On the Regions Formed by the Diagonals of a Convex Polygon

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Abstract. For a positive integer $n \geq 3$, the sides and diagonals of a convex n-gon divide the interior of the convex n-gon into finitely (polynomial in n) many regions bounded by them. Here in this article we associate to every region a unique n-cycle in the symmetric group S_n of a certain type (defined as two standard consecutive cycle) by studying point arrangements in the plane. Then we find that there are more (exponential in n) number of such cycles leading to the conclusion that not every region labelled by a cycle appears in every convex n-gon. In fact most of them do not occur in any given single convex n-gon. Later in main Theorem Ω of this article we characterize combinatorially those cycles (defined as definite cycles) whose corresponding regions occur in every convex n-gon and those cycles (defined as indefinite cycles) whose corresponding regions do not occur in every convex n-gon.

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1. Introduction

The regions of a convex n-gon when divided by the diagonals has been studied in various contexts in the literature. To quote a few occurrences of this, we can refer to J.W. Freeman [2], the chapter 9 of R. Honsberger [5], chapter 3 of J. Herman, R. Kucera, J. Simsa [4] and the OEIS sequence A006522 [6]. Here in this article we consider point arrangements in the plane and using the theory of point arrangements we study the regions of a convex n-gon when divided by diagonals and associate to any region a unique two standard consecutive cycle. The method of associating cycle invariants as a combinatorial model to point arrangements in the plane has already been explored by authors J. E. Goodman and R. Pollack [3]. A similar method is explained in

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chapter 10 of the book [1]. Here in this article, we find that for any n, there are $2^{n-1}-n$ (exponential in n) two standard consecutive cycles where as the number of regions in a convex n-gon in which no three diagonals are concurrent at an interior point of the polygon is given by $\frac{(n-1)(n-2)(n^2-3n+12)}{24}$ which is a polynomial in n. Hence we conclude that not every region occurs in every convex n-gon. This leads to definite (those which occur always) cycles/regions and indefinite (those which do not occur always) cycles/regions. It happens that, for $n \leq 5$ all regions are definite. For $n = 6, 7$ the indefinite regions start to appear. However for $n \geq 8$ both the types can be combinatorial characterized. We characterize them in main Theorem Ω. The main theorem is not stated in the beginning and is stated in Section 6 after the required definitions and motivation for these definitions.

2. Definitions

We begin this section with a few definitions.

Definition 2.1 (Point Arrangement in the Plane). Let $\mathcal{P}_n = \{P_1, P_2, \ldots, P_n\}$ be a finite set of n points in the plane $\mathbb{R}^2$. Then $\mathcal{P}_n$ is said to be a point arrangement in the plane if no three points are collinear.

Definition 2.2 (Isomorphism of Point Arrangements). Let $\mathcal{P}^j_n = \{P^j_1, P^j_2, \ldots, P^j_n\}, j = 1, 2$ be two point arrangements in the plane $\mathbb{R}^2$. Then a bijection $\delta : \mathcal{P}^1_n \longrightarrow \mathcal{P}^2_n$ is an isomorphism if for any four points $A, B, C, D \in \mathcal{P}^1_n$, D is in the interior of the triangle formed by A, B, C if and only if $\delta(D)$ is in the interior of the triangle formed by $\delta(A), \delta(B), \delta(C)$.
We say the isomorphism δ is an orientation preserving isomorphism if for any three points the $A, B, C \in \mathcal{P}^1_n$ the orientation $A \longrightarrow B \longrightarrow C \longrightarrow A$ of the triangle ΔABC and the orientation $\delta(A) \longrightarrow \delta(B) \longrightarrow \delta(C) \longrightarrow \delta(A)$ of the triangle $\Delta\delta(A)\delta(B)\delta(C)$ agree.
We say the isomorphism δ is an orientation reversing isomorphism if for any three points the $A, B, C \in \mathcal{P}^1_n$ the orientation $A \longrightarrow B \longrightarrow C \longrightarrow A$ of the triangle ΔABC and the orientation $\delta(A) \longrightarrow \delta(B) \longrightarrow \delta(C) \longrightarrow \delta(A)$ of the triangle $\Delta\delta(A)\delta(B)\delta(C)$ disagree.

Example 2.3. *Here we mention isomorphism classes of point arrangements $\mathcal{P}_n$ for initial values of $n = 3, 4, 5$. Figure 1 illustrates the isomorphism classes of three, four and five point arrangements in the plane. There are two isomorphism classes of four point arrangements and three isomorphism classes of five point arrangements. However there is only one isomorphism class a triangle arrangement for a three point arrangement in the plane.*

Example 2.4. *In this example we mention about orientation preserving and orientation reversing isomorphisms of point arrangements. Let $\mathcal{P}_n = \{P_1, P_2, \ldots, P_n\}$ be a point arrangement such that $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ form a convex n-gon in this anticlockwise manner. Then any bijection $\delta : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ given by a permutation $\pi \in S_n$ defined as $\delta(P_i) = P_{\pi(i)}$ is an*

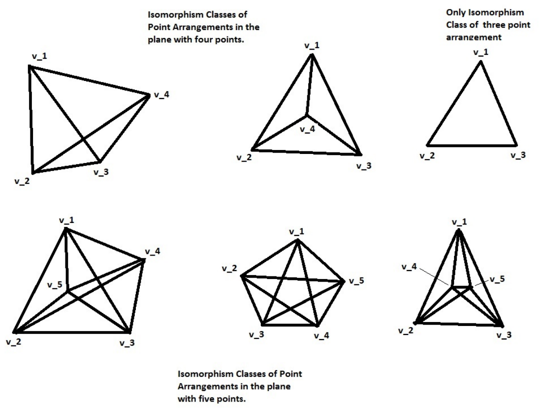

FIGURE 1. Isomorphism Classes of Three, Four, Five Point Arrangements in the Plane

isomorphism of $\mathcal{P}_n$ to itself. However the bijections given by permutations corresponding to the di-hedral group $D_n \subset S_n$ of rotations and reflections on the vertices of a regular n-gon give either an orientation preserving or an orientation reversing isomorphism of $\mathcal{P}_n$ to itself. Remaining permutations induce isomorphisms which are neither orientation preserving nor orientation reversing.

Definition 2.5 (Line Cycle at a Point). Let $\mathcal{P}_n = \{P_1, P_2, \ldots, P_n\}$ be a point arrangement in the plane. Fix $P_i \in \mathcal{P}_n$ for some $1 \leq i \leq n$. Consider the $(n-1)$ lines L^i_j joining P_i, P_j for $1 \leq j \leq n, j \neq i$. An anticlockwise traversal around the point P_i cuts the lines L^i_j in a cycle $\sigma_i \in S_{n-1}$ a symmetric group over the elements $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$. This cycle is defined to be the line cycle at P_i for the arrangement $\mathcal{P}_n$.

Definition 2.6 (Point Cycle at a Point). Let $\mathcal{P}_n = \{P_1, P_2, \ldots, P_n\}$ be a point arrangement in the plane. Fix $P_i \in \mathcal{P}_n$ for some $1 \leq i \leq n$. Consider the $(n-1)$ rays R^i_j joining P_i, P_j for $1 \leq j \leq n, j \neq i$ starting form P_i. An anticlockwise traversal around the point P_i cuts the rays R^i_j in a cycle $\sigma_i \in S_{n-1}$ a symmetric group over the elements $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$. This cycle is defined to be the point cycle at P_i for the arrangement $\mathcal{P}_n$.

3. An Isomorphism Theorem on Point Arrangements in the Plane

First we mention some preliminary observations in Lemma 3.1 and Theorem 3.2 before proving isomorphism Theorem 3.3.

