LUCAS SEQUENCES IN T-UNIFORM SIMPLICIAL COMPLEXES

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ABSTRACT. We introduce t-uniform simplicial complexes and we show that the lengths of spheres in such complexes are the terms of certain Lucas sequences. We find optimal constants for the linear isoperimetric inequality in the hyperbolic case.

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1. INTRODUCTION

Isoperimetric inequalities relate the length of closed curves to the infimal area of the discs which they bound. Every closed loop of length L in the Euclidean plane bounds a disc whose area is less than $L^2/4\pi$, and this bound is optimal. Thus one has a quadratic isoperimetric inequality for loops in Euclidean space. In contrast, loops in real hyperbolic space satisfy a linear isoperimetric inequality: there is a constant C such that every closed loop of length L in hyperbolic space bounds a disc whose area is less than or equal to $C \cdot L$.

With a suitable notion of area, a geodesic space X is δ-hyperbolic if and only if loops in X satisfy a linear isoperimetric inequality (see [BH99], chapter $III.H$, page 417 and page 419). For loops in arbitrary CAT(0) spaces, however, there is a quadratic isoperimetric inequality (see [BH99], chapter $III.H$, page 414). Osajda introduced in [Osa15] a local combinatorial condition called 8-location implying Gromov hyperbolicity of the universal cover (see [Laz15]). A related curvature condition, called 5/9-condition, also implies Gromov hyperbolicity (see [Laz15b]). Both 8-located complexes and 5/9-complexes satisfy therefore, under the additional hypothesis of simply connectedness, a linear isoperimetric inequality.

One can also express curvature using a condition called local k-largeness which was introduced independently by Chepoi [Che00] (under the name of bridged complexes) and by Januszkiewicz-Swiatkowski [JŚ06]. A flag simplicial complex is locally k-large if its links do not contain essential loops of length less than k, $k \geq 4$. Cycles in systolic complexes satisfy a quadratic isoperimetric inequality (see [JŚ06]). In [Els09] explicit constants are provided presenting the optimal estimate on the area of a systolic disc. In systolic complexes the isoperimetric function for 2-spherical cycles (the so called second isoperimetric function) is linear (see [JŚ07]). In [CCHO14] it is shown that meshed graphs (thus, in particular, weakly modular graphs) satisfy a quadratic isoperimetric inequality.

The purpose of the current paper is to show that the lengths of spheres in t-uniform simplicial complexes are the terms of certain Lucas sequences. Using

this result we find a connection between the area and the length of spheres in such complexes. For $t \geq 7$, we find optimal constants for the linear isoperimetric inequality in terms of t. We also study t-uniform simplicial complexes for $t \leq 6$.

We consider certain loops called spheres which are the "roundest" loops. Intuitively, any loop of the same length which is not a sphere contains less area. Therefore we consider only the lengths of spheres, not of all loops. By this choice we can find the best constant for the isoperimetric inequality. So we do not prove the isoperimetric inequality for the simplicial complex. Instead we find the best candidate for the constant of the isoperimetric inequality. The idea is to get a better understanding of how the isoperimetric inequality behaves as the vertices have more or less neighbours.

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2. Preliminaries

2.1. **Simplicial complexes.** Let X be a simplicial complex. We denote by $X^{(k)}$ the k-skeleton of $X, 0 \leq k < \dim X$. A subcomplex L in X is called *full* as a subcomplex of X if any simplex of X spanned by a set of vertices in L, is a simplex of L. For a set $A = \{v_1, ..., v_k\}$ of vertices of X, by $\langle A \rangle$ or by $\langle v_1, ..., v_k \rangle$ we denote the *span* of A, i.e. the smallest full subcomplex of X that contains A. We write $v \sim v'$ if $\langle v, v' \rangle \in X$ (it can happen that $v = v'$). We write $v \nsim v'$ if $\langle v, v' \rangle \notin X$. We call X *flag* if any finite set of vertices which are pairwise connected by edges of X, spans a simplex of X.

A *cycle (loop)* γ in X is a subcomplex of X isomorphic to a triangulation of S^1. A *full cycle* in X is a cycle that is full as a subcomplex of X. A *k-wheel* in X $(v_0; v_1, ..., v_k)$ (where $v_i, i \in \{0, ..., k\}$ are vertices of X) is a subcomplex of X such that $(v_1, ..., v_k)$ is a full cycle and $v_0 \sim v_1, ..., v_k$. The *length* of γ (denoted by $|\gamma|$) is the number of edges of γ.

We define the *metric* on the 0-skeleton of X as the number of edges in the shortest 1-skeleton path joining two given vertices.

Let σ be a simplex of X. The *link* of X at σ, denoted X_σ, is the subcomplex of X consisting of all simplices of X which are disjoint from σ and which, together with σ, span a simplex of X. A *full cycle* in X is a cycle that is full as subcomplex of X. We call a flag simplicial complex *k-large* if there are no full j-cycles in X, for $j < k$. We say X is *locally k-large* if all its links are k-large. We call a vertex of X *k-large* if its link is k-large.

Definition 2.1. A *simplicial map* $f : X \to Y$ between simplicial complexes X and Y is a map which sends vertices to vertices, and whenever vertices $v_0, ..., v_k \in X$ span a simplex σ of X then their images span a simplex τ of Y and we have $f(\sigma) = \tau$. Therefore a simplicial map is determined by its values on the vertex set of X. A simplicial map is called *nondegenerate* if it is injective on each simplex.

Definition 2.2. Let γ be a cycle in X. A *filling diagram* for γ is a simplicial map $f : D \to X$ where D is a triangulated 2-disc, and $f|_{\partial D}$ maps ∂D isomorphically onto γ. We denote a filling diagram for γ by (D, f) and we say it is

- *minimal* if D has minimal area (it consists of the least possible number of 2-simplices among filling diagrams for γ);
- *nondegenerate* if f is a nondegenerate map;

- *locally k-large* if D is a locally k-large simplicial complex.

