zero.

For the present analysis, we retain the leading term and the first correction, like for the bosonic prime partitions in [2], and define

$$\ln Z^F_{as}(\beta) = -\frac{1}{\beta\ln\beta}\int_{0}^{\infty} dy \left[1+\left(\frac{\ln(y)}{\ln(\beta)}\right)\right] \ln\left(1+e^{-y}\right). \tag{17}$$

The integrals may again be evaluated analytically and we obtain

$$\ln Z^F_{as}(\beta) = \frac{1}{\beta\ln(\beta)}\left[-\frac{\pi^2}{12}+\frac{1}{\ln\beta}\left(\frac{C\pi^2}{12}+\sum_k(-1)^{k-1}\frac{\ln(k)}{k^2}\right)\right], \tag{18}$$

where $C = 0.5772156649\ldots$ is the Euler constant.

C. Solution of saddle-point equation and $Q_{as}(n)$

In order to find the saddle point β_0 from Eq. (10), we start from the entropy $S^F(\beta)$ in the asymptotic limit. Using Eqs. (9) and (18) we get up to order $1/(\ln\beta)^2$

$$S^F(E,\beta) = \beta E - \frac{F_1}{\beta\ln(\beta)} + \frac{F_2}{\beta\,(\ln\beta)^2} + \cdots, \tag{19}$$

where

$$F_1 = \frac{\pi^2}{12}, \qquad F_2 = \left[\frac{C\pi^2}{12}+\sum_k(-1)^{k-1}\frac{\ln(k)}{k^2}\right] = 0.3734242774\ldots \tag{20}$$

Eq. (19) is identical in form with that of the bosonic case given in [2]:

$$S^B(E,\beta) = \beta E - \frac{f_1}{\beta\ln(\beta)} + \frac{f_2}{\beta(\ln\beta)^2} + \cdots, \tag{21}$$

where

$$f_1 = \frac{\pi^2}{6}\,, \qquad f_2 = \left[\frac{C\pi^2}{6} + \sum_k \frac{\ln(k)}{k^2}\right] = 1.887029965\ldots \tag{22}$$

The only difference is that the coefficients f_1 and f_2 of [2] are replaced here by the F_1 and F_2, respectively. Therefore we obtain our result simply by replacing the coefficients f_i in the bosonic case by the F_i in the present fermionic case and following the steps outlined in [2].

Thus we can directly give the result for the fermionic case as

$$\rho^F_{as}(E) = \frac{1}{\sqrt{4E^{3/2}[6\ln(E)]^{1/2}}} \exp\left\{2\pi\sqrt{\frac{E}{6\ln(E)}}\left[1 - \frac{1}{2}\frac{\ln[\ln(E)]}{\ln(E)} + b^F\frac{1}{\ln(E)}\right]\right\} \tag{23}$$

with the constant

$$b^F = \left[\frac{F_2}{F_1} + \ln(\pi/\sqrt{6})\right] = 0.7028796287\ldots \tag{24}$$

The asymptotic $Q_{as}(n)$ is then obtained replacing E by n above, so that:

$$Q_{as}(n) = \frac{1}{\sqrt{4n^{3/2}[6\ln(n)]^{1/2}}} \exp\left\{2\pi\sqrt{\frac{n}{6\ln(n)}}\left[1 - \frac{1}{2}\frac{\ln[\ln(n)]}{\ln(n)} + b^F\frac{1}{\ln(n)}\right]\right\}. \tag{25}$$

This is the main result of the present paper. The corresponding result for the bosonic partitions in [2] was

$$P_{as}(n) = \frac{1}{\sqrt{4n^{3/2}[3\ln(n)]^{1/2}}} \exp\left\{2\pi\sqrt{\frac{n}{3\ln(n)}}\left[1 - \frac{1}{2}\frac{\ln[\ln(n)]}{\ln(n)} + b^B\frac{1}{\ln(n)}\right]\right\} \tag{26}$$

with the constant

$$b^B = \left[\frac{f_2}{f_1} + \ln(\pi/\sqrt{3})\right] = 0.7426003995\ldots \tag{27}$$

Note that the leading exponential terms and the denominators of the pre-exponential terms in (25) and (26) differ by a factor $1/\sqrt{2}$. Note that since the $Q(n)$ are a subset of the $P(n)$, their values must be smaller, which asymptotically is brought about by the extra factor $1/\sqrt{2}$ in the leading exponential term. The first correction term in the exponent, namely $-\frac{1}{2}\ln[\ln(n)]/\ln(n)$, is identical in both cases. As far as we know, the above result (25) for the distinct prime partitions has not been given in the literature so far.