Lemma 3.1. *Let $\mathcal{P}_4 = \{P_1, P_2, P_3, P_4\}$ be a point arrangement in the plane. Then the point P_4 lies in the interior of the triangle $\Delta P_1P_2P_3$ if and only if the line cycle and the point cycle of P_4 for the arrangement $\mathcal{P}_4$ are not equal.*

Proof. This is straight forward verification. ■

Theorem 3.2. *Let $\mathcal{P}_n = \{P_1, P_2, \ldots, P_n\}$ be a point arrangement in the plane. Then a point $P_i, 1 \leq i \leq n$ lies on the convex hull boundary if and only if its point cycle and line cycle for the arrangement $\mathcal{P}_n$ are same. Hence a point $P_i, 1 \leq i \leq n$ lies in the interior of the convex hull if and only if its point cycle and line cycle for the arrangement $\mathcal{P}_n$ are not same.*

Proof. This follows from Lemma 3.1. ■

Theorem 3.3 (Isomorphism Theorem of Point Arrangements). *Let $\mathcal{P}^j_n = \{P^j_1, \ldots, P^j_n\}, j = 1, 2$ be two point arrangements in the plane $\mathbb{R}^2$. For $j = 1, 2$ let σ^j_i be the line cycle at the point P^j_i for the arrangement $\mathcal{P}^j_n$ for $1 \leq i \leq n$. Then a bijection $\delta : \mathcal{P}^1_n \longrightarrow \mathcal{P}^2_n$ is an orientation preserving or an orientation reversing isomorphism if and only if there exists a permutation $\pi \in S_n$ such that*

- *$\delta(P^1_i) = P^2_{\pi(i)}$ and*
- *the line cycle $\sigma^2_{\pi(i)} = \pi\sigma^1_i\pi^{-1}$ for all $1 \leq i \leq n$ or $(\sigma^2_{\pi(i)})^{-1} = \pi\sigma^1_i\pi^{-1}$ for all $1 \leq i \leq n$.*

Proof. Suppose δ is an orientation preserving or an orientation reversing isomorphism. Then the permutation π is given by the defining equation $\delta(P^1_i) = P^2_{\pi(i)}$. Now we restrict the point arrangements $\mathcal{P}^j_n, j = 1, 2$ to four point arrangements given by $\{P^1_r, P^1_s, P^1_t, P^1_u\}$ and $\{P^2_{\pi(r)}, P^2_{\pi(s)}, P^2_{\pi(t)}, P^2_{\pi(u)}\}$ for $1 \leq r < s < t < u \leq n$ and observe the orientations of the four correspoding pairs triangles. We obtain that if δ is orientation preserving then $\sigma^2_{\pi(i)} = \pi\sigma^1_i\pi^{-1}$ for all $1 \leq i \leq n$ and if δ is orientation reversing then we obtain $(\sigma^2_{\pi(i)})^{-1} = \pi\sigma^1_i\pi^{-1}$ for all $1 \leq i \leq n$. Now let us prove the converse. Suppose such a permutation π exists. By renumbering the points of one arrangement and using a reflection if necessary we assume without loss of generality that $\sigma^2_i = \sigma^1_i$ for $1 \leq i \leq n$. Now we restrict to four point sub-arrangements say $\{P^j_r, P^j_s, P^j_t, P^j_u\}, j = 1, 2$. There are two possibilities for any four points. Either they form a convex quadrilateral or the convex hull is a triangle with an interior point. From the line cycles we write the consecutive symbols. For example, in first case, suppose some four points Q_r, Q_s, Q_t, Q_u form a quadrilateral and the line cycles be given by $(rst), (stu), (tur), (urs)$. The consecutive symbols are given by $\{rs, rt, st, su, tr, tu, ur, us\}$ consisting of eight elements. We can also read off the anticlockwise order of the points $Q_r \longrightarrow Q_s \longrightarrow Q_t \longrightarrow Q_u \longrightarrow Q_r$. In the second case let the convex hull be a triangle with an interior point and the line cycles be given by $(tsu), (tru), (urs), (srt)$. The consecutive symbols are given by $\{rs, rt, ru, su, sr, ts, tr, ut, ur\}$ consisting of nine elements. So we conclude that the convex hull is a triangle. The interior point Q_r can be read off from the six consecutive symbols $\{rs, sr, rt, tr, ru, ur\}$ out of the nine consecutive symbols. We also find from the line cycle (tsu) of the point Q_r, the points Q_s, Q_t, Q_u from an triangle in $Q_s \longrightarrow Q_t \longrightarrow Q_u \longrightarrow Q_s$ in this anticlockwise order. Hence the map $P_i^1 \longrightarrow P_i^2, 1 \leq i \leq n$ is an orientation preserving isomorphism. This proves the theorem. ∎

Example 3.4. *Suppose in Theorem 3.3 we replace line cycles by point cycles. It is not true that if there exists $\pi \in S_n$ such that the point cycle $\sigma^2_{\pi(i)} = \pi\sigma^1_i\pi^{-1}$ for all $1 \leq i \leq n$ or $(\sigma^2_{\pi(i)})^{-1} = \pi\sigma^1_i\pi^{-1}$ for all $1 \leq i \leq n$ then the map δ defined as $\delta(P_i^1) = P^2_{\pi(i)}$ is an orientation preserving or an orientation reversing isomorphism. In fact δ need not be an isomorphism at all. Figure 2 gives a counter example. The identity map is not an isomorphism but the*

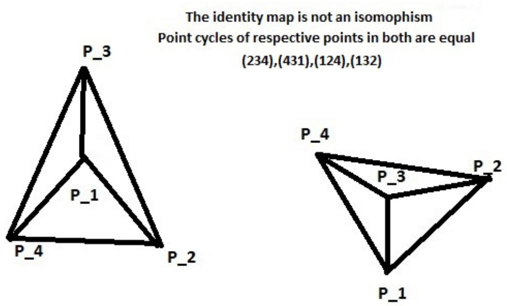

FIGURE 2. Four Point Arrangements with same Point Cycles

point cycles of all the four respective pairs of points are equal. It is necessary to take line cycles in Theorem 3.3, though there is another permutation $\pi \in S_4$ such that $\pi(2) = 1, \pi(3) = 2, \pi(1) = 3, \pi(4) = 4$ for point cycles which gives rise to an isomorphism in Figure 2.

4. The number of regions in a convex n-gon

In this section we define the regions of a convex n-gon which has generic diagonals. We then compute the number of regions formed by the diagonals and sides.

Definition 4.1 (Definition of a side and diagonal of a convex n-gon). Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangements in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$. A side of the convex n-gon is a line P_iP_j with $j \equiv i{\pm}1$ where $1 \leq i \neq j \leq n$. A diagonal of the convex n-gon is a line P_iP_j with $j \not\equiv i{\pm}1$ where $1 \leq i \neq j \leq n$.

Definition 4.2 (Convex n-gon with generic diagonals). Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangements in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$. We say the convex n-gon has generic diagonals if for every three pairs of subscripts $\{i_t, j_t\}, 1 \leq i_t \neq j_t \leq n, 1 \leq t \leq 3$ with

$$\{i_1, j_1\} \cap \{i_2, j_2\} \cap \{i_3, j_3\} = \emptyset,$$

$P_{i_1}P_{j_1}, P_{i_2}P_{j_2}, P_{i_3}P_{j_3}$ do not concur in the plane $\mathbb{R}^2$.

Definition 4.3 (Definition of a Region). Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangements in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. A region is defined to be a connected component of the interior of the convex n-gon when the diagonals and sides are removed.

Now we state the theorem.

Theorem 4.4. *Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Then the number of regions formed by the convex n-gon is given by*

$$\frac{(n-1)(n-2)(n^2-3n+12)}{24}.$$

Proof. For the proof, refer to R. Honsberger [5] chapter 9, pp. 99 – 107 and also J.W.Freeman [2]. ■

5. On the 2-standard consecutive structure of an n-cycle and the Regions of a Convex n-gon

Here in this section, we define i-standard consecutive structure on an n-cycle and observe in Theorem 5.4 that an n-cycle has an unique i-standard consecutive structure for some $1 \leq i \leq n-1$ for $n > 1$. We associate to a region in a convex $n-$gon formed by the diagonals, a 2-standard consecutive cycle. Now we introduce a structure on a permutation as follows.

Definition 5.1. We say an n-cycle $(a_1 = 1, a_2, \ldots, a_n)$ is an i-standard cycle if there exists a way to write the integers $a_i : i = 1, \ldots, n$ as i sequences of inequalities as follows:

$$\begin{gathered}
a_{11} < a_{12} < \ldots < a_{1j_1} \\
a_{21} < a_{22} < \ldots < a_{2j_2} \\
a_{31} < a_{32} < \ldots < a_{3j_3} \\
\vdots \\
a_{i1} < a_{i2} < \ldots < a_{ij_i}
\end{gathered}$$

where $\{a_{st} \mid 1 \leq s \leq i, 1 \leq t \leq j_s\} = \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}, j_1 + j_2 + \ldots + j_i = n$ and i is minimal, that is, there exists no smaller integer with such property and further more that $a_{s(t+1)}$ occurs to the right of a_{st} for every $1 \leq s \leq i$ and $1 \leq t \leq j_s - 1$ in this cycle arrangement $(a_1 = 1, a_2, \ldots, a_n)$.

Define i-standard consecutive structure on an n-cycle as follows.