Lemma 2.1. *Let X be a simplicial complex and let γ be a homotopically trivial loop in X. Then:*

(1) *there exists a filling diagram (D, f) for γ (see* [Che00] *- Lemma* 5.1, [JŚ06] *- Lemma* 1.6 *and* [Pry14] *- Theorem* 2.7*);*

(2) *any minimal filling diagram for γ is simplicial and nondegenerate (see* [Che00] *- Lemma* 5.1, [JŚ06] *- Lemma* 1.6, *Lemma* 1.7 *and* [Pry14] *- Theorem* 2.7*).*

Let D be a simplicial disc. We denote by C the cycle bounding D and by AreaC the area of D. We denote by V_i and V_b the numbers of internal and boundary vertices of D, respectively. Then: AreaC $= 2V_i + V_b - 2 = |C| + 2(V_i - 1)$ (Pick's formula). In particular, the area of a simplicial disc depends only on the numbers of its internal and boundary vertices.

Definition 2.3. Let X be a simplicial complex and let γ be a homotopically trivial loop in X. We call X *t-uniform*, $t \geq 4$, if any minimal filling diagram (D, f) for γ satisfies the following conditions:

(1) for any interior vertex v of D, we have $|D_v| = t$ (i.e. any interior vertex v of D has t neighbours);

(2) each boundary vertex of D is connected either to one interior vertex or to two adjacent interior vertices.

Let X be a t-uniform simplicial complex and let γ be a homotopically trivial loop in X. Let (D, f) be a minimal filling diagram for γ and let v be a vertex of D. We call the *sphere* centered at v of radius n the set of edges spanned by the vertices at distance n from v, $n \geq 0$. We denote it by S_n^t. We call the *area* of a sphere the number of triangles inside the sphere. We denote it by A_n^t. We call the *length* of a sphere the number of edges on the sphere. We denote it by $|S_n^t|$.

If Z is a set of vertices, we denote by $|Z|$ the number of its vertices.

2.2. **Lucas sequences.**

2.2.1. *General considerations.* Given two integer parameters P and Q, the Lucas sequences of the first kind $(U_n(P, Q))_{n \geq 0}$ and of the second kind $(V_n(P, Q))_{n \geq 0}$ are defined by the following recurrence relations (see [Rib00])

- $U_0(P, Q) = 0$,
- $U_1(P, Q) = 1$,
- $U_n(P, Q) = P \cdot U_{n-1}(P, Q) - Q \cdot U_{n-2}(P, Q)$, for $n > 1$

and

- $V_0(P, Q) = 2$,
- $V_1(P, Q) = P$,
- $V_n(P, Q) = P \cdot V_{n-1}(P, Q) - Q \cdot V_{n-2}(P, Q)$, for $n > 1$.

The characteristic equation of the recurrence relation for the Lucas sequences $U_n(P, Q)$ and $V_n(P, Q)$ is

$$x^2 - P \cdot x + Q = 0$$

It has the discriminant $D = P^2 - 4Q$ and the roots

$$a = \frac{P + \sqrt{P^2 - 4Q}}{2}, \quad b = \frac{P - \sqrt{P^2 - 4Q}}{2}$$

We discuss two cases.

- If $D \neq 0$, then a and b are distinct and we have

$$a^n = \frac{V_n + U_n\sqrt{D}}{2}, \quad b^n = \frac{V_n - U_n\sqrt{D}}{2}$$

Then the terms of the Lucas sequences can be expressed in terms of a and b as follows

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = a^n + b^n \tag{1}$$

- If $D = 0$, then $P = 2S$ and $Q = S^2$ for some integer S so that $a = b = S$. In this case we have

$$U_n(P, Q) = U_n(2S, S^2) = nS^{n-1} \tag{2}$$
$$V_n(P, Q) = V_n(2S, S^2) = 2S^n$$

2.2.2. *Fibonacci sequence.* The Fibonacci sequence $(F)_{n\geq 0}$ is a special case of Lucas sequence of the first kind for $P = 1$, $Q = -1$: $F_n = U_n(1, -1)$. The recurrence relation is

$$F_n = F_{n-1} + F_{n-2}$$

The initial terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$

The limit of the ratio of two successive terms of the Fibonacci sequence is the *golden ratio*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1 + \sqrt{5}}{2}$$

2.2.3. *The bisection of the Fibonacci sequence.* The bisection of the Fibonacci sequence $(B)_{n\geq 0}$ contains the terms on even positions of the Fibonacci sequence. It is the Lucas sequence of the first kind for $P = 3$, $Q = 1$: $B_n = U_n(3, 1)$. The recurrence relation is

$$B_n = 3 \cdot B_{n-1} - B_{n-2}$$

The initial terms of the bisection of the Fibonacci sequence are $0, 1, 3, 8, 21, 55, ...$

Expressed in terms of the Fibonacci sequence, we have $B_n = F_{2n}$.

3. Lucas sequences in simplicial complexes

The main goal of this section is to find, in terms of t, the best constant for the isoperimetric inequality for a t-uniform simplicial complex X, $t \geq 6$. The only loops we consider are the spheres inside the disc of a minimal filling diagram associated to a homotopically trivial loop of X. Namely, we compute for $t \geq 7$ the limit of the ratio $\dfrac{A_n^t}{|S_n^t|}$ as n goes to infinity (Theorem 3.4). For $t = 6$ we show that the ratio $\dfrac{A_n^t}{|S_n^t|^2}$ is constant (Theorem 3.5).

