2-NEIGHBORLY 0/1-POLYTOPES OF DIMENSION 7

ALEKSANDR MAKSIMENKO

ABSTRACT. We give a complete enumeration of all 2-neighborly 0/1-polytopes of dimension 7. There are 13 959 358 918 different 0/1-equivalence classes of such polytopes. They form 5 850 402 014 combinatorial classes and 1 274 089 different f-vectors. It enables us to list some of their combinatorial properties. In particular, we have found a 2-neighborly polytope with 14 vertices and 16 facets.

1. Introduction

A *0/1-polytope* is a convex polytope whose set of vertices is subset of $\{0,1\}^d$. For beautiful introduction to the world of 0/1-polytopes, we refer to Ziegler [17]. Some recent results can be found in [10]. The classification of all 1 226 525 different 0/1-equivalence classes of 0/1-polytopes of dimension 5 was done by Aichholzer [1]. Also he completed the classification of 6-dimensional 0/1-polytopes up to 12 vertices. Recently, Chen and Guo [4] computed the numbers of 0/1-equivalence classes of 6-dimensional polytopes for each number of vertices $k \in [13, 64]$. But nowadays it is too hard to list all these $\approx 4.0 \cdot 10^{14}$ classes and investigate their properties explicitly. (If every polytope will ocupy 8 bytes, then all the database will take about 3 petabytes.) Thus, it makes sense to focus on some interesting families of 0/1-polytopes.

A convex polytope P is called *2-neighborly* if any two vertices form a 1-face (i.e. edge) of P. There are at least two reasons for investigation of 2-neighborly 0/1-polytopes:

(1) Let $P_{d,n}$ is a random d-dimensional 0/1-polytope with n vertices. In 2008, Bondarenko and Brodskiy [3] showed, that if $n = O(2^{d/6})$, then the probability $\Pr(P_{d,n}$ is 2-neighborly) tends to 1 as $d \to \infty$. Similar results are obtained by Gillmann [10].
(2) Special 0/1-polytopes, such as the cut polytopes, the traveling salesman polytopes, the knapsack polytopes, the k-SAT polytopes, the 3-assignment polytopes, the set covering polytopes and many others have 2-neighborly faces with superpolynomial (in the dimension) number of vertices [6, 14, 15].

We enumerated and classified all 13 959 358 918 0/1-equivalence classes of 7-dimensional 2-neighborly 0/1-polytopes. It enables us to investigate extremal properties of these polytopes. For example, we have found a 2-neighborly polytope with 14 vertices and 16 facets. This is the first known example of a 2-neighborly polytope (except a simplex) whose number of facets is not greater than the number of vertices plus 2.

In [1], Aichholzer stated the question about the maximal number $N_2(d)$ of vertices of a 2-neighborly d-dimensional 0/1-polytope. He showed that $N_2(6) = 13$, $18 \le N_2(7) \le 24$, $N_2(8) \ge 25$, $N_2(9) \ge 33$, $N_2(10) \ge 44$. We improve these estimations: $N_2(7) = 20$, $28 \le N_2(8) \le 34$, $N_2(9) \ge 38$, $N_2(10) \ge 52$.

The entire database occupies about 1TB. The part of it (in particular, all 6-polytopes) and the list of all f-vectors are available at `https://github.com/maksimenko-a-n/2neighborly-01polytopes`.

Supported by the project № 1.5768.2017/9.10 of P. G. Demidov Yaroslavl State University within State Assignment for Research.

2. Enumeration of 2-neighborly 0/1-polytopes

Every 0/1-polytope is a convex hull of a set $X \subseteq \{0,1\}^d$. Since the natural way for defining a 0/1-polytope is the defining its set of vertices X, in the following we will frequently call X by a "polytope", having in mind the convex hull $\operatorname{conv}(X)$.

We will use the following trivial facts. Every 0/1-polytope $\operatorname{conv}(X)$, $X \subseteq \{0,1\}^d$, is the convex hull of a 0/1-polytope $\operatorname{conv}(X \setminus \{x\})$ and a vector $x \in X$. The same is true for 2-neighborly polytopes. Let P be a 2-neighborly polytope and $X = \operatorname{ext}(P)$ be its set of vertices. Then for every $x \in X$ the polytope $\operatorname{conv}(X \setminus x)$ is also 2-neighborly. Thus, we can enumerate 2-neighborly 0/1-polytopes iteratively, starting with a polytope consisting of a single point and adding one point each time.