Definition 5.2. We say an n- cycle $(a_1 = 1, a_2, \ldots, a_n)$ is a consecutive i-standard cycle or a i-standard consecutive cycle if we have

$$a_{s1} < a_{s2} < \ldots < a_{sj_s}$$

and in addition $a_{st} = a_{s1} + (t-1), 1 \leq t \leq j_s, 1 \leq s \leq i$ where $\{a_{st} \mid 1 \leq s \leq i, 1 \leq t \leq j_s\} = \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}, j_1 + \ldots + j_s = n$ and $a_{s(t+1)}$ occurs to the right of a_{st} for every $s = 1, \ldots, i$ and $1 \leq t \leq j_s - 1$ in this cycle arrangement $(a_1 = 1, a_2, \ldots, a_n)$ and i is minimal, that is, there exists no smaller integer with such property. If the minimal value of i is two then we say that the cycle has 2-standard consecutive structure.

Example 5.3. *For example if we consider the* 5*-cycle* $(1, 4, 5, 2, 3)$ *it is a* 2*-standard consecutive cycle. However it has the following two* 2*-standard structures.*

- $1 < 4 < 5, 2 < 3$ *(not consecutive).*
- $1 < 2 < 3, 4 < 5$ *(consecutive).*

Now we prove Theorem 5.4 on the existence and uniqueness of the i-standard consecutive structure on an n-cycle for $n > 1$.

Theorem 5.4 (Existence and Uniqueness of the Consecutive i-Standard Structure on an n-cycle).
For $n > 1$*, there exists* i*-standard consecutive structure for some* $1 \leq i \leq n-1$ *on an* n*-cycle and is uniquely determined.*

Proof. We prove this by induction on i, n as follows. If $i = n = 1$ then there is nothing to prove. The position of the element n is uniquely determined as it should appear in one of them at the end and $(n-1)$ appears before n if $(n-1)$ appears before n in the n-cycle and appears as a single element of standardness if $(n-1)$ appears after n. Now we remove n from the cycle. The remaining cycle is either i-standard on $(n-1)$-elements or $(i-1)$ standard on $(n-1)$-elements. This proves the theorem.
We can actually build this structure in an unique way for the given n-cycle as follows. Write 1 first. Then write $1 < 2$ as it appears later. Then write 3 next to 2 if it appears after 2 or write as a single element of standardness if it appears before 2 and so on. ∎

Now we prove a theorem below which enables us to identify regions by their two standard consecutive cycles.

Theorem 5.5. *Let* $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ *be a point arrangement in* $\mathbb{R}^2$ *such that the points form a convex* n*-gon in the anticlockwise manner* $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ *and which has generic diagonals. Let* R *be a region and* $P_{n+1} \in R$*. Then the line cycle of* P_{n+1} *for the point arrangement* $\mathcal{P}_n \cup \{P_{n+1}\}$ *is a two standard consecutive structure and the cycle is independent of any point* $\mathcal{P}_{n+1} \in R$ *and depends only on the region* R *and moreover different regions of the convex* n*-gon are associated to different cycles.*

Proof. Since the point $P_{n+1} \in R$ and not on the diagonals the finite set $\mathcal{P}_n \cup \{P_{n+1}\}$ is a point arrangement. An anticlockwise traversal around the point P_{n+1} gives a cycle $(a_1 = 1, a_2, \ldots, a_n)$ which has the property that it is obtained by interlacing in some manner two sequences $1 < 2 < \ldots < i$ and $i+1 < i+2 < \ldots < n$ for some $2 < i < n$ giving rise to a two standard consecutive cycle. It is clear that the line cycle depends only on the region R. If R and S are two different regions then there exist three subscripts $i < j < k$ such that $\Delta P_i P_j P_k$ contains R and $\Delta P_i P_j P_k$ does not contain S. In the cycle associated to R the subscripts i, j, k appear as the sub-cycle (ikj) and in the cycle associated to S the subscripts i, j, k appear as the sub-cycle (ijk) and hence they are different. ■

Definition 5.6 (Two standard consecutive cycle of a region). Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Let R be a region. The line cycle associated to any point $P \in R$ is defined to be the two standard consecutive cycle of the region R. Using Theorem 5.5, the cycle is well defined, unique and has the two standard consecutive structure.

Definition 5.7 (Isomorphism between two regions). For $j = 1, 2$, let $\mathcal{P}^j_n = \{P^j_1, \ldots, P^j_n\}$ be two point arrangements in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P^j_1 \longrightarrow P^j_2 \longrightarrow \ldots \longrightarrow P^j_n \longrightarrow P^j_1$ and both of which has generic diagonals. Let R^j be a region in $\mathcal{P}^j_n, j = 1, 2$ respectively. We say R^1 is isomorphic to R^2 if the line cycle of the regions R^1 and R^2 are same, that is, for any two points $P^j_{n+1} \in R^j, j = 1, 2$ the map $\delta : \mathcal{P}^1_n \cup \{P^1_{n+1}\} \longrightarrow \mathcal{P}^2_n \cup \{P^2_{n+1}\}$ given by $\delta(P^1_i) = P^2_i, 1 \leq i \leq n+1$ is an orientation preserving isomorphism of point arrangements.

Example 5.8. *Here is an example in Figure 3 of two non-isomorphic regions R and S in two hexagons and their line cycles* $(145236), (125634)$ *respectively. Later in Theorem 6.3 we will observe that if two regions R and S of two convex n-gons respectively are isomorphic then the regions R and S still need not have the same number of sides. The number of sides can vary.*

Now we enumerate the two standard consecutive cycles.

Lemma 5.9. *Let $T_n \subset S_n$ be the set of 2-standard consecutive n-cycles in S_n.*

1. *We have*
$$\#(T_n) = 2^{n-1} - n.$$
2. *The number of non-isomorphic regions R in a convex n-gon is also $2^{n-1} - n$.*

Proof. This proof of (1) follows by counting the cardinality of T_n. If the 2-standard consecutive structure is given by

- $1 < 2 < 3 < \ldots < j$.
- $j + 1 < j + 2 < \ldots < n$.

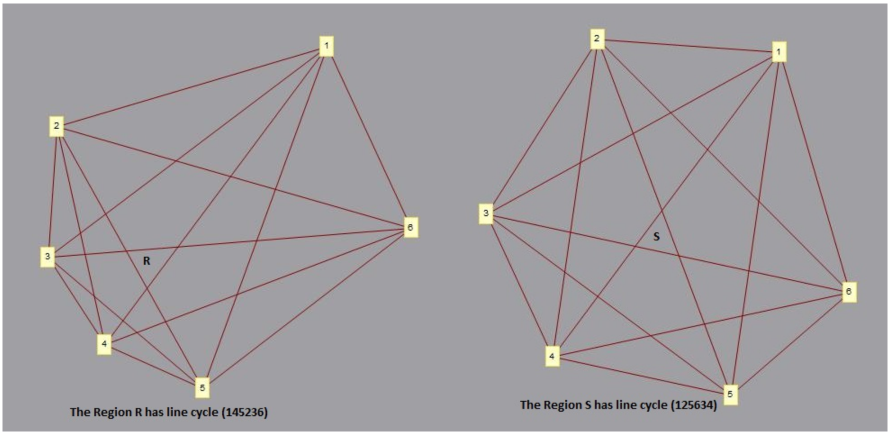

FIGURE 3. Two non-isomorphic regions R and S in hexagons

then the number of such cycles is given by $\binom{n-1}{j-1} - 1$. Hence the total number is given by

$$\sum_{i=2}^{n-1}\left(\binom{n-1}{i-1} - 1\right) = \sum_{i=0}^{n-1}\left(\binom{n-1}{i} - 1\right) = 2^{n-1} - n.$$

Now we construct for every given two standard consecutive cycle, a convex

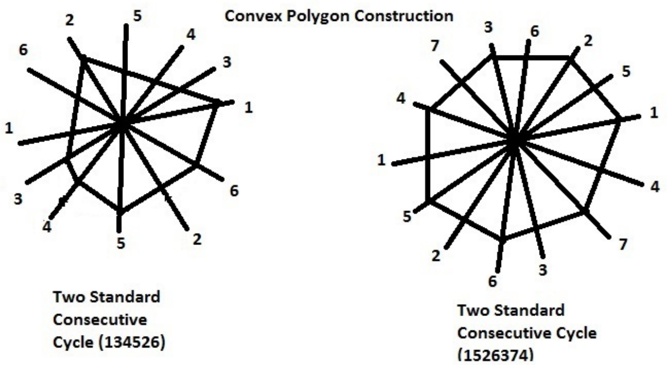

FIGURE 4. Two examples for $n = 6, 7$ and cycles $(134526), (1526374)$ respectively

n-gon and a region R inside it which has the given two standard consecutive cycle. Let $(1 = a_1 a_2 \dots a_n)$ be a given two standard consecutive cycle. The construction is done as follows. Choose a point P in the plane and draw n lines $L_{a_1}, L_{a_2}, \dots, L_{a_n}$ passing through P with increasing angles for L_{a_i} with respect to $L_1 = L_{a_1}$ by assuming L_1 is the X-axis. Let $a_{i_j} = j, 1 \leq j \leq n$.