In section 3.1, we give examples of t-uniform simplicial complexes. In section 3.2 we find the relation between the lengths of spheres in t-uniform simplicial complexes and the terms of certain Lucas sequences (Theorem 3.1). Moreover, we express the area of a sphere in terms of the lengths of certain spheres (Theorem 3.3). In section 3.3 we study the bisection of the Fibonacci sequence in 7-uniform simplicial complexes. In section 3.4 we analyze Lucas sequences for $Q = 1$ in t-uniform simplicial complexes, $t \geq 4$. In section 3.5 we present tables of sequences involving lengths and areas of spheres in t-uniform simplicial complexes, $4 \leq t \leq 10$.
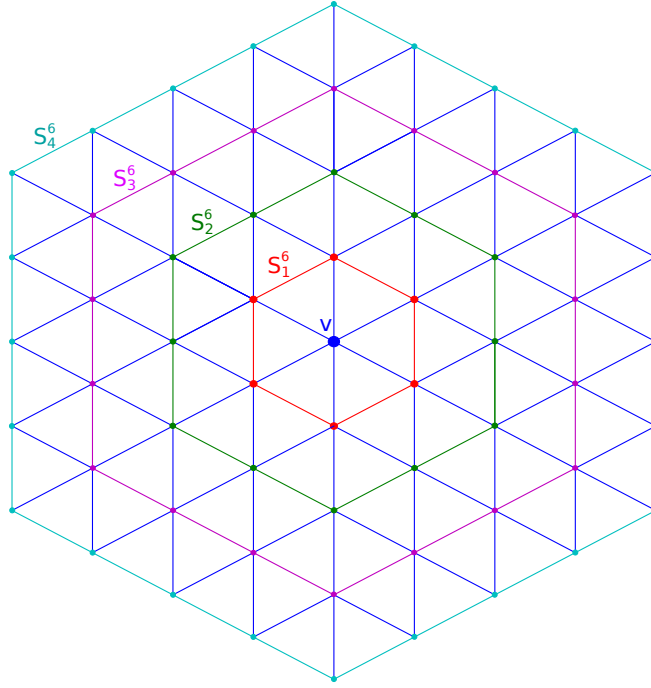
FIGURE 1. A minimal filling diagram of a 6-uniform simplicial complex

3.1. **Examples.** We start by presenting a few examples of t-uniform simplicial complexes. Namely,

- for $t = 4$: an octahedron (Figure 5);
- for $t = 5$: an icosahedron (Figure 6);
- for $t = 6$: the regular tessellation of the Euclidean plane by equilateral triangles (Figure 1);
- for $t \geq 7$: the complex is hyperbolic (Figures 2 and 3);
    - if all triangles are equilateral with an angle measuring $\pi/3$ at each vertex, then the sum of the measures of the angles around each vertex is bigger than 2π;
    - if the sum of the measures of the angles around each vertex is equal to 2π, then each triangle has angles of measure less then $\pi/3$ at each vertex.

3.2. **Lucas sequences in t-uniform simplicial complexes.** We start by establishing, for $t \geq 6$, a connection between the lengths of spheres in t-uniform simplicial complexes and the terms of certain Lucas sequences.

Theorem 3.1. *Let $t \geq 6$ and let X be a t-uniform simplicial complex. Let γ be a homotopically trivial loop of X and let (D, f) be a minimal filling diagram for γ. Let v be an interior vertex of D. Then the lengths of the spheres centered at v are Lucas sequences of the first kind U_n with parameters $P = t - 4$ and $Q = 1$ multiplied by t. Namely,*

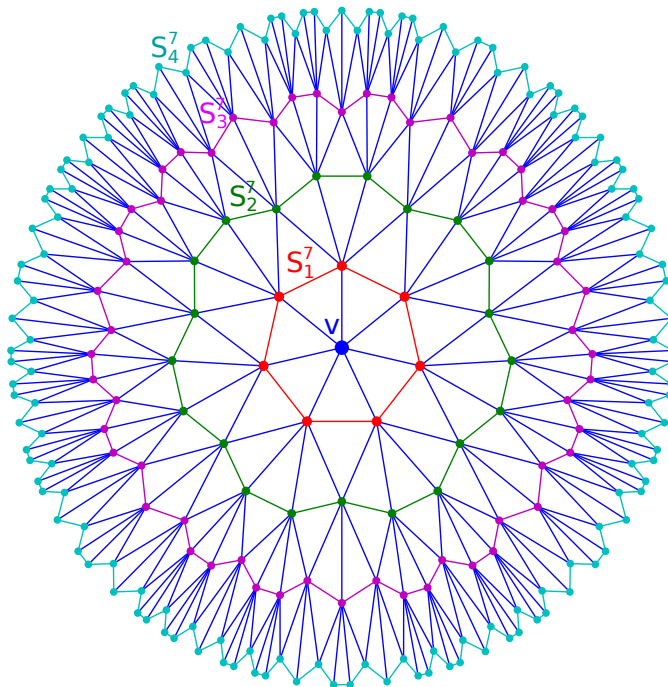$$|S_n^t| = t \cdot U_n(t - 4, 1) \tag{3}$$

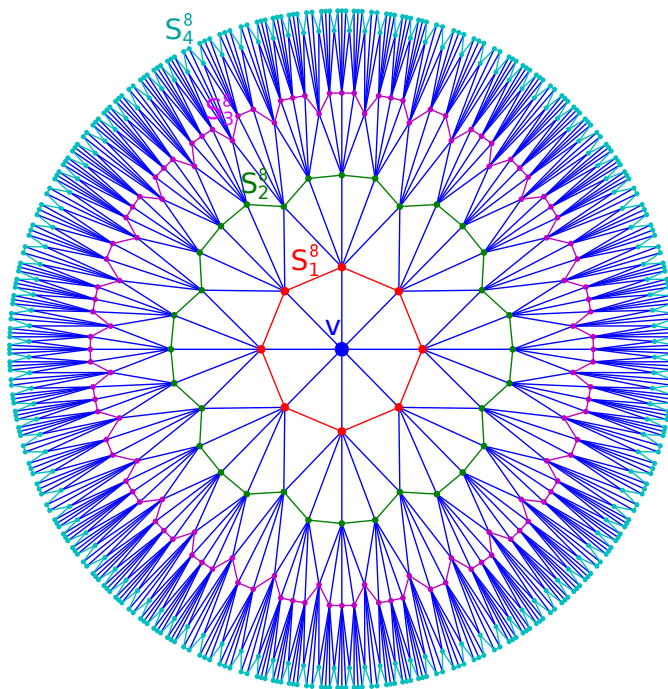FIGURE 2. A minimal filling diagram of a 7-uniform simplicial complex