Two polytopes are *0/1-equivalent* if one can be transformed into the other by a symmetry of the 0/1-cube. More precisely, this transformation means the using of two operations: permuting of coordinates and replacing some coordinates x_i by $1-x_i$ (*switching*). Thus, one 0/1-equivalence class can contain up to $2^d d!$ of 0/1-polytopes of dimension d. The 0/1-equivalence implies affine and combinatorial equivalences [17, Proposition 7].

Every 0/1-vector $x = (x_1, x_2, \dots, x_d) \in \{0,1\}^d$ can be associated with the binary number "$x_1 x_2 \dots x_d$". Thus, every 0/1-polytope $X \subseteq \{0,1\}^d$ can be naturally associated with the sequence of such binary numbers sorted in increasing order. Let $\mathcal{C}$ be a 0/1-equivalence class (a set of polytopes). A polytope $X \in \mathcal{C}$ is called a *representative* if it is lexicografically less than any other polytope $Y \in \mathcal{C}$. For a given 0/1-polytope Y, the appropriate `representative(`Y`)` can be found with a straightforward branch and bound algorithm.

Therefore, for enumeration of 2-neighborly 0/1-polytopes we can iteratively use the algorithm 1. In the first step, T_1 contains only one polytope $\{(0,\dots,0)\}$.

For testing 2-neighborliness of a 0/1-polytope $X \subseteq \{0,1\}^d$ we use the ideas described in [1, Sec. 2.2]. Let $v, w \in X$ and we want to check the adjacency of v and w in $\operatorname{conv}(X)$. First of all, we switch X to $Y = \{x \oplus v \mid x \in X\}$. (Here, $\oplus$ is a coordinatewise XOR operation.) Hence, v will be switched to 0. It is easy to prove that the vertices $0, y \in Y$ form an edge of a polytope Y iff they form an edge of a polytope $Z = \{z \in Y \mid z \wedge y = z\}$. (Here, $\wedge$ is a coordinatewise AND operation.) Thus, we have to check whether y is in the conical hull

$$\operatorname{cone}(Z) = \left\{ \sum_{z \in Z} \lambda_z z \;\middle|\; \lambda_z \geq 0 \right\}.$$

Namely, vertices 0 and y form an edge in Z iff $y \notin \operatorname{cone}(Z)$. The cheking of $y \notin \operatorname{cone}(Z)$ can be done by solving the corresponding linear programming problem. We did it with COIN-OR Linear Programming Solver [5].

We have run this algorithm on the computer cluster of Discrete and computational geometry laboratory of Yaroslavl state university (`https://dcgcluster.accelcomp.org`). The cluster has a hundred 2.9GHz-cores. After several weeks of computations we had got the rezults collected in Table 1b. Every 0/1-vector $x \in \{0,1\}^d$, $d \leq 8$, we store as a 1-byte integer. Thus, a polytope with n vertices occupies n bytes and all the database — about 173GB.

Our results for the dimension 6 coincide with Aichholzer database [1]. In addition, we enumerate all 2-neighborly 0/1-polytopes of dimension 6 with 13 vertices.

3. Evaluating of combinatorial types and f-vectors

It is well known (see e.g. [13]) that the combinatorial type (face lattice) of a polytope P with vertices $\{v_1, \dots, v_n\}$ and facets $\{f_1, \dots, f_k\}$ is uniquely determined by its facet-vertex incidence matrix $M = (m_{ij}) \in \{0,1\}^{k \times n}$, where $m_{ij} = 1$ if facet f_i contains vertex v_j, and $m_{ij} = 0$

Algorithm 1: The enumeration of 2-neighborly 0/1-polytopes

```
Input  : the dimension d, the array T_n of 2-neighborly 0/1-polytopes with n vertices
         (every polytope is an array of n 0/1-vectors)
Output: the array T_{n+1} of 2-neighborly 0/1-polytopes with n + 1 vertices

Function enumerate_2neighborly(d, T_n)
    for X ∈ T_n do
        for v ∈ {0,1}^d \ X do
            if is_2neighborly(X, v) then
                add representative(X ∪ {v}) to T_{n+1};
            end
        end
    end
    sort T_{n+1} and remove duplicates;
    return T_{n+1};
end

// Let conv(X) be 2-neighborly.  Is conv(X ∪ {v}) 2-neighborly?
Function is_2neighborly(X, v)
    Y := ∅;
    // firstly, test edges for v and x ∈ X
    for x ∈ X do add x ⊕ v to Y;                          // switch v to 0
    for y ∈ Y do
        if no_edge_0y(Y, y) then return false;
    end
    // test edges {x, y} ⊆ X
    for x ∈ X do
        w := v ⊕ x;
        Y := ∅;
        for y ∈ X \ x do add y ⊕ x to Y;                  // switch x to 0
        for y ∈ Y do
            if w ∧ y = w then
                if no_edge_0y(Y ∪ {w}, y) then return false;
            end
        end
    end
    return true;
end

// Isn't {0, y} an edge of conv(Y ∪ {0})?
Function no_edge_0y(Y, y)
    Z := ∅;
    for z ∈ Y \ y do
        if z ∧ y = z then add z to Z;
    end
    if y ∈ cone(Z) then return true;
    return false;
end
```