Choose a point P_1 on the positive X-axis. Now traverse anticlockwise around the point P cutting the lines $L_{a_2}, L_{a_3} \ldots, L_{a_{(i_2-1)}}$ and choose a point P_2 on the ray that we have reached on $L_{a_{i_2}} = L_2$ and continue this process till we choose a point P_n on a suitable ray of $L_{a_{i_n}} = L_n$. In this process we make sure that the we obtain a convex n-gon $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ in the anticlockwise manner with generic diagonals. We refer to Figure 4 for two illustrative examples. This proves the lemma.

■

6. Definite and Indefinite Regions and their Characterization

We have seen in Theorem 4.4 that there are polynomial (in n) number of regions in any convex n-gon with generic diagonals. However in Lemma 5.9 we have seen that there are exponential (in n) number of 2-standard consecutive cycles. So we conclude that not every region labeled by 2-standard consecutive cycle occurs in every convex n-gon with generic diagonals. This motivates the following definition.

Definition 6.1 (Definite and Indefinite Two Standard Consecutive n-cycle). We say a two standard consecutive n-cycle $(a_1 = 1a_2 \ldots a_n)$ is definite if there is a corresponding region R which occurs in every convex n-gon with generic diagonals. Otherwise we say the cycle is indefinite.

Example 6.2. *All regions labelled by two standard consecutive cycles of a triangle, quadrilateral and a pentagon are definite. The first occurence of indefinite regions is when $n = 6$. We refer to Figure 3. The indefinite two standard consecutive cycles are given by*

$$(145236), (125634).$$

6.1. Some Properties of Two Standard Consecutive Cycles Associated to Regions

We prove some properties of two standard consecutive cycles associated to regions in this section. We state the following theorem.

Theorem 6.3. *Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n-gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Let R be a region with associated two standard consecutive cycle $(1 = a_1 a_2 \ldots a_n)$. Let the two standard consective structure be given by*

$$1 < 2 < \ldots < l$$
$$(l+1) < (l+2) < \ldots < n.$$

Then

1. *If P_iP_j is a side of the region R then the elements i, j are consecutive in the cycle $(1 = a_1 a_2 \ldots a_n)$.*

2. *The two standard consecutive cycle for the region S on the other side of the region R across P_iP_j is obtained by swapping i,j in the cycle $(1 = a_1a_2 \dots a_n)$.*
3. *Let P_iP_j be the side of the region R and $j \not\equiv i \pm 1 \mod n$. Then either $i \in \{1, 2, \dots, l\}, j \in \{l+1, l+2, \dots, n\}$ or $j \in \{1, 2, \dots, l\}, i \in \{l+1, l+2, \dots, n\}$.*
4. *The converse is not true, that is, if i,j occur consecutively in $(1 = a_1a_2 \dots a_n)$ such that $j \not\equiv i \pm 1 \mod n, 1 \leq i \leq l, l+1 \leq j \leq n$ then P_iP_j need not be a side of the region R.*
5. *The number of sides of a region R with the same cycle $(1 = a_1a_2 \dots a_n)$ may vary from one convex n-gon to another convex n-gon provided it occurs in them.*
6. *The region R is contained in the triangle $\Delta P_iP_jP_k$ with $i < j < k$ if and only if the cycle $(1 = a_1a_2 \dots a_n)$ contains (ikj) as a sub-cycle.*

Proof.

- We prove (1). The line cycle does not change for any point in R. Hence choosing a point close to P_iP_j in the region R we conclude that i,j occurs consecutively in the line cycle. Moreover if we give anticlockwise orientation to the sides of R and the directed side is $\overrightarrow{ij}$ then i,j appear next to each other with j first and then i second in the the cycle $(1 = a_1a_2 \dots a_n)$.
- We prove (2). If we go across the side P_iP_j to a new region S from R then it is clear that there is swap of i,j in the line cycle $(1 = a_1a_2 \dots a_n)$ to obtain the line cycle for S.
- We prove (3). If P_iP_j is a side of the region R with $j \not\equiv i \pm 1 \mod n$ then it is not the side of the convex n-gon. So there is a region S adjacent to R across P_iP_j. The cycle of S is obtained by swapping i,j in $(1 = a_1a_2 \dots a_n)$. Now this is two standard consecutive if and only if $i \in \{1, 2, \dots, l\}, j \in \{l+1, l+2, \dots, n\}$ or vice-versa.
- We prove (4). We consider the example in Figure 5. In this example the region R is a pentagon with cycle (1526374) and the two standard consecutive structure $1 < 2 < 3 < 4; 5 < 6 < 7$ with $l = 4$. The numbers $1, 5$ appear consecutively in the cycle of R, however, P_1P_5 is not the side of the region R.
- We prove (5). Consider the central region of a regular heptagon. It also has cycle (1526374). This central region is heptagon where as the region R in Figure 5 is a pentagon.
- We prove (6). Let $Q_1, Q_2, Q_3 \in \mathcal{P}_n$. Let Q_4 be in the interior of the convex n-gon. Then Q_4 is in the interior of the triangle $\Delta Q_1Q_2Q_3$ oriented anticlockwise if and only if the line cycle of Q_4 for the point arrangement $\{Q_1, Q_2, Q_3, Q_4\}$ is (132).

This completes the proof of the theorem. ■

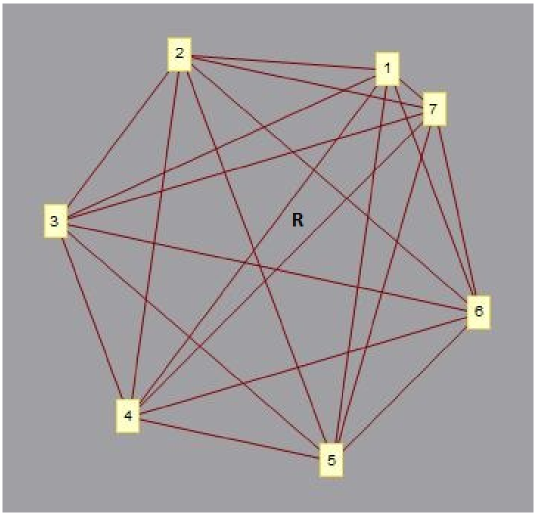

FIGURE 5. A heptagon and a region R with cycle (1526374) respectively

6.2. Characterization of Definite and Indefinite Regions

In this section we characterize the definite and indefinite two standard consecutive n-cycles combinatorially. We begin with a definition.

Definition 6.4 (The diagonal distance of a two standard consecutive cycle). Let $(1 = a_1a_2 \ldots a_n)$ be a two standard consecutive n-cycle. Let the two standard consecutive structure be given by

$$1 < 2 < \ldots < l$$
$$l + 1 < l + 2 < \ldots < n.$$

Consider the set S of all pairs $\{i, j\}$ with $1 \leq i \neq j \leq n$ such that

1. $j \not\equiv i \pm 1 \mod n$,
2. i, j are consecutive in $(1 = a_1a_2 \ldots a_n)$ and
3. $i \in \{1, \ldots, l\}, j \in \{l + 1, \ldots, n\}$ or $j \in \{1, \ldots, l\}, i \in \{l + 1, \ldots, n\}$

The diagonal distance of $(1 = a_1a_2 \ldots a_n)$ is defined as

$$\min_{\{i,j\} \in S} \{(i - j) \bmod n, (j - i) \bmod n\}$$

Here the residue classes mod n are $\{0, 1, \ldots, (n - 1)\}$.

Example 6.5. *Now we consider an illustration of Theorem 6.6 where diagonal distance two cycles are mentioned by taking $n = 6$. There are $2^{6-1} - 6 = 26$ two standard consecutive cycles. There are 24 cycles with diagonal distance two. Now we list the cycles obtained in Case 1 of Theorem 6.6.*

- *Move* 1 $\longrightarrow (134562), (145623), (156234), (162345)$.
- *Move* 2 $\longrightarrow (132456), (134256), (134526), (134562)$.
- *Move* 3 $\longrightarrow (124356), (124536), (124563), (132456)$.

- *Move* $4 \longrightarrow (123546), (123564), (142356), (124356)$.
- *Move* $5 \longrightarrow (123465), (152346), (125346), (123546)$.
- *Move* $6 \longrightarrow (162345), (126345), (123645), (123465)$.