FIGURE 3. A minimal filling diagram of an 8-uniform simplicial complex

Proof. We consider consecutive spheres of D centered at the same vertex v. We count the number of vertices on each sphere. In general the length of a sphere equals the number of vertices on the sphere. The sphere made of a single vertex, however, is an exception (i.e. the sphere S_0 at distance 0). Namely, although the length of the sphere is 0, the number of its vertices is equal to 1. Let $n \geq 2$. In order to count the number of vertices on the sphere S_n^t, we split the vertices on S_{n-1}^t into two sets:

- the set Y_{n-1}^t contains those vertices connected to two interior vertices on S_{n-2}^t which are adjacent,
- the set Z_{n-1}^t contains those vertices connected to one interior vertex on S_{n-2}^t.

The vertices from the set Y_{n-1}^t are connected to two interior vertices on S_{n-2}^t, to two vertices on the same sphere S_{n-1}^t, and, because the complex is t-uniform, to other $t-4$ vertices on the exterior sphere S_n^t. The number of these vertices is equal to the number of edges on the sphere S_{n-2}^t. We note that each vertex in Y_{n-1}^t corresponds to an edge on S_{n-2}^t. So $|Y_{n-1}^t| = |S_{n-2}^t|$. The vertices from the set Z_{n-1}^t are connected to one interior vertex on S_{n-2}^t, to two vertices on S_{n-1}^t, and, because the complex is t-uniform, to other $t-3$ vertices on the exterior sphere S_n^t. Any two vertices spanning an edge on S_{n-1}^t are connected to the same vertex on S_n^t. Therefore, in order to obtain the number of vertices on S_n^t, we have to count one vertex less for each vertex on S_{n-1}^t.

In conclusion, for $n \geq 2$, the number of vertices on S_n^t is equal to

$$\begin{aligned}|S_n^t| &= [(t-4) - 1] \cdot |Y_{n-1}^t| + [(t-3) - 1] \cdot |Z_{n-1}^t| = \\ &= (t-5) \cdot |Y_{n-1}^t| + (t-4) \cdot |Z_{n-1}^t| = \\ &= (t-4) \cdot (|Y_{n-1}^t| + |Z_{n-1}^t|) - |Y_{n-1}^t| = \\ &= (t-4) \cdot |S_{n-1}^t| - |S_{n-2}^t|\end{aligned}$$

The initial terms are $|S_0^t| = 0$ and $|S_1^t| = t$. This implies that the lengths of spheres in t-uniform simplicial complexes are Lucas sequences of the first kind U_n with parameters $P = t-4$ and $Q = 1$ multiplied by t: $|S_n^t| = t \cdot U_n(t-4, 1)$. ∎

Next we express the area between two consecutive spheres in terms of the lengths of these spheres. We denote by $A_{k-1,k}^t$ the area between two consecutive spheres S_{k-1}^t and S_k^t centered at v.

Lemma 3.2. *Let X be a t-uniform simplicial complex and let γ be a homotopically trivial loop in X. Let (D, f) be a minimal filling diagram for γ. Let v be an interior vertex of D. Then the area between two consecutive spheres S_{k-1}^t and S_k^t centered at v is equal to the sum of the lengths of these spheres*

$$A_{k-1,k}^t = |S_{k-1}^t| + |S_k^t| \tag{4}$$

Proof. Assume that S_{k-1}^t has r vertices (u_0, u_1, ..., u_{r-1}), S_k^t has s vertices (v_0, v_1, ..., v_{s-1}), $u_0 \sim v_0$, and that the indices of the two sequences of vertices increase in the same direction. We start counting the edges and the triangles between the spheres with the edge $\langle u_0, v_0 \rangle$ (see Figure 4). This edge is included either in the triangle $\langle u_0, v_0, v_1 \rangle$ which has an edge on S_k^t, or in the triangle $\langle u_0, v_0, u_1 \rangle$ which has an edge on S_{k-1}^t. The figure below illustrates only the first case. We continue the counting either with $\langle u_0, v_1 \rangle$ or with $\langle u_1, v_0 \rangle$. This edge belongs to a triangle,
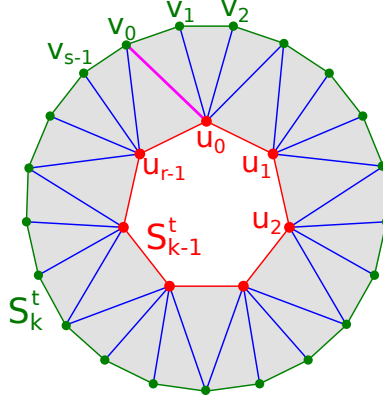
FIGURE 4. Area between two consecutive spheres centered at the
same vertex

which has an edge on S_{k-1}^t or on S_k^t and another edge between vertices on the
two spheres. The counting is complete once we return to the edge $\langle u_0, v_0 \rangle$. For
each edge joining vertices on both spheres S_{k-1}^t and S_k^t, we count one triangle. In
conclusion the number of triangles between the spheres S_{k-1}^t and S_k^t is equal to
the sum of the number of edges on both spheres. The number of these triangles
represents the area $A_{k-1,k}^t$. ∎

Theorem 3.3. *Let X be a t-uniform simplicial complex and let γ be a homotopically
trivial loop in X. Let (D, f) be a minimal filling diagram for γ. Let v be an interior
vertex of D. Then the area of a sphere centered at v is equal to*

$$A_n^t = 2 \left(\sum_{k=0}^{n} |S_k^t| \right) - |S_n^t| \tag{5}$$

Proof. The area of a sphere is equal to the sum of the areas between each pair of
consecutive spheres around v. Namely,

$$A_n^t = \sum_{k=1}^{n} A_{k-1,k}^t$$

Therefore, using (4), we can express this area in terms of the lengths of the spheres
around v as follows:

$$A_n^t = \sum_{k=1}^{n} A_{k-1,k}^t = \sum_{k=1}^{n} (|S_{k-1}^t| + |S_k^t|) = 2 \left(\sum_{k=0}^{n} |S_k^t| \right) - |S_n^t|$$