(A) The dimension 6

vertices	0/1-equivalence classes
1	1
2	6
3	16
4	94
5	445
6	2 528
7	12 359
8	47 445
9	108 220
10	110 032
11	38 221
12	3 222
13	36
total	322 625

(B) The dimension 7

vertices	0/1-equivalence classes
1	1
2	7
3	23
4	191
5	1 510
6	16 373
7	183 209
8	1 985 525
9	18 565 154
10	136 197 421
11	707 274 277
12	2 345 160 234
13	4 456 209 397
14	4 284 931 624
15	1 757 834 961
16	244 831 279
17	8 967 617
18	73 512
19	180
20	3
total	13 962 232 498

TABLE 1. The number of 0/1-equivalence classes of 2-neighborly polytopes of dimensions 6 and 7

otherwise. Thus, polytopes are combinatorially equivalent iff their facet-vertex incidence matrices differ only by column and row permutations.

For every polytope in our database we computed its facet-vertex incidence matrix M by using *lrs* [2]. This evaluation takes about 10 days on the computer cluster with 32 cores. After that, for every matrix M, we computed the canonical form of a vertex-facet digraph of M by using *bliss* [12] (as it was done in [7]). This evaluation takes about 2 days on the computer cluster with 32 cores. Having sorted canonical forms, we have splitted all the polytopes into combinatorial equivalence classes.

For computing f-vector of a polytope from its facet-vertex incidence matrix, we used Kaibel&Pfetsch algorithm [13] and modified it for the case, when the number of vertices is small (an incidence matrix row can be stored in a 32-bit integer). The computing of f-vectors of all polytopes took about two weeks on the cluster.

The results of these computations are collected in Tables 2–4. We enumerate only full-dimensional 0/1-polytopes, since any nonfull-dimensional 0/1-polytope is affinely equivalent to some full-dimensional one [17].

To give an idea of the magnitude of the obtained numbers, we give a couple of examples: f-vector (13, 78, 266, 531, 603, 355, 84) consists of 2 448 144 combinatorial classes; f-vector (9, 36, 82, 114, 97, 48, 12) consists of one combinatorial class with 5 160 979 0/1-equivalence classes.

vertices	**0/1-equivalence classes**	**combinatorial classes**	**f-vectors**
6	237	1	1
7	334	2	2
8	102	8	5
9	10	7	4
10	1	1	1
total	684	19	13

TABLE 2. Full-dimensional 2-neighborly 0/1-polytopes of dimension 5

vertices	**0/1-equivalence classes**	**combinatorial classes**	**f-vectors**
7	9 892	1	1
8	46 813	4	4
9	108 178	81	32
10	110 029	9 651	180
11	38 221	17 782	411
12	3 222	2 730	455
13	36	35	34
total	316 391	30 284	1 117

TABLE 3. Full-dimensional 2-neighborly 0/1-polytopes of dimension 6

vertices	**0/1-equivalence classes**	**combinatorial classes**	**f-vectors**
8	1 456 318	1	1
9	17 588 780	6	6
10	135 330 686	419	108
11	706 996 729	4 790 131	2 090
12	2 345 138 023	271 351 237	17 113
13	4 456 209 206	1 414 858 979	66 929
14	4 284 931 624	2 487 091 476	171 289
15	1 757 834 961	1 431 813 684	303 063
16	244 831 279	231 549 854	382 319
17	8 967 617	8 872 600	282 000
18	73 512	73 444	48 988
19	180	180	180
20	3	3	3
total	13 959 358 918	5 850 402 014	1 274 089

TABLE 4. Full-dimensional 2-neighborly 0/1-polytopes of dimension 7

For every combinatorial type, we store its facet-vertex incidence matrix. If the number of vertices (columns of the matrix) is not greater than 16, one row of the matrix occupies

2 bytes. The average number of facets (rows of the matrix) is 97. Thus, all combinatorial types occupy about 1TB. The part of the database (in particular, all 6-polytopes) is available at `https://github.com/maksimenko-a-n/2neighborly-01polytopes`. The full information can be requested from the author (by e-mail).