These give $n^2 - 3n = 6^2 - 3.6 = 18$ *cycles which are definite. These are the outermost layer regions of the convex* n*-gon where* $n = 6$*. Now we list the cycles obtained in Case* 2 *of Theorem 6.6.*

- *Move* 1 *and swap* $2, 6 \longrightarrow (134526), (145263), (152634), (126345)$.
- *Move* 2 *and swap* $1, 3 \longrightarrow (124563), (142563), (145263), (145623)$.
- *Move* 3 *and swap* $2, 4 \longrightarrow (142356), (142536), (142563), (134256)$.
- *Move* 4 *and swap* $3, 5 \longrightarrow (125346), (125364), (142536), (124536)$.
- *Move* 5 *and swap* $4, 6 \longrightarrow (123645), (152364), (125364), (123564)$.
- *Move* 6 *and swap* $1, 5 \longrightarrow (156234), (152634), (152364), (152346)$.

These give in addition $n^2 - 5n = 6^2 - 5.6 = 6$ *cycles which are definite. These six cycles correspond to the second outermost layer regions of the convex* n*-gon where* $n = 6$*. We also have two indefinite cycles given by* $(145236), (125634)$*. These total to* $18 + 6 + 2 = 26$ *cycles.*

Now in the following theorem we characterize the two standard consecutive cycle which has diagonal distance two.

Theorem 6.6. *Let* $(1 = a_1 a_2 \ldots a_n)$ *be a two standard consecutive* n*-cycle with the two standard consecutive structure given by* $1 < \ldots < l;\ \ l+1 < \ldots < n$*. Then the diagonal distance of the cycle is two if and only if it is obtained from the cycle* $(12\ldots n)$ *in the following two ways.*

1. *Considering the cyclic notation of the cycle* $(12\ldots n)$ *on a circle in an anticlockwise manner, and moving* i *forward (anticlockwise) for some* $1 \le i \le n-1$ *in a finite number of steps to any position after* $(i+1)$ *and before* $(i-1)$*.*
2. *From any cycle obtained in the previous step by moving* i*, we swap the adjacent elements* $(i-1), (i+1)$*.*

Proof. We prove the reverse implication ($\Leftarrow$) first. Any cycle obtained in steps $(1), (2)$ have diagonal distance two since $(i-1), (i+1)$ considered cyclically are consecutive and either $(i-1) \in \{1, \ldots, l\}, i+1 \in \{l+1, l+2, \ldots, n\}$ or $(i+1) \in \{1, \ldots, l\}, i-1 \in \{l+1, l+2, \ldots, n\}$. This proves that the diagonal distance is two.
Now we prove the forward implication. Suppose the diagonal distance is two. Then first we construct a convex n-gon with vertices $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ in this anticlockwise manner such that the cycle $(1 = a_1 a_2 \ldots a_n)$ appears as a region using Figure 4 in Lemma 5.9. Since the diagonal distance is two, there exists $1 \le i \le n$ such that $i-1, i+1$ appear consecutively and either $(i-1) \in \{1, \ldots, l\}, i+1 \in \{l+1, l+2, \ldots, n\}$ or $(i+1) \in \{1, \ldots, l\}, i-1 \in \{l+1, l+2, \ldots, n\}$. Now using the convex n-gon, by applying moves of the type mentioned in step $(1), (2)$ on the cycle, we just cross the regions to reach the outside of convex n-gon via crossing either the side $P_{i-1}P_i$ or P_iP_{i+1}. Here the cycle we arrive at must be the cycle $(12\ldots n)$. Hence this proves the forward implication thereby completing the proof of the theorem. ■

Now we prove a lemma which is used in the proof of Theorem Ω. This lemma describes certain two standard consecutive $(n-1)$-cycles of diagonal distance two which upon adding n gives two standard consecutive n-cycles of diagonal distance two.

Lemma 6.7. *Let $7 \leq n \in \mathbb{N}$. Let $(1 = a_1a_2\ldots a_{n-1})$ be a two standard consecutive $(n-1)$-cycle with diagonal distance two. For some $1 \leq i \leq n-1$ let*

$$(i-1) \bmod (n-1), (i+1) \bmod (n-1) \in \{1, \ldots, (n-1)\}$$

appear consecutively in the cycle and suppose $i \notin \{1, 2, (n-1), (n-2)\}$, that is,

$$\{(i-1) \bmod (n-1), (i+1) \bmod (n-1)\} \notin \{\{1, n-2\}, \{2, n-1\}, \{1, 3\}, \{n-3, n-1\}\}.$$

If we add n to the cycle and obtain a two standard consecutive n-cycle then it also has diagonal distance two.

Proof. If we add n to $(1 = a_1a_2\ldots a_{n-1})$ in the right of both $(i-1) \bmod (n-1), (i+1) \bmod (n-1)$, the cycle still has diagonal distance two unless $\{(i-1) \bmod (n-1), (i+1) \bmod (n-1)\} \in \{\{2, n-1\}, \{1, n-2\}\}$. We cannot add n after 1 and to the left of both of them as the resulting cycle will not be two standard consecutive. Now if we add n in between $(i-1) \bmod (n-1)$ and $(i+1) \bmod (n-1)$ then only the following possibilities occur.

(1) $(1 = a_1a_2\ldots a_{n-1}n) = (1 = a_1\ldots 3n)$ with $a_{n-1} = 3$ or
(2) $(1 = a_1a_2\ldots a_{n-1}n) = (1(n-1)2\ldots(n-2)n)$ with $a_2 = (n-1), a_j = j-1, 3 \leq j \leq n-1$ or
(3) $(1 = a_1a_2\ldots a_{n-1}n) = (12\ldots j(n-1)(j+1)\ldots(n-2)n)$ for some $2 \leq j \leq (n-3)$
(4) $(1 = a_1\ldots(n-1)n2\ldots a_{n-1}) = (1j\ldots(n-2)(n-1)n2\ldots(j-1))$ with $4 \leq j \leq n-2$ or
(5) $(1 = a_1a_2na_3\ldots a_{n-1}) = (1(n-1)n2\ldots(n-2))$ with $a_2 = (n-1), a_j = j-1, 3 \leq j \leq n-1$ or
(6) $(1 = a_1a_2\ldots a_{n-2}na_{n-1}) = (13\ldots(n-1)n2)$ with $a_{n-1} = 2, a_j = (j+1), 2 \leq j \leq n-2$ or
(7) $(1 = a_1a_2\ldots a_{n-4}a_{n-3}na_{n-2}a_{n-1}) = (123\ldots(n-4)(n-1)n(n-3)(n-2))$ with $a_j = j, 2 \leq j \leq n-4, a_{n-3} = n-1, a_{n-2} = n-3, a_{n-1} = n-2$.
(8) $(1 = a_1a_2\ldots a_{n-3}a_{n-2}na_{n-1}) = (1\ldots j(n-2)(j+1)\ldots(n-4)(n-1)n(n-3))$ for some $2 \leq j \leq n-5, a_{n-3} = n-4, a_{n-2} = n-1, a_{n-1} = n-3$.

In these cases we have $\{(i-1) \bmod (n-1), (i+1) \bmod (n-1)\} \in \{\{1, n-2\}, \{2, n-1\}, \{1, 3\}, \{n-1, n-3\}\}$. This proves the lemma. ■

Now we prove the main theorem of the article.

Theorem Ω. Let $(1 = a_1a_2\ldots a_n)$ be a two standard consecutive n-cycle.

(A) Then this cycle is definite if and only if it is obtained from the cycle $(12\ldots n)$ in the following three ways.

(a) Considering the cyclic notation of the cycle $(12\dots n)$ on a circle in an anticlockwise manner, and moving i forward (anticlockwise) for some $1 \leq i \leq n-1$ in a finite number of steps to any position after $(i+1)$ and before $(i-1)$. The diagonal distance of these cycles is two.

(b) From any cycle obtained in the previous step by moving i, we swap the adjacent elements $(i-1), (i+1)$. The diagonal distance of these cycles is also two.

(c) $n = 7$ and $(1 = a_1 a_2 \dots a_n) = (1526374)$. This is the only cycle which has diagonal distance more than two and is definite. This phenomenon occurs only when $n = 7$.