∎

3.3. The bisection of the Fibonacci sequence in 7-uniform simplicial complexes. As a particular case, relation (3) implies that spheres in a 7-uniform simplicial complex are the terms of the bisection of the Fibonacci sequence multiplied
by 7. Namely,

$$|S_n^7| = 7 \cdot U_n(3, 1) = 7 \cdot B_n = 7 \cdot F_{2n} \tag{6}$$

The sum of the first $n+1$ elements from the bisection of the Fibonacci sequence is equal to

$$\sum_{k=0}^{n} B_k = \sum_{k=0}^{n} F_{2k} =$$

$$= F_0 + F_2 + F_4 + ... + F_{2n} =$$

$$= F_1 + F_2 + F_4 + ... + F_{2n} - F_1 =$$

$$= F_3 + F_4 + F_6 + ... + F_{2n} - 1 =$$

$$= F_5 + F_6 + ... + F_{2n} - 1 = ... =$$

$$= F_{2n+1} - 1 \tag{7}$$

Then the relations (6) and (7) imply that

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n} |S_k^7|}{|S_n^7|} = \lim_{n \to \infty} \frac{7 \cdot \sum_{k=0}^{n} F_{2k}}{7 \cdot F_{2n}} = \lim_{n \to \infty} \frac{F_{2n+1} - 1}{F_{2n}} = \varphi \tag{8}$$

From (5) and (8), we get the limit, as n goes to infinity, of the ratio between the area and the length of spheres in 7-uniform simplicial complexes. Namely,

$$\lim_{n \to \infty} \frac{A_n^7}{|S_n^7|} = \lim_{n \to \infty} \left(2 \cdot \frac{\sum_{k=0}^{n} |S_k^7|}{|S_n^7|} - \frac{|S_n^7|}{|S_n^7|} \right) =$$

$$= 2\varphi - 1 = 2 \cdot \frac{1 + \sqrt{5}}{2} - 1 = \sqrt{5} \tag{9}$$

As shown below (Theorem 3.4), the sequence $\left(\frac{A_n^7}{|S_n^7|} \right)_{n \geq 0}$ is strictly increasing. Therefore for spheres in 7-uniform simplicial complexes, the following inequality holds:

$$A_n^7 < \sqrt{5} \cdot |S_n^7| \tag{10}$$

As above we split the vertices on each sphere S_n^7 into two sets:

- the set Y_n^7 contains the vertices connected to two interior vertices on S_{n-1}^7 which are adjacent,
- the set Z_n^7 contains the vertices connected to one interior vertex on S_{n-1}^7.

We note that:

- $|Y_1^7| = 7 * 0$,
- $|Z_1^7| = 7 * 1$,
- $|Y_2^7| = 7 * 1$,
- $|Z_2^7| = 7 * 2$,
- $|Y_3^7| = 7 * 3$,
- $|Z_3^7| = 7 * 5$.

So it turns out that the sequence $(|Y_1^7|, |Z_1^7|, |Y_2^7|, |Z_2^7|, |Y_3^7|, |Z_3^7|, ...)$ is the Fibonacci sequence multiplied by 7.

3.4. **Lucas sequences for Q=1 in t-uniform simplicial complexes.** Using the result from Theorem 3.1, we continue studying Lucas sequences of the first kind $U_n(P, 1)$. The recurrence relation is

$$U_n(P, 1) = P \cdot U_{n-1}(P, 1) - U_{n-2}(P, 1)$$

The characteristic equation of the recurrence relation is

$$x^2 - P \cdot x + 1 = 0$$

It has the discriminant $D = P^2 - 4$ and the roots

$$a = \frac{P + \sqrt{P^2 - 4}}{2}, \quad b = \frac{P - \sqrt{P^2 - 4}}{2} \tag{11}$$

Thus $ab = 1$, and we have $b = \frac{1}{a}$.

We discuss two cases: either $P \neq 2$, or $P = 2$.

• If $P \neq 2$ then $D \neq 0$. Note that the roots a and b are distinct. In this case the terms of the Lucas sequence of the first kind given in (1) can be expressed only in terms of a. Namely,

$$U_n(P, 1) = \frac{a^n - \dfrac{1}{a^n}}{a - \dfrac{1}{a}} =$$

$$= \frac{a}{a^2 - 1} \cdot \frac{(a^n - 1)(a^n + 1)}{a^n}$$

If $P > 2$ then $D > 0$. In this case a and b are distinct real numbers and $a > 1 > b$.
If $P < 2$ then $D < 0$. In this case a and b are distinct complex numbers.
For $P > 2$ the sum of the first $n + 1$ elements of the Lucas sequence is

$$\sum_{k=0}^{n} U_k(P, 1) = \sum_{k=0}^{n} \frac{a}{a^2 - 1} \cdot \left(a^k - \frac{1}{a^k} \right) =$$

$$= \frac{a}{a^2 - 1} \cdot \left(\sum_{k=0}^{n} a^k - \sum_{k=0}^{n} \frac{1}{a^k} \right) =$$

$$= \frac{a}{a^2 - 1} \cdot \left(\frac{a^{n+1} - 1}{a - 1} - \frac{1 - \dfrac{1}{a^{n+1}}}{1 - \dfrac{1}{a}} \right) =$$

$$= \frac{a}{a^2 - 1} \cdot \left(\frac{a^{n+1} - 1}{a - 1} - \frac{a^{n+1} - 1}{(a - 1)a^n} \right) =$$

$$= \frac{a}{a^2 - 1} \cdot \frac{(a^{n+1} - 1)(a^n - 1)}{(a - 1)a^n}$$

Moreover, the sum of the first $n + 1$ elements of the Lucas sequence divided by the last element is