In the next section, we compare numerically our asymptotic result (25) with the exact values $Q(n)$ of the distinct prime partitions.

III. NUMERICAL TEST OF $Q_{as}(n)$

In this section we test our asymptotic result (25) numerically. We have generated the exact $Q(n)$ up to $n = 100\,000$. In Figs. 1 and 2 we show their values by the dots (red) on a logarithmic scale in two regions of n. The dashed line (green) shows the leading-order exponential expression

$$Q_0(n) = \exp\left\{2\pi\sqrt{\frac{n}{6\ln(n)}}\right\}, \tag{28}$$

which is the only asymptotic result that has been given so far in the literature [6], and the solid (blue) line gives our full asymptotic result (25).

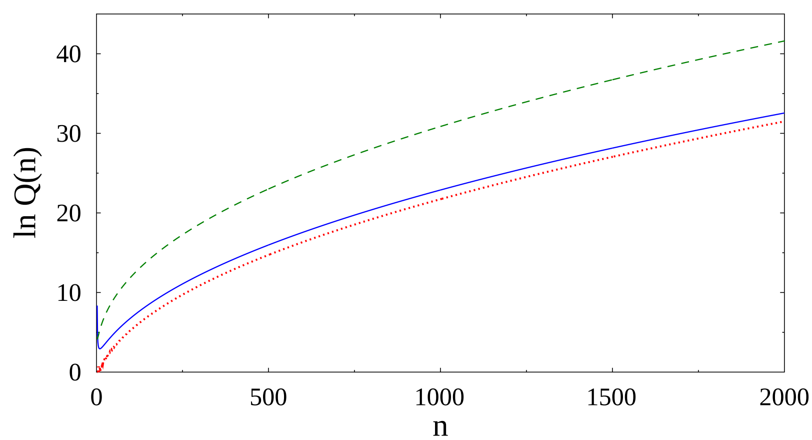

FIG. 1: Exact $\ln Q(n)$ by dots (red), lowest-order asymptotic form $\ln Q_0(n)$ from (28) by the dashed line (green) and our full asymtotic form $\ln Q_{as}(n)$ given by (25) by the solid (blue) line, shown as functions of n up to $n = 2000$.

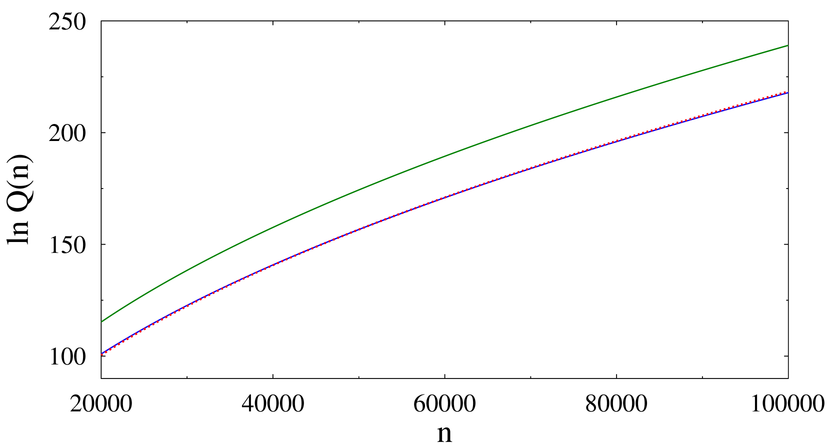

FIG. 2: Same as Fig. 1 in the region $n = 20\,000 - 100\,000$.

A large discrepancy between $Q_0(n)$ and $Q(n)$ is noticed for all n. Our full asymptotic result

$Q_{as}(n)$ (25) approaches the exact $Q(n)$ much better (except in the academic limit $n \to 0$ where it diverges due to the pre-exponential factor). In Fig. 2 for the values $n \geq 20\,000$, the two curves can hardly be distinguished.