(B) Then this cycle is indefinite if and only if there exists

$$1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq n$$

with the following property that if for some $j \in \{0, 1, 2, 3, 4, 5\}$

$$a_{i_{j+1}} = \min\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}\} \text{ with } a_{i_{j+t}} = a_{i_{((j+t) \bmod 6)}}$$

where $((j+t) \bmod 6) \in \{1, 2, 3, 4, 5, 6\}$ then we have either

$$a_{i_{j+1}} < a_{i_{j+4}} < a_{i_{j+5}} < a_{i_{j+2}} < a_{i_{j+3}} < a_{i_{j+6}}$$

or

$$a_{i_{j+1}} < a_{i_{j+2}} < a_{i_{j+5}} < a_{i_{j+6}} < a_{i_{j+3}} < a_{i_{j+4}}.$$

Here in the subscripts the local cycles $(145236), (125634)$ appear which are the indefinite cycles for $n = 6$.

The proof of Theorem Ω is given after the following example.

Example 6.8. *We illustrate Theorem* $\Omega(B)$ *in this example. Clearly the cycles* $(145236), (125634)$ *are indefinite using Theorem* $\Omega(B)$*. Consider the two standard consecutive cycle*

$$(1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8) = (15263748).$$

Choose $i_1 = 1, i_2 = 2, i_3 = 4, i_4 = 5, i_5 = 7, i_6 = 8$*. Choose* $j = 0$*. We have*

$$a_{i_{j+1}} = a_{i_1} = a_1 = 1 < a_{i_{j+4}} = a_{i_4} = a_5 = 3 < a_{i_{j+5}} = a_{i_5} = a_7 = 4 <$$
$$a_{i_{j+2}} = a_{i_2} = a_2 = 5 < a_{i_{j+3}} = a_{i_3} = a_4 = 6 < a_{i_{j+6}} = a_{i_6} = a_8 = 8.$$

So using Theorem $\Omega(B)$ *we have that the cycle is indefinite. Also observe that this can be expressed by the fact that* (815634) *is a sub-cycle.*
Consider the two standard consecutive cycle

$$(1 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8) = (15263784).$$

Choose $i_1 = 2, i_2 = 3, i_3 = 5, i_4 = 6, i_5 = 7, i_6 = 8$*. Choose* $j = 1$*. We have*

$$a_{i_{j+1}} = a_{i_2} = a_3 = 2 < a_{i_{j+2}} = a_{i_3} = a_5 = 3 < a_{i_{j+5}} = a_{i_6} = a_8 = 4 <$$
$$a_{i_{j+6}} = a_{i_1} = a_2 = 5 < a_{i_{j+3}} = a_{i_4} = a_6 = 7 < a_{i_{j+4}} = a_{i_5} = a_7 = 8.$$

So using Theorem $\Omega(B)$ *we have that the cycle is indefinite. Also observe that this can be expressed by the fact that* (452378) *is a sub-cycle.*

Proof of Theorem Ω. First we prove the reverse ($\Leftarrow$) implication in (A). The cycles obtained in $A(a)$ and $A(b)$ exactly correspond to the regions for which one of the following n diagonals

$$P_1P_3, P_2P_4, \ldots, P_{n-1}P_1, P_nP_2$$

is a side and these clearly are definite regions. These cycles have diagonal distance exactly two. There are $2n^2 - 8n$ such cycles with corresponding regions. For $n = 7$ there are $2^{7-1} - 7 = 57$ two standard consecutive cycles. There are $\frac{(7-1)(7-2)(7^2-3*7+12)}{24} = 50$ regions in a heptagon with generic diagonals. Now in Figure 3 we have seen for the hexagon that the cycles (145236) and (125634) are indefinite and mutually exclusive, that is, if one occurs then the other does not occur. Extending this scenario for $n = 7$ we conclude that there are seven pair of mutually exclusive two standard consecutive cycles. They are obtained by cyclically shifting as follows.

(I) First pair $(1523674), (1256347)$. We ignore 1 and the oriented triangle $\Delta\overrightarrow{P_5P_2}\overrightarrow{P_3P_6}\overrightarrow{P_7P_4}$ containing the region (1523674) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_5P_2}\overrightarrow{P_3P_6}\overrightarrow{P_7P_4}$ containing the region (1256347) is anticlockwise. We illustrate this in Figure 6.

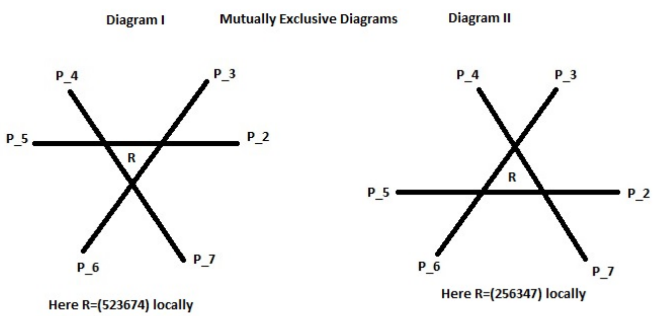

FIGURE 6. Local Triangles ignoring 1 containing mutually exclusive cycles/regions $(1523674), (1256347)$ respectively

(II) Second pair $(1526347), (1236745)$. We ignore 2 and the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_3}\overrightarrow{P_4P_7}$ containing the region (1526347) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_3}\overrightarrow{P_4P_7}$ containing the region (1236745) is anticlockwise.

(III) Third pair $(1263745), (1562347)$. We ignore 3 and the oriented triangle $\Delta\overrightarrow{P_5P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_4}$ containing the region (1263745) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_5P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_4}$ containing the region (1562347) is anticlockwise.

(IV) Fourth pair $(1562374), (1267345)$. We ignore 4 and the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_2}\overrightarrow{P_3P_7}$ containing the region (1562374) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_2}\overrightarrow{P_3P_7}$ containing the region (1267345) is anticlockwise.

(V) Fifth pair $(1526734),(1456237)$. We ignore 5 and the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_3}$ containing the region (1526734) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_3}$ containing the region (1456237) is anticlockwise.

(VI) Sixth pair $(1452637),(1256734)$. We ignore 6 and the oriented triangle $\Delta\overrightarrow{P_1P_4}\overrightarrow{P_5P_2}\overrightarrow{P_3P_7}$ containing the region (1452637) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_1P_4}\overrightarrow{P_5P_2}\overrightarrow{P_3P_7}$ containing the region (1256734) is anticlockwise.

(VII) Seventh pair $(1256374),(1452367)$. We ignore 7 and the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_5}\overrightarrow{P_6P_3}$ containing the region (1256374) is oriented clockwise where as the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_5}\overrightarrow{P_6P_3}$ containing the region (1452367) is anticlockwise.

Now there are $2n^2 - 8n = 2.7^2 - 8 * 7 = 42$ definite cycles whose regions definitely occur using $A(a), A(b)$. These cycles can be written down. Out of the remaining 15 cycles there are seven mutually exclusive pairs. Totally there are 50 regions that occur in a heptagon with generic diagonals. Hence there is one more definite cycle which occurs that is cycle (1526374). This completes the proof of reverse implication $(\Leftarrow)$ of Theorem $\Omega(A)$.

Now we prove the reverse implication $(\Leftarrow)$ in (B). In both the cases mentioned in (B) we can obtain a mutually exclusive cycle for the cycle $(1 = a_1a_2\ldots a_n)$ by orienting the local triangle the other way similar to the diagrams given in Figure 6. We illustrate this in Figure 7. This proves that the cycle $(1 = a_1a_2\ldots a_n)$ is indefinite. This completes the proof of reverse implication $(\Leftarrow)$ of Theorem $\Omega(B)$.

To complete the proof of Theorem Ω it is enough to prove that if for a two standard consecutive cycle $(1 = a_1a_2\ldots a_n)$, the diagonal distance is more than two and $n \geq 8$ (Note: $n \neq 7$) then the cycle is indefinite and satisfies the property mentioned in Theorem $\Omega(B)$.

First we show that for $n = 8$ the only definite cycles are those with diagonal distance two explicitly. For this we list all the 120 two standard consecutive cycles as a union of fifteen orbits each containing eight cycles under the cyclic shift action of $\mathbb{Z}/8\mathbb{Z}$. The cycles in the first eight orbits $1,\ldots,8$ have diagonal distance two and satisfy the conditions of Theorem Ω $A(a), A(b)$. The cycles in the remaining seven orbits $9,\ldots,15$ satisfy the conditions of Theorem $\Omega(B)$. The cycles in these orbits do not have diagonal distance two. We consider the one standard consecutive cycle (12345678) on a circle in anticlockwise cyclic manner.