$$\frac{\sum_{k=0}^{n} U_k(P, 1)}{U_n(P, 1)} = \frac{a}{a^2 - 1} \cdot \frac{(a^{n+1} - 1)(a^n - 1)}{(a - 1)a^n} \cdot \frac{a^2 - 1}{a} \cdot \frac{a^n}{(a^n - 1)(a^n + 1)} =$$

$$= \frac{a^{n+1} - 1}{(a - 1)(a^n + 1)} \tag{12}$$

Since $a > 1$, the limit of the above expression when n goes to infinity is

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} U_k(P,1)}{U_n(P,1)} = \lim_{n\to\infty} \frac{a^{n+1}-1}{(a-1)(a^n+1)} =$$

$$= \lim_{n\to\infty} \frac{a^{n+1}-1}{a^{n+1}-a^n+a-1} =$$

$$= \lim_{n\to\infty} \frac{1 - \dfrac{1}{a^{n+1}}}{1 - \dfrac{1}{a} + \dfrac{1}{a^n} - \dfrac{1}{a^{n+1}}} =$$

$$= \frac{1}{1 - \dfrac{1}{a}} = \frac{a}{a-1}$$

Using (11), we can express the above result in terms of P as follows:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} U_k(P,1)}{U_n(P,1)} = \frac{\dfrac{P+\sqrt{P^2-4}}{2}}{\dfrac{P+\sqrt{P^2-4}}{2}-1} = \frac{P+\sqrt{P^2-4}}{P-2+\sqrt{P^2-4}} =$$

$$= \frac{(P-2-\sqrt{P^2-4})\cdot(P+\sqrt{P^2-4})}{(P-2)^2 - \sqrt{P^2-4}^2} =$$

$$= \frac{P(P-2)+(P-2)\sqrt{P^2-4}-P\sqrt{P^2-4}-\sqrt{P^2-4}^2}{P^2-4P+4-(P^2-4)} =$$

$$= \frac{P^2-2P-2\sqrt{P^2-4}-P^2+4}{-4P+8} =$$

$$= \frac{-2P+4-2\sqrt{P^2-4}}{-4(P-2)} =$$

$$= \frac{P-2+\sqrt{(P-2)(P+2)}}{2(P-2)} =$$

$$= \frac{1+\sqrt{\dfrac{P+2}{P-2}}}{2} \tag{13}$$

• If $P = 2$ then $D = 0$. Note that $a = b = 1$. Hence, $S = \dfrac{P}{2} = 1$. Thus, using (2), we get

$$U_n(2,1) = nS^{n-1} = n \tag{14}$$

Then the sum of the first $n+1$ elements of the Lucas sequence is

$$\sum_{k=0}^{n} U_k(2,1) = \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \tag{15}$$

Theorem 3.4. *In t-uniform simplicial complexes, $t \geq 7$, the following inequality holds:*

$$A_n^t < \sqrt{\frac{t-2}{t-6}} \cdot |S_n^t| \tag{16}$$

In particular, the sequence $\left(\dfrac{A_n^t}{|S_n^t|}\right)_{n \geq 0}$ is strictly increasing.

Proof. Based on (3), the lengths of spheres in a t-uniform simplicial complex are Lucas sequences of parameters $P = t - 4$ and $Q = 1$ multiplied by t. Therefore, using (13) we get the following

$$\lim_{n \to \infty} \frac{\sum_{k=0}^n |S_k^t|}{|S_n^t|} = \lim_{n \to \infty} \frac{\sum_{k=0}^n \left[t \cdot U_k(t-4,1)\right]}{t \cdot U_n(t-4,1)} = \frac{1 + \sqrt{\dfrac{t-2}{t-6}}}{2} \tag{17}$$

Using (5) and (17), for a sphere S_n^t we get the limit, when n goes to infinity, of the ratio between its area and its length. Namely,

$$\lim_{n \to \infty} \frac{A_n^t}{|S_n^t|} = \lim_{n \to \infty} \frac{2\left(\sum_{k=0}^n |S_k^t|\right) - |S_n^t|}{|S_n^t|} =$$

$$= \lim_{n \to \infty} 2 \cdot \frac{\sum_{k=0}^n |S_k^t|}{|S_n^t|} - 1 =$$

$$= 2 \cdot \frac{1 + \sqrt{\dfrac{t-2}{t-6}}}{2} - 1 = \sqrt{\frac{t-2}{t-6}}$$

Also, from (3), (5) and (12), for $n > 0$, we have

$$\frac{A_n^t}{|S_n^t|} - \frac{A_{n-1}^t}{|S_{n-1}^t|} = \frac{2\left(\sum_{k=0}^n |S_k^t|\right) - |S_n^t|}{|S_n^t|} - \frac{2\left(\sum_{k=0}^{n-1} |S_k^t|\right) - |S_{n-1}^t|}{|S_{n-1}^t|} =$$

$$= 2\left[\frac{7 \cdot \sum_{k=0}^n U_k(t-4,1)}{7 \cdot U_n(t-4,1)} - \frac{7 \cdot \sum_{k=0}^{n-1} U_k(t-4,1)}{7 \cdot U_{n-1}(t-4,1)}\right] =$$

$$= 2\left[\frac{a^{n+1} - 1}{(a-1)(a^n + 1)} - \frac{a^n - 1}{(a-1)(a^{n-1} + 1)}\right] =$$

$$= \frac{2}{a-1} \cdot \frac{a^{2n} + a^{n+1} - a^{n-1} - 1 - a^{2n} + 1}{(a^n + 1)(a^{n-1} + 1)}$$

$$= \frac{2(a^{n+1} - a^{n-1})}{(a-1)(a^n + 1)(a^{n-1} + 1)}$$

As $a > 1$, it follows that $\dfrac{A_n^t}{|S_n^t|} - \dfrac{A_{n-1}^t}{|S_{n-1}^t|} > 0$. So, the sequence $\left(\dfrac{A_n^t}{|S_n^t|}\right)_{n \geq 0}$ is strictly increasing.