4. d-Polytopes with $d+3$ vertices

Combinatorial types of d-polytopes with $d+3$ or less vertices can be enumerated by using Gale diagrams [11, Chap. 6]. For $d+1$ vertices there are only one d-polytope — a simplex. For $d+2$ vertices, the number of combinatorial types is equal to the number of tuples $(m_0, \{m_1, m_{-1}\})$, where $m_0, m_1, m_{-1} \in \mathbb{Z}$, $m_0 \geq 0$, $m_1 \geq 2$, $m_{-1} \geq 2$, and $m_0 + m_1 + m_{-1} = d+2$ [11, Sec. 6.3]. The appropriate polytope is 2-neighborly iff $m_1 \geq 3$, $m_{-1} \geq 3$. For small d, these tuples can be easily enumearted by hands.

The combinatorial type of every d-polytope with $d+3$ vertices is defined by the appropriate reduced Gale diagram or wheel-sequence [9]. We don't list here the properties of these interesting objects, since it was done in [11, 9, 16]. The results of enumerating wheel-sequences by a computer are collected in Table 5. They coincide with the first values of the sequence A114289: `https://oeis.org/search?q=A114289` and with the Fukuda–Miyata–Moriyama collection of d-polytopes for $d \leq 6$ [8].

	$d+2$ vertices			**$d+3$ vertices**		
d	all	2-neighborly polytopes	2-neighborly 0/1-polytopes	all	2-neighborly polytopes	2-neighborly 0/1-polytopes
4	4	1	1	31	1	0
5	6	2	2	116	11	8
6	9	4	4	379	85	81
7	12	6	6	1133	423	419

Table 5. Combinatorial types of d-polytopes with $d+2$ and $d+3$ vertices

For $d \leq 7$, every combinatorial type of a 2-neighborly d-polytope with $d+2$ vertices can be represented by a 0/1-polytope. Almost the same is true for polytopes with $d+3$ vertices. The exceptions are 4 polytopes and the pyramids over them. The first one is a cyclic 4-polytope with 7 vertices. The second can be represented by the wheel-sequence (0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1). It has f-vector (8, 28, 50, 44, 16) and its facet-vertex incidence matrix has two columns with 12 ones as opposed to other polytopes with the same f-vector. The third polytope can be represented by the wheel-sequence (0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1). It has f-vector (8, 28, 51, 47, 18) and its facet-vertex incidence matrix has a column with 14 ones as opposed to other polytopes with the same f-vector. The forth polytope is represented by the sequence (0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1). It has f-vector (9, 36, 80, 103, 72, 22) and its incidence matrix has no column with 12 ones as opposed to other polytopes with the same f-vector.

5. Polytopes with a small number of facets

The minimal and the maximal numbers of facets of 2-neighborly 0/1-polytopes listed in Table 6. As can be seen, there is a 2-neighborly 7-polytope $P_{14,16}$ with 14 vertices and 16 facets. In Figure 1, we list vertices of $P_{14,16}$. As far as we know, any other 2-neighborly polytope (except a simplex) has the property (facets $-$ vertices) ≥ 3. The polytope $P_{14,16}$ has several other special properties. It is the only 2-simple polytope in our database. (A d-polytope is *2-simple* if every $(d-3)$-face is incident to exactly three facets.) All its vertex figures are combinatorially equivalent 6-polytopes with 13 vertices and 11 facets. For any vertex figure of any other polytope in our database, the number of facets is not less than the number of vertices.

dimension 5

vertices	6	7	8	9	10
facets min	6	10	12	16	22
facets max	6	12	20	22	22

dimension 6

vertices	7	8	9	10	11	12	13
facets min	7	11	13	14	17	21	26
facets max	7	16	30	47	55	65	76

dimension 7

vertices	8	9	10	11	12	13	14	15	16	17	18	19	20
facets min	8	12	14	15	18	20	16	39	55	67	100	139	219
facets max	8	20	40	70	104	134	163	198	239	254	281	244	228