1. Move element i forward (anticlockwise) by one position in (12345678) for $1 \leq i \leq 8$. This is also same as moving another element i forward by six positions in (12345678) for $1 \leq i \leq 8$.
   $(13456782),(13245678),(12435678),(12354678),$
   $(12346578),(12345768),(12345687),(18234567)$.
2. Move element i forward by two positions in (12345678) for $1 \leq i \leq 8$. This is also same as moving another element i forward by one position

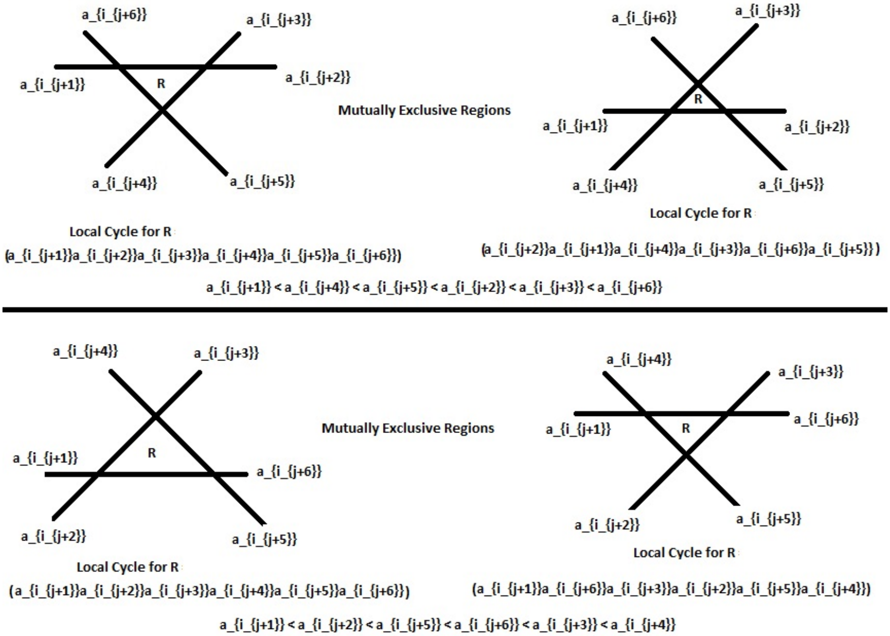

FIGURE 7. Mutually Exclusive Indefinite Regions

in (12345678) and swapping $(i-1),(i+1)$ considered cyclically for $1 \leq i \leq 8$. They are given as:
$(14567823),(13425678),(12453678),(12356478),$
$(12346758),(12345786),(17234568),(12834567).$

3. Move element i forward by three positions in (12345678) for $1 \leq i \leq 8$.
$(15678234),(13452678),(12456378),(12356748),$
$(12346785),(16234578),(12734568),(12384567).$
4. Move element i forward by four positions in (12345678) for $1 \leq i \leq 8$.
$(16782345),(13456278),(12456738),(12356784),$
$(15234678),(12634578),(12374568),(12348567).$
5. Move element i forward by five positions in (12345678) for $1 \leq i \leq 8$.
$(17823456),(13456728),(12456783),(14235678),$
$(12534678),(12364578),(12347568),(12345867).$
6. Move element i forward by two positions in (12345678) and swap $(i-1),(i+1)$ considered cyclically for $1 \leq i \leq 8$. This is also same as moving another element i forward by five positions in (12345678) and swapping $(i-1),(i+1)$ considered cyclically for $1 \leq i \leq 8$. They are given as:
$(14567283),(14256783),(14253678),(12536478),$
$(12364758),(12347586),(17234586),(17283456).$
7. Move element i forward by three positions in (12345678) and swap $(i-1),(i+1)$ considered cyclically for $1 \leq i \leq 8$.

(15672834), (14526783), (14256378), (12536748),
(12364785), (16234758), (12734586), (17238456).

8. Move element i forward by four positions in (12345678) and swap $(i-1),(i+1)$ considered cyclically for $1 \leq i \leq 8$.
(16728345), (14562783), (14256738), (12536784),
(15236478), (12634758), (12374586), (17234856).
9. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12563478) contains the sub-cycle (125634).
(12563478), (12367458), (12347856), (16723458),
(12783456), (14567238), (12567834), (14523678).
10. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12563748) contains the sub-cycle (125634).
(12563748), (12367485), (16234785), (16273458),
(12738456), (15672384), (15267834), (14526378).
11. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12567348) contains the sub-cycle (125634).
(12567348), (12367845), (15623478), (12673458),
(12378456), (15672348), (12678345), (14562378).
12. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12567384) contains the sub-cycle (125634).
(12567384), (15236784), (15263478), (12637458),
(12374856), (16723485), (16278345), (14562738).
13. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12563784) contains the sub-cycle (125634).
(12563784), (15236748), (12634785), (16237458),
(12734856), (16723845), (15627834), (14526738).
14. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12637485) contains the sub-cycle (126745) locally ignoring $3, 8$.
(12637485), (16237485), (16273485), (16273845),
(15627384), (15267384), (15263784), (15263748).
15. The following cycles satisfy the condition of Theorem $\Omega(B)$. For example the cycle (12673845) contains the sub-cycle (126745) locally ignoring $3, 8$.
(12673845), (15623784), (15267348), (12637845),
(15623748), (12673485), (16237845), (15627348).

This completes the proof of Theorem Ω for $n = 8$ the base case of the induction step.

Now we show by induction on $k = n \geq 9$ we show if the cycle is indefinite then the diagonal distance is not two and satisfies the criterion given in Theorem $\Omega(B)$.

Using Lemma 6.7 we first consider cycles of diagonal distance two and hence definite cycles for $k = n - 1 \geq 8$ of the form $(1 = a_1 a_2 \ldots a_{n-1})$ which satisfy one of the following properties.

(a) The cycle contains $2,(n-1)$ consecutively with $(n-1)$ first and 2 next (has diagonal distance two).
(b) The cycle contains $2,(n-1)$ consecutively with 2 first and $(n-1)$ next (has diagonal distance two).
(c) The cycle contains $1,(n-2)$ consecutively with 1 first and $(n-2)$ next to it (which has diagonal distance two).
(d) The cycle contains $1,(n-2)$ consecutively with 1 first and $(n-2)$ at the end (which has diagonal distance two).
(e) The cycle contains $(n-1),(n-3)$ consecutively with $(n-1)$ first and $(n-3)$ next (has diagonal distance two).
(f) The cycle contains $(n-3),(n-1)$ consecutively with $(n-3)$ first and $(n-1)$ next (has diagonal distance two).
(g) The cycle contains $1,3$ consecutively with 1 first and 3 next to it (which has diagonal distance two).
(h) The cycle contains $1,3$ consecutively with 1 first and 3 at the end (which has diagonal distance two).

These cycles upon adding n may or may not remain definite. The remaining definite cycles upon adding n will remain definite using Lemma 6.7. We will later consider indefinite $(n-1)$-cycles. We mention the various cases (i)-(xvi) and prove in each case that upon adding n the resulting two standard consecutive cycle will either have diagonal distance two and hence remain definite or it does not have diagonal distance two and satisfies the criterion of Theorem Ω (so becomes indefinite).

(i) If the cycle $(1=a_1a_2\ldots a_{n-1})$ contains $2,(n-1)$ consecutively with $(n-1)$ first and 2 next (has diagonal distance two) then the cycle is of the following form

$$(1j\ldots(n-2)(n-1)23\ldots(j-1)) \text{ for some } 4\leq j\leq n-2 \text{ or}$$
$$\text{either } (134\ldots(n-1)2) \text{ or } (1(n-1)23\ldots(n-2)).$$

It is clear that upon adding n to $(134\ldots(n-1)2)$ the resulting two standard consecutive cycle has diagonal distance two and hence definite. If we add n in between $(n-1)$ and 2 or just next to 2 in any of these the resulting two standard consecutive cycle also has diagonal distance two and hence definite. If we add n to $(1(n-1)23\ldots(n-2))$ anywhere after 3 then the resulting two standard consecutive cycle also has diagonal distance two and hence definite. If we add n to $(1j\ldots(n-2)(n-1)23\ldots(j-1))$ with $4\leq j\leq n-2$, anywhere after 3 then the resulting two standard consecutive cycle does not have diagonal distance two and is indefinite because it has $(23n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(ii) $2,(n-1)$ appear in this order next to 1 as follows. Now consider the cycle
$$\big(12(n-1)34\ldots(n-4)(n-3)(n-2)\big).$$
If we add n anywhere after $(n-1)$ and before $(n-3)$ the resulting two standard cycle does not have diagonal distance two and is indefinite

because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$. If we add n in between $(n-3)$ and $(n-2)$ or after $(n-2)$ then the cycle is definite and has diagonal distance two.