So, for $t \geq 7$ in t-uniform simplicial complexes, we have $A_n^t < \sqrt{\dfrac{t-2}{t-6}} \cdot |S_n^t|$. ∎

We note that for 7-uniform complexes, Theorem 3.4 ensures that relation (10) is indeed fulfilled: $A_n^7 < \sqrt{5} \cdot |S_n^7|$.

Pick's formula implies that $A_n^t = |S_n^t| + 2 \cdot (V_i - 1)$. We have denoted by V_i the number of interior vertices of the disc enclosed by S_n^t. Because $V_i \geq 1$, for $n \geq 1$, we have $A_n^t \geq |S_n^t|$. Thus $\frac{A_n^t}{|S_n^t|} \geq 1$. When t goes to infinity, we get

$$\lim_{t \to \infty} \lim_{n \to \infty} \frac{A_n^t}{|S_n^t|} = \lim_{t \to \infty} \sqrt{\frac{t-2}{t-6}} = 1$$

Theorem 3.5. *In 6-uniform simplicial complexes the following equality holds:*

$$A_n^6 = \frac{|S_n^6|^2}{6} \tag{18}$$

Proof. Theorem 3.1 implies that $|S_n^6| = 6 \cdot U_n(2,1)$. Then using (5), (14) and (15), we get

$$\frac{A_n^6}{|S_n^6|^2} = \frac{2\left(\sum_{k=0}^{n} |S_k^6|\right) - |S_n^6|}{|S_n^6|^2} = \frac{2 \cdot 6 \cdot \left[\sum_{k=0}^{n} U_k(2,1)\right] - 6 \cdot U_n(2,1)}{6^2 \cdot [U_n(2,1)]^2} =$$

$$= \frac{2 \cdot \dfrac{n(n+1)}{2} - n}{6 \cdot n^2} = \frac{1}{6}$$

So we have $A_n^6 = \frac{|S_n^6|^2}{6}$. ∎

We present a few conjectures.

Conjecture 3.6. For $k \geq 7$, k-large simplicial complexes satisfy a linear isoperimetric inequality. Namely,

$$A < \sqrt{\frac{k-2}{k-6}} \cdot L$$

Conjecture 3.7. 6-large simplicial complexes satisfy a quadratic isoperimetric inequality. Namely,

$$A \leq \frac{L^2}{6}$$

We note that for $t = 4$ and $t = 5$, relation (3) holds only for the initial values. Namely, for $t = 4$ (Figure 5) let the octahedron be an example of 4-uniform simplicial complex. The largest possible sphere in a minimal filling diagram for a homotopically trivial loop in the complex is S_1^4. Each of the four vertices on S_1^4 are connected to 3 vertices (one is the central vertex; the other two are on the same sphere S_1^4). So, because the complex is 4-uniform, these vertices are connected to a single vertex on S_2^4. This implies that S_2^4 contains a single vertex. Such a situation is not possible in a minimal filling diagram because a disc is flat.

One can reason similarly for $t = 5$ (Figure 6). Let the icosahedron with a missing vertex be an example of 5-uniform simplicial complex. The largest possible sphere in a minimal filling diagram for a homotopically trivial loop in the complex is S_2^5. Each of the five vertices on S_1^5 are connected to three vertices (one is the central vertex; the other two are on the same sphere S_1^5). Because the complex is 5-uniform, each of these vertices is connected to two of the five vertices on S_2^5. The vertices on
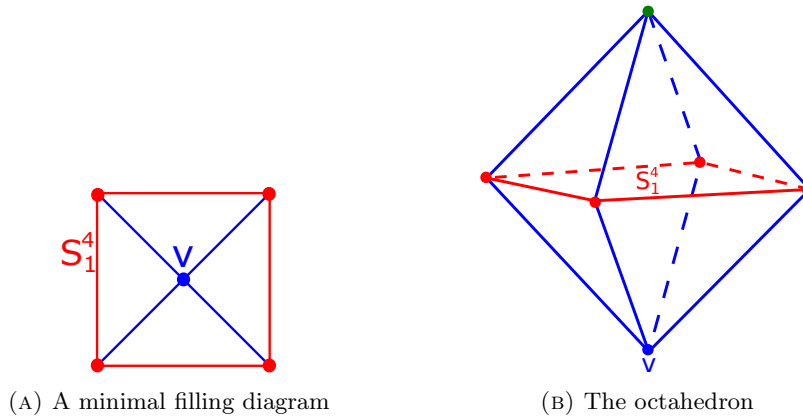
(A) A minimal filling diagram   (B) The octahedron

FIGURE 5. A 4-uniform simplicial complex

S_2^5 are connected to four vertices (two are on S_1^5; the other two are on S_2^5). Because the complex is 5-uniform, these vertices are connected to a single vertex on S_3^5. This implies that the sphere S_3^5 contains a single vertex. This is not possible in a minimal filling diagram because a disc is flat.

(A) A minimal filling diagram   (B) The icosahedron (with a missing vertex)
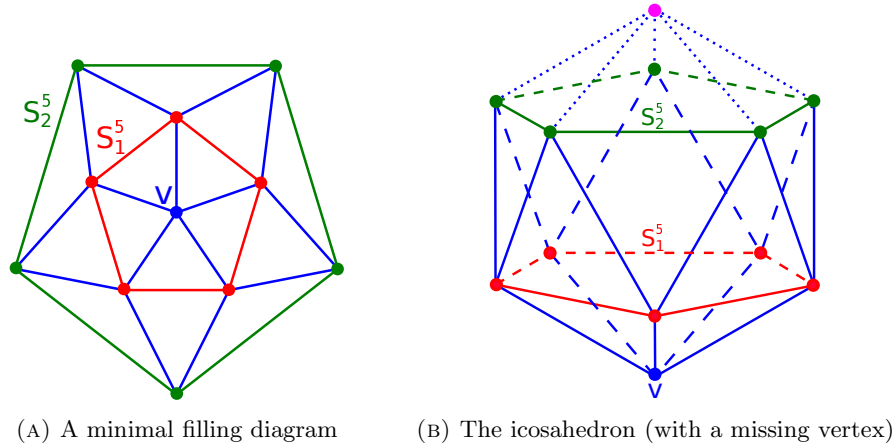
FIGURE 6. A 5-uniform simplicial complex

3.5. **Tables of sequences.** We end by computing sequences of lengths of spheres, sum of lengths of spheres, areas between spheres and areas inside spheres in t-uniform simplicial complexes. The values are divided by t to outline the relation between these sequences and other integer sequences. We introduce a table line to include approximations of the ratio between the area and the length of spheres. For $t = 6$, an extra line represents the quadratic equality between the area and the length of spheres. In the last table column we give links to some related entries in the On-Line Encyclopedia of Integer Sequences (OEIS).