Table 6. The number of facets of a 2-neighborly 0/1-polytope

coordinates (rows: vertices)

0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	0	0	0	1	0
0	0	0	0	1	0	0
0	0	0	1	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	1	1
1	0	0	1	1	0	0
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1
1	1	1	0	0	0	0

Figure 1. The 2-neighborly 0/1-polytope with 14 vertices and 16 facets

6. Polytopes with a big number of vertices

Let $N_2(d)$ be the maximal number of vertices of a 2-neighborly d-dimensional 0/1-polytope. In [1], it was showed that $N_2(d-1)+1 \leq N_2(d) \leq 2N_2(d-1)$ and given some estimations for $d \leq 10$. By using Algorithm 1, we improve these estimations (see Table 7).

Acknowledgments

The author is grateful to Sergey Lichak, who helped to run Algorithm 1 on the computer cluster.

	dimension	7	8	9	10
[1]	best example	18	25	33	44
	upper bound	24	48	96	192
new	best example	20	28	38	52
	upper bound	20	34	68	136

Table 7. The maximal number of vertices of a 2-neighborly 0/1-polytope

References

[1] O. Aichholzer. Extremal properties of 0/1-polytopes of dimension 5. In Polytopes — Combinatorics and Computation. Birkhauser, 2000, pp. 111–130, `http://www.ist.tugraz.at/staff/aichholzer/research/rp/rcs/info01poly`

[2] D. Avis. lrs: A revised implementation of the reverse search vertex enumeration algorithm. In Polytopes — Combinatorics and Computation. Birkhauser, 2000, pp. 177–198, `http://cgm.cs.mcgill.ca/~avis/C/lrs.html`

[3] V.A. Bondarenko, A.G. Brodskiy. On random 2-adjacent 0/1-polyhedra. Discrete Mathematics and Applications, 18(2): 181–186, 2008.

[4] W.Y.C. Chen, P.L. Guo. Equivalence classes of full-dimensional 0/1-polytopes with many vertices. Discrete & Computational Geometry, 52(4): 630–662, 2014.

[5] CLP: COIN-OR linear programming solver: `https://github.com/coin-or/Clp`

[6] M.M. Deza, M. Laurent. Geometry of cuts and metrics, Springer, 1997.

[7] M. Firsching. The complete enumeration of 4-polytopes and 3-spheres with nine vertices, preprint, 22 pages. `https://arxiv.org/abs/1803.05205v2`.

[8] K. Fukuda, H. Miyata, S.Moriyama. *Complete enumeration of small realizable oriented matroids*. Discrete & Computational Geometry, **49**(2), 2013, 359–381. The database is available at `http://www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented_matroids/`

[9] É. Fusy. Counting d-polytopes with $d+3$ vertices. The electronic journal of combinatorics, 13, #R23, 2006.

[10] R. Gillmann. 0/1-Polytopes: typical and extremal properties. PhD Thesis, TU Berlin, 2006.

[11] B. Grünbaum. Convex polytope, 2nd edition (V. Kaibel, V. Klee and G.M. Ziegler, eds.), Springer, 2003.

[12] T. Junttila and P. Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In: Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments (ALENEX07), pages 135-149, SIAM, 2007.
bliss: A Tool for Computing Automorphism Groups and Canonical Labelings of Graphs: version 0.73. `http://www.tcs.hut.fi/Software/bliss/`.

[13] V. Kaibel and M. E. Pfetsch. Computing the face lattice of a polytope from its vertex-facet incidences. Comput.Geom., 23(3): 281–290, 2002.

[14] A. Maksimenko. The common face of some 0/1-polytopes with NP-complete nonadjacency relation. Journal of Mathematical Sciences, 203(6): 823–832, 2014.

[15] A. Maksimenko. Boolean quadric polytopes are faces of linear ordering polytopes. Siberian Electronic Mathematical Reports, 14: 640–646, 2017.

[16] A. Maksimenko. The lower bound for the number of facets of a k-neighborly d-polytope with $d+3$ vertices. arXiv:1509.00362.

[17] G.M. Ziegler. Lectures on 0/1-polytopes. In Polytopes — Combinatorics and Computation, 2000, pp. 1–41.

Laboratory of Discrete and Computational Geometry, P.G. Demidov Yaroslavl State University, ul. Sovetskaya 14, Yaroslavl 150000, Russia

E-mail address: `maximenko.a.n@gmail.com`