(iii) $2,(n-1)$ appear in this order in the middle as follows. Now consider for some $3 \leq a \leq n-4$ the cycle

$$(1(a+2)\dots(n-2)2(n-1)3\dots a(a+1)).$$

If we add n anywhere between $(n-1)$ and a, then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)na(a+1))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$. If we add n in between a and $a+1$ and obtain the two standard consecutive cycle $(15\dots(n-2)2(n-1)3n4)$ for $a=3$ then it does not have diagonal distance two and is indefinite because it has $(23n156)$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$ since $n \geq 8$. This argument does not work for $n=7$. This is because (152634) has diagonal distance two for $k=6$ and if we add $n=7$ in between $(n-1)=6$ and 4 we get (1526374) which does not have (237156) as a sub-cycle. In fact (1526374) does not have diagonal distance two and also does not satisfy the criterion of Theorem $\Omega(B)$. It is a definite cycle for $n=7$ and this is the only exception phenomenon. If we add n after $(a+1)$ or for $a>3$ we add n in between a and $(a+1)$ then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(34n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(iv) $2,(n-1)$ appear in this order at the last but one or at the end positions as follows. Now consider the cycles

$$(14\dots(n-2)2(n-1)3) \text{ and } (134\dots(n-2)2(n-1)).$$

These cycles are definite as $1,3$ appear consecutively and has diagonal distance two.

(v) $1,(n-2)$ appear in the beginning and $(n-1)$ appears at the end as follows. Now consider the cycle

$$\big(1(n-2)23\dots(n-3)(n-1)\big).$$

n can only be added at the end and the resulting two standard cycle is definite and has diagonal distance two and it has $(n-3),(n-1)$ as consecutive.

(vi) $1,(n-2)$ appear in the beginning and $(n-1)$ appears at the last but one position as follows. Now consider the cycle

$$\big(1(n-2)23\dots(n-4)(n-1)(n-3)\big).$$

If n is added at the end then the the resulting two standard cycle is definite and has diagonal distance two and it has $(n-3),(n-1)$ as consecutive. If n is added in between $(n-1)$ and $(n-3)$ then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(23(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(vii) $1,(n-2)$ appear in the beginning and $(n-1)$ appears in between 3 and $(n-4)$ as follows. Now consider the cycle

$$\big(1(n-2)23\dots(n-1)\dots(n-4)(n-3)\big).$$

If we add n at the end then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $((n-4)(n-3)n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$. If we add n after $(n-1)$ and before $(n-3)$ then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(23(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(viii) $1,(n-2)$ appear in the beginning and $(n-1)$ appears in between 2 and 3 as follows. Now consider the cycle

$$\big(1(n-2)2(n-1)34\dots(n-4)(n-3)\big).$$

If we add n anywhere after 4 then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(34n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$. If we add n in between $(n-1)$ and before 4 then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n45)$ (Note: $n \geq 8$) as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$. For $n=7$ this argument does not work. This is because (152634) has diagonal distance two for $k=6$ and if we add $n=7$ in between $(n-1)=6$ and 4 we get (1526374) which does not have (126745) as a sub-cycle. In fact (1526374) does not have diagonal distance two and also does not satisfy the criterion of Theorem $\Omega(B)$. It is a definite cycle for $n=7$ and this is the only exception phenomenon.

(ix) $1,(n-2)$ appear in the beginning and $(n-1)$ appears in between $(n-2)$ and 2 as follows. Now consider the cycle

$$\big(1(n-2)(n-1)23\dots(n-4)(n-3)\big).$$

If we add n anywhere after 3 then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(23n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$. If we add n in between 2 and 3 or in between $(n-1)$ and 2 then the the resulting two standard cycle has diagonal distance two and is definite with $2,n$ as consecutive.

(x) $1,(n-2)$ appear consecutively with 1 first and $(n-2)$ at the end as follows. Now consider the cycle

$$\big(1\dots(n-1)\dots(n-2)\big).$$

If we add n at the end after $(n-2)$ or just before $(n-2)$ then the resulting two standard cycle has diagonal distance two and is definite. If we add n after $(n-1)$ and before $(n-2)$ then the cycles $(1(n-1)2\dots n\dots(n-3)(n-2)),(1(n-1)n2\dots(n-3)(n-2))$ has diagonal distance two and are definite. The cycle $(12\dots(n-1)\dots n\dots(n-3)(n-2))$ does not have diagonal distance two and is indefinite because it has

$(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(xi) $(n-1),(n-3)$ appear consecutively in this order and $(n-2)$ appears after $(n-3)$. Now consider the cycle

$$\big(123\ldots(n-1)(n-3)(n-2)\big).$$

If we add n after $(n-3)$ then the resulting two standard cycle has diagonal distance two and is definite. If we add n in between $(n-1)$ and $(n-3)$ then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(xii) $(n-1),(n-3)$ appear consecutively in this order and $(n-2)$ appears before $(n-3)$. Now consider the cycle

$$\big(1\ldots(n-2)\ldots(n-1)(n-3)\big).$$

If we add n after $(n-3)$ then the resulting two standard cycle has diagonal distance two and is definite. If we add n in between $(n-1)$ and $(n-3)$ then the cycles $\big(1\ldots(n-5)(n-4)(n-2)(n-1)n(n-3)\big)$ and $\big(1\ldots(n-5)(n-2)(n-4)(n-1)n(n-3)\big)$ has diagonal distance two and are definite. The cycle $\big(1\ldots(n-2)\ldots(n-5)(n-4)(n-1)n(n-3)\big)$ does not have diagonal distance two and is indefinite because it has $((n-5)(n-4)(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(xiii) $(n-3),(n-1)$ appear consecutively in this order and $(n-2)$ appears before $(n-3)$. Now consider the cycle

$$\big(1\ldots(n-2)\ldots(n-3)(n-1)\big).$$

Now n has to be added at the end and the resulting two standard cycle has diagonal distance two and is definite.

(xiv) $(n-3),(n-1)$ appear consecutively in this order and $(n-2)$ appears after $(n-3)$. Now consider the cycle

$$\big(1\ldots(n-1)(n-3)(n-2)\big).$$

If n is added after $(n-3)$ then the resulting two standard cycle has diagonal distance two and is definite. If n is added in between $(n-1)$ and $n-3$ then the resulting two standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

(xv) $1,3$ appear consecutive with 1 first and 3 just next to it. Now consider the cycle

$$\big(13\ldots2\ldots(n-1)\big),\big(13\ldots(n-1)2\big).$$

Now n must be added at the end in $\big(13\ldots2\ldots(n-1)\big)$ and the resulting two standard cycle has diagonal distance two and is definite. In $\big(13\ldots(n-1)2\big)$, n must be added after $(n-1)$ and the resulting two standard cycle has diagonal distance two and is definite.

(xvi) $1, 3$ appear consecutive with 1 first and 3 at the end. Now consider the cycle

$$\big(1 \dots 2 \dots (n-1)3\big).$$

If n is added in between $(n-1)$ and 3 then the resulting two standard cycle has diagonal distance two and is definite. If n is added after 3 at the end then the cycles $(1425 \dots (n-1)3n), (1245 \dots (n-1)3n)$ have diagonal distance two and are definite. The cycle $(145 \dots 2 \dots (n-1)3n)$ does not have diagonal distance two and is indefinite because it has $(23n145)$ as a sub-cycle which satisfies the criterion of Theorem $\Omega(B)$.

If a two standard consecutive $(n-1)$-cycle $(1 = a_1a_2 \dots a_{n-1})$ is already indefinite satisfying the criterion of Theorem $\Omega(B)$ then it satisfies the criterion of Theorem $\Omega(B)$ after adding n and obtain a two standard consecutive n-cycle. We will now show that it does not have diagonal distance two. Let $(1 = a_1 \dots a_{n-1})$ be an indefinite $(n-1)$-cycle. By induction it does not have diagonal distance two. After adding n we show that n cannot be adjacent to 2 or $(n-2)$. Suppose the $(n-1)$-cycle is given by $(1 = a_1 \dots (n-2) \dots (n-1) \dots a_{n-1})$. Then n should appear after $(n-1)$ unless the cycle is $(12 \dots (n-3)n(n-2)(n-1))$ or $(12 \dots (n-2)n(n-1))$ which is impossible. Hence n and $(n-2)$ cannot be adjacent. Suppose $(n-2)$ appears after $(n-1)$ in the indefinite $(n-1)$-cycle $(1 = a_1 \dots a_{n-1})$. Then it is given by $(1 = a_1 \dots (n-1) \dots (n-2) = a_{n-1})$ which has diagonal distance two. Hence a contradiction. By a similar reasoning n cannot appear adjacent to 2. So the resulting two standard consecutive n-cycle does not have diagonal distance 2. This completes the proof of main Theorem Ω. ■

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