We start by presenting the tables for $6 \leq t \leq 10$.

n	0	1	2	3	4	5	6	7	8
$\|S_n^6\|$ /6	0	1	2	3	4	5	6	7	8
$\sum_{k=0}^{n}\|S_k^6\|$ /6	0	1	3	6	10	15	21	28	36
$A_{n-1,n}^6$ /6	-	1	3	5	7	9	11	13	15
A_n^6 /6	0	1	4	9	16	25	36	49	64
$\dfrac{A_n^6}{\|S_n^6\|}$	-	1	2	3	4	5	6	7	8
$\dfrac{6\cdot A_n^6}{\|S_n^6\|^2}$	-	1	1	1	1	1	1	1	1

TABLE 1. Sequences in a 6-uniform simplicial complex

n	0	1	2	3	4	5	6	7	8	OEIS
$\|S_n^7\|$ /7	0	1	3	8	21	55	144	377	987	A001906
$\sum_{k=0}^{n}\|S_k^7\|$ /7	0	1	4	12	33	88	232	609	1596	A027941
$A_{n-1,n}^7$ /7	-	1	4	11	29	76	199	521	1364	A002878
A_n^7 /7	0	1	5	16	45	121	320	841	2205	A004146
$\dfrac{A_n^7}{\|S_n^7\|}$	-	1	1.(6)	2	2.14	2.2	2.(2)	2.23	2.234	$\to \sqrt{5}$

TABLE 2. Sequences in a 7-uniform simplicial complex

n	0	1	2	3	4	5	6	7	OEIS
$\|S_n^8\|$ /8	0	1	4	15	56	209	780	2911	A001353
$\sum_{k=0}^{n}\|S_k^8\|$ /8	0	1	5	20	76	285	1065	3976	A061278
$A_{n-1,n}^8$ /8	-	1	5	19	71	265	989	3691	A001834
A_n^8 /8	0	1	6	25	96	361	1350	5041	A092184
$\dfrac{A_n^8}{\|S_n^8\|}$	-	1	1.5	1.(6)	1.71	1.72	1.73	1.731	$\to \sqrt{3}$

TABLE 3. Sequences in an 8-uniform simplicial complex

n	0	1	2	3	4	5	6	OEIS
$\lvert S_n^9 \rvert$ /9	0	1	5	24	115	551	2640	A004254
$\sum_{k=0}^n \lvert S_k^9 \rvert$ /9	0	1	6	30	145	696	3336	A089817
$A_{n-1,n}^9$ /9	-	1	6	29	139	666	3191	A030221
A_n^9 /9	0	1	7	36	175	841	4032	A054493
$\dfrac{A_n^9}{\lvert S_n^9 \rvert}$	-	1	1.4	1.5	1.52	1.526	1.527	$\to \sqrt{\dfrac{7}{3}}$

TABLE 4. Sequences in a 9-uniform simplicial complex

n	0	1	2	3	4	5	OEIS
$\lvert S_n^{10} \rvert$ /10	0	1	6	35	204	1189	A001109
$\sum_{k=0}^n \lvert S_k^{10} \rvert$ /10	0	1	7	42	246	1435	A053142
$A_{n-1,n}^{10}$ /10	-	1	7	41	239	1393	A002315
A_n^{10} /10	0	1	8	49	288	1681	A001108
$\dfrac{A_n^{10}}{\lvert S_n^{10} \rvert}$	-	1	1.(3)	1.4	1.41	1.413	$\to \sqrt{2}$

TABLE 5. Sequences in a 10-uniform simplicial complex

Next we present the tables for $t = 4$ and $t = 5$. For $t = 4$ the only valid values for n are 0 and 1. For $t = 5$ the only valid values for n are 0, 1 and 2. Larger values for n are not valid. Still we present them to see what values we would get in case such spheres would exist. We note that the lengths of spheres would also be the terms of certain Lucas sequences.

n	0	1	2	3	4	5	6	7	8
$\lvert S_n^4\rvert$ /4	0	1	0	-1	0	1	0	-1	0
$\sum_{k=0}^{n}\lvert S_k^4\rvert$ /4	0	1	1	0	0	1	1	0	0
$A_{n-1,n}^4$ /4	-	1	1	-1	-1	1	1	-1	-1
A_n^4 /4	0	1	2	1	0	1	2	1	0
$\dfrac{A_n^4}{\lvert S_n^4\rvert}$	-	1	∞	-1	-	1	∞	-1	-

TABLE 6. Sequences in a 4-uniform simplicial complex

n	0	1	2	3	4	5	6	7	8
$\lvert S_n^5\rvert$ /5	0	1	1	0	-1	-1	0	1	1
$\sum_{k=0}^{n}\lvert S_k^5\rvert$ /5	0	1	2	2	1	0	0	1	2
$A_{n-1,n}^5$ /5	-	1	2	1	-1	-2	-1	1	2
A_n^5 /5	0	1	3	4	3	1	0	1	3
$\dfrac{A_n^5}{\lvert S_n^5\rvert}$	-	1	3	∞	-3	-1	-	1	3

TABLE 7. Sequences in a 5-uniform simplicial complex

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