We have thus achieved a considerable improvement over the simple exponential form (28). A closer look reveals that the curve for $Q_{as}(n)$, which for smaller n overestimates the exact $Q(n)$, crosses the curve of the latter around $n \sim 50\,000$. A similar result was found in [2] for the bosonic prime partitions, where $P_{as}(n)$ crosses $P(n)$ much earlier and then appears to approach it asymptotically from below for $n \to \infty$.

In order to focus on this asymptotic behavior, we show in Fig. 3 the difference of the natural logs relative to the lowest-order term, i.e., the quantity $[\ln Q_{as}(n) - \ln Q(n)]/\ln Q_0(n)$, plotted versus $1/n$ in a region of the largest n available.

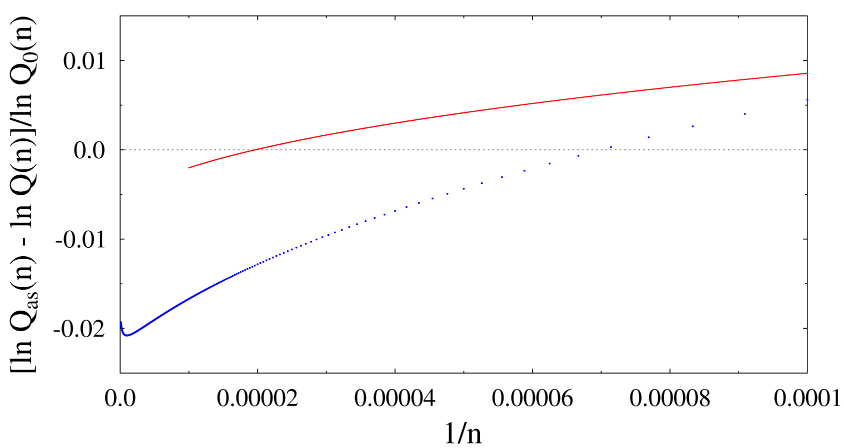

FIG. 3: Relative difference $[\ln Q_{as}(n) - \ln Q(n)]/\ln Q_0(n)$, shown versus $1/n$ by the solid (red) line. The dotted (blue) line shows the corresponding quantity obtained in [2] for the bosonic prime partitions $P(n)$.

The solid (red) curve gives the result obtained with our full asymptotic form (25). For comparison we show in this figure by the dotted (blue) curve also the corresponding quantity obtained in Ref. [2] from the (bosonic) prime partitions $P(n)$ and their respective asymptotic forms. The overall behaviour of the two curves is similar; for the results in [2] we had larger values of n available. There we notice a tendency for the difference to approach zero from below for $1/n \to 0$ (i.e. $n \to \infty$). We have no proof that the same will happen in the present case, but it is well possible.

IV. SUMMARY

In summary, we have shown how an improved asymptotic expression for the function $Q(n)$, which counts the number of distinct prime partitions of an integer n, can be obtained from asymptotic expansions of the partition function $Z^F(\beta)$ in (4) and the corresponding density of states $\rho^F(E)$ in (5). $Z^F(\beta)$ can be understood as the quantum-statistical partition function of a system of N fermions, whose single-particle energy spectrum is given by the primes p, in the limit $N \to \infty$. It is identical to the generating function of the $Q(n)$ known in number theory. The density of states $\rho^F(E)$ is identical to the the density of distinct prime partitions given in Eq. (6). Exploiting the connection between $\rho^F(E)$ and $Q(n)$ using the saddle-point approximation for the inverse Laplace transform (5), we have obtained the asymptotic form $Q_{as}(n)$ in Eq. (25) and shown it numerically to approach the exact $Q(n)$ in the limit $n \to \infty$ far better than the hitherto known expression $Q_0(n)$ given in (28).

We have used the same method as in Ref. [2] where the non-distinct prime partitions $P(n)$ were studied, and have found similar results as there. The asymptotic $Q_{as}(n)$ overestimates the exact $Q(n)$ for smaller n but overshoots it for $n \gtrsim 50,000$. Like in [2], the limit $Q_{as}(n) \to Q(n)$ for $n \to \infty$ cannot be demonstrated numerically, since this would require numerical data for forbiddingly large values of $n \sim 10^8$ or more. Nevertheless, we can state that with our result (25) we have obtained an excellent asymptotic approximation for the distinct prime partitions, which is far superior to the result known so far.

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